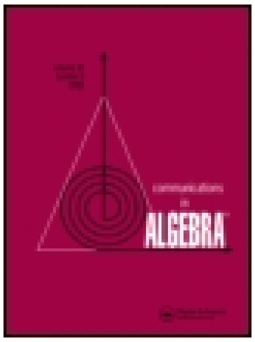

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The radical of a module category of string algebra

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ABSTRACT
We study how to read from the ordinary quiver of a representation-finite string algebra, the minimal lower bound $m \geq 1$ such that the mth power of the radical of its module category vanishes. We also show how to read the degree of any irreducible morphism in such a representation-finite string algebras considering strings.

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1. Introduction

Let A be an a finite-dimensional k-algebra over an algebraically closed field k, and mod A the category of finitely generated left A-modules. For $X, Y \in \text{mod}\, A$, we denote by $\Re(X, Y)$ the set of all morphisms $f : X \to Y$ such that, for all indecomposable A-module M, each pair of morphisms $h : M \to X$ and $h' : Y \to M$ the composition $h'fh$ is not an isomorphism. Inductively, the powers of $\Re(X, Y)$ are defined. By $\Re^{\infty}_A(X, Y)$ we denote the intersection of all powers $\Re^i_A(X, Y)$ of $\Re_A(X, Y)$ with $i \geq 1$.

By [3], if $f : X \to Y$ is a morphism between indecomposable A-modules, then f is irreducible if and only if $f \in \Re_A(X, Y) \backslash \Re^2_A(X, Y)$.

An important research direction towards understanding the structure of a module category is the study of the compositions of irreducible morphisms in relation with the powers of the radical of their module categories, see [8].

In case we deal with a representation finite algebra, it is well known by a result of Auslander that there is a positive integer n such that $\Re^n(\text{mod}\, A) = 0$, see [2, p. 183].

In order to study the relationship between compositions of irreducible morphisms and the powers of the radical of their module categories, in [12], Liu introduced the notion of left and right degree of an irreducible morphism. This notion has been a fundamental tool to determine the above bound in case we deal with a finite-dimensional algebra over an algebraically closed field of finite representation type. More precisely, if $A \simeq kQ_A/I_A$ is representation-finite, in [5], the author showed how to find the minimal lower bound $m \geq 1$ such that $\Re^m(\text{mod}\ A)$ vanishes in terms of the left and right degrees of particular irreducible morphisms.

The aim of this work is to determine the minimal positive integer $m \geq 1$ such that $\Re^{m+1}(X, Y) = 0$ for all modules $X, Y \in \text{mod}\, A$, in case A is a representation-finite string algebra. This bound is given in terms of strings and taking into account their, respectively, ordinary quivers.

Furthermore, we also show how to read the degree of any irreducible morphism of a representation-finite string algebra. This result shows that we have another way to read degrees in case we deal with these algebras, and without considering the Auslander-Reiten quiver to read them.

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This paper is organized as follows. In the first section, we present some notations and preliminary results. Section 2 is dedicated to prove the results concerning how to read from the ordinary quiver of a representation-finite string algebra the nilpotency index of the radical of the module category. In Section 3, we explain how to read the degree of any irreducible morphism taking into account strings.

2. Preliminaries

Throughout this work, by an algebra, we mean a finite-dimensional basic k-algebra over an algebraically closed field, k.

1.1. A *quiver* Q is given by a set of vertices Q_0 and a set of arrows Q_1, together with two maps $s, e : Q_1 \to Q_0$. Given an arrow $\alpha \in Q_1$, we write $s(\alpha)$ the starting vertex of α and $e(\alpha)$ the ending vertex of α. For each arrow $\alpha \in Q_1$, we denote by α^{-1} its formal inverse, where $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$.

A *walk* in Q is a concatenation $c_n \dots c_1$, with $n \geq 1$, such that c_i is either an arrow or the inverse of an arrow, and $e(c_i) = s(c_{i+1})$. We say that $c_n \dots c_1$ is a *reduced walk* provided $c_i \neq c_{i+1}^{-1}$ for each i, $1 \leq i \leq n-1$.

If A is an algebra then there exists a quiver Q_A, called the *ordinary quiver of* A, such that A is the quotient of the path algebra kQ_A by an admissible ideal.

1.2. Let A be an algebra. We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ which consists of one representative of each isomorphism class of indecomposable A-modules.

Let X be a non-projective (non-injective) indecomposable A-module. By $\alpha(X)$ ($\alpha'(X)$, respectively) we denote the number of indecomposable summands in the middle term of an almost split sequence ending (starting, respectively) at X. We say that $\alpha(\Gamma_A) \leq 2$ if $\alpha(X)$ and $\alpha'(X)$ are less than or equal to 2, whenever they are defined.

1.3. A morphism $f : X \to Y$, with $X, Y \in \operatorname{mod} A$, is called *irreducible* provided it does not split and whenever $f = gh$, then either h is a split monomorphism or g is a split epimorphism.

If $X, Y \in \operatorname{mod} A$, the ideal $\Re(X, Y)$ is the set of all the morphisms $f : X \to Y$ such that, for each $M \in \operatorname{ind} A$, each $h : M \to X$ and each $h' : Y \to M$ the composition $h'fh$ is not an isomorphism. For $n \geq 2$, the powers of $\Re(X, Y)$ are defined inductively. By $\Re^\infty(X, Y)$ we denote the intersection of all powers $\Re^i(X, Y)$ of $\Re(X, Y)$, with $i \geq 1$.

By [3], it is known that a morphism $f : X \to Y$, with $X, Y \in \operatorname{ind} A$, is irreducible if and only if $f \in \Re(X, Y) \setminus \Re^2(X, Y)$.

We denote by Γ_A its Auslander-Reiten quiver, by τ the Auslander-Reiten translation and τ^{-1} its inverse.

A path $M_1 \to M_2 \to \dots \to M_n$ of irreducible morphisms with $M_j \in \operatorname{ind} A$ for $j = 1, \dots, n$ and $n \geq 3$ is called *sectional* if for each $j = 3, \dots, n$ we have that $M_{j-2} \not\simeq \tau M_j$.

Following [11], we say that the depth of a morphism $f : M \to N$ in $\operatorname{mod} A$ is infinite if $f \in \Re^\infty(M, N)$; otherwise, the depth of f is the integer $n \geq 0$ for which $f \in \Re^n(M, N)$ but $f \notin \Re^{n+1}(M, N)$. We denote the depth of f by $\operatorname{dp}(f)$.

Next, we state the definition of degree of an irreducible morphism given by S. Liu in [12].

Let $f : X \to Y$ be an irreducible morphism in $\operatorname{mod} A$, with X or Y indecomposable. The *left degree* $d_l(f)$ of f is infinite, if for each integer $n \geq 1$, each module $Z \in \operatorname{ind} A$ and each morphism $g : Z \to X$ with $\operatorname{dp}(g) = n$ we have that $fg \notin \Re^{n+2}(Z, Y)$. Otherwise, the left degree of f is the least natural number m such that there is an A-module Z and a morphism $g : Z \to X$ with $\operatorname{dp}(g) = m$ such that $fg \in \Re^{m+2}(Z, Y)$.

The *right degree* $d_r(f)$ of an irreducible morphism f is dually defined.

For the convenience of the reader we state [7, Theorem 2.26] which will be useful for our further purposes.

Theorem 1.4. *Let A be a finite dimensional k-algebra over an algebraically closed field and $\Gamma \subset \Gamma_A$ be a component with $\alpha(\Gamma) \leq 2$. The following statements hold.*

(1) *If $f = (f_1, f_2) : X \to Y_1 \oplus Y_2$ is an irreducible epimorphism, with $Y_1, Y_2, X \in \Gamma$ then $d_l(f) = d_l(f_1) + d_l(f_2)$.*

(2) *If $f = (f_1, f_2)^t : X_1 \oplus X_2 \to Y$ is an irreducible monomorphism, with $X_1, X_2, Y \in \Gamma$ then $d_r(f) = d_r(f_1) + d_r(f_2)$.*

In order to read the nilpotency index of the radical of any module category, we shall strongly use the following result from [5, Theorem A].

If either $P_a = S_a$ or $I_a = S_a$ then we write $n_a = 0$ and $m_a = 0$, respectively. Otherwise, we consider the irreducible morphisms $\iota_a : \mathrm{rad}(P_a) \hookrightarrow P_a$ and $g_a : I_a \to I_a/\mathrm{soc}(I_a)$ and we write $n_a = d_r(\iota_a)$ and $m_a = d_l(g_a)$.

Theorem 1.5. *Let $A \simeq kQ_A/I_A$ be a finite dimensional algebra over an algebraically closed field and assume that A is of finite representation type. Consider $m = \max\, \{n_a + m_a\}_{a \in Q_0}$. Then $\Re^m(mod\ A) \neq 0$ and $\Re^{m+1}(mod\ A) = 0$.*

1.6. Let A be an algebra such that $A \cong kQ_A/I_A$. The algebra A is called a *string algebra* provided:

(1) Any vertex of Q_A is the starting point of at most two arrows.
(1') Any vertex of Q_A is the ending point of at most two arrows.
(2) Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I_A$.
(2') Given an arrow γ, there is at most one arrow β with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I_A$.
(3) The ideal I_A is generated by a set of paths of Q_A.

Let $A = kQ_A/I_A$ be a string algebra. A *string* in Q_A is either a trivial path ε_v with $v \in Q_0$, or a reduced walk $C = c_n \dots c_1$ of length $n \geq 1$ such that no sub-walk $c_{i+t} \dots c_i$ nor its inverse belongs to I_A. We say that a string $C = c_n \dots c_1$ is *direct (inverse)* provided all c_i are arrows (inverse of arrows, respectively). We consider the trivial walk ε_v a direct as well as an inverse string.

For each string $C = c_n \dots c_1$ in Q_A, an indecomposable string A-module $M(C)$ is defined. Conversely, given M an indecomposable string A-module, there exists a "unique" string C such that $M = M(C) = M(C^{-1})$. The band modules are defined over strings C such that all powers C^n, with $n \in \mathbb{N}$, are defined, see [4]. Every module over a string algebra is defined either as a string module or as a band module, see [4]. Moreover, if A is a representation-finite string algebra, then all the indecomposable A-modules are strings ones.

We say that a string C *starts in a deep (on a peak)* provided there is no arrow β such that $C\beta^{-1}$ ($C\beta$, respectively) is a string. Dually, a string C *ends in a deep (on a peak)* provided there is no arrow β such that βC ($\beta^{-1}C$, respectively) is a string.

1.7. In [4], the authors proved that given a string C not starting (ending) on a peak, then there exists a string $C_h = C\beta C'$ (${}_hC = C''\beta^{-1}C$) with $\beta \in (Q_A)_1$ and C' inverse (C'' direct) such that C_h starts in a deep (${}_hC$ ends in a deep, respectively). Moreover, the canonical embedding $f : M(C) \hookrightarrow M(C_h)$ ($f' : M(C) \hookrightarrow M({}_hC)$) is irreducible, and the cokernel of f (f') is $M(C')$ ($M(C'')$, respectively).

We claim that C_h and ${}_hC$ are unique. In fact, consider C a string not starting on a peak and $\beta \in (Q_A)_1$ such that $C\beta$ is a string. By definition of string algebras the arrow β is unique. If $C\beta$ is a string starting in a deep, then $C_h = C\beta$. Otherwise, there is a unique arrow α_1 such that $C\beta\alpha_1^{-1}$ is a string. If $C\beta\alpha_1^{-1}$ is a string starting in a deep then $C_h = C\beta\alpha_1^{-1}$. Otherwise, we iterate the same argument a finite number of times getting that $C_h = C\beta\alpha_1^{-1} \dots \alpha_r^{-1}$ is a string starting in a deep. Since the arrows β, $\alpha_j, j = 1, ..., r$

are unique then C_h is unique. Similarly, if C is a string not ending on a peak, we can prove that ${}_hC$ is unique.

Dually, if C is a string not starting (ending) in a deep, then there exists a unique string $C_c = C\gamma^{-1}C'$ (${}_cC = C''\gamma C$) with $\gamma \in (Q_A)_1$ and C' direct (C'' inverse) such that C_c (${}_cC$) starts (ends, respectively) on a peak.

By [4] we know that given a string algebra A then $\alpha(\Gamma_A) \leq 2$. Moreover, the authors also described all the almost split sequences of $\operatorname{mod} A$ in terms of strings.

Consider $I(u)$ to be the injective module corresponding to the vertex $u \in (Q_A)_0$. Then, $I(u) = M(D_2D_1)$ where D_1 is a direct string starting on a peak and D_2 is an inverse string ending on a peak. Suppose $D_1 = \gamma_1 \dots \gamma_s$ and $D_2 = \beta_r^{-1} \dots \beta_1^{-1}$. Then, $J_1 = M(\gamma_2 \dots \gamma_s)$ and $J_2 = M(\beta_r^{-1} \dots \beta_2^{-1})$ are the indecomposable direct summands of $I(u)/\operatorname{soc} I(u)$.

Dually, if $P(u)$ is the projective corresponding to $u \in Q_0$ then $P(u) = M(C_2C_1)$ where C_1 is an inverse string and C_2 is a direct string. Moreover, C_2C_1 is a string that starts and ends in a deep.

For a detail account on these algebras, see [4] and for general Auslander–Reiten theory, we refer the reader to [1] and [2].

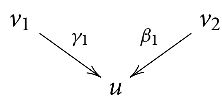
3. On the nilpotency index of the radical of the module category of a string algebra

Throughout this work, we consider A to be a representation-finite string algebra.

In this section, we show how to compute the minimal lower bound $m \geq 1$ such that $\mathfrak{R}^m(\operatorname{mod} A)$ vanishes taking into account the ordinary quiver of A.

Let $A = kQ_A/I_A$ be a representation-finite string algebra. Let I be an indecomposable injective A-module. Throughout all the paper, we may assume that $I = I(u) = M(D_2D_1)$, with $u \in Q_0$, $D_1 = \gamma_1 \dots \gamma_s$ a direct string starting on a peak and $D_2 = \beta_r^{-1} \dots \beta_1^{-1}$ an inverse string ending on a peak, with $e(D_1) = u = s(D_2)$. We denote by v_1 and v_2 the vertices of Q_A such that $v_1 = s(\gamma_1)$ and $v_2 = s(\beta_1)$.

Let $J_1 = M(\gamma_2 \dots \gamma_s)$ and $J_2 = M(\beta_r^{-1} \dots \beta_2^{-1})$ be the indecomposable direct summands of $I/\operatorname{soc} I$. If $I/\operatorname{soc} I$ has exactly two non-zero indecomposable direct summands, then Q_A has a subquiver as follows



Otherwise, if $I/\operatorname{soc} I$ is indecomposable then we consider $D_2 = \varepsilon_u^{-1}$.

We denote by $\overline{J_1}$ (by $\overline{J_2}$, respectively) the A-module such that the quotient of I by $\overline{J_1}$ ($\overline{J_2}$, respectively) is J_2 (J_1, respectively). Observe that $\overline{J_1}$ and $\overline{J_2}$ are indecomposable A-modules. Moreover, $\overline{J_1} = M(D_1)$ and $\overline{J_2} = M(D_2)$.

In a similar way, if P is an indecomposable projective A-module we may assume that $P = P(u) = M(C_2C_1)$, with $u \in Q_0$, $C_1 = \alpha_1^{-1} \dots \alpha_m^{-1}$ an inverse string starting in a deep and $C_2 = \lambda_n \dots \lambda_1$ a direct string ending in a deep, with $e(C_1) = u = s(C_2)$.

Let $R_1 = M(\alpha_2^{-1} \dots \alpha_m^{-1})$ and $R_2 = M(\lambda_n \dots \lambda_2)$ be the indecomposable direct summands of $\operatorname{rad} P$. If $\operatorname{rad} P$ is indecomposable then we consider $C_2 = \varepsilon_u$. We denote by $\overline{R_1}$ (by $\overline{R_2}$, respectively) the A-module which is the quotient of P by R_2 (by R_1, respectively). Again, we observe that $\overline{R_1}$ and $\overline{R_2}$ are indecomposable A-modules. Moreover, $\overline{R_1} = M(C_1)$ and $\overline{R_2} = M(C_2)$.

For each indecomposable A-module Z we define the following sets:

$$\mathcal{M}_Z = \{X \in \operatorname{ind} A \mid Z \text{ is a submodule of } X\}$$

and

$$\mathcal{S}_Z = \{X \in \operatorname{ind} A \mid \exists\ Y \in \operatorname{ind} A \text{ such that } Y \text{ is a submodule of } X \text{ and } X/Y \simeq Z\}.$$

Our next result characterizes the sets $\mathcal{M}_{\overline{J_2}}$ and $\mathcal{S}_{\overline{R_2}}$ in terms of walks of the ordinary quiver of a string algebra. More precisely:

Lemma 2.1. *Let $A = kQ_A/I_A$ be a representation-finite string algebra. Let $I = M(D_2D_1)$ and $P = M(C_2C_1)$ be an indecomposable injective and an indecomposable projective A-module, respectively, where $D_1 = \gamma_1 \dots \gamma_s$ is a direct string starting on a peak, $D_2 = \beta_r^{-1} \dots \beta_1^{-1}$ is an inverse string ending on a peak, $C_1 = \alpha_1^{-1} \dots \alpha_m^{-1}$ is an inverse string starting in a deep and $C_2 = \lambda_n \dots \lambda_1$ is a direct string ending in a deep. Let J_1 and J_2 be the indecomposable direct summands of $I/\mathrm{soc}\, I$ and R_1 and R_2 be the indecomposable direct summands of* rad$P$. *Then,*

(a) $\mathcal{M}_{\overline{J_2}} = \{M(C)$ such that C or C^{-1} belongs to $\mathcal{C}_{D_2}\}$ *where*
$\mathcal{C}_{D_2} = \{D_2D$ where either D is trivial or $D = \gamma_1 D'$ with D' a string$\}$.
(b) $\mathcal{S}_{\overline{R_2}} = \{M(D)$ such that D or D^{-1} belongs to $\mathcal{C}_{C_2}\}$ *where*
$\mathcal{C}_{C_2} = \{C_2C$ where either C is trivial or $C = \alpha_1^{-1}C'$ with C' a string$\}$.

Proof. We only prove statement (a) since (b) follows similarly.

Assume that $X \in \mathcal{M}_{\overline{J_2}}$. Then, $\overline{J_2} \subset X$. If $X = \overline{J_2}$ then $X = M(D_2)$. Otherwise, we consider C a string such that $X = M(C)$. Then, there exists the canonical embedding $M(D_2) \hookrightarrow M(C)$. By [4, p. 166] we have that $C = D_2\alpha D$ for some $\alpha \in Q_1$ and D a string. We claim that $\alpha = \gamma_1$. In fact, since $I = I(u) = M(D_2D_1)$ with $u \in Q_0$ and $u = s(D_2) = e(\alpha)$ then either $\alpha = \beta_1$ or $\alpha = \gamma_1$. If $\alpha = \beta_1$ since $D_2 = \beta_r^{-1} \dots \beta_1^{-1}$ then we get a contradiction to the fact that $C = D_2\beta_1 D$ is a string. Therefore, $\alpha = \gamma_1$. Hence $C \in \mathcal{C}_{D_2}$.

Conversely, it is clear that $M(D_2) = \overline{J_2} \in \mathcal{M}_{\overline{J_2}}$. Without loss of generality, we may consider C a string such that $C = D_2\gamma_1 D$ with D a string. Then, by [4, p. 166] we have that $M(D_2)$ is a submodule of $M(C)$. Therefore, $M(C) \in \mathcal{M}_{\overline{J_2}}$. □

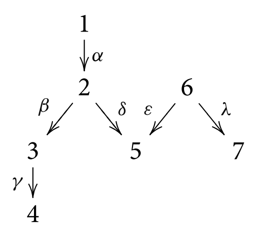
Note that a similar analysis as above can be done for $\overline{J_1}$ and $\overline{R_1}$, since $I = M(D_2D_1) = M(D_1^{-1}D_2^{-1})$ and $P = M(C_2C_1) = M(C_1^{-1}C_2^{-1})$.

Remark 2.2. Given an indecomposable injective A-module I with J_1 and J_2 the direct summands of $I/\mathrm{soc}I$, we have that I and $\overline{J_2}$ are in $\mathcal{M}_{\overline{J_2}}$. Moreover, I is the unique indecomposable injective module that belongs to $\mathcal{M}_{\overline{J_2}}$.

If we consider $C = D_2\gamma_1 D\gamma_1^{-1}D_2^{-1}$ then, D is not trivial, otherwise C is not a reduce walk. Note that C and C^{-1} are different strings in $\mathcal{C}_{D_2}$, but $M(C) \simeq M(C^{-1})$. When $C \neq D_2$, $C \neq D_2\gamma_1 D\gamma_1^{-1}D_2^{-1}$ and $M(C) \in \mathcal{M}_{\overline{J_2}}$ we write $M(C) = M(D_2\gamma_1 D)$ with D a string.

Next, we show an example of how to compute the sets $\mathcal{M}_{\overline{J_2}}$ and $\mathcal{S}_{\overline{R_2}}$.

Example 2.3. Let $A = kQ_A/I_A$ be the string algebra given by the quiver



with $I_A =< \beta\alpha >$. We denote the indecomposable modules by their Loewy series.

Consider $I(5) = M(\varepsilon^{-1}\delta\alpha)$: $\begin{smallmatrix} 1 \\ 2\ 6 \\ 5 \end{smallmatrix}$. Then, $I(5)/\mathrm{soc}I(5) = J_1(5) \oplus J_2(5)$ where $J_1(5) = M(\alpha)$: $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and $J_2(5) = M(\varepsilon_6)$: 6. Hence, $\overline{J_2}(5) = M(\varepsilon^{-1})$: $\begin{smallmatrix} 6 \\ 5 \end{smallmatrix}$. Then, $\mathcal{M}_{\overline{J_2}} = \{X_i\}_{i=1}^{5}$ where $X_1 = \overline{J_2} =$

$M(\varepsilon^{-1}): \begin{array}{c}6\\5\end{array}$, $X_2 = M(\varepsilon^{-1}\delta): \begin{array}{c}2\,6\\5\end{array}$, $X_3 = I(5) = M(\varepsilon^{-1}\delta\alpha): \begin{array}{c}1\\2\,6\\5\end{array}$, $X_4 = M(\varepsilon^{-1}\delta\beta^{-1}): \begin{array}{c}2\,6\\3\,5\end{array}$ and $X_5 = M(\varepsilon^{-1}\delta\beta^{-1}\gamma^{-1}): \begin{array}{c}2\,6\\3\,5\\4\end{array}$.

Now, consider $P(2) = M(\gamma\beta\delta^{-1}): \begin{array}{c}2\\3\,5\\4\end{array}$. Then, rad $P(2) = R_1(2) \oplus R_2(2)$ where $R_1(2) = M(\varepsilon_5): 5$ and $R_2(2) = M(\gamma): \begin{array}{c}3\\4\end{array}$. Hence, $\overline{R_2}(2) = M(\gamma\beta): \begin{array}{c}2\\3\\4\end{array}$. Then $\mathcal{S}_{\overline{R_2}} = \{X_i\}_{i=1}^{4}$ where $X_1 = \overline{R_2} = M(\gamma\beta): \begin{array}{c}2\\3\\4\end{array}$, $X_2 = P(2) = M(\gamma\beta\delta^{-1}): \begin{array}{c}2\\3\,5\\4\end{array}$, $X_3 = M(\gamma\beta\delta^{-1}\varepsilon): \begin{array}{c}2\,6\\3\,5\\4\end{array}$ and $X_4 = M(\gamma\beta\delta^{-1}\varepsilon\lambda^{-1}): \begin{array}{c}2\,6\\3\,5\,7\\4\end{array}$.

In our next proposition, we describe the almost split sequences starting in an indecomposable A-module which belong either to $\mathcal{M}_{\overline{J_2}}$ or to $\mathcal{S}_{\overline{R_2}}$.

Proposition 2.4. *Let $A = kQ_A/I_A$ be a representation-finite string algebra. Let $I = M(D_2D_1)$ and $P = M(C_2C_1)$ be an indecomposable injective and an indecomposable projective A-module, respectively, where $D_1 = \gamma_1 \dots \gamma_s$ is a direct string starting on a peak, $D_2 = \beta_r^{-1} \dots \beta_1^{-1}$ is an inverse string ending on a peak, $C_1 = \alpha_1^{-1} \dots \alpha_m^{-1}$ is an inverse string starting in a deep and $C_2 = \lambda_n \dots \lambda_1$ is a direct string ending in a deep. Let J_1 and J_2 be the indecomposable direct summands of $I/\mathrm{soc}I$ and R_1 and R_2 be the indecomposable direct summands of* rad P.

(a) *Let X be a non-injective module in $\mathcal{M}_{\overline{J_2}}$. Then,*

- (i) *if $X = \overline{J_2}$ then $\alpha'(X) = 1$ and $0 \to \overline{J_2} \xrightarrow{f} X' \xrightarrow{g} Y \to 0$ is an almost split sequence with $X' \in \mathcal{M}_{\overline{J_2}}$.*
- (ii) *if $X = M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$ then $\alpha'(X) = 2$ and $0 \to X \xrightarrow{(f,g)} X_1 \oplus X_2 \xrightarrow{(f',g')^t} Y \to 0$ is an almost split sequence with $X_1, X_2 \in \mathcal{M}_{\overline{J_2}}$ and where f, g, f', g' are epimorphisms, with kernel equal to $\overline{J_2}$.*
- (iii) *if $X \neq \overline{J_2}$ and $X \neq M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$ then $\alpha'(X) = 2$ and $0 \to X \xrightarrow{(f,g)} X_1 \oplus X_2 \xrightarrow{(f',g')^t} Y \to 0$ is an almost split sequence with $X_2 \in \mathcal{M}_{\overline{J_2}}$, where f, g' are epimorphisms with* $\mathrm{Ker}(f) = \mathrm{Ker}(g') = \overline{J_2}$, *and if f', g are epimorphisms then their kernels are not equal to $\overline{J_2}$.*

(b) *Let Y be a non-projective module in $\mathcal{S}_{\overline{R_2}}$. Then,*

- (i) *if $Y = \overline{R_2}$ then $\alpha(Y) = 1$ and $0 \to X \xrightarrow{f} X' \xrightarrow{g} \overline{R_2} \to 0$ is an almost split sequence with $X' \in \mathcal{S}_{\overline{R_2}}$.*
- (ii) *if $Y = M(C_2\alpha_1^{-1}C\alpha_1 C_2^{-1})$ then $\alpha(Y) = 2$ and $0 \to X \xrightarrow{(f,g)} X_1 \oplus X_2 \xrightarrow{(f',g')^t} Y \to 0$ is an almost split sequence with $X_1, X_2 \in \mathcal{S}_{\overline{R_2}}$ and where f, g, f', g' are monomorphisms with cokernel equal to $\overline{R_2}$.*
- (iii) *if $Y \neq \overline{R_2}$ and $Y \neq M(C_2\alpha_1^{-1}C\alpha_1 C_2^{-1})$ then $\alpha(Y) = 2$ and $0 \to X \xrightarrow{(f,g)} X_1 \oplus X_2 \xrightarrow{(f',g')^t} Y \to 0$ is an almost split sequence with $X_1 \in \mathcal{S}_{\overline{R_2}}$, where f, g' are monomorphisms with* $\mathrm{Coker}(f) = \mathrm{Coker}(g') = \overline{R_2}$, *and if f', g are monomorphisms then their cokernels are not equal to $\overline{R_2}$.*

Proof. We only prove Statement (a) since (b) follows dually.

$(a),(i)$. Let $X=\overline{J_2}$. Then, $X=M(D_2)$. Note that D_2 does not start on a peak since $D_2\gamma_1$ is a string. Therefore, $D_2\gamma_1 D$ is defined and it is unique with D an inverse string starting in a deep. By [4, p. 170], there exists an almost split sequence $0\to M(D_2)\xrightarrow{f} M(D_2\gamma_1 D)\xrightarrow{g} M(D)\to 0$ with indecomposable middle term. By Lemma 2.1 (a), the string module $M(D_2\gamma_1 D)$ belongs to $\mathcal{M}_{\overline{J_2}}$.

$(a),(ii)$ Let $C=D_2\gamma_1 D\gamma_1^{-1}D_2^{-1}$ and $X=M(C)$. Since C is a string that starts and ends on a peak, by [4, p. 172] there is an almost split sequence starting in X of the form $0\to M(C)\xrightarrow{(f,g)} M(D_2\gamma_1 D)\oplus M(D\gamma_1^{-1}D_2^{-1})\xrightarrow{(f',g')} M(D)\to 0$. By Lemma 2.1 (a), we have that $M(D_2\gamma_1 D)\in\mathcal{M}_{\overline{J_2}}$ and that $M(D\gamma_1^{-1}D_2^{-1})=M(D_2\gamma_1 D^{-1})\in\mathcal{M}_{\overline{J_2}}$. By [4, p. 168], the morphism $f:M(C)\to M(D_2\gamma_1 D)$ is the canonical projection with $\mathrm{Ker}(f)=\overline{J_2}$. Furthermore, the morphisms g,f',g' are also epimorphisms with kernel $\overline{J_2}$.

$(a),(iii)$ Let $X=M(C)$ such that $X\in\mathcal{M}_{\overline{J_2}}$, $X\neq\overline{J_2}$ and $X\neq M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$. Without loss of generality, we may assume that $C=D_2\gamma_1 D$ with D a string. Since D_2 is a string ending on a peak then either is C. By [4, p. 171], an almost split sequence starting in X depends on the string D. Then, D satisfies one of the following conditions:

(1) D starts on a peak, or
(2) D does not start on a peak.

If D satisfies (1) then D is not a direct string because X is not injective. Therefore, we write $D=D''\alpha^{-1}D'$, with $\alpha\in Q_1$ and D' a direct string. Then, $C=D_2\gamma_1 D''\alpha^{-1}D'$ and by [4, p. 172] there is an almost split sequence $0\to M(C)\to M(D_2\gamma_1 D'')\oplus M(D''\alpha^{-1}D')\to M(D'')\to 0$ with two indecomposable middle terms. By Lemma 2.1 (a), $M(D_2\gamma_1 D'')\in\mathcal{M}_{\overline{J_2}}$.

By [4, p. 166, 168], the morphisms $M(C)\to M(D)$ and $M(D_2\gamma_1 D'')\to M(D'')$ are epimorphisms with kernel equal to $\overline{J_2}$, and the morphisms $M(C)\to M(D_2\gamma_1 D'')$ and $M(D''\alpha^{-1}D')\to M(D'')$ are epimorphisms with kernel equal to $M(D')$. We claim that $M(D')\neq\overline{J_2}$. In fact, if $M(D')=\overline{J_2}$, then either $D'=D_2$ or $D'=D_2^{-1}$.

Assume that $D'=D_2$. Since D' and D_2 are direct and inverse strings, respectively, then D_2 is trivial. We write, $D_2=\varepsilon_u^{-1}$. Then, $C=\varepsilon_u^{-1}\gamma_1 D''\alpha^{-1}\varepsilon_u$. Moreover, since $e(\alpha)=u=e(\gamma_1)$ then $\alpha=\gamma_1$. Therefore, we get a contradiction that $X=M(C)$ is a module of the form $M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$.

Now, if $D'=D_2^{-1}$ then $M(C)=M(D_2\gamma_1 D''\alpha^{-1}D_2^{-1})$ where $\alpha=\beta_1$ because $\alpha\neq\gamma_1$. Then $\alpha^{-1}D_2^{-1}=\beta_1^{-1}D_2^{-1}=\beta_1^{-1}\beta_1\ldots\beta_r$ contradicting that $\alpha^{-1}D_2^{-1}$ is a string. Therefore $M(D')\neq\overline{J_2}$.

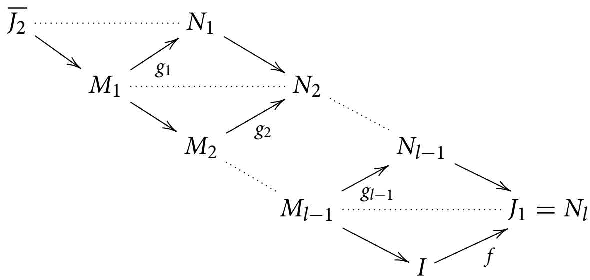
Finally, if D satisfies (2) then D does not start on a peak and neither C does. Then, $D_h=D\alpha D'$ and $C_h=C\alpha D'$ are defined where $\alpha\in Q_1$, see (1.7). Since $C=D_2\gamma_1 D$, by [4, p. 171], there is an almost split sequence $0\to M(C)\to M(D_2\gamma_1 D\alpha D')\oplus M(D)\to M(D\alpha D')\to 0$ with $M(D_2\gamma_1 D\alpha D')\in\mathcal{M}_{\overline{J_2}}$. Moreover, by [4, p. 166, 168], the morphisms $M(C)\to M(D)$ and $M(D_2\gamma_1 D\alpha D')\to M(D\alpha D')$ are epimorphisms with kernel $M(D_2)=\overline{J_2}$, and $M(C)\to M(D_2\gamma_1 D\alpha D')$ and $M(D)\to M(D\alpha D')$ are monomorphisms. □

Given I an indecomposable injective module, the aim of our next result is to determine the left degree of any irreducible morphism $I\to I/\mathrm{soc}\, I$. For such a purpose, we will consider each irreducible epimorphism from I to an indecomposable direct summand of $I/\mathrm{soc}\, I$. We shall apply [9, Lemma 5.1] and prove that the modules involved in the sectional path δ of such a lemma are in $\mathcal{M}_{\overline{J_2}}$. Dually, we can determine the right degree of an irreducible morphism $\mathrm{rad}\, P\to P$.

We also observe that in [6, Proposition 6.1] such a result was generalized for almost pre-sectional paths in artin algebras. Following the above notation we state the next result.

Proposition 2.5. *Let $A\simeq kQ_A/I_A$ be a representation-finite string algebra. The following statements hold.*

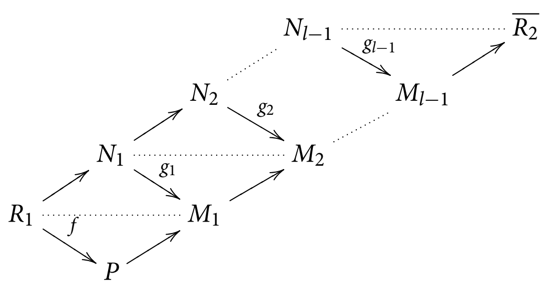
(a) *Let $I=M(D_2D_1)$ be an indecomposable injective A-module and J_1,J_2 be the indecomposable direct summands of $I/\mathrm{soc}\, I$. Let $f:I\to J_1$ be an irreducible epimorphism with $d_l(f)=l\geq 1$. Then, there is* *a configuration of almost split sequences as follows:*



(1)

with $\overline{J_2} \to M_1 \to \cdots \to M_{l-1} \to I$ *a sectional path of length l,* $M_k \in \mathcal{M}_{\overline{J_2}}$ *for* $k = 1, ..., l-1$ *and where* M_k *appears in the sectional path exactly twice if* $M_k = M(D_2\gamma_1 D\gamma_1^{-1} D_2^{-1})$ *and only once if* $M_k \neq M(D_2\gamma_1 D\gamma_1^{-1} D_2^{-1})$*. Moreover, given* $M \in \mathcal{M}_{\overline{J_2}}$ *then either* $M = \overline{J_2}$*,* $M = I$ *or* $M = M_k$ *for some* $k = 1, ..., l-1$*.*

(b) *Let* $P = M(C_2C_1)$ *be an indecomposable projective A-module and* R_1*,* R_2 *be the indecomposable direct summands of* rad P*. Let* $f : R_1 \to P$ *be an irreducible monomorphism with* $d_r(f) = l \geq 1$*. Then, there is a configuration of almost split sequences as follows:*



(2)

with $P \to M_1 \to \cdots \to M_{l-1} \to \overline{R_2}$ *a sectional path of length l,* $M_k \in \mathcal{S}_{\overline{R_2}}$ *for* $k = 1, ..., l-1$ *and where* M_k *appears in the sectional path exactly twice if* $M_k = M(C_2\alpha_1^{-1} C\alpha_1 C_2^{-1})$ *and only once if* $M_k \neq M(C_2\alpha_1^{-1} C\alpha_1 C_2^{-1})$*. Moreover, given* $M \in \mathcal{S}_{\overline{R_2}}$ *then either* $M = P$*,* $M = \overline{R_2}$ *or* $M = M_k$ *for some* $k = 1, ..., l-1$*.*

Proof. We only prove Statement (a) since (b) follows dually.

(a) Let $f : I \to J_1$ be the canonical projection with $d_l(f) = l \geq 1$. By [9, Lemma 5.1], since $\mathrm{Ker}(f) = \overline{J_2}$, there is a configuration of almost split sequences as in (1), where $M_k = \tau N_{k+1}$ for $k = 1, ..., l-1$, $\delta : \overline{J_2} \to M_1 \to \cdots \to M_{l-1} \to I$ is a sectional path of length l such that $f\delta = 0$ and $\alpha'(\overline{J_2}) = 1$.

First, we prove that each M_k belongs to $\mathcal{M}_{\overline{J_2}}$, for $k = 1, ..., l-1$. We prove it by induction on the left degree of f. If $d_l(f) = 1$, then there is an almost split sequence $0 \to \overline{J_2} \to I \xrightarrow{f} J_1 \to 0$ with indecomposable middle term. Since $\overline{J_2} \in \mathcal{M}_{\overline{J_2}}$ then by Proposition 2.4 (a), there is a unique (up to isomorphisms) almost split sequence with indecomposable middle term starting in $\overline{J_2}$ and moreover, with $I \in \mathcal{M}_{\overline{J_2}}$.

Now, if $l > 1$ by inductive hypothesis $M_1, \ldots, M_{l-2}$ belong to $\mathcal{M}_{\overline{J_2}}$. Let us prove that M_{l-1} belongs to $\mathcal{M}_{\overline{J_2}}$. Let $0 \longrightarrow M_{l-2} \xrightarrow{(g_{l-2}, f_{l-1})^T} N_{l-2} \oplus M_{l-1} \xrightarrow{(t_{l-1}, g_{l-1})} N_{l-1} \longrightarrow 0$ be an almost split sequence starting in M_{l-2}. By Proposition 2.4 (a), at least one of the modules N_{l-2} or M_{l-1} belong to $\mathcal{M}_{\overline{J_2}}$. If both modules belong to $\mathcal{M}_{\overline{J_2}}$ then nothing to prove. Otherwise, by Proposition 2.4 (a), (ii), the module M_{l-2} is not of the form $M(D_2\gamma_1 D\gamma_1^{-1} D_2^{-1})$. We write $M_{l-2} = M(D_2\gamma_1 C_{l-2})$.

We claim that $M_{l-1} \in \mathcal{M}_{\overline{J_2}}$. In fact, the morphism $g_{l-2} : M_{l-2} \to N_{l-2}$ is an irreducible epimorphism with kernel $\overline{J_2}$. By [4, p. 166], we have that $M(D_2\gamma_1C_{l-2})/M(D_2) \simeq M(C_{l-2})$ and therefore $Im(g_{l-2}) \simeq M(C_{l-2})$. Hence, $N_{l-2} \simeq M(C_{l-2})$.

Suppose that $N_{l-2} \in \mathcal{M}_{\overline{J_2}}$. By Lemma 2.1 (*a*), we have that $C_{l-2} = D_2C'_{l-2}$ (where either C'_{l-2} is trivial or $C'_{l-2} = \gamma_1C''_{l-2}$ with C''_{l-2} a string) or $C_{l-2} = C'_{l-2}D_2^{-1}$ (where either C'_{l-2} is trivial or $C'_{l-2} = C''_{l-2}\gamma_1^{-1}$ with C''_{l-2} a string). Note that $C_{l-2} \neq C''_{l-2}\gamma_1^{-1}D_2^{-1}$ because by our assumption $M_{l-2} \neq M(D_2\gamma_1C''_{l-2}\gamma_1^{-1}D_2^{-1})$. Moreover, D_2 is not trivial. Indeed, if D_2 is trivial then $D_2 = \varepsilon_u^{-1}$ and the only arrow arriving at the vertex u is γ_1. Since $M_{l-2} = M(D_2\gamma_1C_{l-2})$ then $M_{l-2} = M(\varepsilon_u^{-1}\gamma_1\mathcal{C}_{l-2})$.

Now assume that C_{l-2} is trivial then $C_{l-2} = \varepsilon_u$. If C_{l-2} starts on a peak, then M_{l-2} is injective, since $C_{l-2} = \varepsilon_u$ is a direct string, which is an absurd. If C_{l-2} does not start on a peak, then $(C_{l-2})_h = C_{l-2}\gamma_1D' = \varepsilon_u\gamma_1D'$ is defined and the almost split sequence starting in M_{l-2} is as follows

$$0 \longrightarrow M(\varepsilon_u^{-1}\gamma_1\varepsilon_u) \longrightarrow M(\varepsilon_u^{-1}\gamma_1\varepsilon_u\gamma_1D') \oplus M(\varepsilon_u) \longrightarrow M(\varepsilon_u\gamma_1D') \longrightarrow 0$$

where both indecomposable middle terms belong to $\mathcal{M}_{\overline{J_2}}$, which is not in the hypothesis of this statement.

Now, if C_{l-2} is not trivial then $C_{l-2} = \varepsilon_u^{-1}\gamma_1C''_{l-2}$ and $M_{l-2} = M(\varepsilon_u^{-1}\gamma_1\varepsilon_u^{-1}\gamma_1C''_{l-2})$. With a similar analysis as before we get that either M_{l-2} is injective, (if C''_{l-2} is a direct string starting on a peak), or the almost split sequence starting in M_{l-2} has both indecomposable middle terms in $\mathcal{M}_{\overline{J_2}}$, a contradiction in both cases.

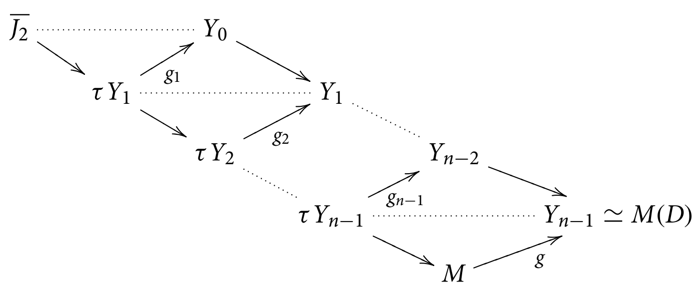
Next, we continue analyzing the other cases assuming that D_2 is not trivial. We will discard them proving that we can construct a band module.

If $C_{l-2} = D_2C'_{l-2}$, since $M_{l-2} = M(D_2\gamma_1C_{l-2})$ then $D_2\gamma_1D_2$ is a string. Therefore, for all the positive integer n, $(D_2\gamma_1)^n$ is defined, getting a band module and contradicting that A is representation-finite.

Now, if $C_{l-2} = D_2^{-1}$ then $D_2\gamma_1D_2^{-1}$ is a string. Since, no sub-walk of $D_2\gamma_1D_2^{-1}$ belongs to I_A, then all the natural powers of the string $D_2\gamma_1D_2^{-1}\gamma_1^{-1}$ are defined, contradicting again that A is representation-finite.

Therefore, we prove that $N_{l-2} \notin \mathcal{M}_{\overline{J_2}}$ and hence $M_{l-1} \in \mathcal{M}_{\overline{J_2}}$.

Now, consider $M_0 = \overline{J_2}$ and $M_l = I$. Let us prove that if $M \in \mathcal{M}_{\overline{J_2}}$ then $M = M_k$ for some $k = 0, ..., l$. By Lemma 2.1, if $M \in \mathcal{M}_{\overline{J_2}}$ then $M = M(D_2)$ or $M = M(D_2\gamma_1D)$, with D a string. In case $M = M(D_2)$, then $M = M_0$ proving the statement. Otherwise, by [4, p. 169] the canonical projection $g : M(D_2\gamma_1D) \to M(D)$ is an irreducible epimorphism with $\mathrm{Ker}(g) = \overline{J_2}$. Since A is representation-finite, then by [10, Theorem A] we have that $d_l(g) = n < \infty$. Moreover, by [6, Proposition 6.1], there exists a configuration of almost split sequences as follows:



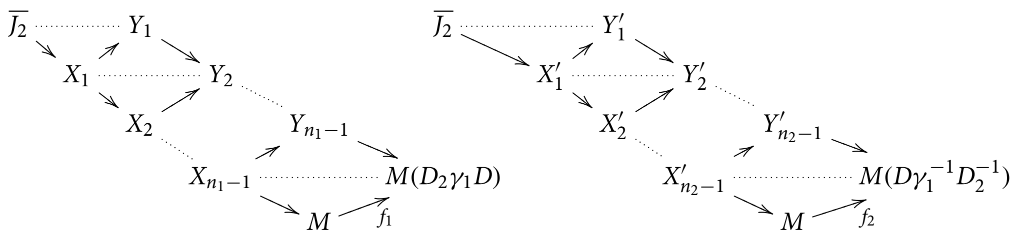
where $\overline{J_2} \to \tau Y_1 \to \cdots \to \tau Y_{n-1} \to M$ is a sectional path.

On the other hand, the irreducible epimorphism $f : I \to J_1$ has $d_l(f) = l$. Moreover, $\mathrm{Ker}(f) = \overline{J_2}$. Then, there is a configuration of almost split sequences as in (1).

We claim that $n \leq l$. Indeed, if $n > l$ then $M_l = I$ and $M_i \simeq \tau Y_i$ for $1 \leq i \leq l$. Since both mentioned configurations involve almost split sequences starting in the same modules, we get to the contradiction that $M_l \simeq \tau Y_l$ but M_l is an injective module. Therefore $n \leq l$. Hence, we prove that $M = M_n$.

Finally, we determine the number of non-isomorphic modules M_k in the sectional path $\overline{J_2} \to M_1 \to \cdots \to M_{n-1} \to I$ in (1). We shall prove that the modules of the form $M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$ appear exactly twice in (1) and that the other modules M_k in (1) are pairwise non-isomorphic.

Consider $M = M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$. By Proposition 2.4 $(a), (ii)$ the irreducible morphisms $f_1 : M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1}) \to M(D_2\gamma_1 D)$ and $f_2 : M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1}) \to M(D\gamma_1^{-1}D_2^{-1})$ are epimorphisms such that $\mathrm{Ker}(f_1) = \mathrm{Ker}(f_2) = \overline{J_2}$. Since A is of finite representation type, then by [10, Theorem A], we have that $d_l(f_1) = n_1 \leq l$ and $d_l(f_2) = n_2 \leq l$. Therefore, there exist two configurations of almost split sequences as follows:



If $n_1 = n_2$, by the uniqueness (up to isomorphisms) of the almost split sequences we infer that $Y_i \simeq Y'_i$ for all $1 \leq i \leq n_1$. But $Y_{n_1} \simeq M(D_2\gamma_1 D) \not\simeq M(D\gamma_1^{-1}D_2^{-1}) \simeq Y'_{n_2}$. Then $n_1 \neq n_2$.

Without loss of generality, we may assume that $n_1 < n_2$. Hence, $M \simeq M_{n_1}$ and $M \simeq M_{n_2}$, with $1 \leq n_1 < n_2 \leq l$, proving that at least M appears twice as a module of the sectional path in (1).

Now suppose that $M \simeq M_k$ for some $k \leq l$, $k \neq n_1$ and $k \neq n_2$. The irreducible epimorphism $g_k : M_k \to N_k$ is such that $d_l(g_k) = k$. Since either $N_k \simeq M(D_2\gamma_1 D)$ or $N_k \simeq M(D\gamma_1^{-1}D_2^{-1})$ then $d_l(g_k) = n_1$ or $d_l(g_k) = n_2$, contradicting our assumption that $k \neq n_1$ and $k \neq n_2$. Therefore, we prove that $M = M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$ appears exactly twice in the sectional path $\overline{J_2} \to M_1 \to \cdots \to M_{l-1} \to I$ of the configuration (1).

Assume that $M \simeq \overline{J_2}$, $M \simeq I$ or $M \simeq M(D_2\gamma_1 D)$. In the first and second case, M appears only once in (1) because there is only one almost split sequence starting in a module in $\mathcal{M}_{\overline{J_2}}$ with indecomposable middle term and there is a unique indecomposable injective module in $\mathcal{M}_{\overline{J_2}}$, respectively. In the third case, $0 \to M \to M' \oplus N' \to N \to 0$ is an almost split sequence with $\alpha'(M) = 2$ and $M' \in \mathcal{M}_{\overline{J_2}}$.

Suppose that $M = M_k$ and $M = M_j$ with $1 \leq k < l$, $1 \leq j < l$ and $k \neq j$. Then, the almost split sequences $0 \to M_k \to M_{k+1} \oplus N_k \to N_{k+1} \to 0$ and $0 \to M_j \to M_{j+1} \oplus N_j \to N_{j+1} \to 0$ are isomorphic. By Proposition 2.4, $(a), (iii)$ we know that $M_{k+1} \to N_{k+1}$ and $M_{j+1} \to N_{j+1}$ are irreducible epimorphisms with kernel equal to $\overline{J_2}$, and also $N_k \to N_{k+1}$ and $N_j \to N_{j+1}$ are either irreducible monomorphisms or if they are epimorphisms then their kernels are not $\overline{J_2}$. Therefore, the morphisms $M_{k+1} \to N_{k+1}$ and $M_{j+1} \to N_{j+1}$ are isomorphic and hence $k = d_l(g_k) = d_l(g_j) = j$, contradicting our assumption that $k \neq j$. Therefore, in these cases M appears only once in (1), proving the result. □

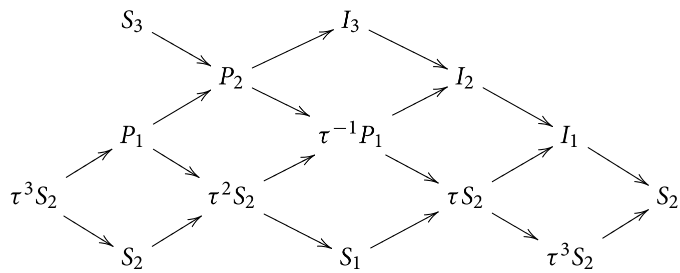
Next, we show two examples where in the above mentioned sectional path some modules appear twice.

Example 2.6.

(a) Consider the string algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\gamma} 3$$

with the relation $\beta\alpha = 0$. The Auslander-Reiten quiver is the following:



where we identify the same modules in the above quiver.

Following the notation of the above proposition, if we consider the injective corresponding to the vertex 3, we have that $I_3 = M(D_2D_1)$ with $D_2 = \varepsilon_3^{-1}$. Hence, the indecomposable projective $P_2 = M(\varepsilon_3^{-1}\gamma\alpha\beta\gamma^{-1}\varepsilon_3)$. Then, as we can see, in the sectional path

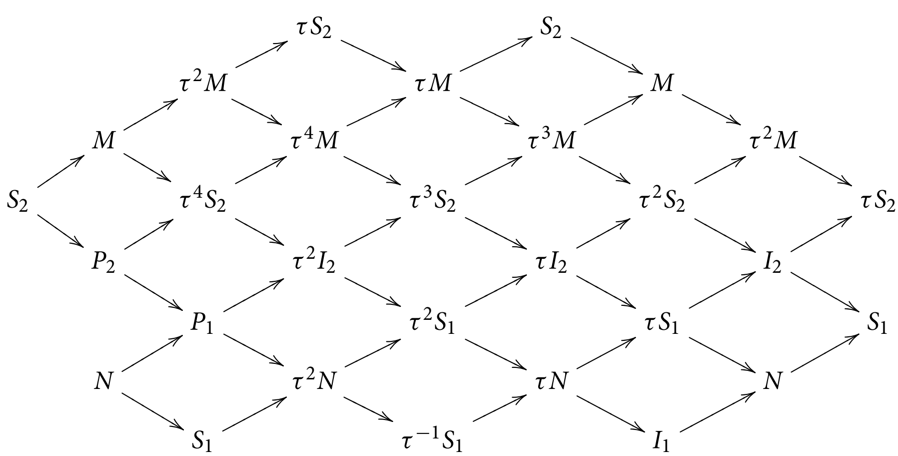
$$S_3 \to P_2 \to \tau^{-1}P_1 \to \tau S_2 \to \tau^3S_2 \to P_1 \to P_2 \to I_3$$

the projective P_2 appears twice and the other modules only once.

(b) Consider the string algebra given by the quiver

$$\alpha \circlearrowleft 1 \xrightarrow{\beta} 2 \circlearrowright \gamma$$

with the relations $\alpha^3 = 0$, $\gamma^2 = 0$ and $\beta\alpha = 0$. The Auslander-Reiten quiver is the following:



where we identify the same modules in the above quiver.

If we consider the injective corresponding to the vertex 1, we have that $I_1 = M(D_2D_1)$ with $D_2 = \varepsilon_1^{-1}$ and $D_1 = \alpha^2$. Then, the sectional path mentioned in the above proposition starts in S_1 and ends in I_1 and is of length 20. Note that the modules $\tau^3S_2 = M(\varepsilon_1^{-1}\alpha^2\beta^{-1}\gamma^{-1}\beta\alpha^{-1}\varepsilon_1)$, $\tau^2I_2 = M(\varepsilon_1^{-1}\alpha^2\beta^{-1}\gamma^{-1}\beta\alpha^{-2}\varepsilon_1)$, $\tau I_2 = M(\varepsilon_1^{-1}\alpha\beta^{-1}\gamma^{-1}\beta\alpha^{-1}\varepsilon_1)$ and $\tau^2S_1 = M(\varepsilon_1^{-1}\alpha\beta^{-1}\gamma^{-1}\beta\alpha^{-2}\varepsilon_1)$ appear twice, while the other modules appear only once.

Now, we are in position to prove one of the main results of this section.

Theorem 2.7. *Let A be a representation-finite string algebra. Let I and P be indecomposable injective and projective A-modules, respectively. Let J_1 and J_2 be indecomposable direct summands of $I/\operatorname{soc} I$ and R_1 and*

R_2 be indecomposable direct summands of rad P. *Then,*
(a) $d_l(I \to J_1) = \text{card}(\mathcal{C}_{D_2}) - 1$ *and* $d_l(I \to J_2) = \text{card}(\mathcal{C}_{D_1}) - 1$.
(b) $d_r(R_1 \to P) = \text{card}(\mathcal{C}_{C_2}) - 1$ *and* $d_r(R_2 \to P) = \text{card}(\mathcal{C}_{C_1}) - 1$.

Proof. The result follows from [9, Lemma 5.1], Proposition 2.5. □

By Remark 2.2, we know that for different strings C and C^{-1} in $\mathcal{C}_{D_2}$ or in $\mathcal{C}_{D_1}$ we get the same string modules in $\mathcal{M}_{\overline{J_2}}$ or in $\mathcal{M}_{\overline{J_1}}$, respectively. The above theorem can be state taking into account the modules instead of the strings as we shown below.

Consider the sets

$$\mathcal{M}_1 = \{M(C) \mid C = D_1^{-1}\beta_1 D\beta_1^{-1}D_1 \text{ with } D \text{ a non-trivial string}\},$$

$$\mathcal{M}_2 = \{M(C) \mid C = D_2\gamma_1 D\gamma_1^{-1}D_2^{-1} \text{ with } D \text{ a non-trivial string}\},$$

$$\mathcal{S}_1 = \{M(C) \mid C = C_1^{-1}\lambda_1^{-1}D\lambda_1 C_1 \text{ with } D \text{ a non-trivial string}\} \quad \text{and}$$

$$\mathcal{S}_2 = \{M(C) \mid C = C_2\alpha_1^{-1}D\alpha_1 C_2^{-1} \text{ with } D \text{ a non-trivial string}\}.$$

Then, we state Theorem 2.7 as follows:

Theorem 2.8. *Let A be a representation-finite string algebra. Let I and P be indecomposable injective and projective A-modules, respectively. Let J_1 and J_2 be indecomposable direct summands of $I/\text{soc}\, I$ and R_1 and R_2 be indecomposable direct summands of rad P. Then,*
(a) $d_l(I \to J_1) = \text{card}(\mathcal{M}_{\overline{J_2}} - \mathcal{M}_2) + 2\ \text{card}(\mathcal{M}_2) - 1$ *and* $d_l(I \to J_2) = \text{card}(\mathcal{M}_{\overline{J_1}} - \mathcal{M}_1) + 2\ \text{card}(\mathcal{M}_1) - 1$.
(b) $d_r(R_1 \to P) = \text{card}(\mathcal{S}_{\overline{R_2}} - \mathcal{S}_2) + 2\ \text{card}(\mathcal{S}_2) - 1$ *and* $d_r(R_2 \to P) = \text{card}(\mathcal{S}_{\overline{R_1}} - \mathcal{S}_1) + 2\ \text{card}(\mathcal{S}_1) - 1$.

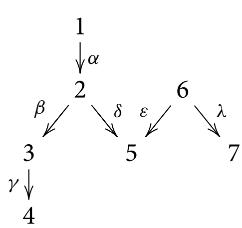
Remark 2.9. In case we do not have a string module of the form $M(C) = M(D_2\gamma_1 D\gamma_1^{-1}D_2^{-1})$ then all modules in the sectional are pairwise non-isomorphic. Then, $d_l(I \to J_1) = \text{card}(\mathcal{M}_{\overline{J_2}}) - 1$.

As an immediate consequence of Theorem 2.7 and [7, Theorem 2.26], we get the following result.

Corollary 2.10. *Let A be a representation-finite string algebra. Let I and P be indecomposable injective and projective A-modules, respectively. Let J_1 and J_2 be indecomposable direct summands of $I/\text{soc}\, I$ and R_1 and R_2 be indecomposable direct summands of radP. Then,*
(a) $d_l(I \to I/\text{soc}\, I) = \text{card}(\mathcal{C}_{D_2}) + \text{card}(\mathcal{C}_{D_1}) - 2$.
(b) $d_r(\text{rad}P \to P) = \text{card}(\mathcal{C}_{C_2}) + \text{card}(\mathcal{C}_{C_1}) - 2$.

Now, we show an example how to compute the nilpotency index of the radical of a module category of a representation finite string algebra, taking into account the ordinary quiver Q_A and [5, Theorem 2.5].

Example 2.11. Let $A = kQ_A/I_A$ be the string algebra given by the quiver



with $I =< \beta\alpha >$.

Let $I(u)$ and $P(u)$ be the injective and projective A-modules corresponding to the vertex u, respectively. Let $J_1(u)$, $J_2(u)$ and $R_1(u)$, $R_2(u)$ be the direct summands of $I(u)/\mathrm{soc}I(u)$ and of $\mathrm{rad}\,P(u)$, respectively.

For $i = 1, 2$ we denote by $m_i(u) = \mathrm{card}(\mathcal{M}_{\overline{J_i(u)}}) = \mathrm{card}(\mathcal{C}_{D_i(u)})$ and by $s_i(u) = \mathrm{card}(\mathcal{S}_{\overline{R_i(u)}}) = \mathrm{card}(\mathcal{C}_{C_i(u)})$. Consider $f_u : I(u) \to I(u)/\mathrm{soc}I(u)$, $g_u : \mathrm{rad}\,P(u) \to P(u)$ and $r(u) = d_l(f_u) + d_r(g_u)$. Computing $m_i(u)$ and $s_i(u)$ for each vertex $u \in Q_0$ we get the following results:

u	$m_1(u)$	$m_2(u)$	$d_l(f_u)$	$s_1(u)$	$s_2(u)$	$d_r(g_u)$	$r(u)$
1	1	1	—	1	5	4	4
2	1	2	1	3	4	5	6
3	1	5	4	1	2	1	5
4	1	6	5	1	1	—	5
5	5	3	6	1	1	—	6
6	1	1	—	6	2	6	6
7	7	1	6	1	1	—	6

Hence, by Theorem 1.5 we get that $\mathfrak{R}^7(\mathrm{mod}\,A) = 0$.

4. How to read degrees from the ordinary quiver

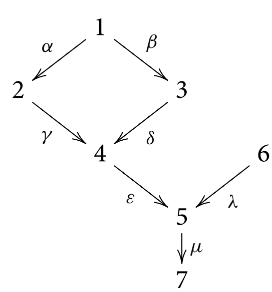
Let $A = kQ/I$ be a representation-finite string algebra. For each $u \in Q_0$ we define the quivers Q_u^e and Q_u^s as follow:

(a) (a) The vertices $(Q_u^e)_0$ are the strings C in Q such that $e(C) = u$, where C is either the trivial walk ε_u or $C = \alpha C'$, with $\alpha \in Q_1$.
 (b) If $a = C$ and $b = C'$ are two vertices of $(Q_u^e)_0$, then there is an arrow from $a \to b$ in Q_u^e if C' is the reduced walk of $C\beta^{-1}$, for some $\beta \in Q_1$.

(b) (a)The vertices of $(Q_u^s)_0$ are the strings C in Q such that $s(C) = u$, where C is either the trivial walk ε_u or $C = C'\alpha$, with $\alpha \in Q_1$.
 (b)If $a = C$ and $b = C'$ are two vertices of $(Q_u^s)_0$, then there is an arrow from $a \to b$ in Q_u^s if C' is the reduced walk of βC, for some $\beta \in Q_1$.

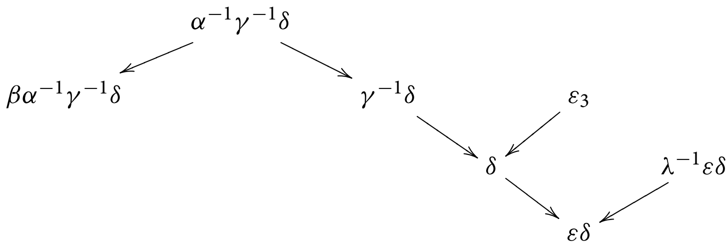
Next, we present an example that shows that these new quivers are not necessarily sub-quivers of Q_A.

Example 3.1. Let $A = kQ/I$ be the string algebra given by the quiver

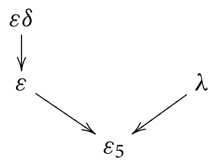


with $I =< \delta\beta, \varepsilon\gamma, \mu\varepsilon >$.

Consider $u = 3$. Then, the quiver Q_3^s is the following

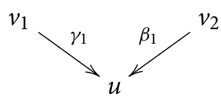


Observe that Q_3^s is not a sub-quiver of Q, but if we consider $u = 5$ we get that Q_5^e is a subquiver of Q as we show below.



Proposition 3.2. *Let $A = kQ/I$ be a representation-finite string algebra. Let $I = I(u)$ and $P = P(u)$ be the injective and the projective A-modules corresponding to the vertex $u \in Q_0$, respectively. Then,*
(a) $d_l(I \to I/\mathrm{soc}\, I) = \mathrm{card}((Q_u^e)_0) - 1$.
(b) $d_r(\mathrm{rad}P \to P) = \mathrm{card}((Q_u^s)_0) - 1$.

Proof. We only prove Statement (a) since (b) follows similarly. We consider the general case, that is, when Q has a subquiver of the form:



The string corresponding to the vertices of Q_u^e are of the form:
(i) $C_0 = \varepsilon_u$,
(ii) $C_1 = \gamma_1 C_1'$ with C_1' a string,
(iii) $C_2 = \beta_1 C_2'$ with C_2' a string.

Observe that there is a bijection between the strings given in *(ii)* and the strings $D_2C_1 \in \mathcal{C}_{D_2} - \{D_2\}$. We also observe that there is a bijection between the strings given in *(iii)* and the strings $D_1C_2 \in \mathcal{C}_{D_1} - \{D_1\}$.

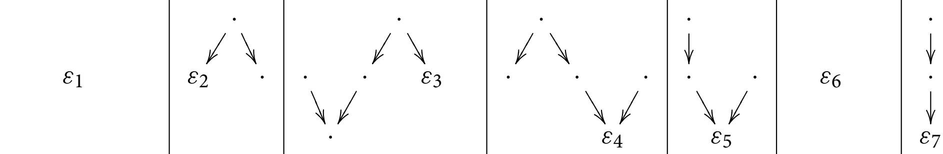
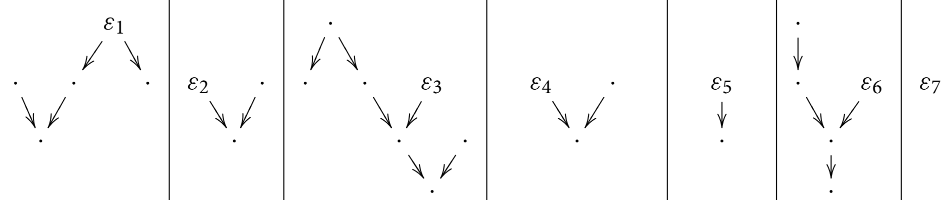
Hence, by Lemma 2.5 we have that

$$\begin{aligned} d_l(I \to I/soc\, I) &= \mathrm{card}(\mathcal{C}_{D_2}) - 1 + \mathrm{card}(\mathcal{C}_{D_1}) - 1 \\ &= \mathrm{card}(\mathcal{C}_{D_2} - \{D_2\}) + \mathrm{card}(\mathcal{C}_{D_1} - \{D_1\}) \\ &= \mathrm{card}((Q_u^e)_0) - 1, \end{aligned}$$

since we are not considering in such a bijection the string $C_0 = \varepsilon_u$. □

Example 3.3. Consider A to be the string algebra given in Example 3.1. By the above result we have that $d_l(I(5) \to I(5)/\mathrm{soc}I(5)) = 3$ since Q_5^e has four vertices, and $d_r(\mathrm{rad}P(3) \to P(3)) = 6$ since Q_3^s has seven vertices.

Next, for each vertex $u \in Q_0$ we show the quivers Q^e_u and Q^s_u. Moreover, we compute the left and right degrees of the irreducible morphisms $f_u : I(u) \to I(u)/\mathrm{soc}I(u)$ and $g_u : \mathrm{rad}\,P(u) \to P(u)$, respectively. We denote by $r(u) = d_l(f_u) + d_r(g_u)$.

u	1	2	3	4	5	6	7
Q^e_u							
$d_l(f_u)$	–	2	4	4	3	–	2
Q^s_u							
$d_r(g_u)$	4	2	6	2	1	4	–
$r(u)$	4	4	10	6	4	4	2

The maximum $\{r(u)\}_{u\in Q_0}$ is given by the vertex $u = 3$. Then, by Theorem 1.5, we infer that $\Re^{11}(\mathrm{mod}\,A) = 0$.

4.1. The degrees of irreducible morphisms in a string algebra

Consider $A \simeq kQ_A/I_A$ a representation-finite string algebra and $I = M(D_2D_1)$ an indecomposable injective A-module with J_1 and J_2 direct summands of $I/\mathrm{soc}\,I$. Assume that $d_l(I \to J_1) = n$. By Theorem 2.7 we have that $\mathrm{card}(\mathcal{C}_{D_2}) = n + 1$.

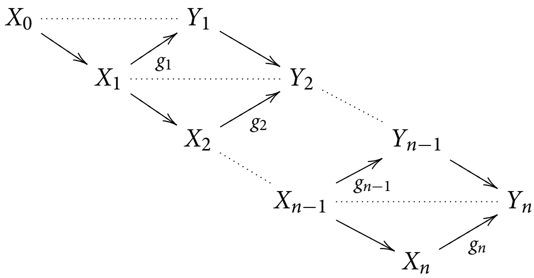
We can give an order to the elements of the set $\mathcal{C}_{D_2} = \{C_0, \dots, C_n\}$. We say that $C_i < C_{i+1}$ for $i = 0, \dots n - 1$, if there is an irreducible morphism from $M(C_i)$ to $M(C_{i+1})$.

We recall that if $C \in \mathcal{C}_{D_2}$ then C is a string ending on a peak.

Let $C_0 = D_2$ and $C_1 = (D_2)_h = D_2\gamma_1C'_1$ with C'_1 an inverse string starting in a deep. We define the following strings inductively.

Consider $C_i = D_2\gamma_1C'_i$. If C'_i does not start on a peak then we choose $C_{i+1} = D_2\gamma_1C'_{i+1}$ with $C'_{i+1} = (C'_i)_h$, that is, $C'_{i+1} = C'_i\beta C''_i$ with β an arrow and C''_i an inverse string starting in a deep, as explained in (1.7). Therefore, by [4] there is an irreducible monomorphism from $M(C_i)$ to $M(C_{i+1})$. If C'_i starts on a peak but it is not a direct string then C'_i is of the form $C'_i = C''_i\alpha^{-1}C'''_i$, where $\alpha \in Q_1$ and C'''_i is a direct string. Then, we choose $C_{i+1} = D_2\gamma_1C'_{i+1}$ with $C'_{i+1} = C''_i$. Again, by [4] there is an irreducible epimorphism from $M(C_i)$ to $M(C_{i+1})$. Otherwise, if C'_i starts on a peak and it is a direct string, then $M(C_i)$ is the injective module of $\mathcal{M}_{\overline{J_2}}$. By Lemma 2.1 we know that in both cases $C_{i+1} \in \mathcal{C}_{D_2}$. Following this construction we have that the last module $M(C_n)$ is the injective module of $\mathcal{M}_{\overline{J_2}}$. Therefore, $M(C_n) = M(D_2D_1)$.

We denote by $M(C_i) = X_i$ for $i = 0, \dots, n$. In this way, we construct the sectional path of Proposition 2.5. Moreover, we have a configuration of almost split sequences as follows:



where $d_l(g_i : X_i \to Y_i) = i$, for $i = 1, \dots, n$.

Now we prove that given an irreducible epimorphism $f : M \to N$ between indecomposable A-modules there is an indecomposable injective A-module such that for some $i = 0, ..., n-1$, in the above configuration of almost split sequences, $f = g_i$.

Lemma 3.4. *Let $A \simeq kQ_A/I_A$ be a representation-finite string algebra and $f : M \to N$ an irreducible epimorphism with $M, N \in \text{ind}\, A$. Then, there exists $u \in (Q_A)_0$ such that $\text{Ker}(f) = \overline{J_2}(u)$, where $I(u)$ is the injective A-module corresponding to the vertex u and $J_1(u)$, $J_2(u)$ are the direct summands of $I/\text{soc}\, I(u)$. Then, $M \in \mathcal{M}_{\overline{J_2}(u)}$.*

Proof. Let $f : M \to N$ be an irreducible epimorphism with $M = M(C)$, $N = M(C')$. If C' is a string starting and ending in a deep, then $C' =_h (C'')_h$ for some string C'' and $M(C')$ can not be the codomain of an irreducible epimorphism, see [4, p.166, p.168]. Then, without loss of generality we may assume that C' is a string not ending in a deep (if not we consider C'^{-1}). Then, by [4, p.169] C is of the form $C = D\alpha C' =_c C'$ with $\alpha \in Q_1$ and D an inverse string ending on a peak. Moreover, $\text{Ker}(f) \simeq M(D)$.

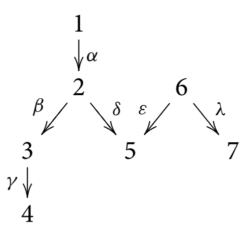
Now, consider $u \in Q_0$ such that $s(D) = u$. The injective A-module corresponding to the vertex u is of the form $I(u) = M(D_2 D_1)$ with D_1 a direct string starting on a peak with $e(D_1) = u$ and D_2 is an inverse string ending on a peak with $s(D_2) = u$. By the uniqueness of such string, $D = D_2$. Moreover, if $J_1(u)$ and $J_2(u)$ are direct summands of $I/\text{soc}\, I(u)$ then $\overline{J_2}(u) = M(D) = \text{Ker}(f)$. Furthermore, by definition $M \in \mathcal{M}_{\overline{J_2}(u)}$, proving the result. □

Remark 3.5. By the above lemma for any irreducible epimorphism between indecomposable A-modules, $f : M \to N$, we have that $M \in \mathcal{M}_{\overline{J_2}(u)}$ for some $u \in Q_0$. If M appears once in the configuration of almost split sequences described above, that is, $M \simeq X_k$ for some $1 \leq k \leq n$, then $d_l(f) = k$. Otherwise, if $M \simeq X_k$ and $M \simeq X_j$ with $1 \leq k < j \leq n$, we have to consider the module N. If $N \simeq X_{k+1}$, then $d_l(f) = j$. Otherwise, $N \simeq Y_k$ and $d_l(f) = k$.

In a similar way we can read the right degree of any irreducible monomorphism $g : M \to N$, giving the same order to the set $\mathcal{C}_{C_2}$, where $M(C_2)$ is the cokernel of g.

Next, we show an example of how to compute the left degree of an irreducible morphism.

Example 3.6. Let $A \simeq kQ_A/I_A$ be the algebra given in Example 2.3.



with $I =< \beta\alpha >$.

Consider the irreducible epimorphism $f : M \to N$, where $M : \begin{array}{c} 2\,6 \\ 3\,5 \end{array}$ and $N : \begin{array}{c} 2 \\ 3 \end{array}$.

We write $M = M(C)$ and $N = M(C')$ with $C = \varepsilon^{-1}\delta\beta^{-1}$ and $C' = \beta^{-1}$. Observe that $C = \varepsilon^{-1}\delta C'$, therefore following the above construction we have that ε^{-1} is the inverse string that ends on a peak and the arrow $\delta = \gamma_1$. Moreover, $\mathrm{Ker}(f) = M(\varepsilon^{-1}) : \begin{array}{c} 6 \\ 5 \end{array}$.

We denote by $\overline{J_2}(5)$ the A-module $M(\varepsilon^{-1})$ and we order the set $\mathcal{M}_{\overline{J_2}(5)}$ as follows; we consider $X_0 = M(\varepsilon^{-1})$ and $X_1 = M(\varepsilon^{-1}\delta C_1)$ with C_1 an inverse string starting in a deep. Therefore, $X_1 = M(\varepsilon^{-1}\delta\beta^{-1}\gamma^{-1})$. Since C_1 starts on a peak but is not a direct string, we write $C_1 = \beta^{-1}\gamma^{-1}\varepsilon_4$, where ε_4 is the direct string. Therefore, we choose $X_2 = M(\varepsilon^{-1}\delta C_2)$ with $C_2 = \beta^{-1}$. Again, C_2 starts on a peak but is not a direct string then we choose $X_3 = M(\varepsilon_1\delta C_3)$ with $C_3 = \varepsilon_2$. Now, C_3 does not start on a peak, then we choose $X_4 = M(\varepsilon^{-1}\delta C_4)$ with $C_4 = \alpha$. Since C_4 is a direct string that starts on a peak, then X_4 is the injective module of $\mathcal{M}_{\overline{J_2}(5)}$, getting the following ordered set:

$$\mathcal{M}_{\overline{J_2}(5)} = \left\{ X_0 : \begin{array}{c} 6 \\ 5 \end{array},\ X_1 : \begin{array}{c} 2\,6 \\ 3\,5 \\ 4 \end{array},\ X_2 : \begin{array}{c} 2\,6 \\ 3\,5 \end{array},\ X_3 : \begin{array}{c} 2\,6 \\ 5 \end{array},\ X_4 : \begin{array}{c} 1 \\ 2\,6 \\ 5 \end{array} \right\}.$$

Since $M = X_2 \in \mathcal{M}_{\overline{J_2}(5)}$, then $d_l(f) = 2$.

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