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Complex Analysis and Operator Theory

A Class of *n*-Entire Schrödinger operators

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Abstract We study singular Schrödinger operators on a finite interval as selfadjoint extensions of a symmetric operator. We give sufficient conditions for the symmetric operator to be in the n-entire class, which was defined in our previous work (Silva and Toloza in J Phys A Math Theor 46:025202, 2013) for some n. As a consequence of this classification, we obtain a detailed spectral characterization for a wide class of radial Schrödinger operators. The results given here make use of de Branges Hilbert space techniques.

Keywords Schrödinger operators · de Branges spaces · Spectral analysis

Mathematics Subject Classification (2010) Primary 47A25 · 47B25; Secondary 46E22 · 47N99

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1 Introduction

This paper deals with the spectral analysis of selfadjoint operators arising from the differential expression

$$-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2}+q(x),\quad x\in(0,1),\quad l\geq-\frac{1}{2},$$

along with separated selfadjoint boundary conditions. These operators describe the radial part of the Schrödinger operator for a particle confined to a ball of finite radius, when the potential is spherically symmetric. Here we show that this kind of differential operators can be related naturally to a class of symmetric operators recently introduced in the literature [32], the so-called n-entire operators. As such, we are able to give a spectral characterization of these operators for a wide class of functions $q(x)$. Namely, we show (for $l>-1/2$) that if $xq(x)$ lies in $L_p(0,1)$, $p>2$, then the spectra of any two selfadjoint realizations satisfy conditions of convergence and have certain regularity in their distribution (for details see Corollary 4.4; this result holds also for $l=-1/2$). To our knowledge, this characterization is new for this class of operators.

This work has been motivated by several relatively new developments. Firstly, recent developments on one-dimensional Schrödinger operators with singular potentials, in particular, the perturbed Bessel operators [2,4,5,7–14,18–21,26,31]. Secondly, the approach by Remling to the inverse spectral analysis of regular Schrödinger operators by means of de Branges space theory [29]. Finally, our generalization [32] of Krein's theory of entire operators [22–24] (for a review see [15]), which in turn was inspired by Woracek's work on n-associated functions of a de Branges space [25,33]. We note in passing that de Branges space techniques have also been used in connection with the inverse spectral analysis of differential operators in [7,8].

Among the classes of n-entire operators, one finds the classes of entire operators and entire operators in the generalized sense. The latter classes were introduced by Krein to treat various problems of analysis, particularly, spectral theory [15]. The construction of n-entire operators was possible due to recent results in de Branges space theory [33].

Our approach for dealing with the singular Schrödinger operators under consideration is based on a particular kind of perturbative method in which the conclusion of Theorem 4.2 is a crucial fact. Due to the functional model given in [32], one has that to any potential $q(x)$ (in the class considered), and any $l\geq-1/2$, there corresponds a de Branges space. Remarkably, as Theorem 4.2 establishes, the set of functions in the de Branges space for the free operator ($q(x)\equiv 0$) equals the set of functions in the de Branges space for the perturbed operator. Since we also show in Theorem 3.1 that the unperturbed radial Schrödinger operator is n-entire as long as $n>\frac{l}{2}+\frac{3}{4}$, Theorem 4.2 allows us to extend this assertion to the whole range of potential functions under consideration (Theorem 4.3). This in turn implies the distinctive distributional properties of the spectra of the associated selfadjoint realizations (Corollary 4.4). We point out that Theorem 4.3 and Corollary 4.4 illustrate an application of the notion of n-entire operators to the direct spectral analysis of Schrödinger operators.

The exposition is organized as follows. In the next section we give the preparatory material, lay out the notation, introduce the relevant notions, and recall some formulae needed later. In Sect. 3 we study the free particle case, that is, $q(x) \equiv 0$. Finally, in Sect. 4, we discuss the main results of this work.

2 Preliminaries

In this section we give a brief recollection of the main theoretical ingredients that will be used in this paper. Concretely, we touch upon the distinct features of n-entire operators, and give a brief account on the theory of de Branges Hilbert spaces.

2.1 de Branges Hilbert Spaces

L. de Branges introduced a class of Hilbert spaces of entire functions which has several distinguishing characteristics. These spaces can be defined in various ways. For our choice of the definition we need two ingredients. The first one is the Hardy space

$$H_2^+ := \left\{ f(z) \text{ is holomorphic in } \mathbb{C}^+ : \sup_{y>0} \int_{\mathbb{R}} |f(x+\mathrm{i}y)|^2 \, dx < \infty \right\},$$

where $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{im} z > 0\}$. The second one is an Hermite-Biehler function, viz., an entire function $e(z)$ satisfying $|e(z)| > |e(\overline{z})|$ for all $z \in \mathbb{C}^+$. Now, define

$$\mathcal{B}(e) := \{f(z) \text{ entire} : f(z)/e(z),\ f^\#(z)/e(z) \in H_2^+\},$$

where $f^\#(z) = \overline{f(\overline{z})}$. Then we call the space $\mathcal{B}(e)$ endowed with the inner product

$$\langle g, f \rangle := \int_{\mathbb{R}} \frac{\overline{g(x)} f(x)}{|e(x)|^2} dx \tag{1}$$

the de Branges Hilbert space generated by the Hermite-Biehler function $e(z)$.

Note that the inner product is well defined since both $f(z)/e(z)$ and $f^\#(z)/e(z)$ belong to H_2^+.

Remark 1 In view of [30, Theorem 5.19], $\mathcal{B}(e)$ is the set of all entire functions f satisfying

$$\int_{\mathbb{R}} \left| \frac{f(x)}{e(x)} \right|^2 dx < \infty \tag{2}$$

and

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} h_i(x) \frac{dx}{x-z} = \begin{cases} h_i(z), & z \in \mathbb{C}^+, \\ 0, & z \in \mathbb{C}^-. \end{cases} \quad i = 1, 2\,, \tag{3}$$

where $h_1(z) = f(z)/e(z)$ and $h_2(z) = f^\#(z)/e(z)$.

Alternative characterizations of de Branges Hilbert spaces are provided in [29, Proposition 2.1] and also [6, Chapter 2]. It is also possible to define de Branges spaces without relying on a given Hermite-Biehler function [6, Problem 50]. However, for all de Branges spaces considered below, there is always an explicit Hermite-Biehler function.

To any de Branges space there corresponds a space of associated functions [17, Definition 4.4]. In this work we are interested in the generalization of this notion given in [25]. For any $\mathcal{B}(e)$ and $n \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, let

$$\operatorname{assoc}_n \mathcal{B}(e) := \mathcal{B}(e) + z\mathcal{B}(e) + \cdots + z^n \mathcal{B}(e). \tag{4}$$

This is the so-called space of n-associated functions that was introduced in the context of intermediate Weyl coefficients.

2.2 n-Entire Operators

Given a separable Hilbert space $\mathcal{H}$, let $\mathscr{S}(\mathcal{H})$ be the class of regular, closed symmetric operators on $\mathcal{H}$, whose deficiency indices are both equal to 1; we recall that a closed operator A is regular if for every $z \in \mathbb{C}$ there exists $k_z > 0$ such that $\|(A - zI)\varphi\| \geq k_z \|\varphi\|$ for every $\varphi \in \operatorname{dom}(A)$. Well known properties of operators of this kind are discussed in [32]. Relevant to this work is that for any operator in $\mathscr{S}(\mathcal{H})$, one can construct a de Branges space [32, Section 2.3].

Definition 2.1 Given $n \in \mathbb{Z}^+$, $A \in \mathscr{S}(\mathcal{H})$ is n-entire if and only if there exist $n + 1$ vectors $\mu_0, \ldots, \mu_n \in \mathcal{H}$ such that

$$\mathcal{H} = \operatorname{ran}(A - zI) \dot{+} \operatorname{span}\{\mu_0 + z\mu_1 + \cdots + z^n \mu_n\}, \tag{5}$$

for all $z \in \mathbb{C}$.

This definition extends the notions of entire operator and entire operator in the generalized sense that were introduced by M. G. Krein as tools to study several problems in classical analysis [15].

The next result is shown in [32, Propositions 2.14, 3.7 and 3.11].

Theorem 2.2 *Let $A \in \mathscr{S}(\mathcal{H})$. Then the following statements are equivalent:*

1. *A is n-entire.*
2. *The space of n-associated functions of the de Branges space related to A (see [32, Section 2.3]) contains a zero-free function.*

3. *There exist $n+1$ vectors $\eta_0, \dots, \eta_n \in \mathcal{H}$ such that*

$$\mathcal{H} = \operatorname{ran}(A - zI) \dot{+} \operatorname{span}\{\eta_0 + z\eta_1 + \cdots + z^n \eta_n\},$$

for all $z \in \mathbb{C}$ with the possible exception of a finite number of points.

4. *Let A_{β_1} and A_{β_2}, $\beta_1 \neq \beta_2$, be canonical selfadjoint extensions of A. Set $\{x_j\}_{j\in\mathbb{N}} = \{x_j^+\}_{j\in\mathbb{N}} \cup \{x_j^-\}_{j\in\mathbb{N}} = \operatorname{spec}(A_{\beta_1})$, where $\{x_j^+\}_{j\in\mathbb{N}}$ and $\{x_j^-\}_{j\in\mathbb{N}}$ are the sequences of positive, respectively non-positive, elements of* $\operatorname{spec}(A_{\beta_1})$, *arranged according to increasing modulus. Then the following assertions hold true:*

   (C1) The limit $\displaystyle\lim_{r\to\infty} \sum_{0<|x_j|\le r} \frac{1}{x_j}$ exists.

   (C2) $\displaystyle\lim_{j\to\infty} \frac{j}{x_j^+} = -\lim_{j\to\infty} \frac{j}{x_j^-} < \infty$.

   (C3) Denoting $\operatorname{spec}(A_\beta) = \{b_j\}_{j\in\mathbb{N}}$ *for an arbitrary canonical selfadjoint extension of A, defin*

$$h_\beta(z) := \begin{cases} \displaystyle\lim_{r\to\infty} \prod_{|b_j|\le r} \left(1 - \frac{z}{b_j}\right) & \text{if } 0 \notin \sigma(A_\beta), \\ \displaystyle z \lim_{r\to\infty} \prod_{0<|b_j|\le r} \left(1 - \frac{z}{b_j}\right) & \text{otherwise.} \end{cases}$$

   The series

$$\sum_{x_j \neq 0} \left| \frac{1}{x_j^{2n} h_{\beta_2}(x_j) h'_{\beta_1}(x_j)} \right| \textit{ is convergent}.$$

Remark 2 In [33, Theorem 3.2], necessary and sufficient conditions for the existence of a real zero-free entire function in the space of n-associated functions of any de Branges space are provided. Therefore, assertions 2 and 4 of the previous theorem are connected via Woracek's result.

The last assertion of the preceding theorem, can be interpreted as a spectral characterization of n-entire operators. We use this characterization for the class of Schrödinger operators discussed in this work, cf. Theorem 4.3.

3 The Radial Schrödinger Operator: Free Particle Case

In this section we consider the spectra of the selfadjoint operators (realized by separated boundary conditions), associated with the differential expression

$$\tau_l := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}, \quad x \in (0,1), \quad l \ge -\frac{1}{2}, \tag{6}$$

from the perspective of Theorem 2.2. In particular, we study conditions for the validity of (C1), (C2), and (C3).

We start by recalling some basic facts. The differential expression τ_l is regular at $x = 1$. At $x = 0$ it is in the limit point case for $l \geq 1/2$ and limit circle case for $l \in [-1/2, 1/2)$. In the latter case it is common to add the boundary condition

$$\lim_{x \to 0} x^l \left[(l+1)\varphi(x) - x\varphi'(x) \right] = 0. \tag{7}$$

A comprehensive investigation of possible boundary conditions (at $x = 0$) can be found in [3].

The differential expression τ_l, supplemented with the boundary condition (7) when required, gives rise to a closed, regular, symmetric operator with deficiency indices both equal to 1. We will denote this operator as H_l. The associated one-parameter family of canonical selfadjoint extensions $H_{l,\beta}$, with $\beta \in [0, \pi)$, is described by boundary conditions at the right endpoint [2,4]. Namely,

$$\operatorname{dom}(H_{l,\beta}) := \left\{ \begin{array}{l} \varphi(x) \in L^2(0,1) \cap \mathrm{AC}^2(0,1] : (\tau_l \varphi)(x) \in L^2(0,1), \\ \varphi(1)\cos\beta = \varphi'(1)\sin\beta \end{array} \right\}$$

plus the boundary condition (7) for $l \in [-1/2, 1/2)$. Of course, one defines $(H_{l,\beta}\varphi)(x) = (\tau_l \varphi)(x)$ in its domain.

The equation

$$-\varphi''(x) + \frac{l(l+1)}{x^2}\varphi(x) = z\varphi(x)$$

has a solution in $L^2(0,1)$, namely,

$$\xi_l(z, x) := z^{-\frac{2l+1}{4}} \sqrt{\frac{\pi x}{2}} J_{l+\frac{1}{2}}(\sqrt{z}x), \tag{8}$$

where $J_m(z)$ is the Bessel function of order m. In what follows all cut branches are chosen along the negative real axis. By [1, 9.1.10], $\xi_l(z, x)$ turns out to be an entire function (for every value of x). In fact,

$$\xi_l(z, x) = \sqrt{\pi} \left(\frac{x}{2} \right)^{l+1} \underbrace{\sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4} z x^2 \right)^k}{k! \Gamma(l + k + \frac{3}{2})}}_{=:\, g_l(z,x)}; \tag{9}$$

note that it satisfies (7) for all $z \in \mathbb{C}$. It is worth to mention that $\xi_l(z, x)$ is a non vanishing element of $\ker(H_l^* - zI)$ for all $z \in \mathbb{C}$.

Theorem 3.1 *Given $l \geq -\frac{1}{2}$, the operator H_l is n-entire, $n \in \mathbb{Z}^+$, if and only if $n > \frac{l}{2} + \frac{3}{4}$.*

Proof In view of Theorem 2.2, it suffices to find two selfadjoint realizations of (6) whose spectra satisfy conditions (C1), (C2) and (C3). According to [18],

$$\operatorname{spec}(H_{l,\beta}) = \{\text{zeros of } \xi_l(z,1)\cos\beta - \xi_l'(z,1)\sin\beta\},$$

where the prime denotes derivative with respect to x. For $\beta = 0$ this becomes

$$\operatorname{spec}(H_{l,0}) = \left\{\text{zeros of } J_{l+\frac{1}{2}}(\sqrt{z})\right\} = \left\{(j_{l+\frac{1}{2},n})^2 : n \in \mathbb{N}\right\};$$

here we are using the notation of [1, Chapter 9]. For $\beta \in (0,\pi)$, we can write

$$\operatorname{spec}(H_{l,\beta}) = \{\text{zeros of } \xi_l'(z,1) - \xi_l(z,1)\cot\beta\}.$$

Now,

$$\xi_l'(z,x) = \frac{1}{2}\sqrt{\frac{\pi}{2}}z^{-\frac{2l+1}{4}}x^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\sqrt{z}x) + \sqrt{\frac{\pi}{2}}z^{-\frac{2l-1}{4}}x^{\frac{1}{2}}J_{l+\frac{1}{2}}'(\sqrt{z}x).$$

By invoking the recurrence relation $J_m'(z) = -J_{m+1}(z) + mz^{-1}J_m(z)$, we obtain

$$\xi_l'(z,1) = \sqrt{\frac{\pi}{2}}(l+1)z^{-\frac{2l+1}{4}}J_{l+\frac{1}{2}}(\sqrt{z}) - \sqrt{\frac{\pi}{2}}z^{-\frac{2l-1}{4}}J_{l+\frac{3}{2}}(\sqrt{z}).$$

Thus,

$$\begin{aligned}\xi_l'(z,1) - \xi_l(z,1)\cot\beta = &\sqrt{\frac{\pi}{2}}(l+1-\cot\beta)z^{-\frac{2l+1}{4}}J_{l+\frac{1}{2}}(\sqrt{z})\\ &-\sqrt{\frac{\pi}{2}}z^{-\frac{2l-1}{4}}J_{l+\frac{3}{2}}(\sqrt{z}).\end{aligned}$$

The zeros of this expression are easy to describe if $\beta = \beta_l$, where β_l satisfies $\cot\beta_l = l+1$. In this case we can write

$$\xi_l'(z,1) - \xi_l(z,1)\cot\beta_l = -\sqrt{\frac{\pi}{2}}z\qquad\underbrace{z^{-\frac{2l+3}{4}}J_{l+\frac{3}{2}}(\sqrt{z})}_{\text{proportional to } g_{l+1}(z,1)}\qquad,$$

where $g_l(z,x)$ is defined as in (9) and clearly $g_l(0,1) \neq 0$. Therefore,

$$\operatorname{spec}(H_{l,\beta_l}) = \left\{\text{zeros of } zg_{l+1}(z)\right\} = \left\{0\right\} \cup \left\{(j_{l+\frac{3}{2},n})^2 : n \in \mathbb{N}\right\}.$$

In what follows we will consider $\operatorname{spec}(H_{l,0})$ and $\operatorname{spec}(H_{l,\beta_l})$. At this point it is convenient to recall the asymptotic formulae [1, 9.2.1 and 9.5.12],

$$J_{l+\frac{3}{2}}(u) = \sqrt{\frac{2}{\pi u}}\left[\cos\left(u - \tfrac{\pi}{2}l - \pi\right) + O\left(u^{-1}\right)\right], \quad u \to +\infty, \tag{10}$$

$$\sqrt{x_m} = j_{l+\frac{1}{2},m} = \frac{\pi}{2}(2m + l) + O\left(m^{-1}\right), \quad m \to +\infty. \tag{11}$$

Clearly, the latter implies that (C1) and (C2) hold irrespective of the value of l. In order to analyze the validity of (C3), let us compute the following functions associated with the spectra of $H_{l,0}$ and H_{l,β_l},

$$h_0(z) := \prod_{n=1}^{\infty}\left[1 - \frac{z}{(j_{l+\frac{1}{2},n})^2}\right], \quad h_{\beta_l}(z) := z\prod_{n=1}^{\infty}\left[1 - \frac{z}{(j_{l+\frac{3}{2},n})^2}\right].$$

Using [1, 9.5.10], we obtain

$$h_0(z) = 2^{l+\frac{1}{2}}\Gamma(l + \tfrac{3}{2})z^{-\frac{2l+1}{4}}J_{l+\frac{1}{2}}(\sqrt{z}), \;\; h_{\beta_l}(z) = 2^{l+\frac{3}{2}}\Gamma(l + \tfrac{5}{2})z^{-\frac{2l-1}{4}}J_{l+\frac{3}{2}}(\sqrt{z}).$$

We also need the derivative of $h_0(z)$. Using [1, 9.1.30],

$$\frac{d}{dz}\left[z^{-\frac{2l+1}{4}}J_{l+\frac{1}{2}}(\sqrt{z})\right] = -\frac{1}{2}z^{-\frac{2l+3}{4}}J_{l+\frac{3}{2}}(\sqrt{z}).$$

Therefore,

$$h_0'(z) = -2^{l-\frac{1}{2}}\Gamma(l + \tfrac{3}{2})z^{-\frac{2l+3}{4}}J_{l+\frac{3}{2}}(\sqrt{z}).$$

We want to analyze the convergence of the series,

$$\sum_{x_m \in \operatorname{spec}(H_{l,\beta})} \frac{1}{x_m^{2n}\left|h_{\beta_l}(x_m)h_0'(x_m)\right|}, \quad n \in \mathbb{Z}^+.$$

Since

$$x_m^{2n}\left|h_{\beta_l}(x_m)h_0'(x_m)\right| = 2^{2l+1}\Gamma(l + \tfrac{3}{2})\Gamma(l + \tfrac{5}{2})x_m^{2n-l-\frac{1}{2}}\left[J_{l+\frac{3}{2}}(\sqrt{x_m})\right]^2,$$

this problem reduces to the convergence of

$$\sum_{x_m \in \operatorname{spec}(H_{l,\beta})} \frac{1}{x_m^{2n-l-\frac{1}{2}}\left[J_{l+\frac{3}{2}}(\sqrt{x_m})\right]^2}. \tag{12}$$

Due to (10) and (11), it follows on one hand that

$$\left[J_{l+\frac{3}{2}}(\sqrt{x_m})\right]^2 = \frac{4}{\pi^2(2m+l)} \underbrace{\cos^2(\pi(m-1))}_{\equiv 1} + O(m^{-2}).$$

On the other hand,

$$x_m^{2n-l-\frac{1}{2}} = (j_{l+\frac{1}{2},m})^{4n-2l-1} = \left[\tfrac{\pi}{4}(2m+l)\right]^{4n-2l-1} \left[1 + O(m^{-2})\right].$$

Thus the convergence of (12) becomes the convergence of

$$\sum_{m=M_0}^{\infty} \frac{1}{(2m+l)^{4n-2l-2}\left[1+O(m^{-1})\right]},$$

for some M_0 large enough, which in turn yields our assertion. □

Remark 3 1. It is clear that no symmetric operator H_l associated with τ_l can be 0-entire (or entire according to Krein's definition).

2. As is readily seen from (6) and (7), the case with $l = 0$ corresponds to (minus) the Laplacian operator in [0, 1] with Dirichlet boundary condition at the left endpoint. The operator acting the same but with Neumann boundary condition is discussed in [32]. As in the present work, there we obtain a 1-entire operator.

4 The Radial Schrödinger Operator: Adding a Perturbation

In this section we prove the main result of this work, namely, a generalization of Theorem 3.1 to include a potential energy function. In other words, we now discuss selfadjoint operators with separated boundary conditions that arise from the differential expression

$$\tau := \tau_l + q(x) = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x). \quad x \in (0,1), \quad l \geq -\frac{1}{2}. \tag{13}$$

To start with, we assume that $q(x)$ is a real function such that $\widetilde{q}(x) \in L_1(0,1)$, where

$$\widetilde{q}(x) := \begin{cases} xq(x) & l > -\frac{1}{2}, \\ x(1-\log x)q(x) & l = -\frac{1}{2}. \end{cases} \tag{14}$$

Under this hypothesis, it is shown in [18, Theorem 2.4] that τ is regular at $x = 1$, and in the limit point case (resp. limit circle case) at $x = 0$ for $l \geq 1/2$ (resp. $l \in [-1/2, 1/2)$). As in the preceding section, we impose the boundary condition (7) in the latter case. In this way, τ gives rise to a family of selfadjoint operators H_β, for $\beta \in [0,\pi)$, associated to the boundary conditions $\varphi(1)\cos\beta = \varphi'(1)\sin\beta$.

These operators are the canonical selfadjoint extensions of a certain closed, regular, symmetric operator H, having deficiency indices both equal to 1.

According to [18, Lemma 2.2], the equation $\tau\varphi(x) = z\varphi(x)$ has a solution $\xi(z, x)$, entire with respect to z, that lies in $L_2(0, 1)$. This function satisfies the following estimates [18, Equations 2.18 and 2.10],

$$|\xi(z, x) - \xi_l(z, x)| \le C \left(\frac{x}{1 + \sqrt{|z|}x}\right)^{l+1} e^{|\operatorname{im}(\sqrt{z})|x} \int_0^x \frac{|\widetilde{q}(y)|}{1 + \sqrt{|z|}y} dy, \qquad (15)$$

$$\left|\xi'(z, x) - \xi_l'(z, x)\right| \le C \left(\frac{x}{1 + \sqrt{|z|}x}\right)^{l} e^{|\operatorname{im}(\sqrt{z})|x} \int_0^x \frac{|\widetilde{q}(y)|}{1 + \sqrt{|z|}y} dy, \qquad (16)$$

where $\xi_l(z, x)$ is given by (8). Moreover, $\xi(z, x)$ lies in $\ker(H^* - zI)$ for all $z \in \mathbb{C}$. It also yields the spectrum of H_β since

$$\operatorname{spec}(H_\beta) = \{\text{zeros of } \xi(z, 1)\cos\beta - \xi'(z, 1)\sin\beta\}.$$

Lemma 4.1 *Suppose* $\widetilde{q}(x) \in L_s(0, 1)$*, with* $1 \le s \le \infty$*. Then, for* r *such that* $s^{-1} + r^{-1} = 1$*,*

$$\left\|\xi(w^2, \cdot) - \xi_l(w^2, \cdot)\right\|_2 = e^{|\operatorname{im} w|} \begin{cases} O\left(|w|^{-l-1-1/r}\right), & s < \infty, \\ O\left(|w|^{-l-2}\log|w|\right), & s = \infty, \end{cases}$$

for $w \in \mathbb{C}$*, and* $|w| \to \infty$*.*

Proof Consider the case $1 < s < \infty$ since the remaining cases can be treated analogously.

Let $v > 0$. The Hölder inequality yields

$$\int_0^1 \frac{|\widetilde{q}(x)|}{1 + vx} dx \le \|\widetilde{q}\|_s \left[\int_0^1 \frac{dx}{(1 + vx)^r}\right]^{1/r}.$$

Integrating the second factor we readily obtain the estimate

$$\int_0^1 \frac{|\widetilde{q}(x)|}{1 + vx} dx = O\left(v^{-1/r}\right), \quad v \to \infty. \qquad (17)$$

Now, in view of (15) and the latter estimate, we have

$$\begin{aligned}
&\left\|\xi(w^2,\cdot)-\xi_l(w^2,\cdot)\right\|_2^2\\
&\leq C^2 e^{2|\operatorname{im} w|}\int_0^1 \underbrace{\left(\frac{x}{1+|w|\,x}\right)^{2(l+1)}}_{=:\,|a(x)|}\underbrace{\left(\int_0^x \frac{|\widetilde{q}(y)|}{1+|w|\,y}dy\right)^2}_{=:\,|b(x)|}dx\\
&\leq C^2 e^{2|\operatorname{im} w|}\,\|a\|_1\,\|b\|_\infty\\
&\leq C^2 e^{2|\operatorname{im} w|}\underbrace{\int_0^1 \left(\frac{x}{1+|w|\,x}\right)^{2(l+1)}dx}_{=\,O\left(|w|^{-2(l+1)}\right)}\underbrace{\left(\int_0^1 \frac{|\widetilde{q}(y)|}{1+|w|\,y}dy\right)^2}_{=\,O\left(|w|^{-2/r}\right)}.
\end{aligned}$$

This completes the proof. □

Let us consider the following transform of a function $\varphi(x)\in L^2(0,1)$,

$$\varphi(x)\mapsto\widehat{\varphi}(z):=\int_0^1 \xi(z,x)\varphi(x)dx. \tag{18}$$

According to [8, Theorem 3.2], the linear set

$$\mathcal{B}:=\left\{\widehat{\varphi}(z):\varphi(x)\in L^2(0,1)\right\}$$

coincides with the de Branges space $\mathcal{B}(e)$ generated by the Hermite-Biehler function

$$e(z):=\xi(z,1)+\mathrm{i}\xi'(z,1),\quad z\in\mathbb{C}.$$

Since moreover $\xi(z,\cdot)\in\ker(H^*-zI)$, it is immediate to see that this construction is a particular instance of the abstract functional model discussed in [32].

The proof of our next theorem makes use of an analogue of the Paley-Wiener Theorem for the Hankel transform of a function with compact support, a result due to Griffith [16]. For the sake of convenience we recall it (taken from [34] with minor changes).

Theorem (Griffith) *Let $z=x+\mathrm{i}y$ and assume $l>-1$. A function $f(z)$ has the representation*

$$f(z)=\int_0^b \sqrt{zx}\,J_{l+\frac{1}{2}}(zx)\varphi(x)dx$$

with $b > 0$ and $\varphi(x) \in L^2(0, b)$ if and only if $f(x) \in L^2(0, \infty)$, $z^{-l-1} f(z)$ is an even entire function, and there exists a constant $C > 0$ such that $|f(z)| \le Ce^{b|y|}$ for all $z \in \mathbb{C}$.

We also recall the following estimate, taken from [18, Lemma A.1],

$$|\xi_l(z, x)| \le C \left(\frac{x}{1 + \sqrt{|z|}x} \right)^{l+1} e^{\left|\operatorname{im}(\sqrt{z})\right|x}. \tag{19}$$

This bound holds for $l \ge -1/2$.

Theorem 4.2 *Assume $\widetilde{q}(x) \in L_s(0, 1)$, with $2 < s \le \infty$. Then, $\mathcal{B} = \mathcal{B}_0$ (as sets), where $\mathcal{B}_0$ is given by*

$$\mathcal{B}_0 = \left\{ \widehat{\varphi}(z) = \int_0^1 \xi_l(z, x)\varphi(x)dx : \varphi(x) \in L^2(0, 1) \right\},$$

with $\xi_l(z, x)$ given by (8).

Proof To simplify the exposition, we restrict the proof to $2 < s < \infty$. The argumentation for $s = \infty$ is essentially the same. We follow the arguments of the proof of [29, Theorem 4.1].

We prove first that $\mathcal{B} \subset \mathcal{B}_0$. Consider $f(z) \in \mathcal{B}$ so

$$f(z) = \int_0^1 \xi(z, x)\varphi(x)dx, \quad \varphi(x) \in L^2(0, 1).$$

Define $g(w) := f(w^2)$. This function is clearly entire and even. Moreover,

$$\begin{aligned} |g(w)| &\le \int_0^1 \left|\xi(w^2, x)\varphi(x)\right| dx \\ &\le \left(\left\|\xi(w^2, \cdot) - \xi_l(w^2, \cdot)\right\|_2 + \left\|\xi_l(w^2, \cdot)\right\|_2 \right) \|\varphi\|_2 \end{aligned}$$

By applying Lemma 4.1 to the first term of the estimate, and using (19) to obtain an upper bound for the second term, one obtains

$$|g(w)| \le C \|\varphi\|_2 \, e^{|w|},$$

thus $g(w)$ is of exponential type not greater than 1. Also, (15) along with Lemma 4.1 yields

$$
\begin{aligned}
g(w) &= \int_0^1 \xi_l(w^2, x)\varphi(x)dx + \int_0^1 \left[\xi(w^2, x) - \xi_l(w^2, x)\right]\varphi(x)dx \\
&= \sqrt{\frac{\pi}{2}} w^{-l-1} \int_0^1 \sqrt{wx}\, J_{l+\frac{1}{2}}(wx)\varphi(x)dx + O\left(\frac{1}{|w|^{l+1+1/r}}\right),
\end{aligned}
$$

for $w \in \mathbb{R},\ w \to \infty$, and $r < 2$. This implies, by virtue of Griffith's Theorem, that the function $h(w) = w^{l+1}g(w)$ lies in $L^2(0, \infty)$. Since clearly $w^{-l-1}h(w)$ is even and entire, again by Griffith's Theorem there exists $\eta(x) \in L^2(0, 1)$ such that

$$
h(w) = \sqrt{\frac{\pi}{2}} \int_0^1 \sqrt{wx}\, J_{l+\frac{1}{2}}(wx)\eta(x)dx.
$$

Therefore,

$$
f(z) = \int_0^1 \xi_l(z, x)\eta(x)dx,
$$

as desired.

We now turn to the converse inclusion, that is, $\mathcal{B}_0 \subseteq \mathcal{B}$. To show this, we check the conditions stated in Remark 1. First, we verify that (2) holds for all $f(z) \in \mathcal{B}_0$. For this it will suffice to prove that

$$
\liminf_{x \to \pm\infty} \left| \frac{e(x)}{e_0(x)} \right| > 0. \tag{20}
$$

Here,

$$
e(z) := \xi(z, 1) + \mathrm{i}\xi'(z, 1), \qquad e_0(z) := \xi_l(z, 1) + \mathrm{i}\xi_l'(z, 1),
$$

are the Hermite-Biehler functions associated to the de Branges spaces $\mathcal{B}$ and $\mathcal{B}_0$ respectively. Note that, because of (15), (16) and (17), one has

$$
\left| \frac{e(x)}{e_0(x)} \right|^2 = \frac{\left|\xi_l(x, 1) + O\left(|x|^{-\frac{l+1+1/r}{2}}\right)\right|^2 + \left|\xi_l'(x, 1) + O\left(|x|^{-\frac{l+1/r}{2}}\right)\right|^2}{|\xi_l(x, 1)|^2 + \left|\xi_l'(x, 1)\right|^2}, \tag{21}
$$

for $x \in \mathbb{R}$. We will analyze the cases $x \to -\infty$ and $x \to \infty$ separately.

Set $x = -w^2$ with $w \in \mathbb{R}^+$. Recalling (9),

$$\xi_l(-w^2,1) = \frac{\sqrt{\pi}}{2^{l+1}} g_l(-w^2,1),$$
$$\xi_l'(-w^2,1) = (l+1)\frac{\sqrt{\pi}}{2^{l+1}} g_l(-w^2,1) + \frac{\sqrt{\pi}}{2^{l+1}} g_l'(-w^2,1),$$

where $g_l(z,x)$ is given by (9) so it readily follows that $\xi_l(-w^2,1) \to \infty$ and $\xi_l'(-w^2,1) \to \infty$ as $w \to \infty$. Hence we obtain

$$\lim_{w\to\infty} \left| \frac{e(-w^2)}{e_0(-w^2)} \right|^2 = 1. \tag{22}$$

Now consider $x = w^2$ with $w \in \mathbb{R}^+$. By [1, 9.2.1],

$$J_m(w) = \sqrt{\frac{2}{\pi w}} \left[\cos\left(w - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi\right) + O\left(w^{-1}\right) \right], \quad w \to \infty,$$

thus it follows that

$$\xi_l(w^2,1) = w^{-l-1} \sin\left(w - \tfrac{1}{2}\pi l\right) + O\left(w^{-l-2}\right), \quad \text{and}$$
$$\xi_l'(w^2,1) = w^{-l} \cos\left(w - \tfrac{1}{2}\pi l\right) + O\left(w^{-l-1}\right),$$

as $w \to \infty$. Inserting these expressions into (21) one gets

$$\left| \frac{e(w^2)}{e_0(w^2)} \right|^2 = \frac{\left|\sin\left(w - \frac{\pi l}{2}\right) + O\left(w^{-\frac{1}{r}}\right)\right|^2 + \left|w \cos\left(w - \frac{\pi l}{2}\right) + O\left(w^{1-\frac{1}{r}}\right)\right|^2}{\left|\sin\left(w - \frac{\pi l}{2}\right) + O\left(w^{-1}\right)\right|^2 + \left|w \cos\left(w - \frac{\pi l}{2}\right) + O\left(1\right)\right|^2}. \tag{23}$$

Suppose there exists a positive unbounded sequence $\{w_n\}_{n=1}^{\infty}$ such that (23) goes to zero as $n \to \infty$. Necessarily, $w_n \cos\left(w_n - \frac{1}{2}\pi l\right)$ is unbounded since otherwise it would follow that $\cos\left(w_n - \frac{1}{2}\pi l\right) \to 0$ as $n \to \infty$, along with $\sin\left(w_n - \frac{1}{2}\pi l\right) \to 1$, implying that (23) is bounded below away from zero. But if $w_n \cos\left(w_n - \frac{1}{2}\pi l\right)$ is unbounded, then (23) goes to 1, again contradicting our assumption. Thus, we obtain

$$\liminf_{w\to\infty} \left| \frac{e(w^2)}{e_0(w^2)} \right|^2 > 0,$$

which, together with (22), gives (20). Thus, we have established (2).

To complete the proof, we now check (3). To this end, we calculate

$$\int_{\Gamma_A} \frac{f(s)}{e(s)(s-z)} ds, \quad z \in \mathbb{C}^+, \quad f(z) \in \mathcal{B}_0,$$

where $\Gamma_A = [-A, A] \cup \{Ae^{\mathrm{i}\phi} : \phi \in [0, \pi]\}$, and attempt to let $A \to \infty$. For the function $f^{\#}(z)$ a similar argument can be carried out. By a standard argumentation, to prove (3), it suffices to show that

$$\lim_{n\to\infty} \int_0^{\pi} \left| \frac{f(A_n e^{i\phi})}{e(A_n e^{i\phi})} \right| d\phi = 0 \tag{24}$$

for some sequence $A_n \to \infty$.

Now, we implement the change of variable $z = R^2 e^{\mathrm{i}2\theta}$, $0 \le \theta \le \pi/2$. According to [28, Chapter 4, Section 9], it holds true that

$$J_m(Re^{\mathrm{i}\theta}) = \sqrt{\frac{2}{\pi R}} e^{-\mathrm{i}\theta/2} \left[\cos\left(Re^{\mathrm{i}\theta} - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi \right) + e^{R\sin\theta} O_\epsilon\left(R^{-1}\right) \right], \tag{25}$$

as $R \to \infty$, provided that

$$0 \le \theta \le \pi/2 - \epsilon\,, \qquad \epsilon > 0. \tag{26}$$

Note that we have stressed the dependence on ϵ of the constant implicit in the asymptotic remainder by means of a subscript.

Next, due to (15), (16), (17) and (25), the following holds in the sector (26):

$$\begin{aligned}
e(R^2 e^{\mathrm{i}2\theta}) &= \xi_l(R^2 e^{\mathrm{i}2\theta}, 1) + \mathrm{i}\xi_l'(R^2 e^{\mathrm{i}2\theta}, 1) + e^{R\sin\theta} O\left(\frac{1}{R^{l+1/r}}\right) \\
&= R^{-(l+1)} e^{\mathrm{i}(l+1)\theta} \left[\sin\left(Re^{\mathrm{i}\theta} - \frac{l\pi}{2} \right) + e^{R\sin\theta} O_\epsilon\left(\frac{1}{R}\right) \right] \\
&\quad + \mathrm{i}(l+1) R^{-(l+1)} e^{-\mathrm{i}\theta(l+1)} \left[\sin\left(Re^{\mathrm{i}\theta} - \frac{l\pi}{2} \right) + e^{R\sin\theta} O_\epsilon\left(\frac{1}{R}\right) \right] \\
&\quad + \mathrm{i}R^{-l} e^{-\mathrm{i}l\theta} \left[\cos\left(Re^{\mathrm{i}\theta} - \frac{l\pi}{2} \right) + e^{R\sin\theta} O_\epsilon\left(\frac{1}{R}\right) \right] \\
&\quad + e^{R\sin\theta} O\left(\frac{1}{R^{l+1/r}}\right)
\end{aligned}$$

as $R \to \infty$. Notice that $1 < r < 2$ so the latter error term is the dominant one. Thus, asymptotically,

$$\left| e(R^2 e^{\mathrm{i}2\theta}) \right| \ge R^{-l} \left| \left| \cos\left(Re^{\mathrm{i}\theta} - l\pi/2 \right) \right| - e^{R\sin\theta} O_\epsilon\left(\frac{1}{R^{l+1/r}} \right) \right|. \tag{27}$$

We want to obtain lower bounds for $\left|e(R^2 e^{\mathrm{i}2\theta})\right|$, with the help of (27), separately for large angles that satisfy

$$e^{-2R\sin\theta} < 1 - \delta \tag{28}$$

but within (26), and small ones

$$e^{-2R\sin\theta} > 1-\delta \tag{29}$$

for some small $\delta > 0$.

Let us first consider (28). Inequality (27) can be rewritten as

$$\begin{aligned}\left|e(R^2e^{i2\theta})\right| &\geq \frac{1}{2}e^{R\sin\theta}R^{-l}\\ &\quad\times\left|\left|1+e^{i2(R\cos\theta-l\pi/2)}e^{-2R\sin\theta}\right| - O_\epsilon\left(\frac{1}{R^{l+1/r}}\right)\right|.\end{aligned}$$

Then there exists R_ϵ such that, for all $R > R_\epsilon$,

$$\left|e(R^2e^{i2\theta})\right| \geq \frac{1}{4}e^{R\sin\theta}R^{-l}\left|1+e^{i2(R\cos\theta-l\pi/2)}e^{-2R\sin\theta}\right|.$$

This implies

$$\left|e(R^2e^{i2\theta})\right| \geq \frac{1}{4}e^{R\sin\theta}R^{-l}\left(1-e^{-2R\sin\theta}\right)$$

so that, under (28), one obtains

$$\left|e(R^2e^{i2\theta})\right| \geq \frac{1}{4}\delta R^{-l}e^{R\sin\theta} \tag{30}$$

for $R > R_\epsilon$.

Let us now consider small angles, that is (29), keeping $R \geq R_\epsilon$. Taking into account that

$$\begin{aligned}\cos(Re^{i\theta}-l\pi/2) &= \frac{1}{2}e^{R\sin\theta}\\ &\quad\times\left[\cos(R\cos\theta-l\pi/2)+e^{i(R\cos\theta-l\pi/2)}(e^{-2R\sin\theta}-1)\right],\end{aligned}$$

one deduces from (27) the following asymptotic estimate

$$\left|e(R^2e^{i2\theta})\right| \geq \frac{1}{2}R^{-l}e^{R\sin\theta}\left|\cos(R\cos\theta-l\pi/2)\right| + O_\epsilon\left(\frac{1}{R^l}\right).$$

Consider the sequence $\{R_n\}_{n=1}^\infty$ given by $R_n := \pi(n+l/2)$. Noticing that

$$\begin{aligned}\cos(R_n\cos\theta-l\pi/2) &= \cos(n\pi\cos\theta)+(\cos\theta-1)l\pi/2)\\ &= \cos(n\pi+\underbrace{O(n\sin^2\theta))}_{O(n^{-1})\text{ due to (29)}},\end{aligned}$$

we arrive at

$$\left|e(R_n^2 e^{i2\theta})\right| \geq \frac{1}{4}\delta R_n^{-l} e^{R_n \sin\theta} \tag{31}$$

for n sufficiently large.

On the other hand, for $f(z) \in \mathcal{B}_0$,

$$\begin{aligned}\left|f(R^2 e^{i2\theta})\right| &\leq \int_0^1 |\varphi(x)| \left|\xi_l(R^2 e^{i2\theta}, x)\right| dx \\ &\leq C \|\varphi\|_2 \left(\frac{1}{1+R}\right)^{l+1} e^{R\sin\theta},\end{aligned} \tag{32}$$

where $\varphi(x)$ is such that $f(z) = \widehat{\varphi}(z)$ and we have used (19). Taking into account (30) and (31), the last estimate implies, for some constant $C' > 0$ and all $\theta \in [0, \pi/2 - \epsilon)$,

$$\left|\frac{f(R^2 e^{i2\theta})}{e(R^2 e^{i2\theta})}\right| \leq C' \frac{1}{R}, \quad \left|\frac{f(R_n^2 e^{i2\theta})}{e(R_n^2 e^{i2\theta})}\right| \leq C' \frac{1}{R_n}, \tag{33}$$

where $R > R_\epsilon$ and $R_n > R_\epsilon$, for large and small angles, respectively.

We claim that the first estimate in (33) holds also for $\theta \in [\pi/2 - \epsilon, \pi/2]$. For (33) implies that

$$\left|\sqrt{z}\frac{f(z)}{e(z)}\right| \tag{34}$$

is bounded along the ray $\theta = \pi/2 - \epsilon$ and, since (2) has been verified, one concludes that (34) is also bounded along the ray $\theta = \pi/2$. Then, the Phragmén-Lindelöf principle [27, Theorem 21] implies that (34) is bounded inside the angle. In this argument, a suitable branch of $\sqrt{z}$ is assumed.

Finally, using that (33) holds for $\theta \in [0, \pi/2]$, one obtains (24) for $A_n = R_n^2$. Clearly, the same argumentation used above leads to the same result for $f^{\#}(z)/e(z)$ since, in view of (18), inequality (32) holds when one substitutes $f(z)$ by $f^{\#}(z)$ and $\varphi(x)$ by $\overline{\varphi(x)}$. Thus, we have shown (3) to be true. □

Theorem 4.3 *Let $l \geq -\frac{1}{2}$. Let H be the regular, symmetric operator with deficiency indices both equal to* 1 *that is associated with the formal differential expression* (13)*, and the boundary condition* (7) *whenever* $[-1/2, 1/2)$. *Assume that $\widetilde{q}(x)$, given by* (14)*, belongs to $L_p(0, 1)$ for some $p > 2$. Then the operator H is n-entire if and only if $n > \frac{l}{2} + \frac{3}{4}$.*

Proof Due to Theorem 4.2 the de Branges spaces associated with the operators H_l and H are the same (note that $\mathcal{B}_0$ is the de Branges space associated with H_l). Then the corresponding spaces of n-associated functions also coincide. The assertion thus follows from Theorem 3.1 and Theorem 2.2. □

Combining this theorem with Theorem 2.2, it follows that the spectra of the selfadjoint realizations H_β have certain asymptotic properties:

Corollary 4.4 *Assume that* $l \geq -\frac{1}{2}$ *and* $\widetilde{q}(x)$ *lies in* $L_p(0,1)$*, with* $p > 2$*. Then the spectra of two canonical selfadjoint extensions* H_{β_1}*,* H_{β_2} *of* H *satisfy the conditions (C1), (C2), (C3), of Theorem 2.2 with* $n > \frac{l}{2} + \frac{3}{4}$*.*

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