Sufficient Cohesion over atomic toposes

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Résumé. Soit $(\mathcal{D}, J_{at})$ un site atomique et $j : \mathbf{Sh}(\mathcal{D}, J_{at}) \to \widehat{\mathcal{D}}$ le topos des faisceaux associé. Tout foncteur $\phi : \mathcal{C} \to \mathcal{D}$ induit un morphisme géométrique $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ et, en prenant le produit fibré le long de j, un morphisme géométrique $q : \mathcal{F} \to \mathbf{Sh}(\mathcal{D}, J_{at})$. Nous donnons une condition suffisante sur ϕ pour que q satisfasse le Nullstellensatz et la Cohésion Suffisante au sens de la Cohésion Axiomatique. Ceci est motivé par les exemples où $\mathcal{D}^{\mathrm{op}}$ est une catégorie d'extensions finies d'un corps.

Abstract. Let $(\mathcal{D}, J_{at})$ be an atomic site and $j : \mathbf{Sh}(\mathcal{D}, J_{at}) \to \widehat{\mathcal{D}}$ be the associated sheaf topos. Any functor $\phi : \mathcal{C} \to \mathcal{D}$ induces a geometric morphism $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ and, by pulling-back along j, a geometric morphism $q : \mathcal{F} \to \mathbf{Sh}(\mathcal{D}, J_{at})$. We give a sufficient condition on ϕ for q to satisfy the Nullstellensatz and Sufficient Cohesion in the sense of Axiomatic Cohesion. This is motivated by the examples where $\mathcal{D}^{\mathrm{op}}$ is a category of finite field extensions.

Keywords. Axiomatic Cohesion, topos, algebraic geometry.

Mathematics Subject Classification (2010). 18F10, 18F20.

1. Introduction and outline

The first paragraph of Section II in [13] explains that the contrast of cohesion with non-cohesion (expressed by a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ with certain special properties) can be made relative, so that $\mathcal{S}$ may be an 'arbitrary' topos. The inverted commas should be taken seriously because reasonable hypotheses on the geometric morphism p imply strong restrictions on the base $\mathcal{S}$. Having said this, the base is not forced to be the category $\mathbf{Set}$ of sets and functions. As an example, it is proposed loc. cit. that in the case of algebraic geometry the base topos $\mathcal{S}$ may be usefully taken as the Galois topos of Barr-atomic sheaves on finite extensions of the ground field. What does 'usefully' mean here? To give a concrete idea let $\mathcal{E}$ be the (Gros)

Zariski topos of a field k. If k is algebraically closed, the canonical geometric morphism $\mathcal{E} \to \mathbf{Set}$ satisfies certain simple intuitive axioms (formalized in Definitions 1.1 and 1.3 below). These axioms do not hold if k is not algebraically closed, but may be restored by changing the base as suggested.

The purpose of the present paper is to give a detailed construction of sufficiently cohesive pre-cohesive toposes over Galois bases. We recall some of the basic definitions and results but the reader is assumed to be familiar with [13]. (See also [12, 9].) For general background on topos theory see [16, 7] and for atomic toposes in particular see also [3].

Let $\mathcal{E}$ and $\mathcal{S}$ be cartesian closed extensive categories.

Definition 1.1. The category $\mathcal{E}$ is called *pre-cohesive (relative to $\mathcal{S}$)* if it is equipped with a string of adjoint functors

$$
\begin{array}{c}
\mathcal{E} \\
p_! \downarrow \quad \uparrow p^* \quad \downarrow p_* \quad \uparrow p^! \\
\mathcal{S}
\end{array}
$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$ and such that:

1. $p^* : \mathcal{S} \to \mathcal{E}$ is full and faithful.

2. $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.

3. (Nullstellensatz) The canonical natural transformation $\theta : p_* \to p_!$ is (pointwise) epi.

For brevity we will say that $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive. The notation is devised to be consistent with that for geometric morphisms. Indeed, if $\mathcal{E}$ and $\mathcal{S}$ are toposes then the functors above determine a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ with direct image p_*. On the other hand, if $p : \mathcal{E} \to \mathcal{S}$ is a geometric morphism between toposes then we call p *pre-cohesive* if the adjunction $p^* \dashv p_*$ extends to one $p_! \dashv p^* \dashv p_* \dashv p^!$ making $\mathcal{E}$ pre-cohesive over $\mathcal{S}$.

Definition 1.2. A pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is called *cohesive* if the canonical natural $p_!(X^{p^*W}) \to (p_!X)^W$ is an iso for all X in $\mathcal{E}$ and W in $\mathcal{S}$. (This is the 'continuity' property in Definition 2 in [13].)

We still do not fully understand the Continuity property defining cohesive categories and for this reason we introduce and concentrate on pre-cohesive ones. It is relevant to stress that most of the results in [13] hold for pre-cohesive p; Theorem 1 loc. cit. being the most important exception.

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive. An object X in $\mathcal{E}$ is called *connected* if $p_! X = 1$. An object Y in $\mathcal{E}$ is called *contractible* if Y^A is connected for all A.

Definition 1.3. The pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is called *sufficiently cohesive* if for every X in $\mathcal{E}$ there exists a monic $X \to Y$ with Y contractible. (We may also say that p satisfies *Sufficient Cohesion*.)

Useful intuition about sufficiently cohesive categories is gained by contrasting them with an opposing class of pre-cohesive categories.

Definition 1.4. The pre-cohesive p is a *quality type* if $\theta : p_* \to p_!$ is an iso. (See Definition 1 in [13].)

In other words, p is a quality type if the (full) reflective subcategory $p^* : \mathcal{S} \to \mathcal{E}$ is a quintessential localization in the sense of [6]. Quality types and sufficiently cohesive categories are contrasting in the precise sense given by Proposition 3 in [13]: if $p : \mathcal{E} \to \mathcal{S}$ is both sufficiently cohesive and a quality type, then $\mathcal{S}$ is inconsistent. (Although stated for cohesive categories, it is clear from the proof that it also holds for pre-cohesive ones.) Loosely speaking, Sufficient Cohesion positively ensures that $\mathcal{E}$ and $\mathcal{S}$ are decidedly different. In particular, assuming that $0 \to 1$ is not an iso in $\mathcal{S}$, Sufficient Cohesion implies that $p_* : \mathcal{E} \to \mathcal{S}$ cannot be an equivalence.

There are many examples of sufficiently cohesive pre-cohesive toposes over $\mathbf{Set}$, including the topos of simplicial sets and the Zariski toposes determined by algebraically closed fields. As already mentioned in the first paragraph, the main contribution of the present paper is the detailed construction of a class of sufficiently cohesive pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ over toposes $\mathcal{S}$ different from $\mathbf{Set}$, namely the Galois toposes of (non algebraically closed) perfect fields. The construction will make evident what is the connection between the Nullstellensatz condition in Definition 1.1 and Hilbert's classical result. The reader will see that each of these geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ is induced by the inclusion of the category of finite extensions of a given field into a category of finitely presented algebras over the same field. It is then reasonable to expect that the same examples can be more directly constructed using a characterization of the morphisms of sites that induce sufficiently cohesive pre-cohesive geometric morphisms; but since we do not have such a characterization at present, we take a more indirect route using some results from [8] which studies the Nullstellensatz in the context of connected and locally connected geometric morphisms.

Notice that any string of adjoint functors $p_! \dashv p^* \dashv p_* : \mathcal{E} \to \mathcal{S}$ with fully faithful $p^* : \mathcal{E} \to \mathcal{S}$ determines a canonical natural $\theta : p_* \to p_!$ and then it is fair to say that the string of adjoints satisfies the Nullstellensatz if θ is epi. We will need to use this generality for such a string of adjoints given by a connected essential geometric morphism $p : \mathcal{E} \to \mathcal{S}$. (Recall that p is *connected* if p^* is full and faithful and it is *essential* if p^* has a left adjoint, typically denoted by $p_! : \mathcal{E} \to \mathcal{S}$.)

It is also relevant to briefly explain the relation with local connectedness. Recall that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *locally connected* if p^* has an $\mathcal{S}$-indexed left adjoint $p_! : \mathcal{E} \to \mathcal{S}$. Such geometric morphisms are, of course, essential. Theorem 3.4 and Proposition 3.5 in [8] imply that if $\mathcal{S}$ has a natural number object (nno) and $p : \mathcal{E} \to \mathcal{S}$ is bounded, connected, locally connected and satisfies the Nullstellensatz then p is pre-cohesive. (Connected locally connected geometric morphisms satisfying the Nullstellensatz are called 'punctually locally connected' in [8] but we will stick to the terminology of [13].) Reorganizing the hypotheses of these results we obtain the following fact.

Corollary 1.5. *If $\mathcal{S}$ has a nno and $p : \mathcal{E} \to \mathcal{S}$ is bounded, connected and locally connected, then p is pre-cohesive if and only if p satisfies the Nullstellensatz.*

In the case that $\mathcal{S} = \mathbf{Set}$ there is of course a stronger result because $p_!$ is automatically indexed. Recall that a site $(\mathcal{C}, J)$ is *locally connected* if every covering sieve is connected (as a subcategory of the corresponding slice). Such a site is called *connected* if $\mathcal{C}$ has a terminal object.

Proposition 1.6. *If $p : \mathcal{E} \to \mathbf{Set}$ is bounded then the following are equivalent:*

1. *p is pre-cohesive,*

2. *p is connected, essential and satisfies the Nullstellensatz,*

3. *$\mathcal{E}$ has a connected and locally connected site of definition $(\mathcal{C}, J)$ such that every object of $\mathcal{C}$ has a point.*

Proof. By Corollary 1.5 above and Proposition 1.4 in [8]. $\square$

We now outline the main results in the paper. In Section 2.2 we prove the following characterization of sufficiently cohesive pre-cohesive toposes over $\mathbf{Set}$.

Corollary 1.7. *Let $(\mathcal{C}, J)$ be a connected and locally connected site such that every object has a point and let $p : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Set}$ be the induced pre-cohesive geometric morphism. Then p is sufficiently cohesive if and only if there is an object in $\mathcal{C}$ with (at least) two distinct points.*

There is a precedent to both results above. In the last paragraph of p. 421 of [11] Lawvere states that it follows from a remark in Grothendieck's 1983 Pursuing Stacks that product preservation of $p_!$ and Sufficient Cohesion "will be satisfied by a topos of M-actions if the generic individual I ($= M$ acting on itself) has at least two distinct points".

A little trick will allow us to apply Corollary 1.7 to prove Sufficient Cohesion over other bases; so it remains to explain how to build pre-cohesive toposes over bases that are not $\mathbf{Set}$. In order to sketch the main ideas fix a geometric morphism $p : \mathcal{E} \to \mathcal{S}$, a Lawvere-Tierney topology j in $\mathcal{S}$ and consider the following pullback

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{E} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{S}_j & \xrightarrow[j]{} & \mathcal{S} \end{array}$$

of toposes. We are interested in conditions on p and j implying that q is pre-cohesive. Assume for simplicity that all toposes involved are Grothendieck and that p is connected and locally connected. Then q is also connected and locally connected by Theorem C3.3.15 in [7]. Corollary 1.5 leads us to consider conditions on p and j implying that q satisfies the Nullstellensatz.

The pullback stability result for locally connected geometric morphisms also shows that the (Beck-Chevalley) natural transformation $p^* j_* \to i_* q^*$ is an iso. Taking left adjoints we obtain an iso $j^* p_! \to q_! i^*$; pre-composing with i_* we get another iso $j^* p_! i_* \to q_! i^* i_*$ and we can use the counit of $i^* \dashv i_*$ to get the canonical iso $j^* p_! i_* \to q_! i^* i_* \to q_!$ that appears in the next result.

Lemma 1.8. *Given the pullback diagram above, the following diagram*

$$\begin{tikzcd}
j^* p_* i_* \arrow[r, "="] \arrow[d, "j^*\theta_{i_*}"'] & j^* j_* q_* \arrow[r, "\text{counit}"] & q_* \arrow[d, "\theta'"] \\
j^* p_! i_* \arrow[r, "\cong"'] & q_! i^* i_* \arrow[r, "\cong"'] & q_!
\end{tikzcd}$$

commutes, where $\theta' : q_* \to q_!$ is the natural transformation associated to the connected essential q.

This result is probably folklore but we give a detailed proof in Section 3.1.

As suggested in [15], we denote the image of the map $\theta_X : p_* X \to p_! X$ by $HX \to p_! X$. This is an "invariant of objects in the bigger category, recorded in the smaller".

Definition 1.9. Let us say that p satisfies the Nullstellensatz *relative to j* if for every X in $\mathcal{E}$, the mono $HX \to p_! X$ is j-dense.

Combining the above we obtain the following fact.

Lemma 1.10. *If, in the pullback diagram above, $p : \mathcal{E} \to \mathcal{S}$ satisfies the Nullstellensatz relative to j then $q : \mathcal{F} \to \mathcal{S}_j$ is pre-cohesive.*

Proof. By hypothesis, the image $H(i_* X) \to p_!(i_* X)$ of the canonical map $\theta_{i_* X} : p_*(i_* X) \to p_!(i_* X)$ is j-dense; so the canonical $\theta' : q_* X \to q_! X$ is epi by Lemma 1.8. □

In the examples that motivate this work, $\mathcal{S}$ is the topos $\widehat{\mathcal{D}}$ of presheaves on a category $\mathcal{D}$ that can be equipped with the atomic topology (inducing a Lawvere-Tierney topology j in $\widehat{\mathcal{D}}$) and p is induced by a functor $\phi : \mathcal{C} \to \mathcal{D}$ that has a fully faithful right adjoint ι. Also, the fact that p satisfies the Nullstellensatz relative to j naturally follows from a more concrete related condition that holds for the adjunction $\phi \dashv \iota$.

Definition 1.11. A full reflective subcategory $\iota : \mathcal{D} \to \mathcal{C}$ is said to satisfy the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there exists a map $\iota D \to C$ for some D in $\mathcal{D}$.

For example, if $\mathcal{C}$ has a terminal object then the inclusion $\iota : 1 \to \mathcal{C}$ of the terminal object is reflective and it satisfies the primitive Nullstellensatz if and only if every object of $\mathcal{C}$ has a point. In contrast notice that if $\mathcal{C}$ has initial object then the inclusion $\iota : 1 \to \mathcal{C}$ of the initial object trivially satisfies that for every C in $\mathcal{C}$ there exists a map $\iota D \to C$ for some D in $\mathcal{D}$, but the subcategory is not reflexive (unless $\mathcal{D}$ is trivial). In other words, the requirement of a left adjoint to ι excludes the situation just described from the examples of the primitive Nullstellensatz.

We now discuss how the primitive Nullstellensatz relates to Hilbert's classical result. Lawvere suggests that the relation is better explained by the conjunction of two facts: "traditionally, the heart of Hilbert's result is the existence of points, and that is merely a consequence of Zorn's Lemma"; the other fact is that that fields k have, as rings, the property that finitely-generated k-algebras that happen to be fields are in fact finitely-generated k-modules. (See also Tholen's analysis in [17], which is particularly well suited for our puroposes.)

Fix a field k. A classical commutative algebra textbook may formulate the two facts above as follows.

Lemma 1.12. *Let A be a k-algebra.*

1. *If A is not trivial then it has at least one maximal ideal.*
2. *If A is finitely generated as a k-algebra and $M \subseteq A$ is a maximal ideal then $k \to A \to A/M$ is a finite algebraic extension.*

Proof. The first item is proved in Theorem 1.3 in [2] as a "standard application of Zorn's lemma". The second item is Corollary 7.10 in [2] and it is referred to as the 'weak' version of Hilbert's Nullstellensatz. □

A k-algebra is called *connected* if it has exactly two idempotents. Let $\mathbf{Con}$ be the category of finitely presented and connected k-algebras. Denote the full subcategory of separable extensions of k by $\mathbf{Ext} \to \mathbf{Con}$.

Lemma 1.13. *The full inclusion* $\mathbf{Ext} \to \mathbf{Con}$ *has a right adjoint.*

Proof. This does not seem to be very well-known so we recall the proof taken from Proposition I, §4, 6.5 in [5]. Let A in $\mathbf{Con}$ and choose a maximal ideal $M \subseteq A$. Since A is connected every separable sub(k-)algebra $K \to A$ is a field and $[K : k] \leq [A/M : k]$. That is, the degrees of all possible such K are bounded; so the filtered system of such $K \subseteq A$ must have a maximum. □

We recall this, of course, because the primitive Nullstellensatz holds as explained below.

Example 1.14. Assume that k is perfect to avoid complications with separable extensions. Lemma 1.12 implies that for any A in $\mathbf{Con}$ there exists a map $A \to A/M$ with A/M in the subcategory $\mathbf{Ext} \to \mathbf{Con}$. This means that the full reflective $\mathbf{Ext}^{\mathrm{op}} \to \mathbf{Con}^{\mathrm{op}}$ satisfies the primitive Nullstellensatz. If k is algebraically closed then this says that every object of $\mathbf{Con}^{\mathrm{op}}$ has a point.

Fix a small category $\mathcal{C}$ and a full reflective subcategory $\iota : \mathcal{D} \to \mathcal{C}$ with reflector $\phi : \mathcal{C} \to \mathcal{D}$. The geometric morphism $\phi : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ induced by the reflector is essential, connected and local and so induces a string of functors

$$\begin{tikzcd}[column sep=small] \widehat{\mathcal{C}} \arrow[d, shift right=9, "\phi_!"'] \arrow[d, shift left=3, "\phi_*"] \\ \widehat{\mathcal{D}} \arrow[u, shift right=3, "\phi^*"'] \arrow[u, shift left=9, "\phi^!"] \end{tikzcd}$$

with $\phi_! \dashv \phi^* \dashv \phi_* \dashv \phi^!$ and $\phi^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ fully faithful. That is, a structure analogous to that in Definition 1.1 except that $\phi_!$ need not preserve products and the Nullstellensatz may not hold.

Assume now that $\mathcal{D}$ satisfies the (right) Ore condition so that it can be equipped with the atomic topology J_{at}. Denote the resulting Lawvere-Tierney topology on $\widehat{\mathcal{D}}$ by j_{at}. In Section 3.2 we prove the following.

Lemma 1.15. *If $\phi \dashv \iota : \mathcal{D} \to \mathcal{C}$ satisfies the primitive Nullstellensatz then the geometric morphism $\phi : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ satisfies the Nullstellensatz relative to j_{at}.*

Lemmas 1.10 and 1.15 imply the first part of the next result. The second part will be proved in Section 3.2.

Proposition 1.16. *Let the following diagram be a pullback*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i"] \arrow[d, "q"'] & \widehat{\mathcal{C}} \arrow[d, "\phi"] \\ \mathbf{Sh}(\mathcal{D}, J_{at}) \arrow[r, "j"'] & \widehat{\mathcal{D}} \end{tikzcd}$$

of toposes. If $\phi : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is locally connected and $\phi \dashv \iota : \mathcal{D} \to \mathcal{C}$ satisfies the primitive Nullstellensatz then $q : \mathcal{F} \to \mathbf{Sh}(\mathcal{D}, J_{at})$ is pre-cohesive. If, moreover, $\mathcal{C}$ has a terminal object and some object with two distinct points then q is sufficiently cohesive.

In Section 4 we discuss how to apply Proposition 1.16 to Example 1.14 and we also give a presentation of the theory classified by $\mathcal{F}$ in the case of $k = \mathbb{R}$.

2. Sufficient Cohesion

Here we characterize sufficiently cohesive pre-cohesive toposes $\mathcal{E} \to \mathbf{Set}$. The strategy to analyse Sufficient Cohesion is suggested by the following result.

Proposition 2.1. *Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive topos. Then p is sufficiently cohesive if and only if the subobject classifier of $\mathcal{E}$ is connected (i.e. $p_!\Omega = 1$).*

Proof. Simply observe that the proof of Proposition 4 in [13] does not need the Continuity condition. □

Let $p : \mathcal{E} \to \mathcal{S}$ be an essential geometric morphism. As usual we denote the left adjoint to p^* by $p_! : \mathcal{E} \to \mathcal{S}$, the subobject classifier of $\mathcal{E}$ by Ω and the top and bottom elements of its canonical lattice structure by $\top, \bot : 1 \to \Omega$.

Lemma 2.2. *If $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products then $p_!\Omega = 1$ if and only if the maps $p_!\top, p_!\bot : p_!1 \to p_!\Omega$ are equal.*

Proof. One direction is trivial (and does not require that $p_!$ preserves finite products). On the other hand, if $p_!$ preserves products then $p_!\Omega$ is equipped with a lattice structure with $p_!\top$ and $p_! \bot$ as top and bottom elements respectively. Since they are equal, $p_!\Omega = 1$. □

So the consideration of Sufficient Cohesion naturally leads to essential geometric morphisms whose leftmost adjoint preserves finite products. For example, recall that a small category $\mathcal{D}$ is *sifted* if and only if the colimit functor $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}$ preserves finite products [1] and that this holds if and only if $\mathcal{D}$ is nonempty and the diagonal $\mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is final. So, if we let $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ be the (essential) canonical geometric morphism then $p_!$ preserves finite products if and only if $\mathcal{C}$ is cosifted. To characterize those such p that satisfy $p_!\Omega = 1$ the following terminology will be useful.

Definition 2.3. A cospan $A \to B \leftarrow C$ in a category is said to be *disjoint* if it cannot be completed to a commutative square.

The next source of examples will also be relevant. (See the proof of Proposition 1.6(iii) in [8] for details.)

Lemma 2.4. *If $\mathcal{C}$ has a terminal object and every object of $\mathcal{C}$ has a point then $\mathcal{C}$ is cosifted.*

2.1 The case of presheaf toposes

Let $\mathcal{C}$ be a small category and $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ the canonical (essential) geometric morphism. Let us recall a description of $p_! : \mathcal{E} \to \mathbf{Set}$.

Fix a presheaf P in $\widehat{\mathcal{C}}$. A cospan $C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$ is said to *connect* the elements $x \in PC$ and $x' \in PC'$ if there is a $y \in PU$ such that $x = y \cdot \sigma_l$ and $x' = y \cdot \sigma_r$. In this case we may denote the situation by the following diagram

$$x \longleftarrow\!\!\shortmid\; y \longmapsto x'$$

$$C \xrightarrow[\sigma_l]{} U \xleftarrow[\sigma_r]{} C'$$

or simply write $x\sigma x'$.

A *path* from C to C' is a sequence of cospans $\sigma_1, \sigma_2, \dots, \sigma_n$ as below

$$C_0 \xrightarrow[\sigma_{1,l}]{} U_1 \xleftarrow[\sigma_{1,r}]{} C_1 \xrightarrow[\sigma_{2,l}]{} U_2 \xleftarrow[\sigma_{2,r}]{} C_2 \cdots\cdots C_{n-1} \xrightarrow[\sigma_{n,l}]{} U_n \xleftarrow[\sigma_{n,r}]{} C_n$$

with $C_0 = C$ and $C_n = C'$. Such a path *connects* elements $x \in PC$ and $x' \in PC'$ if there exists a sequence $(x_i \in PC_i \mid 0 \leq i \leq n)$ of elements such that $x_0 = x \in PC$, $x_n = x' \in PC'$ and for every $1 \leq i \leq n$, $x_{i-1}\sigma_i x_i$. We say that $x \in PC$ and $x' \in PC'$ are *connectable* if there is a path from C to C' that connects x and x'. An element in $p_!P$ is given by a 'tensor' $x \otimes C$ with $x \in PC$. Two such tensors $x \otimes C$ and $x' \otimes C'$ are equal if and only if they are connectable.

We now concentrate on the set $p_!\Omega$ whose elements are of the form $S \otimes C$ with S a sieve on C. Let M_C be the maximal sieve on C and Z_C be the empty sieve on C. We will sometimes write M instead of M_C and similarly for Z_C.

Lemma 2.5. *A cospan* $C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$ *is disjoint if and only if it connects* $M \in \Omega C$ *and* $Z \in \Omega C'$.

Proof. If the cospan is disjoint, the sieve on U generated by σ_l witnesses the fact that the cospan connects M_C and $Z_{C'}$. Conversely, if S is a sieve on U such that $S \cdot \sigma_l = M_C$ and $S \cdot \sigma_r = Z_{C'}$ then $\sigma_l \in S$ and there is no map $h : D \to C'$ such that $\sigma_r h$ is in S. In particular, there is no h such that $\sigma_r h$ factors through σ_l. So the cospan in the statement is disjoint. $\square$

A path $\sigma_1, \ldots, \sigma_n$ as above is called *singular at* i (for some $1 \leq i \leq n$) if the cospan

$$C_{i-1} \xrightarrow{\sigma_{i,l}} U_i \xleftarrow{\sigma_{i,r}} C_i$$

is disjoint. We say that the path is *singular* if it is singular at some i.

Lemma 2.6. *If the cospan* $C \xrightarrow{\sigma_l} U \xleftarrow{\sigma_r} C'$ *connects a non-empty sieve* $S \in \Omega C$ *and the empty sieve* $Z_{C'} \in \Omega C'$ *then there exists a singular path from* C *to* C'.

Proof. By hypothesis there is a sieve T on U as in the diagram below

$$S \longleftarrow\!\!\dashv T \longmapsto Z$$

$$C \xrightarrow[\sigma_l]{} U \xleftarrow[\sigma_r]{} C'$$

and, since S is non-empty, T is also non-empty. Let $\tau : D \to U$ a map in T. Since, $T \cdot \sigma_r = Z$, the cospan (τ, σ_r) is disjoint and so, the path below

$$C \xrightarrow[\sigma_l]{} U \xleftarrow[\tau]{} D \xrightarrow[\tau]{} U \xleftarrow[\sigma_r]{} C'$$

from C to C' is singular. $\square$

The main technical fact of the section is the following.

Lemma 2.7. *For any C and C', M_C is connectable with $Z_{C'}$ if and only if there exists a singular path from C to C'.*

Proof. Consider a path $\sigma_1, \ldots, \sigma_n$ from C to C'. Assume first that this path is singular at i. By Lemma 2.5, the cospan σ_i connects the maximal sieve on C_{i-1} with the empty sieve on C_i. Now observe that any path connects the maximal sieves on its 'extremes', and it also connects the empty sieves on its extremes. In particular, the path $\sigma_1, \ldots, \sigma_{i-1}$ connects M_C with $M_{C_{i-1}}$ and the path $\sigma_{i+1}, \ldots, \sigma_n$ connects Z_{C_i} with $Z_{C'}$. So the whole path $\sigma_1, \ldots, \sigma_n$ connects M_C and $Z_{C'}$.

For the converse assume that the path $\sigma_1, \ldots, \sigma_n$ connects M_C and $Z_{C'}$. Then there exist sieves $S_0, \ldots, S_n$ such that $S_0 = M$, $S_n = Z$ and for every $1 \leq i \leq n$, $S_{i-1}\sigma_i S_i$. So there exists a k such $S_k = Z$ and S_{k-1} is non-empty. By Lemma 2.6 there exists a singular path from C_{k-1} to C_k. Of course, this path can be extended to (a singular) one from C to C'. $\square$

If C is an object of $\mathcal{C}$ and we let the terminal object 1 in $\widehat{\mathcal{C}}$ be such that $1C = \{*\}$ then the morphisms $p_!\top, p_!\perp : p_!1 \to p_!\Omega$ map $* \otimes C$ to $M_C \otimes C$ and $Z_C \otimes C$ respectively.

Proposition 2.8. *If $\mathcal{C}$ is connected then the maps $p_!\top, p_!\perp : p_!1 \to p_!\Omega$ are equal if and only if $\mathcal{C}$ contains a disjoint cospan.*

Proof. As $\mathcal{C}$ is connected, there is an object C in $\mathcal{C}$ and also: $\mathcal{C}$ has a disjoint cospan if and only if there is a singular path from C to C. Now, the maps $p_!\top, p_!\perp : 1 \to p_!\Omega$ are equal if and only if $M_C \otimes C = Z_C \otimes C$. By Lemma 2.7, this holds if and only if there exists a singular path from C to C. $\square$

Since cosifted categories are connected the next result follows.

Corollary 2.9. *Let $\mathcal{C}$ be cosifted and $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ the canonical geometric morphism. Then $p_!\Omega = 1$ if and only if $\mathcal{C}$ contains a disjoint cospan.*

We can now characterize the sufficiently cohesive pre-cohesive presheaf toposes. For this it is convenient to state the presheaf version of Proposition 1.6 and, in fact, it is worth sketching a direct proof.

Proposition 2.10. *Let $\mathcal{C}$ be a small category whose idempotents split. The canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive if and only if $\mathcal{C}$ has a terminal object and every object of $\mathcal{C}$ has a point.*

Proof. The canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is essential and $p_!C = 1$ for every representable C in $\widehat{\mathcal{C}}$. Example C3.6.3(b) in [7] shows that p is local if and only is $\mathcal{C}$ has a terminal object. In this case, of course, p is connected. So we can assume that $\mathcal{C}$ has a point and then $p_*X = \widehat{\mathcal{C}}(1, X) = X1$ for every X in $\widehat{\mathcal{C}}$. If the Nullstellensatz holds then $\mathcal{C}(1, C) = p_*C \to p_!C = 1$ is epi and so every object of $\mathcal{C}$ has a point. For the converse assume that every object of $\mathcal{C}$ has a point and let P in $\widehat{\mathcal{C}}$. Recall that an element of $p_!P$ may be described as a 'tensor' $x \otimes C$ with $x \in PC$. The natural transformation $\theta : p_*P \to p_!P$ sends each $y \in P1$ to the tensor $y \otimes 1$. Since every C in $\mathcal{C}$ has a point, any tensor $x \otimes C$ is equal to one of the form $y \otimes 1$. Finally, $p_!$ preserves finite products by Lemma 2.4. $\square$

If $\mathcal{C}$ has a terminal object and every object has a point then the existence of a disjoint cospan is equivalent to the existence of an object with two distinct points, so the next result follows from Corollary 2.9 and Proposition 2.10.

Corollary 2.11. *Let $\mathcal{C}$ be a small category whose idempotents split and let $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ be pre-cohesive. Then p is sufficiently cohesive if and only if there is an object in $\mathcal{C}$ with two distinct points.*

2.2 The case of sheaves

Proposition 1.3 in [8] proves a characterization of the bounded locally connected $p : \mathcal{E} \to \mathbf{Set}$ such that $p_!$ preserves finite products. In this section we characterize, among these, those which satisfy $p_!\Omega = 1$. Some key ingredients may be isolated as basic facts about dense subtoposes and we treat them first.

Recall that a subtopos $i : \mathcal{F} \to \mathcal{E}$ is *dense* if $i_* : \mathcal{F} \to \mathcal{E}$ preserves the initial object 0 (see A4.5.20 in [7]). For any subtopos $i : \mathcal{F} \to \mathcal{E}$ consider the split mono $i_*\Omega_{\mathcal{F}} \to \Omega_{\mathcal{E}}$ presenting the subobject classifier of $\mathcal{F}$ as a retract of that of $\mathcal{E}$. The diagram on the left below

$$\begin{tikzcd} i_*1 \arrow[r, "i_*\top"] \arrow[d, "\cong"'] & i_*\Omega_\mathcal{F} \arrow[d] \\ 1 \arrow[r, "\top"'] & \Omega_\mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} i_*1 \arrow[r, "i_*\bot"] \arrow[d, "\cong"'] & i_*\Omega_\mathcal{F} \arrow[d] \\ 1 \arrow[r, "\bot"'] & \Omega_\mathcal{E} \end{tikzcd}$$

always commutes. On the other hand, the square on the right above commutes if and only if the subtopos is dense.

Lemma 2.12. *Let $p : \mathcal{E} \to \mathcal{S}$ be an essential geometric morphism. If the geometric $i : \mathcal{F} \to \mathcal{E}$ is a dense subtopos then the maps on the left below*

$$\begin{tikzcd} p_!i_*1 \arrow[r, shift left, "p_!(i_*\top)"] \arrow[r, shift right, "p_!(i_*\bot)"'] & p_!(i_*\Omega_\mathcal{F}) \end{tikzcd} \qquad \begin{tikzcd} p_!1 \arrow[r, shift left, "p_!\top"] \arrow[r, shift right, "p_!\bot"'] & p_!\Omega_\mathcal{E} \end{tikzcd}$$

are equal if and only if the ones on the right above are.

Proof. Since $i : \mathcal{F} \to \mathcal{E}$ is dense, the map $\bot\colon 1 \to \Omega_\mathcal{E}$ factors through the retract $i_*\Omega_\mathcal{F} \to \Omega_\mathcal{E}$. Then the diagram below

$$\begin{tikzcd} p_!1 \arrow[r, shift left, "p_!(i_*\top)"] \arrow[r, shift right, "p_!(i_*\bot)"'] \arrow[rr, bend left, "p_!\top"] \arrow[rr, bend right, "p_!\bot"'] & p_!(i_*\Omega_\mathcal{F}) \arrow[r] & p_!\Omega_\mathcal{E} \end{tikzcd}$$

commutes and the result follows because $p_!(i_*\Omega_\mathcal{F}) \to p_!\Omega_\mathcal{E}$ is (split) mono. □

This is applied in the next result where the subtopos is dense as a result of a stronger condition.

Lemma 2.13. *Consider a diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i"] \arrow[dr, "q"'] & \mathcal{E} \arrow[d, "p"] \\ & \mathcal{S} \end{tikzcd}$$

with i an inclusion. If p^* factors through i_* (in the sense that the canonical $p^* \to i_* i^* p^* = i_* q^*$ is an iso) then i is a dense subtopos and q is essential. If, moreover, $p_!$ preserves finite products then so does $q_!$. Also, in this case, $p_! \Omega_{\mathcal{E}} = 1$ if and only if $q_! \Omega_{\mathcal{F}} = 1$.

Proof. Start with the iso $p^* \to i_* q^*$. Since p^* and q^* preserve 0 then so does i_*. It is straightforward to check that the functor $p_! i_* : \mathcal{F} \to \mathcal{S}$ is left adjoint to $q^* : \mathcal{S} \to \mathcal{F}$ so q is essential and we can define $q_! = p_! i_* : \mathcal{F} \to \mathcal{S}$. Clearly, if $p_!$ preserves finite products then so does $q_!$. It remains to prove that $p_! \Omega_{\mathcal{E}} = 1$ if and only if $q_! \Omega_{\mathcal{F}}$. By Lemma 2.2 it is enough to prove that $p_! \top = p_! \bot : 1 \to p_! \Omega_{\mathcal{E}}$ if and only if $q_! \top = q_! \bot : 1 \to q_! \Omega_{\mathcal{F}}$. Since $q_! = p_! i_*$ the result follows from Lemma 2.12. $\square$

One of the equivalences in Proposition 1.3 of [8] states that if the canonical $p : \mathcal{E} \to \mathbf{Set}$ is bounded and locally connected then, $p_!$ preserves finite products if and only if $\mathcal{E}$ has a locally connected site of definition $(\mathcal{C}, J)$ with $\mathcal{C}$ cosifted.

Proposition 2.14. *Let $(\mathcal{C}, J)$ be a locally connected site with $\mathcal{C}$ cosifted and $q : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Set}$ be the induced geometric morphism. Then $q_! \Omega = 1$ if and only if $\mathcal{C}$ contains a disjoint cospan.*

Proof. We have a diagram

$$\begin{tikzcd} \mathcal{F} = \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "i"] \arrow[dr, "q"'] & \widehat{\mathcal{C}} = \mathcal{E} \arrow[d, "p"] \\ & \mathbf{Set} \end{tikzcd}$$

where p and q are locally connected, $p_!$ and $q_!$ preserve finite products and $i : \mathbf{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is a subtopos. In the proof of Proposition 1.3 in [8] it is observed that if a site $(\mathcal{C}, J)$ is locally connected then constant presheaves on $\mathcal{C}$ are J-sheaves. That is, $p^* : \mathbf{Set} \to \widehat{\mathcal{C}}$ factors through the embedding $\mathbf{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$, so Lemma 2.13 applies. Therefore $q_! \Omega_{\mathcal{F}} = 1$ if and only if $p_! \Omega_{\mathcal{E}} = 1$. The result follows from Corollary 2.9. $\square$

Corollary 1.7 follows from Proposition 2.14 and Lemma 2.4.

3. The Nullstellensatz

In Section 3.1 we prove Lemma 1.8 and then the proof of Lemma 1.10 will be complete. In Section 3.2 we show Lemma 1.15 and complete the proof of Proposition 1.16.

3.1 Proof of Lemma 1.8

As already mentioned in Section 1, this result is probably folklore. It should follow from 2-categorical generalities about morphisms of adjunctions, but I have failed to find the necessary machinery in the material I have access to, so I give here a simple minded proof. I try to keep the notation in Section 2 of [8].

Let $F \dashv R : \mathcal{E} \to \mathcal{S}$ and denote its unit and counit by η and ε respectively. In parallel, consider another adjunction $F' \dashv R' : \mathcal{E}' \to \mathcal{S}'$ with unit and counit denoted by η' and ε'. Fix also a commutative diagram

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "i_*"] \arrow[d, "R'"'] & \mathcal{E} \arrow[d, "R"] \\ \mathcal{S}' \arrow[r, "j_*"'] & \mathcal{S} \end{tikzcd}$$

with i_* and j_* having left adjoints denoted by i^* and j^* respectively. We denote the unit and counit of $i^* \dashv i_*$ by u and c, and those of $j^* \dashv j_*$ by u' and c'.

Because left adjoints are essentially unique there exists a canonical isomorphism $\varphi : i^*F \to F'j^*$ such that the following diagram

$$\begin{tikzcd} Id \arrow[r, "\eta"] \arrow[d, "u'"'] & RF \arrow[r, "Ru_F"] & Ri_*i^*F \arrow[d, "Ri_*\varphi"] \\ j_*j^* \arrow[r, "j_*\eta'_{j^*}"'] & j_*R'F'j^* \arrow[r, "="'] & Ri_*F'j^* \end{tikzcd}$$

commutes. (The top map is the unit of the composite adjunction $i^*F \dashv Ri_*$.)

The transposition of φ is the composite

$$Fj_* \xrightarrow{u_{Fj_*}} i_*i^*Fj_* \xrightarrow{i_*\varphi_{j_*}} i_*F'j^*j_* \xrightarrow{i_*F'c'} i_*F'$$

and will be denoted by $\zeta : Fj_* \to i_* F'$. We call it the *Beck-Chevalley natural transformation*. Trival calculations show the following.

Lemma 3.1. *The diagrams*

$$\begin{tikzcd} i^*F \arrow[r, "i^*Fu'"] \arrow[drr, "\varphi"'] & i^*Fj_*j^* \arrow[r, "i^*\zeta_{j^*}"] & i^*i_*F'j^* \arrow[d, "c_{F'j^*}"] \\ & & F'j^* \end{tikzcd} \qquad \begin{tikzcd} j_* \arrow[r, "\eta"] \arrow[drr, "j_*\eta'"'] & RFj_* \arrow[r, "R\zeta"] & Ri_*F \arrow[d, "="] \\ & & j_*RF \end{tikzcd}$$

commute.

Assume from now on that F has a left adjoint $L : \mathcal{E} \to \mathcal{S}$ and denote the unit of $L \dashv F$ by $\alpha : Id \to FL$.

Lemma 3.2. *If i_* is full and faithful then the following diagram*

$$\begin{tikzcd} i_*i^*i_* \arrow[rrr, "i^*i_*\alpha_{i_*}"] \arrow[d, "i_*c"'] & & & i_*i^*FLi_* \arrow[d, "i_*\varphi_{Li_*}"] \\ i_* \arrow[r, "\alpha_{i_*}"'] & FLi_* \arrow[r, "Fu'_{Li_*}"'] & Fj_*j^*Li_* \arrow[r, "\zeta_{j^*Li_*}"'] & i_*F'j^*Li_* \end{tikzcd}$$

commutes.

Proof. The transposition of the top-right map is

$$i^*i_*i^*i_* \xrightarrow{c_{i^*i_*}} i^*i_* \xrightarrow{i^*\alpha_{i_*}} i^*FLi_* \xrightarrow{\varphi_{Li_*}} F'j^*Li_*$$

while that of the left-bottom one is

$$i^*i_*i^*i_* \xrightarrow{i^*i_*c} i^*i_* \xrightarrow{i^*\alpha_{i_*}} i^*FLi_* \xrightarrow{\varphi_{Li_*}} F'j^*Li_*$$

by Lemma 3.1. But $c_{i^*i_*} = i^*i_*c : i^*i_*i^*i_* \to i^*i_*$ because $c : i^*i_* \to Id$ is an iso by hypothesis. $\square$

We say that the *Beck-Chevalley condition* holds if $\zeta : Fj_* \to i_*F'$ is an iso. (See A4.1.16 in [7].)

Lemma 3.3. *Assume the Beck-Chevalley condition holds and that i_*, j_* and F are full and faithful. Then F' is full and faithful and has a left adjoint defined by $L' = j^*Li_* : \mathcal{E} \to \mathcal{S}$.*

Proof. First calculate:

$$\mathcal{E}'(F'X, F'Y) \cong \mathcal{E}(i_*F'X, i_*F'Y) \cong \mathcal{E}(Fj_*X, Fj_*Y) \cong \mathcal{S}'(X, Y)$$

to show that F' is full and faithful. To prove that $L' \dashv F'$ notice that:

$$\mathcal{S}'(L'X, S) \cong \mathcal{E}(i_*X, Fj_*S) \cong \mathcal{E}(i_*X, i_*F'S)$$

by adjointness and Beck-Chevalley. So $\mathcal{S}'(LX, S) \cong \mathcal{E}'(X, F'S)$ because i_* is full and faithful. $\square$

Assume from now on that the hypotheses of Lemma 3.3 hold and that $L' : \mathcal{E}' \to \mathcal{S}$ is defined as in that statement. Moreover, let α' denote the unit of $L' \dashv F'$.

Lemma 3.4. *The composition*

$$Id \xrightarrow{c^{-1}} i^*i_* \xrightarrow{i^*\alpha_{i_*}} i^*FLi_* \xrightarrow{\varphi_{Li_*}} F'j^*Li_* = F'L'$$

equals the unit $\alpha' : Id \to F'L'$ of $L' \dashv F'$.

Proof. If we chase the identity $L' \to L'$ in the proof of Lemma 3.3 then we obtain that the unit of $L' \dashv F'$ is the top-right composition in the diagram below:

$$\begin{tikzcd}
Id \arrow[r, "c^{-1}"] & i^*i_* \arrow[r, "i^*\alpha_{i_*}"] & i^*FLi_* \arrow[r, "i^*Fu'_{Li_*}"] \arrow[drr, "\varphi_{Li_*}"'] & i^*Fj_*j^*Li_* \arrow[r, "i^*\zeta_{j^*Li_*}"] & i^*i_*F'j^*Li_* \arrow[d, "c"] \\
 & & & & F'j^*Li_*
\end{tikzcd}$$

and the triangle commutes by Lemma 3.1. $\square$

The units α and α' may be related as follows.

Lemma 3.5. *The following diagram*

$$\begin{tikzcd} i_* \arrow[r, "i_*\alpha'"] \arrow[d, "\alpha_{i_*}"'] & i_*F'L' \arrow[r, "\zeta_{L'}^{-1}"] & Fj_*L' \arrow[d, "="] \\ FLi_* \arrow[rr, "Fu'_{Li_*}"'] & & Fj_*j^*Li_* \end{tikzcd}$$

commutes.

Proof. Post-composing with $\zeta_{L'}$ and replacing α' with its expression given in Lemma 3.4 the statement is equivalent to the commutativity of the diagram

$$\begin{tikzcd} i_* \arrow[r, "i_*c^{-1}"] \arrow[d, "\alpha_{i_*}"'] & i_*i^*i_* \arrow[r, "i_*i^*\alpha_{i_*}"] & i_*i^*FLi_* \arrow[d, "i_*\varphi_{Li_*}"] & & \\ FLi_* \arrow[r, "Fu'_{Li_*}"'] & Fj_*j^*Li_* \arrow[r, "\zeta_{j^*Li_*}"'] & i_*F'j^*Li_* & = & i_*F'L' \end{tikzcd}$$

but pre-composing with $i_*c : i_*i^*i_* \to i_*$ this is equivalent to Lemma 3.2. □

Following [8] define $\theta = (\eta_L)^{-1}(R\alpha) : R \to L$ and $\theta' : R' \to L'$ analogously.

Lemma 3.6. *The diagram*

$$\begin{tikzcd} j^*Ri_* \arrow[r, "="] \arrow[d, "j^*\theta_{i_*}"'] & j^*j_*R' \arrow[r, "c'_{R'}"] & R' \arrow[d, "\theta'"] \\ j^*Li_* \arrow[rr, "="'] & & L' \end{tikzcd}$$

commutes.

Proof. Start from the top-right and calculate:

$$\begin{tikzcd} j^*Ri_* \arrow[r, "="] \arrow[d, "j^*Ri_*\alpha'"'] & j^*j_*R' \arrow[r, "c'_{R'}"] \arrow[d, "j^*j_*R'\alpha'"'] & R' \arrow[d, "R'\alpha'"] \\ j^*Ri_*F'L' \arrow[r, "="] \arrow[d, "j^*R(\zeta_{L'})^{-1}"'] & j^*j_*R'F'L' \arrow[r, "c'"] \arrow[d, "j^*j_*(\eta'_{L'})^{-1}"] & R'F'L' \arrow[d, "(\eta'_{L'})^{-1}"] \\ j^*RFj_*L' \arrow[r, "j^*(\eta_{j_*L'})^{-1}"'] & j^*j_*L' \arrow[r, "c_{L'}"'] & L' \end{tikzcd}$$

where the bottom-left square commutes by Lemma 3.1. Now observe that, by Lemma 3.5, the left-edge equals the composition

$$j^*Ri_* \xrightarrow{j^*R\alpha_{i_*}} j^*RFLi_* \xrightarrow{j^*RFu'_{Li_*}} j^*RFj_*L'$$

which, followed by the bottom edge, equals $j^*\theta_{i_*}$. $\square$

To complete the proof of Lemma 1.8 just observe that the pullback diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i"] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{S}_j \arrow[r, "j"'] & \mathcal{S} \end{tikzcd}$$

discussed there satisfies all the hypotheses used in this section: we have already mentioned that, by Theorem C3.3.15 in [7], q is connected and locally connected and the square is Beck-Chevalley; also, i is a subtopos by Example A4.15.14(e) loc. cit.

3.2 Proof of Proposition 1.16

Here we prove Lemma 1.15 and Proposition 1.16. Fix small categories $\mathcal{C}$ and $\mathcal{D}$.

Definition 3.7. A functor $\iota : \mathcal{D} \to \mathcal{C}$ is said to satisfy the *(right) Ore condition* if for every C in $\mathcal{C}$ and diagram as on the left below

$$\begin{tikzcd} & C \arrow[d, "f"] \\ \iota D_1 \arrow[r, "\iota g"'] & \iota D_0 \end{tikzcd} \qquad \begin{tikzcd} \iota D_2 \arrow[r, "h"] \arrow[d, "\iota f'"'] & C \arrow[d, "f"] \\ \iota D_1 \arrow[r, "\iota g"'] & \iota D_0 \end{tikzcd}$$

in $\mathcal{C}$, there exists a map $f' : D_2 \to D_1$ in $\mathcal{D}$ and a map $h : \iota D_2 \to C$ in $\mathcal{C}$ such that the diagram on the right above commutes.

Clearly, a category $\mathcal{D}$ satisfies the right Ore condition in the usual sense if and only if the identity functor $\mathcal{D} \to \mathcal{D}$ does so in the sense of Definition 3.7. We now relate this condition to the one defining the primitive Nullstellensatz (Definition 1.11).

Lemma 3.8. *If $\iota : \mathcal{D} \to \mathcal{C}$ is full and satisfies that for every C in $\mathcal{C}$ there is a map $\iota D \to C$ for some D in $\mathcal{D}$ then the first item below:*

1. *$\mathcal{D}$ satisfies the Ore condition in the usual sense,*
2. *ι satisfies the Ore condition in the sense of Definition 3.7,*

implies the second. If, moreover, ι is faithful then the converse holds.

Proof. Consider a diagram as on the left below

$$
\begin{tikzcd}
 & C \arrow[d, "f"] \\
\iota D_1 \arrow[r, "\iota g"'] & \iota D_0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\iota D \arrow[r, "h"] \arrow[dr, "\iota t"'] & C \arrow[d, "f"] \\
 & \iota D_0
\end{tikzcd}
$$

in $\mathcal{D}$. By hypothesis there is a map $h : \iota D \to C$ for some D and, because ι is full, there is a map $t : D \to D_0$ in $\mathcal{D}$ such that the diagram on the right above commutes. As $\mathcal{D}$ satisfies the Ore condition, there is a diagram as on the left below

$$
\begin{tikzcd}
D_2 \arrow[r, "w"] \arrow[d, "t'"'] & D \arrow[d, "t"] \\
D_1 \arrow[r, "g"'] & D_0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\iota D_2 \arrow[r, "\iota w"] \arrow[d, "\iota t'"'] & \iota D \arrow[r, "h"] \arrow[dr, "\iota t"'] & C \arrow[d, "f"] \\
\iota D_1 \arrow[rr, "\iota g"'] & & \iota D_0
\end{tikzcd}
$$

in $\mathcal{D}$. The diagram on the right above shows that ι satisfies the Ore condition.

For the converse consider a cospan $g : D \to E \leftarrow D' : g'$ in $\mathcal{D}$. As ι satisfies the Ore condition there is an $f' : D_2 \to D$ in $\mathcal{D}$ and an $h : \iota D_2 \to \iota D'$ in $\mathcal{C}$ such that the diagram on the left below

$$
\begin{tikzcd}
\iota D_2 \arrow[r, "h"] \arrow[d, "\iota f'"'] & \iota D' \arrow[d, "\iota g'"] \\
\iota D \arrow[r, "\iota g"'] & \iota E
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
D_2 \arrow[r, "h'"] \arrow[d, "f'"'] & D' \arrow[d, "f"] \\
D \arrow[r, "g"'] & E
\end{tikzcd}
$$

commutes. Because ι is full there is an $h' : D_2 \to D'$ such that $\iota h' = h$ and, since ι is faithful, the diagram on the right above commutes. $\square$

We can now prove Lemma 1.15. Let $\mathcal{D}$ be a small category satisfying the right Ore condition and let $(\mathcal{D}, J_{at})$ be the resulting atomic site. Fix a full reflective subcategory $\phi \dashv \iota : \mathcal{D} \to \mathcal{C}$ satisfying the primitive Nullstellensatz. Lemma 1.15 states that the induced (essential connected) geometric morphism $\phi : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ satisfies the Nullstellensatz relative to the Lawvere-Tierney topology in $\widehat{\mathcal{D}}$ induced by J_{at}. Concretely this means that for any P in $\widehat{\mathcal{C}}$, the image $HP \to \phi_! P$ of θ_P is J_{at}-dense. This holds if and only if the map $\theta_P : \phi_* P \to \phi_! P$ is locally surjective. (Recall that a morphism $\alpha : F \to G$ in $\widehat{\mathcal{D}}$ is *locally surjective* w.r.t. J_{at} if for each D in $\mathcal{D}$ and each $y \in GD$, there is map $e : D' \to D$ such that $y \cdot e$ is in the image of $\alpha_{D'}$. See Corollary III.7.6 in [16].)

Proof of Lemma 1.15. For any P in $\widehat{\mathcal{C}}$ and D in $\mathcal{D}$, $(\phi_! P)D$ may be expressed as the following coequalizer:

$$\textstyle\sum_{C,C'} PC \times \mathcal{C}(C', C) \times \mathcal{D}(D, \phi C') \underset{r}{\overset{l}{\rightrightarrows}} \sum_C PC \times \mathcal{D}(D, \phi C) \longrightarrow (\phi_! P)D$$

where for $x \in PC$, $u : C' \to C$ and $a' \in \mathcal{D}(D, \phi C')$, $l(x, u, a') = (x \cdot u, a')$ and $r(x, u, a') = (x, (\phi u)a')$. The equivalence class determined by a pair (x, a) with $x \in PC$ and $a : D \to \phi C$ will be denoted by $x \otimes a \in (\phi_! P)D$. (Theorem VII.2.2 in [16].) Also, $(\phi_* P)D = P(\iota D)$ for any P in $\widehat{\mathcal{C}}$ and D in $\mathcal{D}$, and $\theta : \phi_* P \to \phi_! P$ assigns to each $x \in (\phi_* P)D = P(\iota D)$ the element $(x \otimes \varepsilon^{-1}) \in (\phi_! P)D$ where $\varepsilon : \phi(\iota D) \to D$ is the counit of $\phi \dashv \iota$.

As explained above we must prove that the map $\theta_P : \phi_* P \to \phi_! P$ is locally surjective. So let $x \otimes d \in (\phi_! P)D$ with $d : D \to \phi C$ and $x \in PC$. By Lemma 3.8 the functor $\iota : \mathcal{D} \to \mathcal{C}$ satisfies the right Ore condition. So there exists a diagram in $\mathcal{C}$ as below

$$\begin{tikzcd} \iota D' \arrow[r, "h"] \arrow[d, "\iota e"'] & C \arrow[d, "\eta"] \\ \iota D \arrow[r, "\iota d"'] & \iota(\phi C) \end{tikzcd}$$

where η is the unit of $\phi \dashv \iota$. We claim that $(x \otimes d) \cdot e = x \otimes (de)$ in $(\phi_! P)D$ equals $\theta(x \cdot h) = (x \cdot h) \otimes \varepsilon^{-1} = x \otimes ((\phi h)\varepsilon^{-1})$. For this, it is enough to prove that $de = (\phi h)\varepsilon^{-1}$ in $\mathcal{D}$. Since the counit is an iso, it is enough to prove that $de\varepsilon = \phi h$. So apply ϕ to the square above, post-compose with ε to obtain

$$\begin{tikzcd}
\phi(\iota D') \arrow[r, "\phi h"] \arrow[d, "\phi(\iota e)"'] & \phi C \arrow[d, "\phi\eta"] \arrow[dr, "id"] & \\
\phi(\iota D) \arrow[r, "\phi(\iota d)"'] & \phi(\iota(\phi C)) \arrow[r, "\varepsilon"'] & \phi C
\end{tikzcd}$$

and observe that the left-bottom composition equals $de\varepsilon$. $\square$

To complete the proof of Proposition 1.16 assume that the connected geometric morphism $\phi : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is locally connected so that if we take the pullback

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "i"] \arrow[d, "q"'] & \widehat{\mathcal{C}} \arrow[d, "\phi"] \\
\mathbf{Sh}(\mathcal{D}, J_{at}) \arrow[r, "j"'] & \widehat{\mathcal{D}}
\end{tikzcd}$$

of toposes then $q : \mathcal{F} \to \mathbf{Sh}(\mathcal{D}, J_{at})$ is connected and locally connected. Lemmas 1.10 and 1.15 imply that q is pre-cohesive. So it remains to show that if $\mathcal{C}$ has a terminal object and has an object with two distinct points then q is sufficiently cohesive. Denote $\widehat{\mathcal{C}}$ by $\mathcal{E}$ and its subobject classifier by $\Omega_{\mathcal{E}}$.

Lemma 3.9. *If $\phi_!\top : 1 = \phi_! 1 \to \phi_!\Omega_{\mathcal{E}}$ is j-dense then $q_!\Omega_{\mathcal{F}} = 1$.*

Proof. By Lemma 3.3 we can assume that $q_! = j^*\phi_! i_* : \mathcal{F} \to \mathbf{Sh}(\mathcal{D}, J_{at})$. We know that $i_*\Omega_{\mathcal{F}}$ is a retract of $\Omega_{\mathcal{E}}$ so $j^*(\phi_!(i_*\Omega_{\mathcal{F}})) = q_!\Omega_{\mathcal{F}}$ is a retract of $j^*(\phi_!\Omega_{\mathcal{E}})$. Hence, $j^*(\phi_!\Omega_{\mathcal{E}}) = 1$ implies $q_!\Omega_{\mathcal{F}} = 1$. $\square$

Now recall that a mono in $\widehat{\mathcal{D}}$ is dense (for the atomic topology) in $\mathcal{D}$ if and only if it is locally surjective.

Lemma 3.10. *Let $f : \widehat{\mathcal{D}} \to \mathbf{Set}$ be the canonical geometric morphism to $\mathbf{Set}$. For any $\alpha : X \to Y$ in $\widehat{\mathcal{D}}$, if $f_!\alpha : f_! X \to f_! Y$ is epi in $\mathbf{Set}$ then α is locally surjective in $\widehat{\mathcal{D}}$.*

Proof. Let $y \in YD$. Then $(y \otimes D) \in f_! Y$ and, by hypothesis, there exists an $(x \otimes E) \in f_! X$ such that $(f_!\alpha)(x \otimes E) = (\alpha_E x) \otimes E = (y \otimes D) \in f_! Y$. Because of the Ore condition this is equivalent to the existence of a span

$$E \xleftarrow{\;l\;} A \xrightarrow{\;r\;} D$$

in $\mathcal{D}$ such that $(\alpha_E x) \cdot l = y \cdot r \in YA$. So $\alpha_A(x \cdot l) = y \cdot r$, showing that y is locally in the image of α. $\square$

Finally let $g : \widehat{\mathcal{C}} \to \mathbf{Set}$ be the canonical geometric morphism, so that $f_! \phi_! = g_! : \mathcal{E} = \widehat{\mathcal{C}} \to \mathbf{Set}$. If $\mathcal{C}$ is cosifted and has a disjoint cospan then $g_! \top = f_!(\phi_! \top) : 1 \to f_!(\phi_! \Omega_{\mathcal{E}})$ is an iso by Corollary 2.9, $\phi_! \top : 1 \to \phi_! \Omega_{\mathcal{E}}$ is locally surjective by Lemma 3.10 and hence $q_! \Omega_{\mathcal{F}} = 1$ by Lemma 3.9. That is, q is sufficiently cohesive, as we needed to prove.

4. Sufficient Cohesion over Galois toposes

Let k be a field. Let $\mathbf{Con}$ be the category of finitely presented connected k-algebras and $\ell : \mathbf{Ext} \to \mathbf{Con}$ the full subcategory of separable extensions of k. Lemma 1.13 shows that ℓ has a right adjoint $\rho : \mathbf{Con} \to \mathbf{Ext}$. It is now relevant to mention a related fact. Let $\mathbf{Alg}$ be the category of finitely presented k-algebras and $\overline{\ell} : \mathbf{Sep} \to \mathbf{Alg}$ the full subcategory of separable k-algebras. It is clear that $\ell : \mathbf{Ext} \to \mathbf{Con}$ is the restriction of $\overline{\ell}$ along the inclusion $\mathbf{Ext} \to \mathbf{Sep}$ as displayed in the following diagram

$$\begin{tikzcd} \mathbf{Con} \arrow[r] & \mathbf{Alg} \\ \mathbf{Ext} \arrow[u, "\ell"] \arrow[r] & \mathbf{Sep} \arrow[u, "\overline{\ell}"'] \end{tikzcd}$$

and that ρ extends to a right adjoint $\overline{\rho} : \mathbf{Alg} \to \mathbf{Sep}$ to $\overline{\ell}$.

Proposition 4.1. *For any A in $\mathbf{Alg}$ and K in $\mathbf{Ext}$, the canonical map $(\overline{\rho} A) \otimes_k K \to \overline{\rho}(A \otimes_k K)$ is an iso. In other words if the square on the left below*

$$\begin{tikzcd} A \arrow[r, "in_0"] & A \otimes_k K \\ k \arrow[u, "j"] \arrow[r, "b"'] & K \arrow[u, "in_1"'] \end{tikzcd} \qquad \begin{tikzcd} \overline{\rho} A \arrow[r, "\overline{\rho}(in_0)"] & \overline{\rho}(A \otimes_k K) \\ k \arrow[u, "\overline{\rho}(j)"] \arrow[r, "b"'] & K \arrow[u, "\overline{\rho}(in_1)"'] \end{tikzcd}$$

is a pushout in $\mathbf{Alg}$ then the square on the right is a pushout in $\mathbf{Sep}$.

Proof. This is Proposition I, §4, 6.7 in [5]. $\square$

Assume for the moment that $\overline{\rho}(j) : k \to \overline{\rho}A$ is an iso in the right square above. In particular, the largest separable subalgebra of A does not have idempotents, so A is connected. Of course, $\overline{\rho}(in_1) : K \to \overline{\rho}(A \otimes_k K)$ is also an iso and, again, this implies that $A \otimes_k K$ is connected. Let us stress this fact, if $A \in \mathbf{Con}$, $\rho A = k$ and $K \in \mathbf{Ext}$ then the algebra $A \otimes_k K$ is also in $\mathbf{Con}$ and $\rho(A \otimes_k K) = K$. Moreover, this is for every k.

Lemma 4.2. *The geometric morphism* $[\mathbf{Con}, \mathbf{Set}] \to [\mathbf{Ext}, \mathbf{Set}]$ *induced by* $\rho : \mathbf{Con} \to \mathbf{Ext}$ *is connected and locally connected.*

Proof. As we have already mentioned, connectedness follows from the fact that ρ has a full and faithful left adjoint. To prove local connectedness we use a sufficient condition proved in [7]. This condition involves a category $\mathcal{X}_\rho = \mathcal{X}$ of so called *ρ-extracts*. In general, its objects would be 4-tuples (U, V, r, i) with U in the domain of ρ, V in the codomain, $r : \rho U \to V$ a map and $i : V \to \rho U$ a section of r; and maps $(U, V, r, i) \to (U', V', r', i')$ would be pairs $(a : U \to U', b : V \to V')$ such that $r'(\rho a) = br$ and $i'b = (\rho a)i$. In our concrete case, every map in the codomain of $\rho : \mathbf{Con} \to \mathbf{Ext}$ is mono and ρ has a full and faithful left adjoint ℓ so each object (U, V, r, i) is completely determined by a map $j : \ell V \to U$ such that $\rho j : \rho(\ell V) \to \rho U$ is an iso. It is convenient to drop ℓ from the notation and denote objects in $\mathbf{Ext}$ with decorated K's. Then the category $\mathcal{X}$ of ρ-extracts may be described as follows: its objects are triples $(U, K, j : K \to U)$ with U in $\mathbf{Con}$ such that $\rho j : K \to \rho U$ is an iso; and a map $a : (U, K, j) \to (U', K', j')$ is just a map $a : U \to U'$ in $\mathbf{Con}$. There is an obvious functor $g : \mathcal{X} \to \mathbf{Ext}$ that sends (U, K, j) to K and $a : (U, K, j) \to (U', K', j')$ to the unique map $ga : K \to K'$ making the following square

$$\begin{array}{ccc} \rho U & \xrightarrow{\rho a} & \rho U' \\ {\scriptstyle \rho j}\uparrow & & \uparrow{\scriptstyle \rho j'} \\ K & \xrightarrow[ga]{} & K' \end{array}$$

commute. For any K in $\mathbf{Ext}$ write $\mathcal{X}(K)$ for the fibre of g over K. Now, for each $b : K \to K'$ in $\mathbf{Ext}$ and lifting of K to an object (U, K, j) in $\mathcal{X}$ define the category $\mathcal{Y}_{U,K,j,b} = \mathcal{Y}$ whose objects are liftings of b to a morphism of $\mathcal{X}$ with domain (U, K, j) and whose morphisms are morphisms of $\mathcal{X}(K')$

forming commutative triangles. Lemma C3.3.8 of [7] implies that: if for each b and (U, K, j) as above, the associated category $\mathcal{Y}$ is connected then $[\mathbf{Con}, \mathbf{Set}] \to [\mathbf{Ext}, \mathbf{Set}]$ is locally connected. Let us first prove that $\mathcal{Y}$ is nonempty. For this consider the pushout on the left below

$$\begin{tikzcd} U \arrow[r, "in_0"] & U \otimes_K K' \\ K \arrow[u, "j"] \arrow[r, "b"'] & K' \arrow[u, "in_1"'] \end{tikzcd} \qquad \begin{tikzcd} \rho U \arrow[r, "\rho(in_0)"] & \rho(U \otimes_K K') \\ K \arrow[u, "\rho(j)"] \arrow[r, "b"'] & K' \arrow[u, "\rho(in_1)"'] \end{tikzcd}$$

calculated in the category of k-algebras. Since $\rho(j)$ is iso by hypothesis (recall that (U, K, j) is in $\mathcal{X}$) Proposition 4.1 implies that $U \otimes_K K'$ is connected and that $\rho(in_1) : K' \to \rho(U \otimes_K K')$ is an iso. Hence, the map $in_0 : (U, K, j) \to (U \otimes_K K', K', in_1)$ is an object in $\mathcal{Y}$. Finally, consider any object $a : (U, K, j) \to (U', K', j')$ in $\mathcal{Y}$ as displayed on the left below

$$\begin{tikzcd} U \arrow[r, "a"] & U' \\ K \arrow[u, "j"] \arrow[r, "b"'] & K' \arrow[u, "j'"'] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "in_0"] \arrow[dr, "a"'] & U \otimes_K K' \arrow[d, "h"] & K' \arrow[l, "in_1"'] \arrow[dl, "j'"] \\ & U' & \end{tikzcd}$$

and notice that the pushout property determines a unique $h : U \otimes_K K' \to U'$ such that the triangles on the right above commute. So h is a map in $\mathcal{Y}$ from $in_0 : (U, K, j) \to (U \otimes_K K', K', in_1)$ to $a : (U, K, j) \to (U', K', j')$. It follows that $\mathcal{Y}$ is indeed connected. $\square$

After the proof of Lemma 4.2 we stress that we do not claim to have found the most efficient way to present the examples. It is to be expected that in a near future there will be simpler ways to explain how the inclusion $\mathbf{Ext} \to \mathbf{Con}$ determines a pre-cohesive topos. In any case, we have the following result.

Proposition 4.3. *If k is perfect, $p : [\mathbf{Con}, \mathbf{Set}] \to [\mathbf{Ext}, \mathbf{Set}]$ is the geometric morphism induced by the coreflector $\rho : \mathbf{Con} \to \mathbf{Ext}$ and the following diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i"] \arrow[d, "q"'] & {[\mathbf{Con}, \mathbf{Set}]} \arrow[d, "p"] \\ \mathbf{Sh}(\mathbf{Ext}^{\mathrm{op}}, J_{at}) \arrow[r] & {[\mathbf{Ext}, \mathbf{Set}]} \end{tikzcd}$$

is a pullback of toposes then $q : \mathcal{F} \to \mathbf{Sh}(\mathbf{Ext}^{\mathrm{op}}, J_{at})$ is pre-cohesive and sufficiently cohesive.

Proof. The category $\mathcal{D} = \mathbf{Ext}^{\mathrm{op}}$ satisfies the right Ore condition (see example 7 in [3]). Let $\mathcal{C} = \mathbf{Con}^{\mathrm{op}}$, $\iota : \mathcal{D} \to \mathcal{C}$ the obvious full inclusion and $\phi = \rho^{\mathrm{op}} : \mathcal{C} \to \mathcal{D}$ its left adjoint. Example 1.14 shows that the reflective subcategory $\iota : \mathcal{D} \to \mathcal{C}$ satisfies the primitive Nullstellensatz and $\phi : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is connected and locally connected by Lemma 4.2. Finally, the category $\mathcal{C}$ has a terminal object and the two maps $k[x] \to k$ in $\mathbf{Con}$ that send x to 0 and 1 in k respectively show that there is an object in $\mathcal{C}$ with two distinct points. So we can apply Proposition 1.16. $\square$

The construction of examples in this section naturally leads to the following questions. Let $\mathcal{C}$ be an extensive category with finite products and let $\mathcal{C}_s \to \mathcal{C}$ is its full subcategory of separable/decidable/unramified objects [10, 4]. When is this category reflective? Assuming that $\mathcal{C}$ is small, when is it the case that the left adjoint $\phi : \mathcal{C} \to \mathcal{C}_s$ induces a locally connected $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}_s}$? To prove this for our examples we used Proposition 4.1 which highlights a special behaviour of tensor products in the category of k-algebras for a field k. So we are led to a more specific problem. Consider a coextensive algebraic category $\mathcal{V}$ (such as those discussed in [14]) and let K be an object in $\mathcal{V}$. The category $K/\mathcal{V}$ is also algebraic and coextensive. If we let $\mathcal{C}$ be the opposite of the category of finitely presentable objects in $K/\mathcal{V}$ then it would be interesting to understand those K that make $\mathcal{C}_s \to \mathcal{C}$ reflective etc.

If $k = \mathbb{C}$ then $\mathbf{Ext}$ is terminal so the horizontal maps in the pullback in the statement of Proposition 4.3 are equivalences and (the canonical) $p : [\mathbf{Con}, \mathbf{Set}] \to [\mathbf{Ext}, \mathbf{Set}] = \mathbf{Set}$ is pre-cohesive. But we stress that, in general, the canonical geometric morphism $\mathcal{F} \to \mathbf{Set}$ is not pre-cohesive. This can be seen even in the simple case of $k = \mathbb{R}$ as we show in the next section.

4.1 The case of the real field

Of course, Galois groups need not be finite. Moreover, if Galois theory is to be done in an arbitrary ambient topos, then Galois groups are not internal groups of automorphisms in the naive sense [18]. Having said this, I believe that it is useful to illustrate the results in the previous sections in the simplest possible non trivial (although finite) case over sets.

Indeed, let us consider the case of $k = \mathbb{R}$ in $\mathbf{Set}$, so that $\mathbf{Con}$ is the category of finitely presented connected $\mathbb{R}$-algebras and $\ell : \mathbf{Ext} \to \mathbf{Con}$ is the (finite) full subcategory determined by finite extensions of k. Of course, this full subcategory is equivalent to that determined by the (the initial object) $\mathbb{R}$ and $\mathbb{C}$. The right adjoint $\rho : \mathbf{Con} \to \mathbf{Ext}$ may be described as follows. For A in $\mathbf{Con}$, ρA is the $\mathbb{R}$-subalgebra generated the square roots of -1. Notice that $\rho A \cong \mathbb{R}$ if A does not have square roots of -1 and $\rho A \cong \mathbb{C}$ otherwise. To check that this is well-defined observe that if $i^2 = -1 = j^2$ then $j = i$ or $j = -i$. (This follows from connectedness and the fact that $\frac{ij+1}{2}$ is idempotent in A.)

The atomic topology J_{at} on $\mathcal{D} = \mathbf{Ext}^{\mathrm{op}}$ has essentially one non-trivial sieve: that generated by the unique map $\mathbb{R} \to \mathbb{C}$ in $\mathbf{Ext}$. Also, since $\mathcal{D}$ is essentially finite and all its idempotents are identities, J_{at} is rigid in the sense C2.2.8 in [7] and $\mathbf{Sh}(\mathcal{D}, J_{at})$ is equivalent to the topos of presheaves on the full subcategory of $\mathcal{D}$ determined by those objects which only have trivial covers. That is, $\mathbf{Sh}(\mathcal{D}, J_{at}) \cong [\mathbf{C}_2, \mathbf{Set}]$ where $\mathbf{C}_2 \to \mathbf{Ext}$ is the full subcategory determined by those objects iso to $\mathbb{C}$. Of course, $\mathbf{C}_2$ is equivalent to the cyclic group C_2 of order two.

Let $\mathbf{Con}' \to \mathbf{Con}$ be the full subcategory determined by those connected $\mathbb{R}$-algebras A such that $\rho A \cong \mathbb{C}$ or, equivalently, there is an $\mathbb{R}$-algebra map $\mathbb{C} \to A$. The following diagram

$$\begin{tikzcd} \mathbf{Con}' \arrow[r] \arrow[d] & \mathbf{Con} \arrow[d] \\ \mathbf{C}_2 \arrow[r] & \mathbf{Ext} \end{tikzcd}$$

is a pullback of categories an the next result shows that it is preserved when passing to toposes of $\mathbf{Set}$-valued functors.

Lemma 4.4. *If we let* $[\mathbf{Con}', \mathbf{Set}] \to [\mathbf{C}_2, \mathbf{Set}]$ *be the geometric morphism induced by the full inclusion* $\mathbf{C}_2 \to \mathbf{Con}'$ *then the following diagram*

$$\begin{tikzcd} {[\mathbf{Con}', \mathbf{Set}]} \arrow[r, "i"] \arrow[d] & {[\mathbf{Con}, \mathbf{Set}]} \arrow[d, "p"] \\ {[\mathbf{C}_2, \mathbf{Set}]} \arrow[r] & {[\mathbf{Ext}, \mathbf{Set}]} \end{tikzcd}$$

is a pullback of toposes. (So $[\mathbf{Con}', \mathbf{Set}] \to [\mathbf{C}_2, \mathbf{Set}]$ *is pre-cohesive and sufficiently cohesive.)*

Proof. The subtopos $[\mathbf{C}_2, \mathbf{Set}] \to [\mathbf{Ext}, \mathbf{Set}]$ is open. Indeed, the sieve in $\mathcal{D}$ generated by the unique morphism $\mathbb{R} \to \mathbb{C}$ in $\mathbf{Ext}$ determines a subobject $U \to 1$ in the topos $[\mathbf{Ext}, \mathbf{Set}]$. More explicitly, $U\mathbb{R} = \emptyset$ and $U\mathbb{C} = 1$; and $[\mathbf{C}_2, \mathbf{Set}] \cong [\mathbf{Ext}, \mathbf{Set}]/U \to [\mathbf{Ext}, \mathbf{Set}]$. Since open subtoposes are closed under pullback it follows that the subtopos $\mathcal{F} \to [\mathbf{Con}, \mathbf{Set}]$ in Proposition 4.3 is equivalent to $[\mathbf{Con}, \mathbf{Set}]/p^*U \to [\mathbf{Con}, \mathbf{Set}]$ and hence $\mathcal{F}$ must be a presheaf topos, say, of the form $[\mathbf{Con}', \mathbf{Set}]$ for some essentially small $\mathbf{Con}'$ determined by $V = p^*U$ in $[\mathbf{Con}, \mathbf{Set}]$. In order to describe $\mathbf{Con}'$ explicitly we first apply the general construction (see e.g. Proposition A1.1.7.). The objects of $\mathbf{Con}'$ are pairs (x, C) with $x \in VC$ and $C \in \mathbf{Con}$. A map $f : (x, C) \to (x', C')$ in $\mathbf{Con}'$ is a morphism $f : C \to C'$ in $\mathbf{Con}$ such that $(Vf)x = x'$. But $VC = (p^*U)C = U(\rho C)$ for each connected $\mathbb{R}$-algebra C. In other words, $VC = (p^*U)C$ is terminal or initial depending on whether there is an $\mathbb{R}$-algebra map $\mathbb{C} \to C$ or not. $\square$

In order to give an explicit description of the Grothendieck topology on $\mathbf{Con}^{\mathrm{op}}$ inducing $\mathcal{F} = [\mathbf{Con}', \mathbf{Set}]$ we first isolate the following basic fact (clearly related to the far more general Proposition 4.1).

Lemma 4.5. *If the* $\mathbb{R}$*-algebra* A *is connected and without square roots of* -1 *then* $A[i] = A \otimes_{\mathbb{R}} \mathbb{C}$ *is connected.*

Proof. Let $a + bi$ in $A[i]$ be idempotent. Then $a^2 - b^2 = a$ and $2ab = b$ in A. Now calculate

$$b^2 = 4a^2b^2 = 4(a + b^2)b^2 = 4ab^2 + 4b^4 = 2b^2 + 4b^4$$

an record that $b^2 + 4b^4 = 0$. So $u = b^2$ satisfies $4u^2 = -u$ in A. Then $(4u)^2 = 16u^2 = -4u$ and so $c = 4u$ satisfies the equality $c^2 = -c$. But then $(c+1)^2 = c^2 + 2c + 1 = -c + 2c + 1 = c + 1$. That is, $c + 1$ is idempotent in A which means, under our hypotheses, that either $c + 1 = 0$ or $c + 1 = 1$; so $c = -1$ or $c = 0$. If $-1 = c = 4u = 4b^2 = (2b)^2$ then we reach a contradiction (since we are assuming that A does not have a square root of -1). If $0 = c = 4b^2$ then $b^2 = 0$ so $a^2 = a$. Since A is connected $a = 0$ or $a = 1$. If $a = 0$ then $b = 2ab = 0$. If $a = 1$ then $b = 2b$ so $b = 0$. Altogether, $a + bi$ is either 0 or 1. $\square$

We can now define a basis K for a Grothendieck topology on $\mathbf{Con}^{\mathrm{op}}$ (in the sense of Exercise III.3 in [16]). We do this in terms of cocovers in $\mathbf{Con}$. First we state that the cocovering families consist of exactly one map, so it is enough to say what maps cocover. First all isos cocover. Also, if $\rho A \cong \mathbb{R}$ then a map $A \to A'$ also cocovers if it is iso over A to the canonical $A \to A[i]$. (This makes sense by Lemma 4.5.)

Lemma 4.6. *The function K that sends A in $\mathbf{Con}^{\mathrm{op}}$ to the collection of covering maps with codomain A is a basis and $\mathbf{Sh}(\mathbf{Con}^{\mathrm{op}}, K) \cong [\mathbf{Con}', \mathbf{Set}]$ as subtoposes of $[\mathbf{Con}, \mathbf{Set}]$.*

Proof. It is easy to check that K is indeed a basis. The main ingredient is that if $A \in \mathbf{Con}$ is such that $\rho A \cong \mathbb{R}$ and $A \to A'$ is in $\mathbf{Con}$ then there exists a cocovering map $A' \to B$ and a commutative square as below

$$\begin{tikzcd} A[i] \arrow[r] & B \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

in $\mathbf{Con}$. Indeed, if $\rho A' \cong \mathbb{C}$ then we can take $B = A'$ and $A' \to B$ to be the identity. On the other hand, if $\rho A' \cong \mathbb{R}$ then we can take $B = A'[i]$ and the canonical $A' \to A'[i] = B$.

To prove that $\mathbf{Sh}(\mathbf{Con}^{\mathrm{op}}, K) = [\mathbf{Con}', \mathbf{Set}]$ we use the notation in the proof of Lemma 4.4. So the subobject $U \to 1$ is the image of the map $\mathbf{Ext}(\mathbb{C}, \_) \to \mathbf{Ext}(\mathbb{R}, \_) = 1$ in $[\mathbf{Ext}, \mathbf{Set}]$ and we denote the map $p^*U \to 1$ by $V \to 1$ in $[\mathbf{Con}, \mathbf{Set}]$. Recall that VC is terminal or initial depending on whether there is an $\mathbb{R}$-algebra map $\mathbb{C} \to C$ or not. For general reasons, the dense subobjects for the associated open topology in $[\mathbf{Con}, \mathbf{Set}]$ are those monos $X' \to X$ such that the projection $\pi_0 : X \times V \to X$ factors through $X' \to X$. In particular, for any $\mathbb{R}$-algebra A in $\mathbf{Con}$ and cosieve $S \to \mathbf{Con}(A, \_)$, S is dense if and only if for every A' in $\mathbf{Con}$ such that $VA' = 1$ (that is, $\rho A' \cong \mathbb{C}$), every $A \to A'$ is in the cosieve S. Notice that if $VA = 1$ then the identity on A must be in S. In other words, if $VA = 1$ then the maximal cosieve is the only (co)covering one. On the other hand, if $VA = 0$ (i.e. $\rho A = \mathbb{R}$) then, S is cocovering if and only if the map $A \to A[i]$ is S. Altogether, a sieve on A is dense with respect to the open topology determined by $V \to 1$ if and only if it contains a cocovering map. $\square$

Now let $\mathbf{Alg}$ be the category of finitely presented $\mathbb{R}$-algebras. The extensive $\mathbf{Alg}^{\mathrm{op}}$ may be equipped with the Gaeta topology and it is well-known (see [14]) that the resulting topos of sheaves is equivalent to $[\mathbf{Con}, \mathbf{Set}]$. It is also well-known that the Gaeta topology is subcanonical and that the restricted Yoneda embedding $\mathbf{Alg}^{\mathrm{op}} \to [\mathbf{Con}, \mathbf{Set}]$ into the Gaeta topos preserves finite coproducts.

Lemma 4.7. *The restricted Yoneda embedding* $\mathbf{Alg}^{\mathrm{op}} \to [\mathbf{Con}, \mathbf{Set}]$ *factors through the subtopos inclusion* $\mathcal{F} \to [\mathbf{Con}, \mathbf{Set}]$ *and the factorization* $\mathbf{Alg}^{\mathrm{op}} \to \mathcal{F}$ *preserves finite coproducts.*

Proof. Let A in $\mathbf{Alg}$. It is fair to write $\mathbf{Con}(A, \_)$ for the non-representable associated object in the Gaeta topos $[\mathbf{Con}, \mathbf{Set}]$. It is enough to prove that every such $\mathbf{Con}(A, \_)$ is a K-sheaf for the basis discussed in Lemma 4.6. We need only worry about objects that have non-trivial covers so let C in $\mathbf{Con}$ be such that $\rho C = \mathbb{R}$ and consider the cocovering $C \to C[i]$. A compatible family consists of a map $f : A \to C[i]$ satisfying that for any pair of maps $g, h : C[i] \to D$ in $\mathbf{Con}$ such that the diagram on the left below commutes

$$C \longrightarrow C[i] \underset{h}{\overset{g}{\rightrightarrows}} D \qquad\qquad A \xrightarrow{f} C[i] \underset{h}{\overset{g}{\rightrightarrows}} D$$

the diagram on the right above commutes too. But $C \to C[i]$ is the equalizer (in $\mathbf{Alg}$) of the identity on $C[i]$ and conjugation. Hence there exists a unique map $f' : A \to C$ factoring f through $C \to C[i]$. This implies that $\mathbf{Con}(A, \_)$ is a sheaf. To confirm that the factorization $\mathbf{Alg}^{\mathrm{op}} \to \mathcal{F}$ preserves finite coproducts just observe that since $1 + 1$ in the Gaeta topos $[\mathbf{Con}, \mathbf{Set}]$ is actually in the image of $\mathbf{Alg}^{\mathrm{op}} \to [\mathbf{Con}, \mathbf{Set}]$ then it is also in the subtopos $\mathcal{F} \to [\mathbf{Con}, \mathbf{Set}]$. □

In short, the geometric morphism $\mathcal{F} = [\mathbf{Con}', \mathbf{Set}] \to [\mathbf{C}_2, \mathbf{Set}]$ makes $\mathcal{F}$ into a sufficiently cohesive pre-cohesive topos embedding the category of 'affine $\mathbb{R}$-schemes' $\mathbf{Alg}^{\mathrm{op}}$ in such a way that finite coproducts are preserved. In contrast, the canonical geometric morphism $f : \mathcal{F} \to \mathbf{Set}$ is not pre-cohesive. It is certainly locally connected because $\mathcal{F}$ is a pre-sheaf topos but the leftmost adjoint $f_! : \mathcal{F} \to \mathbf{Set}$ does not preserve finite products (and hence the Nullstellensatz must fail). The simplest way to see this may be the following.

Example 4.8. The object $X = \mathbf{Con}'(\mathbb{C}, \_)$ in $\mathcal{F}$ is connected in the sense that $f_! X = 1$ because it is representable but $f_!(X \times X) = 2$ as the next calculation shows. Since there are enough maps to $\mathbb{C}$, $f_!(X \times X)$ is a quotient of $(X \times X)\mathbb{C} = \mathbf{Con}'(\mathbb{C}, \mathbb{C}) \times \mathbf{Con}'(\mathbb{C}, \mathbb{C}) \cong C_2 \times C_2$. If $\kappa : \mathbb{C} \to \mathbb{C}$ denotes conjugation then the pairs (id, id) and (κ, κ) induce the same element in $f_!(X \times X)$. Similarly, (id, κ) and (κ, id) induce the same element; but (id, id) and (id, κ) cannot be equivalent.

It seems relevant at this point to compare $\mathcal{F}$ with the Zariski topos. Let Z be the basis on $\mathbf{Alg}^{\mathrm{op}}$ determined by declaring that the cocovering families are (up to iso) those of the form $(A \to A[s^{-1}] \mid s \in S)$ with $S \subseteq A$ a finite subset not contained in any proper ideal of A in $\mathbf{Alg}$. (See III.3 in [16] or A2.1.11(f) in [7].) Denote the Zariski topos $\mathbf{Sh}(\mathbf{Alg}^{\mathrm{op}}, Z)$ by $\mathcal{Z}$. Clearly the basis Z contains the Gaeta one so the inclusion $\mathcal{Z} \to [\mathbf{Alg}, \mathbf{Set}]$ factors through the Gaeta subtopos $[\mathbf{Con}, \mathbf{Set}] \to [\mathbf{Alg}, \mathbf{Set}]$. The basis Z is also subcanonical but we stress that the subtoposes $\mathcal{Z} \to [\mathbf{Con}, \mathbf{Set}]$ and $\mathcal{F} \to [\mathbf{Con}, \mathbf{Set}]$ are incomparable. This is clear if we contrast the basis K of Lemma 4.6 with the Zariski basis defined above. Certainly, the Grothendieck topology generated by K does not contain most of the sieves generated by the 'open' covers of Z. On the other hand, $\mathbb{R}$ in $\mathbf{Con}^{\mathrm{op}}$ does not have a non-trivial Z-cocover. Hence, the composite

$$\mathcal{Z} \to [\mathbf{Con}, \mathbf{Set}] \to [\mathbf{Ext}, \mathbf{Set}]$$

does not factor through the subtopos $[\mathbf{C}_2, \mathbf{Set}] \to [\mathbf{Ext}, \mathbf{Set}]$.

The discussion above suggests considering the intersection of $\mathcal{F}$ and $\mathcal{Z}$ over $[\mathbf{Con}, \mathbf{Set}]$. Hopefully, the resulting topos would combine the benefits of a pre-cohesive topos with the colimit preservation properties of the embedding $\mathbf{Alg}^{\mathrm{op}} \to \mathcal{Z}$. Alternatively, one can consider in $\mathcal{F}$ the algebra object $R = \mathbf{Con}(\mathbb{R}[x], \_)$ and the least Lawvere-Tierney topology that makes the subobject

$$\{a \in R \mid (\exists b \in R)(ab = 1) \vee (\exists b \in R)((1-a)b = 1)\} \longrightarrow R$$

dense. The two subtoposes of $\mathcal{F}$ suggested above may turn out to be the same but, in any case, this will have to be treated elsewhere.

Still in the case that $k = \mathbb{R}$; what does $\mathcal{F} = [\mathbf{Con}', \mathbf{Set}]$ classify? Assume a standard presentation of the theory of $\mathbb{R}$-algebras extending the usual presentation of the theory of rings. The theory of connected $\mathbb{R}$-algebras may be presented by adding the axioms

$$0 = 1 \vdash \bot \qquad \text{and} \qquad x^2 = x \vdash_x (x = 0) \vee (x = 1)$$

and it is well-known (see [14]) that this induces the Gaeta topology on $\mathbf{Alg}^{\mathrm{op}}$ so the resulting topos of sheaves is equivalent to $[\mathbf{Con}, \mathbf{Set}]$.

Lemma 4.9. *The theory classified by $\mathcal{F}$ can be presented by adding the axiom*

$$\vdash (\exists x)(x^2 = -1)$$

to the presentation of the theory of connected $\mathbb{R}$-algebras described above.

Proof. To prove this is convenient to use the presentation of $\mathcal{F}$ given in Lemma 4.6 because the basis K on $\mathbf{Con}^{\mathrm{op}}$ is clearly generated by the map $\mathbb{R} \to \mathbb{C}$ in $\mathbf{Con}$; and this sieve covers if and only if the theory classified by $\mathcal{F}$ satisfies the evident axiom.

Alternatively, one can start with the theory presented as in the statement and regard it as a 'quotient' of the presentation of the theory of $\mathbb{R}$-algebras. It is well-known (see e.g. D3.1.10 in [7]) that one can construct the classifying topos as the topos of sheaves on a site whose underlying category is the opposite of the category of finitely presented algebras. Following this path (and factoring through the Gaeta site) one arrives at the site $(\mathbf{Con}^{\mathrm{op}}, K)$. $\square$

For an arbitrary field the subtopos $\mathbf{Sh}(\mathcal{D}, J_{at}) \to [\mathbf{Ext}, \mathbf{Set}]$ will not be open but the description of the theory classified by $\mathcal{F}$ can probably be modified by adding an appropriate sequent for each map in $\mathbf{Ext}$.

Lawvere suggested to discuss the classifying role of $\mathcal{F}$ over its natural base. To do this recall (Theorem VIII.2.7 in [16]) that the base $[\mathbf{C}_2, \mathbf{Set}]$ classifies C_2-torsors, where C_2 is cyclic group of order 2. For brevity let us define a *(C_2-)torsored topos* as a pair $(\mathcal{T}, T)$ given by a topos $\mathcal{T}$ an a C_2-torsor T in it. A morphism $g : (\mathcal{T}, T) \to (\mathcal{T}', T')$ of torsored toposes is a geometric morphism $g : \mathcal{T} \to \mathcal{T}'$ such that $g^* T' \cong T$.

Definition 4.10. A *torsored algebra* in a torsored topos $(\mathcal{T}, T)$ is an internal $\mathbb{R}$-algebra A in $\mathcal{T}$ together with a map $T \to A$ such that the following diagram

$$\begin{tikzcd}
T \arrow[rr] \arrow[d] & & 1 \arrow[d, "-1"] \\
A \arrow[r, "\Delta"'] & A \times A \arrow[r, "\cdot"'] & A
\end{tikzcd}$$

is a pullback; where Δ is the diagonal and $\cdot$ is the multiplication of the algebra A.

The pre-cohesive $\mathcal{F} \to [\mathbf{C}_2, \mathbf{Set}]$ makes $\mathcal{F}$ into a torsored topos $(\mathcal{F}, F)$ and the object $R = \mathbf{Con}(\mathbb{R}[x], \_)$ is a $\mathbb{R}$-algebra in $\mathcal{F} = \mathbf{Sh}(\mathbf{Con}^{\mathrm{op}}, K)$.

Proposition 4.11. *The $\mathbb{R}$-algebra R in $\mathcal{F}$ may be equipped with a torsored algebra structure in $(\mathcal{F}, F)$ and it is the generic one. That is, $(\mathcal{F}, F, R)$ classifies torsored algebras among torsored toposes.*

Proof. The underlying object of the generic C_2-torsor is the representable $\mathbf{C}_2(\mathbb{C}, \_)$ in $[\mathbf{C}_2, \mathbf{Set}]$. The inverse image of the pre-cohesive $\mathcal{F} \to [\mathbf{C}_2, \mathbf{Set}]$ sends $\mathbf{C}_2(\mathbb{C}, \_)$ to $\mathbf{C}_2(\mathbb{C}, \rho(\_)) \cong \mathbf{Con}(\mathbb{C}, \_) = F$. The unique $\mathbb{R}$-algebra map $\mathbb{R}[x] \to \mathbb{C}$ sending x to i determines a morphism $F \to R$ and since the diagram below

$$\begin{tikzcd}
x \arrow[rr, mapsto] \arrow[dd, mapsto] & & x^2 & \\
& \mathbb{R}[x] \arrow[r] \arrow[d] & \mathbb{R}[x] \arrow[d] & x \arrow[d, mapsto] \\
-1 & \mathbb{R} \arrow[r] & \mathbb{C} & i
\end{tikzcd}$$

is a pushout in $\mathbf{Con}$, the map $F \to R$ in $\mathcal{F}$ makes R into a torsored algebra. To prove that it is the generic one let $(\mathcal{T}, T)$ be a torsored topos and let A be a torsored algebra in $\mathcal{T}$. The unique map $T \to 1$ is epi because T is a torsor and so, the condition defining torsor algebras implies that $\vdash (\exists x)(x^2 = -1)$ holds in $\mathcal{T}$. By Lemma 4.9 there exists an essentially unique geometric morphism $g : \mathcal{T} \to \mathcal{F}$ such that $g^*R = A$. Since g^* preserves finite limits it must be the case that $g^*F \cong T$ so g is a morphism of torsored toposes. $\square$

Acknowledgments

I would like to thank F. W. Lawvere for patiently explaining many of his ideas about Axiomatic Cohesion, R. Pettigrew and J. Mayberry for hosting the Bristol 2009 seminar, W. Tholen, J. Funk and P. T. Johnstone for responding to my technical inquiries, and also M. Morgan and W. Phoa for their generous support which allowed to present some of the results in this paper at the Coimbra 2012 Conference in honour of G. Janelidze, on the occasion of his 60th birthday.

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