Extended q-Gaussian and q-exponential distributions from gamma random variables

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The family of q-Gaussian and q-exponential probability densities fit the statistical behavior of diverse complex self-similar nonequilibrium systems. These distributions, independently of the underlying dynamics, can rigorously be obtained by maximizing Tsallis "nonextensive" entropy under appropriate constraints, as well as from superstatistical models. In this paper we provide an alternative and complementary scheme for deriving these objects. We show that q-Gaussian and q-exponential random variables can always be expressed as a function of two statistically independent gamma random variables with the same scale parameter. Their shape index determines the complexity q parameter. This result also allows us to define an extended family of asymmetric q-Gaussian and modified q-exponential densities, which reduce to the standard ones when the shape parameters are the same. Furthermore, we demonstrate that a simple change of variables always allows relating any of these distributions with a beta stochastic variable. The extended distributions are applied in the statistical description of different complex dynamics such as log-return signals in financial markets and motion of point defects in a fluid flow.

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I. INTRODUCTION

Long-range interparticle interaction, long-term microscopic or mesoscopic memory, fractal or multifractal occupation in phase space, cascade transfer of energy or information, and intrinsic fluctuations of some dynamical system parameters are some of the properties that nowadays are related with complexity. One of the emergent properties related with these phenomena is the power-law statistics of the corresponding nonequilibrium states. While there exist different underlying formalisms for tackling these issues, maximization of *Tsallis "nonextensive" entropy* [1–4] provides an alternative basis over which complexity can be analyzed and studied in a broad class of systems. Introducing a generalized second moment constraint [3,4], this formalism leads to a generalization of standard normal probability densities, known as q-Gaussian distributions. In terms of a generalized exponential function, those densities can be written as $P(x)=(\sqrt{\beta}/\mathcal{N}_q)\exp_q(-\beta x^2)$, where the parameter $q\in$ Re defines different complexity classes. Explicitly, these statistical objects read

$$P(x)=\frac{\sqrt{\beta}}{\mathcal{N}_q}[1-(1-q)\beta x^2]^{1/(1-q)},\quad -\infty<q<1,\quad (1)$$

where the variable of interest x is restricted to the domain $0\leqslant(1-q)\beta x^2\leqslant 1$. On the other hand,

$$P(x)=\frac{\sqrt{\beta}}{\mathcal{N}_q}\left[\frac{1}{1+(q-1)\beta x^2}\right]^{1/(q-1)},\quad 1<q<3,\quad (2)$$

where now x is allowed to run over the real line. The restriction $q<3$ follows from the normalization condition $\int_{-\infty}^{+\infty}dx\,P(x)=1$, which is guaranteed by the dimensionless constant $\mathcal{N}_q$. The parameter $\sqrt{\beta}$ measures the width of the distributions. As is well known [3], in the limit $q\to 1$ both expressions reduce to the standard Gaussian distribution. These generalizations allow one to describe variables restricted to a finite domain [Eq. (1)] as well as power-law statistics [Eq. (2)].

q-Gaussian distributions also arise as solutions of nonlinear Fokker-Planck equations [5,6] as well as in the formulation of central limit theorems with highly correlated random variables [7]. Furthermore, they fulfill a generalized fluctuation relation symmetry [8]. A wide class of systems obeys their statistics [3], such as in fluid flows [9–11], optical lattices [12], trapped ions interacting with a classical gas [13], in granular mixtures [14], anomalous diffusion in dusty plasma [15] or cellular aggregates [16], avalanche sizes in earthquakes [17], in astrophysical variables [18], as well as in econophysics [19–23].

When introducing a first moment constraint, Tsallis entropy leads to a q-exponential distribution [Eqs. (1) and (2) under the replacement $x^2\to x$ with $-\infty<q<1$ and $1<q<2$, respectively], which in the limit $q\to 1$ recovers the standard exponential probability density of a positive random variable. These densities, for example, allow one to fit high energy collisions [24], quark matter statistics [25], solar flares [26], and momentum distributions of charged hadrons [27]. q-exponential functions also fit anomalous power-law dipolar relaxation [28] as well as spin-glass relaxation [29]. More recently, a kind of generalized q-gamma probability density was introduced for describing stock trading volume flow in financial markets [30–32].

It is remarkable that all quoted probability densities can also be obtained from a *superstatistical* modeling [9], where a parameter of an underlying probability measure becomes a (positive) random variable characterized by a gamma distribution [33–36]. This is the case of q-Gaussian densities, where the underlying distribution is a normal one [9], while for q-exponential densities it is an exponential function [24]. For generalized q-gamma variables the underlying distribution is a gamma density, while the random parameter is distributed according to an inverse gamma distribution [30].

The main goal of this paper is to present a complementary and alternative scheme to those provided by entropy extremization and superstatistics. We show that random variables described by any of the quoted families of q distributions can be written as a function of two independent (positive) random gamma variables [33–36]. Their scale parameter is assumed the same, while their shape indexes determine the complexity q parameter. When the shape indexes are different, a class of extended asymmetric q-gaussian and modified q-exponential distributions are obtained. Generation of q-distributed random numbers is straightforward from these results [37,38]. We also show that simple transformation of variables allow relating any of the obtained densities with a beta distribution. Interestingly, this statistical function has been applied to model a wide variety of problems arising in different disciplines [33–35]. On the other hand, a q-triplet [39,40] for the probabilities densities is obtained. The usefulness of the extended distributions in the context of financial signals [21] and motion of point defects in fluid flows [11] is demonstrated. These systems are characterized by highly asymmetric distributions. This feature is absent in previous approaches, being recovered by the present one.

The paper is organized as follows. In Sec. II we review the properties of gamma random variables and introduce the main assumption over which the present scheme relies. In Sec. III asymmetric q-gaussian distributions are obtained, while Sec. IV is devoted to modified q-exponential densities. In Sec. V the properties of the proposed scheme as well as its applications are discussed. In Sec. VI the Conclusions are provided.

II. MODEL

A stochastic random variable Y is gamma distributed [33–36] if its probability density is

$$P(y) = y^{\alpha-1}\frac{e^{-y/\theta}}{\theta^{\alpha}\Gamma(\alpha)}, \quad y > 0, \quad 0 < \alpha < \infty, \tag{3}$$

where $\Gamma(\alpha)$ is the gamma function. This distribution is characterized by the scale parameter $\theta > 0$ and its shape parameter α. In terms of these parameters its mean value reads $\langle Y\rangle = \int_0^\infty yP(y)dy = \alpha\theta$, with variance $\mathrm{var}[Y] = \langle Y^2\rangle - \langle Y\rangle^2 = \alpha\theta^2$. The underlying stochastic process that leads to this statistic involves a cascadelike mechanism [33–36]. In fact, this property is evident from the Laplace transform $P(u) = [\theta^{-1}/(u+\theta^{-1})]^{\alpha}$ where $P(u) = \int_0^\infty dt P(t)e^{-ut}$. Hence, when α is natural it reduces to a convolution of exponential functions, which can be read as a cascade of consecutive random steps.

The present approach relies on two independent gamma random variables Y_1, Y_2. Their joint probability density then reads

$$P(y_1,y_2) = y_1^{\alpha-1}y_2^{\alpha'-1}\frac{e^{-(y_1+y_2)/\theta}}{\theta^{\alpha+\alpha'}\Gamma(\alpha)\Gamma(\alpha')}. \tag{4}$$

Here, we assumed that both scale parameters θ are the same, while α and α' are the shape parameters of Y_1 and Y_2, respectively. The main ingredient of the present scheme is the ansatz

$$X = f(Y_1,Y_2), \tag{5}$$

where the new random variable X, depending on the function $f(y_1,y_2)$, develops different statistics. We will show that a wide class of q distributions arises from nonlinear functions, which in turn are asymmetric in their arguments (see Secs. III and IV). Nevertheless, in all cases they fulfill a very simple symmetry (see Sec. V).

The probability distribution of X is completely determined by the joint probability (4) and the function $f(y_1,y_2)$. In fact, it follows from an elementary change of variables [36]. For closing the problem, we introduce an extra random variable Z defined by the addition

$$Z = (Y_1 + Y_2). \tag{6}$$

Therefore, the joint probability of X and Z is given by

$$P(x,z) = P(y_1,y_2)|\det(J)|, \tag{7}$$

where J is the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial z} \\ \frac{\partial y_2}{\partial x} & \frac{\partial y_2}{\partial z} \end{pmatrix}. \tag{8}$$

The probability of each variable follows by partial integration

$$P(x) = \int_0^\infty dz P(x,z), \quad P(z) = \int_{-\infty}^\infty dx P(x,z). \tag{9}$$

As Z is defined by the addition of two independent gamma variables with the same scale factor, it follows

$$P(z) = z^{\alpha+\alpha'-1}\frac{e^{-z/\theta}}{\theta^{\alpha+\alpha'}\Gamma(\alpha+\alpha')}. \tag{10}$$

Hence, Z is also a gamma variable [$Z > 0$; see Eq. (3)] where its shape index is $(\alpha+\alpha')$ [33–36].

III. ASYMMETRIC *q*-GAUSSIAN DISTRIBUTIONS

In order to motivate the election of the function $f(y_1,y_2)$ that leads to q-Gaussian statistics we may think (in a rough way) about a Brownian particle that interacts with a complex reservoir. Y_1 and Y_2 are the (positive and negative) impulse moments induced by the bath fluctuations. In addition, the complexity of the system-environment interaction is taken into account by a system response function $M^{-1}(Y_1,Y_2)$ that depends on both Y_1 and Y_2. Therefore, this contribution can be read as a random-dissipative-like mechanism. The particle fluctuation is finally written as $X \approx (Y_1 - Y_2)/M(Y_1,Y_2)$. One may also think about an economical agent that, from the available information, predicts that a future price may increases a quantity Y_1 or decreases Y_2. The weight of the available information is then measured by $M^{-1}(Y_1,Y_2)$, leading to the same kind of dependence.

In order to close the model, we assume that $M(Y_1,Y_2)$ is given by a kind of average or mean value between the two random values Y_1 and Y_2. Specifically, we take

$$M(Y_1,Y_2) = \left[\tfrac{1}{2}\left(Y_1^{\mu} + Y_2^{\mu}\right)\right]^{1/\mu}, \tag{11}$$

where $\mu \in \mathrm{Re}$. Therefore, we write the X random variable [Eq. (5)] as

$$X = \frac{1}{\sqrt{\beta}}\frac{Y_1 - Y_2}{2\left[\frac{1}{2}(Y_1^{\mu} + Y_2^{\mu})\right]^{1/\mu}}. \tag{12}$$

For convenience we introduced a factor one-half. On the other hand, the additional parameter $\sqrt{\beta} > 0$ scales and gives the right units of X. In fact, notice that the remaining contribution in Eq. (12) is dimensionless. Taking different values of the real parameter μ a wide class of probability distributions arises, which in turn may also depend on the parameters θ, α, and α' that determine the joint probability density (4).

A. Arithmetic mean value

The arithmetic mean value corresponds to $\mu = 1$, implying that

$$X = \frac{1}{\sqrt{\beta}} \frac{Y_1 - Y_2}{Y_1 + Y_2}. \tag{13}$$

Notice that, for any possible value of Y_1 and Y_2, the random variable X assumes bounded values in the domain $(-1/\sqrt{\beta}, +1/\sqrt{\beta})$. Taking into account the Z variable [Eq. (6)], we obtain the following inverted relations:

$$Y_1 = \frac{Z}{2}(1 + \sqrt{\beta}X), \quad Y_2 = \frac{Z}{2}(1 - \sqrt{\beta}X), \tag{14}$$

which in turn imply that $|\det(J)| = \sqrt{\beta}z/2$. Equations (4) and (7) lead to $P(x,z) = P(x)P(z)$, where $P(z)$ is given by Eq. (10). Therefore, the random variables X and Z are statistically independent. Furthermore, X obeys the statistics given by the probability density

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}}(1 + \sqrt{\beta}x)^{\alpha-1}(1 - \sqrt{\beta}x)^{\alpha'-1}, \tag{15}$$

where the normalization constant reads $\mathcal{N}_{\alpha\alpha'} = 2^{\alpha+\alpha'-1}\Gamma(\alpha)\Gamma(\alpha')/\Gamma(\alpha+\alpha')$. Notice that $P(x)$ does not depend on the scale parameter θ [see Eq. (4)]. It only depends on the shape indexes α, α', and the scale parameter β.

The distribution (15) develops a maximum located at

$$x_M = \frac{1}{\sqrt{\beta}} \frac{\alpha - \alpha'}{(\alpha + \alpha' - 2)}, \tag{16}$$

when $\alpha > 1$, $\alpha' > 1$, or at $x_M = \pm 1/\sqrt{\beta}$ in any other case. Its average value reads

$$\langle X \rangle = \frac{1}{\sqrt{\beta}} \frac{\alpha - \alpha'}{\alpha + \alpha'}, \tag{17}$$

while the variance $\mathrm{var}[X] = \langle X^2 \rangle - \langle X \rangle^2$ is given by

$$\mathrm{var}[X] = \frac{1}{\beta} \frac{4\alpha\alpha'}{(\alpha + \alpha')^2(1 + \alpha + \alpha')}. \tag{18}$$

q-Gaussian distributions

Equation (15) can be rewritten as

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}}(1 - \beta x^2)^{[(\alpha+\alpha')/2]-1}\left(\frac{1 + \sqrt{\beta}x}{1 - \sqrt{\beta}x}\right)^{(\alpha-\alpha')/2}. \tag{19}$$

Hence, we name this function as an asymmetric Poissonian q-Gaussian distribution $G^{p}_{<1}(x|q,a,\beta)$ with index q, and asymmetry parameter a,

$$q = 1 - \left[\frac{\alpha + \alpha'}{2} - 1\right]^{-1}, \quad a = \frac{\alpha - \alpha'}{2}. \tag{20}$$

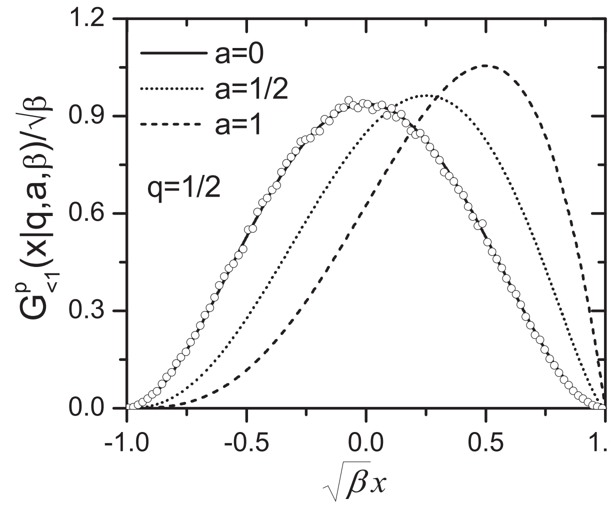

FIG. 1. Poissonian q-Gaussian probability density, Eq. (19), for different values of the asymmetry parameter a, Eq. (20). The circles correspond to a numerical simulation based on Eq. (13).

From the positivity of α and α' the asymmetry index must satisfy $|a| < (2-q)/(1-q)$. In the symmetric case, $a = 0$, $\alpha' = \alpha$, we get

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha}}(1 - \beta x^2)^{\alpha-1}, \tag{21}$$

where $\mathcal{N}_{\alpha} = 2^{2\alpha-1}\Gamma^2(\alpha)/\Gamma(2\alpha)$. Therefore, under the association $\beta \to \beta(1-q)$, with $\alpha > 1$, we recover Eq. (1). Over the domain $\alpha \in (1,\infty)$, the nonextensive parameter runs in the interval $q \in (-\infty, 1)$.

In Fig. 1 we plot the function (19) for different values of the asymmetric factor a, Eq. (20). For increasing $a > 0$, the distribution accumulates around $\sqrt{\beta}x \approx 1$. For $a < 0$, a reflected accumulation around $\sqrt{\beta}x \approx -1$ is developed. The distribution with $a = 0$ corresponds to Tsallis nonextensive thermodynamics.

B. Geometric mean value

In Eq. (11) the geometric mean value corresponds to $\lim \mu \to 0$, which satisfies $\lim_{\mu\to 0}[\frac{1}{2}(Y_1^{\mu} + Y_2^{\mu})]^{1/\mu} = \sqrt{Y_1 Y_2}$. Therefore, we get the random variable [Eq. (12)]

$$X = \frac{1}{\sqrt{\beta}} \frac{Y_1 - Y_2}{2\sqrt{Y_1 Y_2}}. \tag{22}$$

Notice that X takes values over the entire real number line, $X \in \mathrm{Re}$. In this case, the inverted relations are

$$Y_1 = \frac{Z}{2}\left(1 + \frac{\sqrt{\beta}X}{\sqrt{1+\beta X^2}}\right), \quad Y_2 = \frac{Z}{2}\left(1 - \frac{\sqrt{\beta}X}{\sqrt{1+\beta X^2}}\right), \tag{23}$$

implying that $|\det(J)| = \sqrt{\beta}(z/2)(1+\beta x^2)^{-3/2}$, which also leads to $P(x,z) = P(x)P(z)$, but here

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}}\left(\frac{1}{1+\beta x^2}\right)^{(\alpha+\alpha'+1)/2}(\sqrt{1+\beta x^2} + \sqrt{\beta}x)^{\alpha-1} \times (\sqrt{1+\beta x^2} - \sqrt{\beta}x)^{\alpha'-1}. \tag{24}$$

As in the previous case, this distribution is independent of the rate parameter θ. The normalization constant is the same, $\mathcal{N}_{\alpha\alpha'} = 2^{\alpha+\alpha'-1}\Gamma(\alpha)\Gamma(\alpha')/\Gamma(\alpha+\alpha')$.

Equation (24) develops a maximum, which occurs at

$$x_M = \frac{\alpha - \alpha'}{\sqrt{(1+2\alpha)(1+2\alpha')\beta}}. \tag{25}$$

In the limit $\sqrt{\beta}x \gg 1$, a power-law behavior arises:

$$\lim_{x\to\infty} P(x) \approx \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}} 2^{\alpha-\alpha'} \left(\frac{1}{\sqrt{\beta}x}\right)^{2\alpha'+1}, \tag{26}$$

while for $\sqrt{\beta}x \ll -1$ we obtain

$$\lim_{x\to-\infty} P(x) \approx \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}} 2^{\alpha'-\alpha} \left(\frac{1}{-\sqrt{\beta}x}\right)^{2\alpha+1}. \tag{27}$$

Due to the previous asymptotic behaviors the moments are not defined for any value of the characteristic shape parameters. When $\alpha > 1/2$ and $\alpha' > 1/2$, the average value reads

$$\langle X\rangle = \frac{1}{\sqrt{\beta}}(\alpha-\alpha')\frac{\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma\left(\alpha'-\frac{1}{2}\right)}{2\Gamma(\alpha)\Gamma(\alpha')}, \tag{28}$$

while the second moment, for $\alpha > 1$ and $\alpha' > 1$, is

$$\langle X^2\rangle = \frac{1}{\beta}[(\alpha-\alpha')^2 + (\alpha+\alpha'-2)]\frac{\Gamma(\alpha-1)\Gamma(\alpha'-1)}{4\Gamma(\alpha)\Gamma(\alpha')}. \tag{29}$$

Outside the previous intervals the first two moments are not defined.

q-Gaussian distributions

Equation (24) can be rewritten as

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}}\left(\frac{1}{1+\beta x^2}\right)^{(\alpha+\alpha'+1)/2}\left(\frac{\sqrt{1+\beta x^2}+\sqrt{\beta}x}{\sqrt{1+\beta x^2}-\sqrt{\beta}x}\right)^{(\alpha-\alpha')/2}. \tag{30}$$

As in the previous case, we name this function as an asymmetric Poissonian q-Gaussian distribution $G^{p}_{>1}(x|q,a,\beta)$ with index q, and asymmetry parameter a,

$$q = 1 + \left[\frac{\alpha+\alpha'}{2} + \frac{1}{2}\right]^{-1}, \quad a = \frac{\alpha-\alpha'}{2}. \tag{31}$$

Hence, $1 < q < 3$ and the asymmetry factor satisfies the restriction $|a| < (3-q)/[2(q-1)]$. In the symmetric case, $a = 0$, $\alpha' = \alpha$, $P(x)$ reduces to

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha}}\left(\frac{1}{1+\beta x^2}\right)^{\alpha+(1/2)}, \tag{32}$$

where $\mathcal{N}_\alpha = 2^{2\alpha-1}\Gamma^2(\alpha)/\Gamma(2\alpha)$. Under the association $\beta \to \beta(q-1)$, we recover Eq. (2). In the interval $\alpha \in (0,\infty)$ the nonextensive parameter runs in the interval $q \in (3,1)$.

In Fig. 2 we plot the function (30) for different values of the asymmetric factor a, Eq. (31). For increasing $a > 0$, the maximum of the distribution is shifted towards higher values. For $a < 0$ the extremum develops for negative values of x. The symmetric case $a = 0$ corresponds to the q-Gaussian distribution arising from Tsallis entropy.

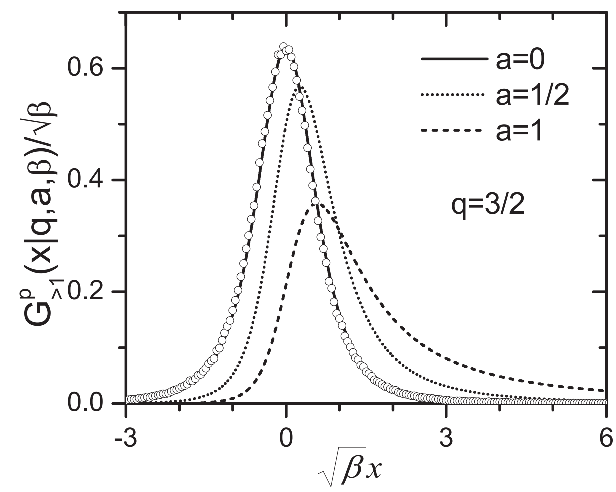

FIG. 2. Poissonian q-Gaussian probability density, Eq. (30), for different values of the asymmetry parameter a, Eq. (31). The circles correspond to a numerical simulation based on Eq. (22).

C. Relation between both cases

Given X determined by relation (13), the random variable X' defined as

$$X' = \frac{X}{\sqrt{1-\beta X^2}}, \tag{33}$$

recovers Eq. (22). This simple expression demonstrates that there exists a one to one mapping between q-Gaussian variables in the different domains of the complexity parameter q. In fact, if we define $\sqrt{\beta}X = \sin(\phi) \in (-1,1)$ for $q \in (-\infty,1)$, hence $\sqrt{\beta}X' = \tan(\phi) \in (-\infty,+\infty)$, where X' has associated the index $q \in (1,3)$.

Alternatively, if X is given by Eq. (22), the inverse transformation

$$X' = \frac{X}{\sqrt{1+\beta X^2}} \tag{34}$$

leads to Eq. (13). While these relations are known for symmetric q-Gaussian distributions ($a = 0$) [3], here we showed that they are also valid for the asymmetric densities ($a \neq 0$) previously introduced.

IV. MODIFIED *q*-EXPONENTIAL DISTRIBUTIONS

Variables distributed according to a q-exponential density are positive. Therefore, the previous scheme does not apply, but a similar one can be implemented. We name the emerging distributions as modified q-exponential densities. Taking into account the notation of Refs. [30–32] they can also be called generalized q-gamma densities. Nevertheless, it seems that they do not satisfy the same entropic properties as standard gamma distributions [41]. On the other hand, we remark that some properties of the following distributions are known and can be found under different denominations [34,35].

A. Bounded domain

For getting a positive variable, we introduce the following functional dependence:

$$X = \frac{1}{\sqrt{\beta}} \frac{Y_2}{Y_1 + Y_2}. \tag{35}$$

Notice that this assumption is very similar to Eq. (13), but here X is a bounded positive stochastic variable, $0 < \sqrt{\beta}X < 1$. Using the approach defined in Sec. II, here we obtain the inverse relations

$$Y_1 = Z(1 - \sqrt{\beta}X), \quad Y_2 = Z\sqrt{\beta}X, \tag{36}$$

while $|\det(J)| = \sqrt{\beta}z$, leading again to a statistical independence of X and Z, that is, $P(x,z) = P(x)P(z)$. The density of interest here is

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}} (\sqrt{\beta}x)^{\alpha'-1}(1 - \sqrt{\beta}x)^{\alpha-1}, \tag{37}$$

where $\mathcal{N}_{\alpha\alpha'} = \Gamma(\alpha)\Gamma(\alpha')/\Gamma(\alpha+\alpha')$. When $\alpha > 1$ and $\alpha' > 1$, $P(x)$ reaches a maximal value located at

$$x_M = \frac{1}{\sqrt{\beta}} \frac{\alpha' - 1}{(\alpha + \alpha' - 2)}. \tag{38}$$

Its first moment reads

$$\langle X \rangle = \frac{1}{\sqrt{\beta}} \frac{\alpha'}{\alpha + \alpha'}, \tag{39}$$

while the variance is given by

$$\mathrm{var}[X] = \frac{1}{\beta} \frac{\alpha\alpha'}{(\alpha + \alpha')^2(1 + \alpha + \alpha')}. \tag{40}$$

q-exponential densities

The distribution (37) may be named as a modified Poissonian q-exponential distribution $E^{p}_{<1}(x|q,d,\beta)$ with index q, and "distortion parameter" d,

$$q = 1 - \frac{1}{\alpha - 1}, \quad d = \alpha' - 1. \tag{41}$$

Therefore, $-\infty < q < 1$ and $d > -1$. When $d = 0$, that is $\alpha' = 1$, Eq. (37) becomes

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha}} (1 - \sqrt{\beta}x)^{\alpha-1}, \tag{42}$$

where $\mathcal{N}_{\alpha} = \Gamma(\alpha)/\Gamma(\alpha+1)$. Therefore, under the extra association $\sqrt{\beta} \to \sqrt{\beta}(1-q)$, we obtain a standard q-exponential density. For $\alpha \in (1,\infty)$, it follows $q \in (-\infty,1)$.

In Fig. 3 we plotted the function (37) for different values of the distortion parameter d, Eq. (41). For $d < 0$, the density diverges around the origin. This property is inherited from the gamma distribution, Eq. (4). On the other hand, for $d > 0$ the density vanishes at the origin and for increasing d it accumulates around $\sqrt{\beta}x \approx 1$. The plot for $a = 0$ is the q-exponential distribution arising from Tsallis entropy.

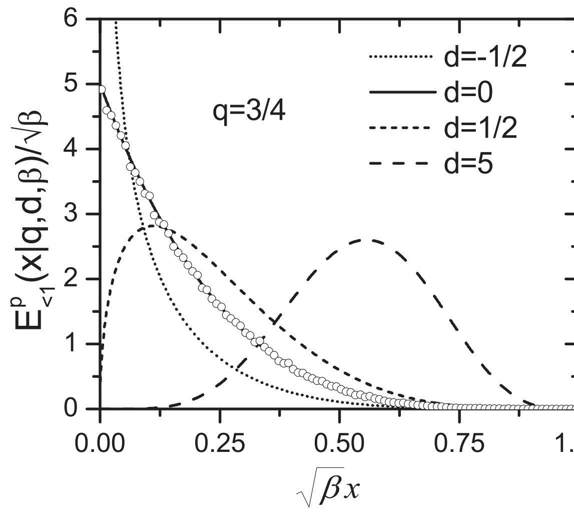

FIG. 3. Poissonian q-exponential probability density, Eq. (37), for different values of the distortion parameter d, Eq. (41). The circles correspond to a numerical simulation based on Eq. (35).

B. Unbounded domain

An unbounded positive variable ($0 < X < \infty$) is obtained from the relation

$$X = \frac{1}{\sqrt{\beta}} \frac{Y_2}{Y_1}, \tag{43}$$

where as in the previous cases $\sqrt{\beta}$ scales the random variable X. Here, the inverse relations are

$$Y_1 = \frac{Z}{1 + \sqrt{\beta}X}, \quad Y_2 = \frac{Z\sqrt{\beta}X}{1 + \sqrt{\beta}X}, \tag{44}$$

while $|\det(J)| = \sqrt{\beta}z/(1 + \sqrt{\beta}x)^2$, leading to $P(x,z) = P(x)P(z)$, where

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}} \frac{(\sqrt{\beta}x)^{\alpha'-1}}{(1 + \sqrt{\beta}x)^{\alpha+\alpha'}}, \tag{45}$$

with $\mathcal{N}_{\alpha\alpha'} = \Gamma(\alpha)\Gamma(\alpha')/\Gamma(\alpha+\alpha')$. This distribution develops a maximum that is located at ($\alpha' > 1$)

$$x_M = \frac{1}{\sqrt{\beta}} \frac{\alpha' - 1}{(\alpha + 1)}. \tag{46}$$

For $\sqrt{\beta}x \gg 1$, it follows the asymptotic power-law behavior

$$\lim_{x\to\infty} P(x) \approx \frac{\sqrt{\beta}}{\mathcal{N}_{\alpha\alpha'}} \left(\frac{1}{\sqrt{\beta}x}\right)^{\alpha+1}. \tag{47}$$

In consequence, the moments are not defined for any value of the shape parameter α. For $\alpha > 1$, the mean value reads

$$\langle X \rangle = \frac{1}{\sqrt{\beta}} \frac{\Gamma(\alpha-1)\Gamma(\alpha'+1)}{\Gamma(\alpha)\Gamma(\alpha')}, \tag{48}$$

while the second moment is only defined for $\alpha > 2$,

$$\langle X^2 \rangle = \frac{1}{\beta} \frac{\Gamma(\alpha-2)\Gamma(\alpha'+2)}{\Gamma(\alpha)\Gamma(\alpha')}. \tag{49}$$

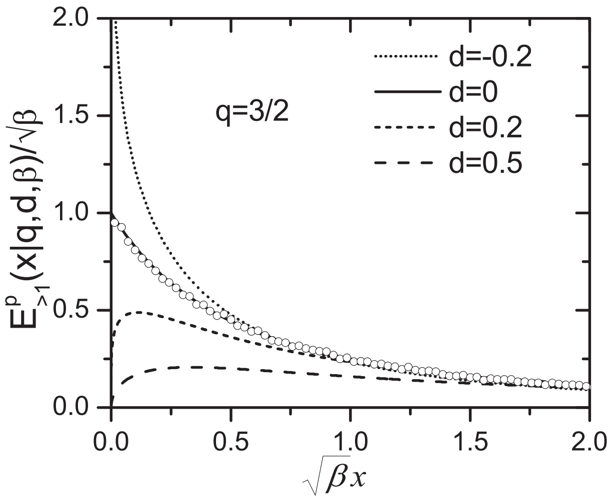

FIG. 4. Poissonian q-exponential probability density, Eq. (45), for different values of the distortion parameter d, Eq. (50). The circles correspond to a numerical simulation based on Eq. (43).

q-exponential densities

The distribution (37) may also be named as a modified Poissonian q-exponential distribution $E^{p}_{>1}(x|q,d,\beta)$ with index q, and distortion parameter d,

$$q = 1 + \frac{1}{\alpha + \alpha'}, \quad d = \alpha' - 1. \tag{50}$$

In consequence, $1 < q < \infty$, and the distortion parameter satisfies $(2 - q)/(q - 1) > d > -1$. When the distortion is null, $\alpha' = 1$, Eq. (45) becomes

$$P(x) = \frac{\sqrt{\beta}}{\mathcal{N}_\alpha} \frac{1}{(1 + \sqrt{\beta}x)^{\alpha+1}}, \tag{51}$$

where $\mathcal{N}_\alpha = \Gamma(\alpha)/\Gamma(1+\alpha)$. Thus, under the extra association $\sqrt{\beta} \to \sqrt{\beta}(1-q)$ we get a q-exponential probability density. In this case, for $\alpha \in (0,\infty)$, it follows $q \in (2,1)$.

The function (45) is plotted in Fig. 4 for different values of the distortion parameter d, Eq. (50). For $d < 0$, the density diverges around the origin. On the other hand, for $d > 0$, the density vanishes at the origin. In all cases a power-law behavior is obtained for $\sqrt{\beta}x \gg 1$. The plot for $d = 0$ is the q-exponential distribution arising from Tsallis entropy.

C. Relation between both cases

Given X determined by Eq. (35), the random variable X' defined as

$$X' = \frac{X}{1 - \sqrt{\beta}X}, \tag{52}$$

is given by Eq. (43). Alternatively, if X is given by Eq. (43), the inverse transformation

$$X' = \frac{X}{1 + \sqrt{\beta}X} \tag{53}$$

leads to Eq. (35). These conjugate relations are valid for both the unmodified ($d = 0$) as well as the modified ($d \neq 0$) q-exponential densities.

D. Stretched *q*-exponential densities

Introducing the change of variables

$$\sqrt{\tilde{\beta}}\tilde{X} = (\sqrt{\beta}X)^{1/\nu}, \tag{54}$$

defined by the extra parameter $\nu \in \mathrm{Re}$, if X is distributed according to Eq. (37), it follows $[P(\tilde{x})d\tilde{x} = P(x)dx]$

$$P(\tilde{x}) = \frac{\nu}{\mathcal{N}_{\alpha\alpha'}} \left(\sqrt{\tilde{\beta}}\tilde{x}\right)^{\nu\alpha'-1} \left[1 - \left(\sqrt{\tilde{\beta}}\tilde{x}\right)^{\nu}\right]^{\alpha-1}. \tag{55}$$

On other hand, if X is distributed according to Eq. (45), we get

$$P(\tilde{x}) = \frac{\nu\sqrt{\tilde{\beta}}}{\mathcal{N}_{\alpha\alpha'}} \frac{(\sqrt{\tilde{\beta}}\tilde{x})^{\nu\alpha'-1}}{[1 + (\sqrt{\tilde{\beta}}\tilde{x})^{\nu}]^{\alpha+\alpha'}}. \tag{56}$$

In both cases, imposing the condition $\nu\alpha' = 1$, the previous two expressions become stretched q-exponential densities, $P(x) = (\sqrt{\beta}/\mathcal{N}_q)\exp_q[-(\sqrt{\beta}x)^{\nu}]$ $(x > 0, \ \nu > 0)$. Hence, these distributions can also be covered with the present approach [Eqs. (35) and (43) under the change of variables (54)].

V. PROPERTIES AND APPLICATIONS

In the previous two sections we demonstrated that the assumption (5) allows us to recover and to define an extended family of q-Gaussian and q -exponential densities. Here, we discuss some general properties of the approach as well as some applications of the extended distributions.

A. *q*-distributed random numbers

Numerical generation of random numbers obeying q-Gaussian statistics was explored previously by introducing a generalized Box-Muller method [37]. Generation of Levy distributed numbers was also established [38]. On the other hand, numerical generation of gamma random numbers is also well established [34,35]. Therefore, the present scheme defines an alternative and solid basis for obtaining q-distributed random numbers by generating two independent gamma random numbers. Using this method, in Figs. 1–4 we explicitly show (circles) the recovering of the symmetric and unmodified distributions, all of them corresponding to Tsallis entropy formalism.

B. Symmetries

While the underlying joint statistics of the gamma variables depends on the scale parameter θ, Eq. (4), the distributions of X do not depend on it. This is not the only symmetry of the proposed scheme. In fact, it is simple to check that the symmetry

$$f(Y_1,Y_2) = f\left(\frac{1}{Y_2},\frac{1}{Y_1}\right) \tag{57}$$

is fulfilled, where $f(Y_1,Y_2)$ defines the X random variable, Eq. (5). In fact, this property is valid for the q-Gaussian case [Eq. (12) for any μ] as well as for the q-exponential variables [Eqs. (35) and (43)]. We notice that $f(Y_1,Y_2) = g(Y_1/Y_2)$, for arbitrary functions $g(y)$, always satisfies the relation (57).

Extra structures can be established by introducing an arbitrary change of variables.

The symmetry (57) implies that the same results arise if instead of gamma distributed variables one takes inverse gamma variables, that is, $Y' = 1/Y$ where Y is gamma distributed [Eq. (3)]. Using that $P(y)dy = P(y')dy'$, it follows

$$P(y') = \frac{1}{(y')^{\alpha+1}} \frac{e^{-1/y'\theta}}{\theta^\alpha \Gamma(\alpha)}, \quad 0 < \alpha < \infty. \tag{58}$$

C. Relation with beta distributions

As all functions $f(Y_1, Y_2)$ fulfill the condition (57), it is clear that any of the corresponding variables X are always related by a change of variables between them. Therefore, it does not make sense to affirm that one of them generates or is more fundamental than the others. Nevertheless, here we want to emphasize that any of the probability densities obtained previously can be related to the well-known beta distribution [33–35]. It reads

$$P(w) = \frac{\Gamma(\alpha + \alpha')}{\Gamma(\alpha)\Gamma(\alpha')} w^{\alpha-1}(1-w)^{\alpha'-1}, \tag{59}$$

where the domain of its variable is $w \in (0,1)$. Furthermore, its shape parameters α and α' are positive.

Defining the change of variables $w = w(x)$, all obtained q distributions become equal to Eq. (59). Alternatively, defining a new variable $x = x(w)$, from the beta distribution it is possible to obtain the q densities. Explicitly, for the asymmetric distribution, Eq. (15) [or Eq. (19)], the change of variables reads

$$w = \tfrac{1}{2}(1 + \sqrt{\beta}x), \qquad \sqrt{\beta}x = 2\left(w - \tfrac{1}{2}\right). \tag{60}$$

Therefore, all (asymmetric and symmetric) q-Gaussian distributions with $-\infty < q < 1$ are related to a beta variable by a shifting of their arguments.

For the q-Gaussian defined by Eq. (24) [or Eq. (30)], where $1 < q < 3$, the change of variables is

$$w = \frac{1}{2}\left(1 + \frac{\sqrt{\beta}x}{\sqrt{1+\beta x^2}}\right), \quad \sqrt{\beta}x = \frac{(w - \frac{1}{2})}{\sqrt{w(1-w)}}. \tag{61}$$

For Eq. (37), the relations are

$$w = 1 - \sqrt{\beta}x, \quad \sqrt{\beta}x = 1 - w, \tag{62}$$

that is, the modified (and standard) q-exponential densities in the interval $-\infty < q < 1$ arise from an axe inversion of a beta distribution. Finally, in the interval $1 < q < \infty$, Eq. (45), the transformations are

$$w = \frac{1}{1 + \sqrt{\beta}x}, \quad \sqrt{\beta}x = \frac{1-w}{w}. \tag{63}$$

The previous relations can be enlightened by using that a variable W obeying the beta statistics (59) can also be written in terms of two independent gamma variables (Y_1 and Y_2) [33–35] [Eq. (4)],

$$W = \frac{Y_1}{Y_1 + Y_2}, \quad W' = \frac{Y_2}{Y_1 + Y_2}, \tag{64}$$

where the additional variable W' is also beta distributed. In fact, $W + W' = 1$. After a straightforward manipulation, the random variables associated with the q-Gaussian distributions, Eqs. (13) and (22), can respectively be rewritten as

$$\sqrt{\beta}X = (W - W'), \quad \sqrt{\beta}X = \frac{(W - W')}{2\sqrt{WW'}}, \tag{65}$$

while for the q-exponential densities, Eqs. (35) and (43), respectively it follows

$$\sqrt{\beta}X = W', \quad \sqrt{\beta}X = \frac{W'}{W}. \tag{66}$$

Both Eq. (65) and Eq. (66) show the close relation between all the generalized q distributions and beta random variables. In fact, any stochastic variable defined by a function satisfying the symmetry (57) can be related by a transformation of variables with a beta distribution, Eq. (59).

One may also take a complementary point of view and explore whether the previous densities can be obtained from Tsallis entropy under a more general constraint. In fact, any of the extended distributions can be rewritten as $P(x) = (\sqrt{\beta}/\mathcal{N}_q)\exp_q[-\beta V(x)]$. This structure emerges from Tsallis entropy by using a constraint based on a generalized mean value of $V(x)$ [3]. Nevertheless, here the resulting functions $V(x)$ depend on the parameter q and also on the asymmetry and distortion factors. Therefore, a relation between Tsallis entropy and the asymmetric and modified distributions cannot be established in this way.

D. q-triplet for probability densities

In the context of nonextensive thermodynamics three different values of the complexity parameter, named as q-triplet, are associated with different physical properties such as the statistics of metastable or quasistationary states, sensitivity to initial conditions, and time decay of observable correlations [39,40]. Here, we show that the symmetric and unmodified probability distributions can be indexed with only three different values of q. We remark that not any direct relation can be postulated between both triplets, because here it is established for normalizable objects, $\int_{-\infty}^{+\infty} P(x)dx = 1$.

We denote by $q^g_{<1}$ and $q^g_{>1}$ the complexity indexes of the q-Gaussian distributions, Eqs. (20) and (31), respectively. Furthermore, $q^e_{<1}$ and $q^e_{>1}$ denote the indexes of the q-exponentials, Eqs. (41) and (50), respectively. The four indexes are given by

$$q^g_{<1} = 1 - \frac{1}{\alpha - 1}, \quad q^g_{>1} = 1 + \frac{1}{\alpha + 1/2}, \tag{67a}$$

$$q^e_{<1} = 1 - \frac{1}{\alpha - 1}, \quad q^e_{>1} = 1 + \frac{1}{\alpha + 1}. \tag{67b}$$

We notice that $q^g_{<1} = q^e_{<1}$ [see Eqs. (20) and (41)]. This equality is expectable because q-Gaussian and q-exponential distributions for $-\infty < q < 1$ are related by a linear change of variables [see Eqs. (60) and (62)] with a beta distribution. Therefore, the complete family of analyzed distributions can be indexed with only three q parameters: $(q^e_{<1}, q^g_{>1}, q^e_{>1})$. The previous expressions are equivalent to

$$\frac{1}{1 - q^g_{<1}} = \alpha - 1, \qquad \frac{1}{q^g_{>1} - 1} = \alpha + \frac{1}{2}, \tag{68a}$$

$$\frac{1}{1-q^e_{<1}}=\alpha-1,\quad \frac{1}{q^e_{>1}-1}=\alpha+1. \tag{68b}$$

From here we realize that there exist simple relations between any of the q-triplet parameters.

E. Applications of the extended distributions

The assumption (5) defines a broad class of probability densities, which in turn covers the most used probability densities arising from Tsallis entropy maximization. Hence, besides it theoretical interest, we ask about the possible applications of the asymmetric and modified distributions.

From the previous analysis, we arrived at the conclusion that asymmetric q-Gaussian and modified q-exponential distributions in the interval $-\infty<q<1$ [Eqs. (19) and (37)] are related by a linear change of variables to a beta distribution. Therefore, these functions fall in a wide range of applicability of this distribution [33–35]. For example, (unnormalized) beta distributions emerge in the statistical description of quark matter [see Eq. (102) in Ref. [25]].

The modified q-exponential function, Eq. (45), was used in the description of stock trading volume flow in financial markets [30–32] (named as generalized q-gamma probability density). This distribution is also known as a Pearson Type VI distribution or, alternatively, beta-prime distribution [34] (see also [35]).

To our knowledge, asymmetric q-Gaussian distributions, Eq. (30), were not used previously. In the present approach, the asymmetry of these probability densities has a clear dynamical origin. In fact, associating a cascade process to each gamma variable, asymmetries arise whenever the cascades have a different shape index (α and α'). We discuss below the application of these kinds of distributions as a fitting tool in the context of financial signals [21] and movement of defects in fluid flow [11].

1. Log-return signals on large time windows. From the price signal $y(t)$ in a financial market it is possible to define the stochastic process $\tilde{y}(t)=\ln[y(t+\Delta t)/y(t)]$, where Δt is a constant time interval. This signal gives a simple way of representing returns in the market. Usually it is studied using the *normalized log-return* $Z_{\Delta t}(t)=\ln[\tilde{y}(t)-\langle\tilde{y}(t)\rangle]/\sigma_{\Delta t}$, where $\langle\tilde{y}(t)\rangle$ is the average and $\sigma_{\Delta t}$ gives the standard derivation of $\tilde{y}(t)$ for a given Δt. Daily closing price values of the S&P index for a period of 20 years were analyzed by Ausloos and Ivanova in Ref. [21]. Assuming a stationary signal, $Z_{\Delta t}(t)\to Z_{\Delta t}$, for *large time windows* ($\Delta t\geqslant 1$ day), the authors fitted the experimental data with a q-Gaussian-like distribution

$$p(z_{\Delta t})=\frac{\sqrt{\beta_q}}{\mathcal{N}_q}[1+(\sqrt{\beta_q}|z_{\Delta t}|)^{2\tilde{\alpha}}]^{-[1/(q-1)]}, \tag{69}$$

where $(1/\mathcal{N}_q)=\tilde{\alpha}\Gamma(\frac{1}{q-1})/[\Gamma(\frac{1}{q-1}-\frac{1}{2\tilde{\alpha}})\Gamma(\frac{1}{2\tilde{\alpha}})]$ and $\sqrt{\beta_q}$ depends on the parameters q and $\tilde{\alpha}$ [see Eqs. (4) and (5) in [21]]. This distribution can be obtained from a superstatistical model assuming, for example, that Brownian particles diffuse in a potential $U(x)=C|x|^{2\tilde{\alpha}}$ [9]. On the other hand, Eq. (69) can also be recovered from the present approach based on random Poisson variables. In fact, it follows by extending symmetrically ($x\to|x|$) the stretched q-exponential distribution (56), with $\alpha'=1/\nu$, and the following replacements: $\tilde{x}\to|z_{\Delta t}|$, $\tilde{\beta}\to\beta_q$, $\nu\to 2\tilde{\alpha}$, and $\alpha\to(\frac{1}{q-1}-\frac{1}{2\tilde{\alpha}})$. Random number generation is achieved by introducing an extra stochastic variable that with probability one-half defines their sign (positive or negative).

Equation (69) develops an asymptotic ($\sqrt{\beta_q}|z_{\Delta t}|\gg 1$) power-law behavior. Nevertheless, the authors also find that the experimental data are not consistent with the symmetry $p(z_{\Delta t})=p(-z_{\Delta t})$. In particular, the power-law behaviors have different exponents for positive and negative values. Similar *asymmetries* were found previously in Ref. [42].

Here, we may associate the observed asymmetry of the data with two cascade mechanisms, each one being represented by a gamma random variable. In a rough way, the difference between both variables can be associated with different network properties related to the propagation of information that support an increased or decreased future value. The complete system response is defined by Eq. (22), which is a geometric mean value of the driving fluctuations. Hence, instead of using Eq. (69), we propose fitting the probability density of the log-return with the asymmetric q-Gaussian distribution (30) under the shifting $P(x)\to P(z_{\Delta t}+\langle X\rangle)$, where $\langle X\rangle$ is given by Eq. (28).

In order to check this proposal, here we analyze the daily closing price values of the S&P index [43] for the period between Jan. 3, 1950 and Dec. 3, 2014, which provides a 16 336 data base. In Fig. 5(a) we show the "experimental" probability distribution (circles) for $\Delta t=30$ days. The data are clearly asymmetric, which in fact are fitted by Eq. (30) (full line). Its characteristic parameters α, α', and β were determined by minimizing the global error. Based on the quadratic global error $\sum(p_{\rm exp}-p_{\rm theory})^2/p_{\rm exp}$, we checked that for a wide range of Δt the asymmetric distribution provides a better fitting than Eq. (69).

In Fig. 5(b) we plot α and α' as functions of Δt. In the limit $\Delta t\to 0$ the asymmetry vanishes ($\alpha\simeq\alpha'$). Furthermore, in the plotted interval, both shape parameters increase linearly with Δt. For higher values of Δt an irregular-logarithmic-like growth behavior is observed (not shown). For $\Delta t\gtrsim 500$ the distributions approach normal Gaussian ones. This limit is consistent with the growth of α and α' [see Eq. (31)]. On the other hand, we find that $1/\beta$ also increases linearly with Δt, Fig. 5(c). This (diffusive) behavior is also found for intervals Δt less than a day [23].

Extra analysis and elements are necessary for explaining the linear behaviors shown in Figs. 5(b) and 5(c). On the other hand, Fig. 5(a) shows that the proposed probability density provides a reasonable and alternative fitting to that based on Eq. (69), which in turn is able to capture the observed asymmetries.

2. Defect velocities in inclined layer convection. In Ref. [11], Daniels *et al.* studied the motion of point defects in thermal convection patterns in an inclined fluid layer (heated from below and cooled from above), a variant of Rayleigh-Bénard convection. Due to the inclination the system is anisotropic. The (experimental) probability distribution of the (positive and negative) defect velocities is different in the transverse ($\hat{x}$, across rolls) and longitudinal ($\hat{y}$, along rolls, uphill-downhill) directions. In the transverse direction the velocity (v_x) can be fit with a symmetric q-Gaussian distribution

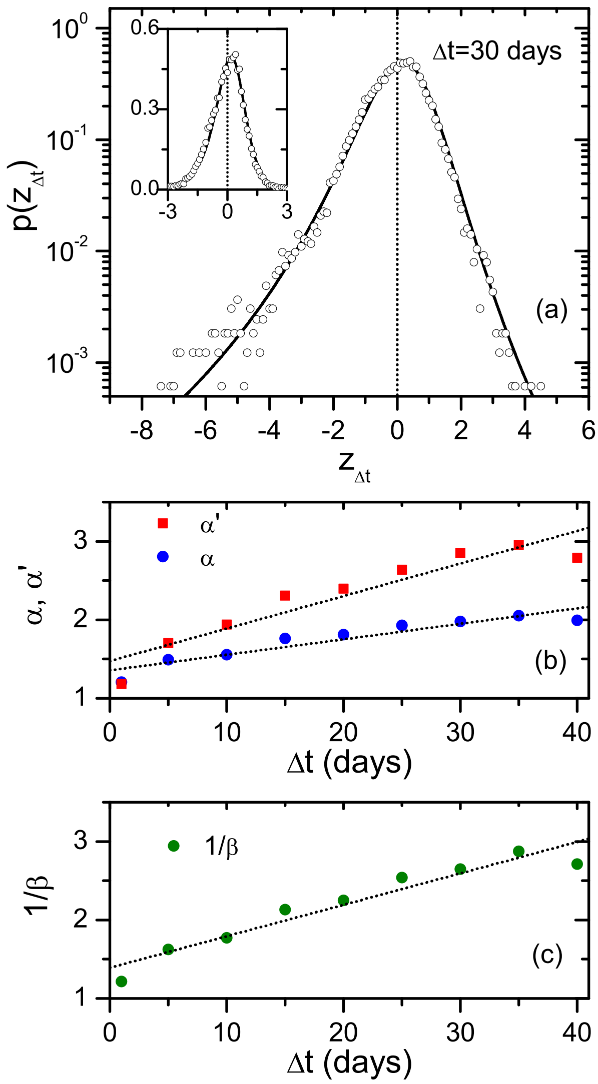

FIG. 5. (Color online) (a) Probability density $p(z_{\Delta t})$ of normalized log-return for the S&P index (circles). The full line corresponds to Eq. (24) (see text). The inset shows the peak region. The parameters are $\alpha = 1.97$, $\alpha' = 2.85$, and $\beta = 0.37$. From (31) it follows that $q = 1.34$ and $a = -0.87$. (b) Dependence with Δt of the shape parameters α (blue circles data) and α' (red square data). Dotted line, linear fit. (c) Parameter β.

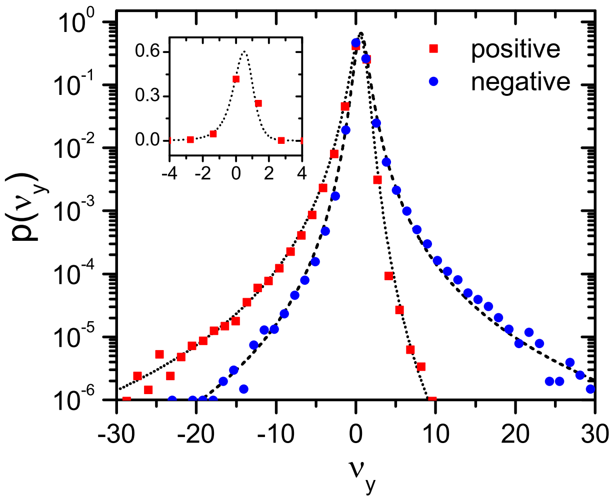

FIG. 6. (Color online) Probability density $p(\nu_y)$ of the normalized velocity $\nu_y = v_y/\sqrt{\langle v_y^2\rangle - \langle v_y\rangle^2}$ for positive and negative defects in inclined layer convection, $\varepsilon = 0.08$ (see [11] for details). The fittings (dotted and dashed lines) correspond to Eq. (30) (see text). For positive defects (red squares, experimental data) the parameters are $\Delta = 0.62$, $\alpha = 1.60$, and $\alpha' = 2.71$, which lead to $q = 1.37$ and asymmetry $a = -0.55$. For negative defects (blue circles, experimental data) the parameters are $\Delta = 0.58$, $\alpha = 1.67$, and $\alpha' = 1.38$, which lead to $q = 1.49$ and $a = 0.14$. The inset shows the peak region for positive defects.

($q \simeq 1.4$). Nevertheless, in the longitudinal direction (v_x) the distribution, depending on a dimensionless temperature ε (see details in [11]), develops strong asymmetries. In that situation, Tsallis distributions, even defined with a cubic potential, are unable to fit the experimental data [see Fig. 4(a) in [11]]. As shown in the next figure, these asymmetries can be fitted with the probability densities introduced previously.

In Fig. 6 we show a set of experimental data ($\varepsilon = 0.08$) [44], corresponding to the probability density $p(\nu_y)$ of the dimensionless velocity $\nu_y \equiv v_y/\sqrt{\langle v_y^2\rangle - \langle v_y\rangle^2}$ of positive and negative defects [see also Fig. 4(a) in [11]]. We find that these data can be very well fitted with the distribution, Eq. (30), under the associations $x \to \nu_y - \Delta/\sqrt{\beta} = (v_y/\sqrt{\langle v_y^2\rangle - \langle v_y\rangle^2}) - \Delta/\sqrt{\beta}$. Hence, $p(\nu_y) = \sigma P(\sigma\nu_y - \Delta/\sqrt{\beta})$, where $\sigma \equiv \sqrt{\langle X^2\rangle - \langle X\rangle^2}$ follows from Eqs. (28) and (29). Furthermore, we introduced an extra dimensionless parameter Δ that only shifts the complete distribution. Due to the previous rescaling, $p(\nu_y)$ does not depend on the parameter β. The parameters α, α', and Δ were determined in such a way that the global error is minimized. Even when the asymmetry is appreciable, the maximum of the distribution [Eq. (25)] is around the origin. In fact, the influence of the shift introduced by Δ is only appreciable around the origin (note that in both cases $|\Delta| < 1$). For both positive and negative defects a very good fitting is obtained. We also checked that a similar fitting is obtained for higher values of the dimensionless temperature, $\varepsilon = 0.17$, where the distribution asymmetry is smaller than that shown in Fig. 6 [see Fig. 4(b) in [11]]. Hence, we conclude that the defect dynamics may be represented by two cascade mechanism with different statistical properties, such as that defined by Eq. (22). While a rigorous derivation of this interpretation is not developed here, the quality of the fitting gives consistent support to the proposed theoretical frame.

VI. CONCLUSIONS

The present approach relies on expressing the variable of interest, associated with a given complex system, as a function of two independent gamma random variables. These variables represent intrinsic fluctuations that drive the system. In addition, the complexity of the dynamics is represented by a random-system response that also depends, in a nonlinear way, on the fluctuations. Writing the system response in terms of a generalized mean value, Eq. (12), we showed that the arithmetic and geometric cases allow us to introduce a class of asymmetric q-Gaussian distributions. In the symmetric case, for any value of the complexity parameter q, they recover densities that follow from Tsallis entropy maximization. A similar approach applies for q-exponential distributions, which become defined in terms of a distortion parameter. We also showed that the complete family of obtained distributions can be related via a change of variables with a beta distribution. A q-triplet was derived for the symmetric and unmodified distributions.

These results define an alternative numerical tool for random number generation obeying the previous statistical behaviors. On the other hand, the present approach may provide an alternative and very simple basis for understanding statistical behaviors in complex dynamics. Of special interest is the possibility of relating any asymmetry in the probability distributions with different underlying cascade mechanisms. We have shown that, in fact, asymmetric Poissonian q-Gaussian densities ($1 < q < 3$) provide a very good fitting to the statistical distribution of log-return signals in financial markets (Fig. 5) as well as the probability distribution of the velocity of moving defects in inclined layer convection [11] (Fig. 6). Therefore, the derivation of the present approach from deeper microscopic or mesoscopic descriptions is an issue that with certainty deserves extra analysis. The possibility of recovering asymmetric distributions or the beta statistics from nonextensive thermodynamics also remains as an open problem.

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