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Parametrizing projections with selfadjoint operators

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ABSTRACT

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be an orthogonal decomposition of a Hilbert space, with E_+, E_- the corresponding projections. Let A be a selfadjoint operator in $\mathcal{H}$ which is codiagonal with respect to this decomposition (i.e. $A(\mathcal{H}_+) \subset \mathcal{H}_-$ and $A(\mathcal{H}_-) \subset \mathcal{H}_+$). We consider three maps which assign a selfadjoint projection to A:

1. The graph map Γ: $\Gamma(A) =$ projection onto the graph of $A|_{\mathcal{H}_+}$.
2. The exponential map of the Grassmann manifold $\mathcal{P}$ of $\mathcal{H}$ (the space of selfadjoint projections in $\mathcal{H}$) at E_+: $\exp(A) = e^{i\frac{\pi}{2}A}E_+e^{-i\frac{\pi}{2}A}$.
3. The map p, called here the Davis' map, based on a result by Chandler Davis, characterizing the selfadjoint contractions which are the difference of two projections.

The ranges of these maps are studied and compared.
Using Davis' map, one can solve the following operator matrix completion problem: given a contraction $a : \mathcal{H}_- \to \mathcal{H}_+$, complete the matrix

$$\begin{pmatrix} * & a/2 \\ a^*/2 & * \end{pmatrix}$$

to a projection P, in order that $\|P - E_+\|$ is minimal.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed orthogonal decomposition of the Hilbert space $\mathcal{H}$, and denote by E_+ and E_- the projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. If X is a selfadjoint operator in $\mathcal{H}$ which is codiagonal with respect to the decomposition (i.e. $X(\mathcal{H}_+) \subset \mathcal{H}_-$ and $X(\mathcal{H}_-) \subset \mathcal{H}_+$) and $x = X|_{\mathcal{H}_+}$, then $G_x = \{\xi \oplus x\xi : \xi \in \mathcal{H}_+\}$ is a closed subspace of $\mathcal{H}$. This is the usual way to obtain local charts for $\mathcal{P}$. Note that the trivial operator corresponds to the subspace $\mathcal{H}_+$. By means of the one-to-one map $\mathcal{S} \mapsto P_{\mathcal{S}}$ (= orthogonal projection onto $\mathcal{S}$), we identify $\mathcal{P}$ with the set of orthogonal projections in $\mathcal{H}$.

The graph map is the mapping which sends X to the orthogonal projection onto G_x.

The tangent space of the manifold $\mathcal{P}$ at E_+ (or $\mathcal{H}_+$) identifies with the space of selfadjoint codiagonal operators [7]. Thus the exponential map of $\mathcal{P}$ (from $T\mathcal{P}$ to $\mathcal{P}$) can also be regarded as a map from codiagonal selfadjoint operators to orthogonal projections.

The third map is obtained from a result by Chandler Davis [8], and will be defined below. Let us fix some notation.

Let J be the symmetry in $\mathcal{H}$ given by the decomposition of $\mathcal{H}$. J is a selfadjoint unitary operator ($J^* = J = J^{-1}$) whose spectral spaces are $\mathcal{H}_+$ and $\mathcal{H}_-$:

$$\mathcal{H}_+ = \{\xi \in \mathcal{H} : J\xi = \xi\} \quad \text{and} \quad \mathcal{H}_- = \{\xi \in \mathcal{H} : J\xi = -\xi\}.$$

The corresponding projections are

$$E_+ = \frac{1}{2}(J+1) \quad \text{and} \quad E_- = \frac{1}{2}(1-J).$$

Let $\mathcal{B}_J$ be the space of selfadjoint operators which are codiagonal. A simple calculation shows that X is codiagonal if and only if it anti-commutes with J: $XJ = -JX$.

Let $\mathcal{D}_J$ be the unit ball of $\mathcal{B}_J$.

In Theorem 6.1 in [8] Chandler Davis proved that a contraction A which anti-commutes with the symmetry J (i.e. $A \in \mathcal{D}_J$) is a difference of projections: $A = P_J - Q_J$ (with explicit formulas for P_J and Q_J). We define Davis' map as $p_J(A) = P_J$.

We shall consider the following subsets of $\mathcal{D}_J$. Let

$$\mathcal{D}_J^p = \mathcal{D}_J \cap \mathcal{B}_p(\mathcal{H}),$$

where $\mathcal{B}_p(\mathcal{H})$ is the ideal of p-Schatten operators ($1 \le p \le \infty$), with $\mathcal{B}_\infty(\mathcal{H})$ the ideal of compact operators. In [4] and [1] the notion of Fredholm pairs of projections was defined. Namely, a pair (P, Q) of projections is a Fredholm pair if

$$QP|_{R(P)} : R(P) \to R(Q)$$

is a Fredholm operator. The index of the pair is the index of this operator. In [4] it was proved that the pair (P, Q) is a Fredholm pair if and only if 1 and -1 are isolated points in the spectrum of $A = P - Q$, with finite multiplicity. The index of the pair equals

$$\dim N(A-1) - \dim N(A+1).$$

Let

$$\mathcal{D}_J^F = \{A = P - Q \in \mathcal{D}_J : (P, Q) \text{ is a Fredholm pair}\}.$$

Note the fact that $JAJ = -A$ implies that $\dim N(A-1) = \dim N(A+1)$. Thus the differences A in $\mathcal{D}_J^F$ correspond to zero index pairs.

Note that $\mathcal{D}_J^p \subset \mathcal{D}_J^F \subset \mathcal{D}_J$, and the inclusions are proper if $\mathcal{H}$ is infinite dimensional.

Let us define the maps studied in this paper:

- The exponential map:

$$\exp_J : \mathcal{D}_J \to \mathcal{P}, \qquad \exp_J(X) = e^{i\frac{\pi}{2}X} E_+ e^{-i\frac{\pi}{2}X}.$$

- Davis' map:

$$p_J : \mathcal{D}_J \to \mathcal{P}, \qquad p_J(X) = \frac{1}{2}\{1 + X + J(1 - X^2)^{1/2}\}.$$

- The graph map:

$$\Gamma_J : \mathcal{B}_J \to \mathcal{P}, \qquad \Gamma_J(X) = (J + X)(1 + X^2)^{-1} + E_-.$$

That this is indeed the map described above will be made clear in Section 4.

The map p_J has a left inverse: $a_J : \mathcal{P} \to \mathcal{D}_J$, $a_J(P) = P - JPJ$. The fact that $a_J(p_J(X)) = X$, implies that the composition $\Delta_J = p_J \circ a_J : \mathcal{P} \to \mathcal{P}$ is a retraction. The characterization of the range of p_J in Section 3, implies that Δ_J retracts $\mathcal{P}$ onto the ball $\{P \in \mathcal{P} : \|P - E_+\| \le \frac{1}{\sqrt{2}}\}$. This property means that $\Delta_J(P)$ is the answer to the matrix completion problem stated in the abstract.

The contents of the paper are the following. In Section 2 we introduce the facts concerning the exponential map. In Section 3 we study Davis' map. In Section 4 we study the graph map. The ranges of these maps are characterized and compared. Also when the maps are restricted to the subsets of $\mathcal{D}_J$ described above (compact, Fredholm). In Section 5 we study the retraction Δ_J.

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2. The exponential map of the Grassmann manifold

Corach, Porta and Recht [7] studied the geometry of

$$\mathcal{P} = \{P \in \mathcal{B}(\mathcal{H}) : P^2 = P^* = P\}.$$

The tangent space $(T\mathcal{P})_P$ at a point P consists of all selfadjoint operators which are P-codiagonal, i.e. are codiagonal with respect to the decomposition $\mathcal{H} = R(P) \oplus N(P)$ ($R(P)$ is the range of P and $N(P)$ its nullspace). The exponential map at P is given by

$$X \mapsto e^{iX} P e^{-iX}.$$

Geodesics starting at P are curves of the form $\delta(t) = e^{itX} P e^{-itX}$. If one measures the length of a differentiable curve $\gamma(t)$ $(t \in I)$ in $\mathcal{P}$ by means of

$$\int_I \|\dot{\gamma}(t)\| dt,$$

the geodesics have minimal length along their paths up to $|t| \leq \frac{\pi}{2\|X\|}$ [10,7].

Thus the set $\mathcal{D}_J$ has a geometric interpretation in terms of $\mathcal{P}$. Namely, the closed unit ball of the tangent space of $\mathcal{P}$ at E_+:

$$\{X \in T\mathcal{P}_{E_+} : \|X\| \leq 1\} = \mathcal{D}_J.$$

Thus

$$\exp_J : \mathcal{D}_J \to \mathcal{P}, \qquad \exp_J(X) = e^{i\frac{\pi}{2}X} E_+ e^{-i\frac{\pi}{2}X}$$

is (a rescaling of) the exponential map of $\mathcal{P}$ at E_+. Note that $\exp_J$ takes values in the unitary orbit of E_+, which coincides with the connected component of E_+ in $\mathcal{P}$.

Two projections P and Q are in *generic* position [9], if

$$R(P) \cap R(Q) = R(P) \cap N(Q) = N(P) \cap R(Q) = N(P) \cap N(Q) = \{0\}.$$

In general, let us denote by $\mathcal{H}_{P,Q}$ (the generic part of P and Q) the orthogonal complement of the sum of the above four subspaces,

$$\mathcal{H}_{P,Q} = \{(R(P) \cap R(Q)) \oplus (R(P) \cap N(Q)) \oplus (N(P) \cap R(Q)) \oplus (N(P) \cap N(Q))\}^{\perp}.$$

Proposition 2.1. *The range* $\exp_J(\mathcal{D}_J)$ *of* $\exp_J$ *consists of all projections* P *which satisfy*

$$\dim(R(P) \cap \mathcal{H}_-) = \dim(N(P) \cap \mathcal{H}_+).$$

Moreover, there exists a unique $X \in \mathcal{D}_J$ *such that* $\exp_J(X) = P$ *if and only if* $R(P) \cap \mathcal{H}_- = N(P) \cap \mathcal{H}_+ = \{0\}$.

Proof. In [2] it was proven that P and E_+ can be joined by a minimal geodesic if and only if

$$\dim\big(R(P)\cap\mathcal{H}_-\big)=\dim\big(N(P)\cap\mathcal{H}_+\big).$$

In this case there exists $Z^*=Z$ which is E_+-codiagonal, with $\|Z\|\le\pi/2$, such that $P=e^{iZ}E_+e^{-iZ}$. This is precisely our first claim, taking $X=\frac{2}{\pi}Z\in\mathcal{D}_J$.

In [3] it was shown that two projections in generic position are joined by a unique minimal geodesic. We claim here that this uniqueness result holds under the weaker condition

$$R(P)\cap\mathcal{H}_-=N(P)\cap\mathcal{H}_+=\{0\},$$

i.e. $R(P)\cap\mathcal{H}_+$ or $N(P)\cap\mathcal{H}_-$ need not be trivial.

Suppose that $R(P)\cap\mathcal{H}_-=N(P)\cap\mathcal{H}_+=\{0\}$ and let $X\in\mathcal{D}_J$ such that $\exp_J(X)=P$. In [2] it was noted that this implies that (besides being E_+-codiagonal) X is also P-codiagonal. Then

$$X\big(R(P)\cap\mathcal{H}_+\big)\subset N(P)\cap\mathcal{H}_-\quad\text{and}\quad X\big(N(P)\cap\mathcal{H}_-\big)\subset R(P)\cap\mathcal{H}_+.$$

Since $R(P)\cap\mathcal{H}_-=N(P)\cap\mathcal{H}_+=\{0\}$, we have

$$\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_{P,E_+},$$

where $\mathcal{H}_1=(R(P)\cap\mathcal{H}_+)\oplus(N(P)\cap\mathcal{H}_-)$. Clearly P,E_+ and X are simultaneously reduced by this decomposition. On the generic part $\mathcal{H}_{P,E_+}$, X is uniquely determined by P and E_+. Denote by $X_1=X|_{\mathcal{H}_1}$ (as an operator in $\mathcal{H}_1$). It is codiagonal with respect to the decomposition $\mathcal{H}_1=(R(P)\cap\mathcal{H}_+)\oplus(N(P)\cap\mathcal{H}_-)$,

$$i\frac{\pi}{2}X_1=\begin{pmatrix}0 & B\\ -B^* & 0\end{pmatrix}\begin{matrix}R(P)\cap\mathcal{H}_+\\ N(P)\cap\mathcal{H}_-\end{matrix}.$$

Note that in $\mathcal{H}_1$, both projections P and E_+ coincide

$$P|_{\mathcal{H}_1}=E_+|_{\mathcal{H}_1}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{matrix}R(P)\cap\mathcal{H}_+\\ N(P)\cap\mathcal{H}_-\end{matrix}.$$

Clearly $e^{i\frac{\pi}{2}X_1}E_+|_{\mathcal{H}_1}e^{-i\frac{\pi}{2}X_1}=E_+|_{\mathcal{H}_1}$ (i.e. $e^{i\frac{\pi}{2}X_1}$ and $E_+|_{\mathcal{H}_1}$ commute). This exponential can be computed, in matrix form

$$e^{i\frac{\pi}{2}X_1}=\begin{pmatrix}\cos(|B^*|) & -B\,\mathrm{sinc}(|B|)\\ -B^*\,\mathrm{sinc}(|B^*|) & \cos(|B|)\end{pmatrix},$$

where sinc denotes the cardinal sine. A simple computation shows that the fact that this exponential commutes with $E_+|_{\mathcal{H}_1}$, implies that

$$B\,\mathrm{sinc}\big(|B|\big) = 0 \quad \text{in } \mathcal{H}_1.$$

Note that $|||B||| = \frac{\pi}{2}\|X\| \leq \frac{\pi}{2}$, and thus

$$\sigma\big(\mathrm{sinc}\big(|B|\big)\big) \subset \left\{\frac{\sin(t)}{t} : t \in (0, \pi/2]\right\} \cup \{1\},$$

a set which does not contain 0. Thus $\mathrm{sinc}(|B|)$ is invertible in $\mathcal{H}_1$, and therefore $B = 0$, i.e. $X_1 = 0$. Thus there is a unique minimal geodesic joining P and E_+.

Conversely, it was shown in [3] that if $\dim(R(P) \cap \mathcal{H}_-) = \dim(N(P) \cap \mathcal{H}_+) \neq 0$, there are infinitely many geodesics joining P and E_+ with length $\pi/2$. □

For $1 \leq p \leq \infty$, let $\mathcal{B}_p(\mathcal{H})$ be the p-Schatten class of $\mathcal{H}$. Denote by $\mathcal{D}_J^p = \mathcal{D}_J \cap \mathcal{B}_p(\mathcal{H})$. Let

$$\mathcal{P}_J^p = \big\{P \in \mathcal{P} : P - E_+ \in \mathcal{B}_p(\mathcal{H})\big\}$$

This set is the connected component of E_+ in the so-called *p-reduced* Grassmannian relative to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ [11,5]. It is known [11,5] that

$$\mathcal{P}_J^p = \big\{UE_+U^* : U \in \mathcal{U}(\mathcal{H}),\ U - 1 \in \mathcal{B}_p(\mathcal{H})\big\}.$$

The connected component $\mathcal{P}_J^p$ of E_+ in the restricted Grassmannian is also characterized by an index condition [11]: it consists of projections P such that

$$E_-|_{R(P)} \in \mathcal{B}_p\big(R(P), \mathcal{H}_-\big)$$

and

$$E_+P|_{\mathcal{H}_+} : \mathcal{H}_+ \to \mathcal{H}_+$$

is invertible modulo $\mathcal{B}_p(\mathcal{H}_+)$, and has index zero [11].

This index coincides with the index of the pair of projections (P, E_+) [4,1], and thus

$$\dim\big(R(P) \cap \mathcal{H}_-\big) - \dim\big(N(P) \cap \mathcal{H}_+\big) = 0.$$

One has the following:

Corollary 2.2. *The restriction*

$$\exp_J|_{\mathcal{D}_J^p} : \mathcal{D}_J^p \to \mathcal{P}_J^p$$

is onto.

Let F_+ be the conditional expectation

$$F_+ : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \qquad F_+(B) = E_+BE_+ + E_-BE_-.$$

Let us denote by $\mathcal{F}(\mathcal{H})$ the set of Fredholm operators in $\mathcal{H}$.

Proposition 2.3.

$$\exp_J\big(\mathcal{D}_J^F\big) = \big\{P \in \mathcal{P} : \dim R(P) \cap \mathcal{H}_- = \dim N(P) \cap \mathcal{H}_+ \ \textit{and}\ F_+(P - E_-) \in \mathcal{F}(\mathcal{H})\big\}.$$

Proof. Let $X \in \mathcal{D}_J^F$. This means that $1 - X$ and $1 + X$ are Fredholm operators. Then $1 - X^2$ is a Fredholm operator. In matrix form,

$$X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}, \qquad 1 - X^2 = \begin{pmatrix} 1 - |x^*|^2 & 0 \\ 0 & 1 - |x|^2 \end{pmatrix}.$$

Then $1 - |x^*|^2$ and $1 - |x|^2$ are Fredholm operators in $\mathcal{H}_+$ and $\mathcal{H}_-$ (respectively). This means that 1 is isolated in the spectra of $|x^*|$ and $|x|$, and that $N(1-|x^*|)$ and $N(1-|x|)$ are finite dimensional. It follows that -1 is isolated in the spectra of $\cos(\pi|x^*|)$ and $\cos(\pi|x|)$, and that -1 has finite multiplicity for both operators, i.e. $1 + \cos(\pi|x^*|)$ and $1 + \cos(\pi|x|)$ are Fredholm operators in $\mathcal{H}_+$ and $\mathcal{H}_-$. By straightforward computations,

$$\begin{aligned} P = \exp_J(X) &= \frac{1}{2}\big\{e^{i\frac{\pi}{2}X} J e^{-i\frac{\pi}{2}X} + 1\big\} = \frac{1}{2}\big\{e^{i\pi X} J + 1\big\} \\ &= \frac{1}{2}\begin{pmatrix} \cos(\pi|x^*|) + 1 & * \\ * & -\cos(\pi|x|) + 1 \end{pmatrix}. \end{aligned}$$

Then

$$F_+(P - E_-) = F_+(P) - E_- = \frac{1}{2}\begin{pmatrix} \cos(\pi|x^*|) + 1 & 0 \\ 0 & -\cos(\pi|x|) - 1 \end{pmatrix}.$$

Since both diagonal entries are Fredholm operators in $\mathcal{H}_+$ and $\mathcal{H}_-$, it follows that $F_+(P - E_-)$ is a Fredholm operator. The equality $\dim(R(P) \cap \mathcal{H}_-) = \dim(N(P) \cap \mathcal{H}_+)$ was shown to characterize projections in the rank of $\exp_J$.

Conversely, if P satisfies $\dim(R(P) \cap \mathcal{H}_-) = \dim(N(P) \cap \mathcal{H}_+)$, then $P = \exp_J(X)$, and reasoning as above,

$$\cos\big(\pi\big|x^*\big|\big) + 1 \quad \text{and} \quad \cos\big(\pi|x|\big) + 1$$

are Fredholm operators in $\mathcal{H}_+$ and $\mathcal{H}_-$. By the same spectral argument as above, this means that $1 - |x^*|^2$ and $1 - |x|^2$ are Fredholm in $\mathcal{H}_+$ and $\mathcal{H}_-$, and thus $X \in \mathcal{D}_J^F$. □

3. Davis' map

In [8] (Theorem 6.1) Chandler Davis characterized operators A which are the difference of two projections: $A = P - Q$. He observed that P and Q are non-unique in general, and are parametrized by symmetries which anti-commute with A.

In our case, the selfadjoint contractions $A \in \mathcal{D}_J$, anti-commute with the fixed symmetry J. Using a formula obtained by Davis, we consider the map p_J, which we call the Davis' map:

$$p_J : \mathcal{D}_J \to \mathcal{P}, \qquad p_J(A) = \frac{1}{2}\left\{1 + A + J\left(1 - A^2\right)^{1/2}\right\}.$$

This map is apparently continuous. Clearly $\mathcal{D}_J$ is arcwise connected, and then the image of p_J is arcwise connected. Note that

$$p_J(0) = \frac{1}{2}\{1 + J\} = E_+.$$

Thus the image of p_J is contained in the unitary orbit of E_+.

Proposition 3.1. *The map p_J is one-to-one. It has a left inverse*

$$a_J : \mathcal{P} \to \mathcal{D}_J, \qquad a_J(P) = P - JPJ,$$

i.e. $a_J(p_J(A)) = A$ for $A \in \mathcal{D}_J$.

Proof. Direct computation: $JAJ = -A$, A^2 commutes with J,

$$Jp_J(A)J = \frac{1}{2}\left\{1 - A + \left(1 - A^2\right)^{1/2} J\right\} = \frac{1}{2}\left\{1 - A + J\left(1 - A^2\right)^{1/2}\right\}$$

and thus

$$a_J\left(p_J(A)\right) = \frac{1}{2}\left\{1 + A + J\left(1 - A^2\right)^{1/2}\right\} - \frac{1}{2}\left\{1 - A + J\left(1 - A^2\right)^{1/2}\right\} = A. \qquad \square$$

Let us characterize the range of p_J. Given $P \in \mathcal{P}$, since p_J is one-to-one, if there exists $A \in \mathcal{D}_J$ such that $p_J(A) = P$, it must be $A = a_J(P) = P - JPJ$.

In matrix form, in terms of the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, any given projection P is of the form

$$P = \begin{pmatrix} b/2 & a/2 \\ a^*/2 & c/2 \end{pmatrix}.$$

The fact that P is a projection implies the following relations between its matrix entries:

$$2b - b^2 = \left|a^*\right|^2, \qquad 2c - c^2 = |a|^2, \quad 0 \le b, c \le 2$$

and consequently $\|a\|^2 = \max\{2t - t^2 : t \in \sigma(c)\} \le 1$.

Note that

$$A = P - JPJ = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

and

$$p_J(A) = \begin{pmatrix} \frac{1}{2}\{1 + (1 - |a^*|^2)^{1/2}\} & \frac{a}{2} \\ \frac{a^*}{2} & \frac{1}{2}\{1 - (1 - |a|^2)^{1/2}\} \end{pmatrix}. \tag{1}$$

Thus, P belongs to the range of p_J if

$$b = 1 + \left(1 - \left|a^*\right|^2\right)^{1/2} \quad \text{and} \quad c = 1 - \left(1 - |a|^2\right)^{1/2}, \tag{2}$$

or equivalently, combining the relations above

$$b = 1 + \left((b-1)^2\right)^{1/2} = 1 + |b-1| \quad \text{and} \quad c = 1 - \left((1-c)^2\right)^{1/2} = 1 - |1-c|. \tag{3}$$

Thus we have:

Proposition 3.2. *Let P be a selfadjoint projection,*

$$P = \begin{pmatrix} b/2 & a/2 \\ a^*/2 & c/2 \end{pmatrix}.$$

Then P belongs to the range of p_J if and only if

$$1_{\mathcal{H}_+} \le b \quad \textit{and} \quad c \le 1_{\mathcal{H}_-}.$$

Proof. By the above computations, P belongs to the range of p_J if and only if $b - 1 = |b-1|$ and $1 - c = |1 - c|$, i.e. $b - 1 \ge 0$ in $\mathcal{H}_+$ and $1 - c \ge 0$ in $\mathcal{H}_-$. □

Theorem 3.3.

$$\operatorname{Im} p_J = \left\{ P \in \mathcal{P} : \|P - E_+\| \le \frac{1}{\sqrt{2}} \right\}. \tag{4}$$

Proof. First note that for any $A \in \mathcal{D}_J$, one has

$$\left\|p_J(A) - E_+\right\| \le \frac{1}{\sqrt{2}}.$$

Indeed, note that

$$p_J(A) - E_+ = \frac{1}{2}\left\{1 - 2E_+ + A + J\left(1 - A^2\right)^{1/2}\right\} = \frac{1}{2}\left\{(A - J) + J\left(1 - A^2\right)^{1/2}\right\}.$$

Using that $JA = -AJ$, the square of this (selfadjoint) operator above is,

$$\frac{1}{2}\{1-(1-A^2)^{1/2}\}.$$

Note that for $|t| \le 1$, $0 \le 1-(1-t^2)^{1/2} \le 1$. Thus our first claim follows.

Conversely, let $P \in \mathcal{P}$, with $\|P - E_+\| \le \frac{1}{\sqrt{2}}$. Then

$$(P-E_+)^2 = P + E_+ - PE_+ - E_+P = \begin{pmatrix} 1-b/2 & 0 \\ 0 & c/2 \end{pmatrix}.$$

Then $\|P - E_+\|^2 < 1/2$ implies that $1 - b/2 \le 1/2$ and $c/2 \le 1/2$, i.e. $1 \le b$ and $c \le 1$. □

Remark 3.4.

1. Projections P in the image of p_J such that $\|P - E_+\| = \frac{1}{\sqrt{2}}$ come from elements $A \in \mathcal{D}_J$ with $\|A\| = 1$. Indeed, in this case 1 (and -1) belong to the spectrum of A, and thus

$$\frac{1}{2} = \sup\left\{\frac{1}{2}\{1-(1-t^2)^{1/2}\} : t \in \sigma(A)\right\} = \frac{1}{2}\|1-(1-A^2)^{1/2}\| = \|p_J(A) - E_+\|^2.$$

2. Apparently, the restriction of p_J to the interior of $\mathcal{D}_J$ is one-to-one onto the open ball of radius $\frac{1}{\sqrt{2}}$ of $\mathcal{P}$:

$$p_J : \{A \in \mathcal{D}_J : \|A\| < 1\} \xrightarrow{\sim} \left\{P \in \mathcal{P} : \|P - E_+\| < \frac{1}{\sqrt{2}}\right\}.$$

Proposition 3.5. *The restriction of p_J to the p-Schatten part $\mathcal{D}^p_J$ of $\mathcal{D}_J$ ($1 \le p \le \infty$), takes values in the restricted Grassmannian $\mathcal{P}^p_J$:*

$$p_J|_{\mathcal{D}^p_J} : \mathcal{D}^p_J \to \mathcal{P}^p_J.$$

Proof. Suppose that $A \in \mathcal{D}^p_J = \mathcal{D}_J \cap \mathcal{B}_p(\mathcal{H})$. Note that

$$2(p_J(A) - E_+) = A + J(1-A^2)^{1/2} - J = A + J(1-(1-A^2)^{1/2}).$$

Thus it suffices to show that $1-(1-A^2)^{1/2} \in \mathcal{B}_p(\mathcal{H})$. Consider

$$f : [0,1] \to \mathbb{R}, \qquad f(t) = 1-(1-t^2)^{1/2}.$$

Then $f(0) = 0$, and if (λ_n) is a sequence of non-negative numbers which is p-summable (resp. tends to 0), then $(f(\lambda_n))$ is $p/2$ summable (resp. tends to 0). It follows that if

$A \in \mathcal{B}_p(\mathcal{H})$, then $f(A) = 1 - (1 - A^2)^{1/2} \in \mathcal{B}_{p/2}(\mathcal{H}) \subset \mathcal{B}_p(\mathcal{H})$. Therefore $p_J(A) - E_+ \in \mathcal{B}_p(\mathcal{H})$. $\square$

Restricted to $\mathcal{D}_J^p$, p_J is one-to-one onto the ball of $\mathcal{P}_J^p$ centered at E_+ with radius $\frac{1}{\sqrt{2}}$.

Proposition 3.6.

$$p_J(\mathcal{D}_J^p) = \left\{ P \in \mathcal{P}_J^p : \|P - E_+\| \leq \frac{1}{\sqrt{2}} \right\}.$$

Proof. Clearly, by the above proposition, $p_J(\mathcal{D}_J^p) \subset \{P \in \mathcal{P}^p : \|P - E_+\| \leq \frac{1}{\sqrt{2}}\}$. Let $P \in \mathcal{P}_J^p$ with $\|P - E_+\| \leq \frac{1}{\sqrt{2}}$. Then, by Theorem 3.3,

$$P = p_J(P - JPJ).$$

Note that if $P \in \mathcal{P}_J^p$ (i.e. $P - E_+ \in \mathcal{B}_p(\mathcal{H})$) then $A \in \mathcal{B}_p(\mathcal{H})$. Indeed

$$P - JPJ = P - E_+ + E_+ - JPJ = P - E_+ + J(E_+ - P)J \in \mathcal{B}_p(\mathcal{H}). \qquad \square$$

Corollary 3.7. *For any $A \in \mathcal{D}_J$, $p_J(A)$ belongs to the image of* $\exp_J$. *More precisely, it belongs to the subset of $\mathcal{P}$ where* $\exp_J$ *is one-to-one. The same statement holds for the restrictions of p_J and* $\exp_J$ *to $\mathcal{D}_J^p$ ($1 \leq p \leq \infty$).*

Let us now characterize the image of the Fredholm differences. Recall the conditional expectation F_+ defined in Section 2.

Proposition 3.8.

$$p_J(\mathcal{D}_J^F) = \left\{ P \in \mathcal{P} : \|P - E_+\| \leq \frac{1}{\sqrt{2}},\ \textit{and}\ \frac{1}{2} - F_+(P)\ \textit{is Fredholm} \right\}.$$

Proof. If $\frac{1}{2} - F_+(P)$ is Fredholm, writing

$$\frac{1}{2} - F_+(P) = \frac{1}{2}E_+ - E_+PE_+ + \frac{1}{2}E_- - E_-PE_-,$$

it follows that $\frac{1}{2}E_+ - E_+PE_+$ is Fredholm in $\mathcal{H}_+$ and $\frac{1}{2}E_- - E_-PE_-$ is Fredholm in $\mathcal{H}_-$. In matrix terms, we have that $\frac{1}{2} - b/2$ is Fredholm in $\mathcal{H}_+$, thus 1 is isolated in the spectrum of b, and has finite multiplicity. Similarly for c in $\mathcal{H}_-$. The fact that $\|P - E_+\|^2 \leq \frac{1}{2}$, means that $b \geq 1$ and $1 \geq c \geq 0$. If as usual $A = P - JPJ$, since

$$1 - A^2 = \begin{pmatrix} (b-1)^2 & 0 \\ 0 & (c-1)^2 \end{pmatrix},$$

then $N(b-1)\oplus N(c-1)=N(1-A^2)$ is finite dimensional. Thus $1-A^2$ is Fredholm, i.e. $A\in\mathcal{D}_J^F$.

Conversely, by the same computations, the fact that $1-A^2$ is Fredholm implies that $(b-1)^2$ and thus $b-1$, are Fredholm in $\mathcal{H}_+$. Similarly for $c-1$ in $\mathcal{H}_-$. Thus $\frac{1}{2}(1-b)=\frac{1}{2}E_+-E_+PE_+$ is Fredholm in $\mathcal{H}_+$ and $\frac{1}{2}E_--E_-PE_-$ is Fredholm in $\mathcal{H}_-$. Then

$$\frac{1}{2}-F_+\big(p_J(A)\big)=\frac{1}{2}E_+-E_+PE_++\frac{1}{2}E_--E_-PE_-$$

is Fredholm in $\mathcal{H}$. □

4. The graph map

A way to parametrize subspaces or projections, is by considering graphs of operators between two given orthogonal subspaces (see for instance [9], or [6] for more recent developments). In our case, given the decomposition $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$, any bounded linear operator $x:\mathcal{H}_+\to\mathcal{H}_-$ induces a closed subspace

$$G_x=\{\xi+x\xi:\xi\in\mathcal{H}_+\},$$

the graph of x. Operators $x\in\mathcal{B}(\mathcal{H}_+,\mathcal{H}_-)$ are in correspondence with codiagonal selfadjoint operators X which anti-commute with J: the map

$$\mathcal{B}(\mathcal{H}_+,\mathcal{H}_-)\to\mathcal{B}_J,\qquad x\mapsto X=\begin{pmatrix}0 & x^*\\ x & 0\end{pmatrix}$$

is a real linear isometric isomorphism. We define the graph map Γ_J on selfadjoint codiagonal operators. Denote by

$$\mathcal{B}_J=\big\{X\in\mathcal{B}(\mathcal{H}):X^*=X,XJ=-JX\big\}\quad\text{and}\quad\mathcal{B}_J^p=\mathcal{B}_J\cap\mathcal{B}_p(\mathcal{H}).$$

Note that $\mathcal{D}_J$ is the closed unit ball of $\mathcal{B}_J$. Put

$$\Gamma_J:\mathcal{B}_J\to\mathcal{P},\qquad\Gamma_J(X)=P_{G_x}.$$

This map is clearly continuous and one-to-one. It is also smooth. Note that an idempotent whose range is G_x is given by the matrix (in terms of $\mathcal{H}_+\oplus\mathcal{H}_-$)

$$Q=\begin{pmatrix}1 & 0\\ x & 0\end{pmatrix}.$$

One way to obtain the orthogonal idempotent with the same range as Q is the known formula

$$P_{R(Q)} = QQ^*\big(1 - \big(Q - Q^*\big)^2\big)^{-1},$$

which after straightforward computation gives

$$P_{G_x} = \begin{pmatrix} (1+|x|^2)^{-1} & x^*(1+|x^*|^2)^{-1} \\ x(1+|x|^2)^{-1} & |x^*|^2(1+|x^*|^2)^{-1} \end{pmatrix}. \tag{5}$$

In terms of X, note that the above $Q = E_+ + XE_+$. So that

$$QQ^* = (1+X)\frac{J+1}{2}(1+X) = \frac{1}{2}\big\{(1+X)^2 + J\big(1-X^2\big)\big\}$$

and (using that $JX = -XJ$)

$$1 - \big(Q - Q^*\big)^2 = 1 - [X, E_+]^2 = 1 - \frac{1}{2}[X, J]^2 = 1 + X^2.$$

Therefore

$$\Gamma_J(X) = \frac{1}{2}\big\{(1+X)^2 + J\big(1-X^2\big)\big\}\big(1+X^2\big)^{-1} = (J+X)\big(1+X^2\big)^{-1} + E_-.$$

Proposition 4.1. *The map Γ_J maps $\mathcal{D}_J^p$ into $\mathcal{P}_J^p$*

Proof. Note that

$$\begin{aligned} \Gamma_J(X) - E_+ &= (J+X)\big(1+X^2\big)^{-1} + E_- - E_+ = (J+X)\big(1+X^2\big)^{-1} - J \\ &= J\big\{\big(1+X^2\big)^{-1} - 1\big\} + X\big(1+X^2\big)^{-1}. \end{aligned}$$

Since $f(t) = \frac{1}{1+t^2} - 1 = \frac{t^2}{t^2+1} = o(t^2)$,

$$\big(1+X^2\big)^{-1} - 1 \in \mathcal{B}_{p/2}(\mathcal{H}).$$

The second term $X(1+X^2)^{-1}$ clearly belongs to $\mathcal{B}_p(\mathcal{H})$. $\square$

The main result in [6] characterizes the image of this map. Adapted to our notation, K.Y. Chung proved [6]:

$$P = \Gamma_J(X) \quad \text{if and only if} \quad \|P - E_+\| < 1.$$

Therefore the image of Γ_J contained in the image of $\exp_J$. More precisely, it is contained in the subset of $\mathcal{P}$ where $\exp_J$ is one-to-one.

Let us relate the images of Γ_J and p_J. The first fact is apparent, due to the result by Chung [6], and Theorem 3.3:

Remark 4.2.

$$\operatorname{Im} p_J \subset \operatorname{Im} \Gamma_J,$$

and the inclusion is proper.

Remark 4.3. The inverse of Γ_J can be computed explicitly. Its domain is $\{P \in \mathcal{P} : \|P - E_+\| < 1\}$. If P lies in this (open) subset of $\mathcal{P}$, then E_+PE_+ is invertible as an operator in $\mathcal{B}(\mathcal{H}_+)$:

$$\|E_+PE_+ - E_+\| \le \|P - E_+\| < 1.$$

In matrix terms, as above

$$P = \Gamma_J(X) = \begin{pmatrix} (1+|x|^2)^{-1} & x^*(1+|x^*|^2)^{-1} \\ x(1+|x|^2)^{-1} & |x^*|^2(1+|x^*|^2)^{-1} \end{pmatrix},$$

and thus

$$\Gamma_J^{-1}(P) = X = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} = E_-PE_+(E_+PE_+)^{-1} + (E_+PE_+)^{-1}E_+PE_-.$$

Since $\|E_+PE_+ - E_+\| < 1$, the inverse of E_+PE_+ in $\mathcal{B}(\mathcal{H}_+)$ can be computed with the series $(E_+PE_+)^{-1} = \sum_{k=0}^{\infty}(E_+ - E_+PE_+)^k$.

Using estimates obtained in [6], or by a direct computation, one can characterize the image of Γ_J restricted to the open and closed unit balls of $\mathcal{B}_J$.

Proposition 4.4.

$$\Gamma_J\big(\{A \in \mathcal{B}_J : \|A\| < 1\}\big) = \left\{P \in \mathcal{P} : \|P - E_+\| < \frac{1}{\sqrt{2}}\right\},$$

and

$$\Gamma_J\big(\{A \in \mathcal{B}_J : \|A\| \le 1\}\big) = \left\{P \in \mathcal{P} : \|P - E_+\| \le \frac{1}{\sqrt{2}}\right\}.$$

Proof. Let $A \in \mathcal{D}_J$ with $\|A\| < 1$, then

$$\Gamma_J(A) - E_+ = (J + A)\big(1 + A^2\big)^{-1} - J.$$

The square of this selfadjoint operator is $1 - (1 + A^2)^{-1}$. Thus

$$\big\|\Gamma_J(A) - E_+\big\|^2 = \max\left\{\frac{t^2}{1+t^2} : t \in \sigma(A)\right\} < 1/2.$$

Conversely, by the results in the previous section, any projection P with $\|P-E_+\| < \frac{1}{\sqrt{2}}$, satisfies

$$P \in \operatorname{Im} p_J \subset \operatorname{Im} \Gamma_J.$$

Thus $P = \Gamma_J(X)$. Since

$$1/2 > (P - E_+)^2 = \big(\Gamma_J(X) - E_+\big)^2 = 1 - \big(1 + X^2\big)^{-1},$$

it follows that $1/2 < (1 + X^2)^{-1}$, i.e. $X^2 < 1$. The proof of the other statement is similar. □

Corollary 4.5.

$$\Gamma_J\big(\mathcal{B}_J^p\big) = \big\{P \in \mathcal{P}^p : \|P - E_+\| < 1\big\}$$

and

$$\Gamma_J\big(\mathcal{D}_J^p\big) = \left\{P \in \mathcal{P}^p : \|P - E_+\| \le \frac{1}{\sqrt{2}}\right\}.$$

Proof. Apparently

$$\Gamma_J\big(\mathcal{B}_J^p\big) \subset \big\{P \in \mathcal{P}^p : \|P - E_+\| < 1\big\} \quad \text{and} \quad \Gamma_J\big(\mathcal{D}_J^p\big) \subset \left\{P \in \mathcal{P}^p : \|P - E_+\| \le \frac{1}{\sqrt{2}}\right\}.$$

Let $P \in \mathcal{P}^p$ such that $\|P - E_+\| < 1$. Then $P = \Gamma_J(X)$ for some $X \in \mathcal{B}_J$, by the result in [6]. Since $P - E_+ \in \mathcal{B}_p(\mathcal{H})$, by the computation above,

$$X^2\big(1 + X^2\big)^{-1} = 1 - \big(1 + X^2\big)^{-1} = \big(\Gamma_J(X) - E_+\big)^2 = (P - E_+)^2 \in \mathcal{B}_{p/2}(\mathcal{H}).$$

Thus $X^2 \in \mathcal{B}_{p/2}(\mathcal{H})$, i.e. $X \in \mathcal{B}_p(\mathcal{H})$, because X is selfadjoint. The proof of the remaining inclusion is analogous. □

Let us show that $\Gamma_J(\mathcal{D}_J^F) = p_J(\mathcal{D}_J^F)$.

Proposition 4.6.

$$\Gamma_J\big(\mathcal{D}_J^F\big) = \left\{P \in \mathcal{P} : \|P - E_+\| \le \frac{1}{\sqrt{2}},\ \frac{1}{2} - F_+(P) \text{ is Fredholm}\right\}.$$

Proof. Suppose that $A \in \mathcal{D}_J^F$, i.e. $1 - A^2$ is a Fredholm operator. In matrix terms,

$$A = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}, \qquad 1 - A^2 = \begin{pmatrix} 1 - |x|^2 & 0 \\ 0 & 1 - |x^*|^2 \end{pmatrix}.$$

Then $1-|x|^2$ and $1-|x^*|^2$ are Fredholm operators in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. If $P=\Gamma_J(A)$, after straightforward computations,

$$\begin{aligned}\frac{1}{2}-F_+(P) &= \begin{pmatrix} 2(1+|x|^2)^{-1}(1-|x|^2) & 0 \\ 0 & 2(1+|x^*|^2)^{-1}(|x^*|^2-1) \end{pmatrix} \\ &= \begin{pmatrix} 2(1+|x|^2)^{-1} & 0 \\ 0 & 2(1+|x^*|^2)^{-1} \end{pmatrix} \begin{pmatrix} 1-|x|^2 & 0 \\ 0 & |x^*|^2-1 \end{pmatrix},\end{aligned}$$

which is an invertible operator times a Fredholm operator, thus a Fredholm operator.

Conversely, if $\frac{1}{2}-F_+(P)$ is a Fredholm operator, by the above factorization, $1-|x|^2$ and $|x^*|^2-1$ are Fredholm operators in $\mathcal{H}_+$ and $\mathcal{H}_-$. Thus $1-A^2=(1-|x|^2)\oplus(1-|x^*|^2)$ is a Fredholm operator in $\mathcal{H}_+\oplus\mathcal{H}_-=\mathcal{H}$, i.e. $A\in\mathcal{D}_J^F$. □

5. A retraction in $\mathcal{P}$

As seen in Section 3, the map $p_J:\mathcal{D}_J\to\mathcal{P}$ has a left inverse, $a_J:\mathcal{P}\to\mathcal{D}_J$, $a_J(P)=P-JPJ$. Thus one can define a retraction

$$\Delta_J:\mathcal{P}\to\mathcal{P},$$
$$\Delta_J(P)=p_J\big(a_J(P)\big)=\frac{1}{2}\big\{1+P-JPJ+J\big(1-(P-JPJ)^2\big)^{1/2}\big\}.$$

In matrix form, in terms of $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$,

$$\Delta_J(P)=\begin{pmatrix} \frac{1}{2}\{1+(1-aa^*)^{1/2}\} & a/2 \\ a^*/2 & \frac{1}{2}\{1-(1-a^*a)^{1/2}\} \end{pmatrix},$$

if $a/2:\mathcal{H}_-\to\mathcal{H}_+$ denotes the $1,2$ entry of P.

By the results in Section 3, the range of Δ_J is $\{P\in\mathcal{P}:\|P-E_+\|\le\frac{1}{\sqrt{2}}\}$. Clearly Δ_J is the identity on this ball: if $P\in\{P\in\mathcal{P}:\|P-E_+\|\le\frac{1}{\sqrt{2}}\}$, then $P=p_J(A)$ for some $A\in\mathcal{D}_J$, and then

$$\Delta_J(P)=p_J\big(a_J\big(p_J(A)\big)\big)=p_J(A)=P.$$

Thus Δ_J is a retraction from $\mathcal{P}$ onto $\{P\in\mathcal{P}:\|P-E_+\|\le\frac{1}{\sqrt{2}}\}$.

Let us examine the fibers of this retraction. For $E\in\{P\in\mathcal{P}:\|P-E_+\|\le\frac{1}{\sqrt{2}}\}$

$$\Delta_J^{-1}(E)=\big\{P\in\mathcal{P}:\Delta_J(P)=E\big\}.$$

The fibers of Δ_J can vary from one point to the set of projections in the commutant of E_+ (the fiber over E_+).

Remark 5.1. $\Delta_J^{-1}(E) = \{P \in \mathcal{P} : P - E \text{ commutes with } J\}$.

If $\Delta_J(P) = E$, then

$$a_J(E) = a_J\big(\Delta_J(P)\big) = a_J\big(p_J\big(a_J(P)\big)\big) = a_J(P),$$

i.e. $E - JEJ = P - JPJ$, which means that $JE - EJ = JP - PJ$, i.e. $P - E$ commutes with J. Conversely, if $P - E$ commutes with J, then $a_J(P) = a_J(E)$, and thus $\Delta_J(P) = \Delta_J(E) = E$.

Note that in particular, the fiber over E_+ consists of all projections in the commutant of E_+.

If we write

$$E = \begin{pmatrix} b/2 & a/2 \\ a^*/2 & c/2 \end{pmatrix}, \qquad A = a_J(E) = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

one has that $\Delta_J(P) = E$ if and only if

$$P = \begin{pmatrix} x/2 & a/2 \\ a^*/2 & y/2 \end{pmatrix}.$$

Thus the fiber $\Delta_J^{-1}(E)$ consists of all possible completions of the matrix

$$\begin{pmatrix} * & a/2 \\ a^*/2 & * \end{pmatrix} \tag{6}$$

which give rise to a selfadjoint projection. These completions consist of all pairs of operators $x \in \mathcal{B}(\mathcal{H}_+)$, $y \in \mathcal{B}(\mathcal{H}_-)$ satisfying:

$$0 \le x, y \le 2, \quad aa^* = 2x - x^2, \qquad a^*a = 2y - y^2 \quad \text{and} \quad xa + ay = 2a. \tag{7}$$

Remark 5.2.

1. Completions of (6) to a selfadjoint projection always exist. There is one special solution to this problem. Namely, given a contraction $a \in \mathcal{B}(\mathcal{H}_-, \mathcal{H}_+)$, put

$$A = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}.$$

   Then $P_a = p_J(A)$ is a completion.
2. For any pair P_1, P_2 of completions of (6), one has

$$\Delta_J(P_1) = \Delta_J(P_2) = P_a.$$

   The proof of this fact is a straightforward computation.

Theorem 5.3. *Given a contraction $a \in \mathcal{B}(\mathcal{H}_-, \mathcal{H}_+)$, let P be a completion of the matrix (6)*

$$\begin{pmatrix} * & a/2 \\ a^*/2 & * \end{pmatrix}$$

to an orthogonal projection. Then $\Delta_J(P) = P_a$ is the completion of (6) which is closest in norm to E_+, i.e.

$$\|P_a - E_+\| \leq \|P - E_+\|,$$

with equality only if $P = \Delta_J(P)$.

Proof. We know that $\|\Delta_J(P) - E_+\| \leq \frac{1}{\sqrt{2}}$. If $\|P - E_+\| > \frac{1}{\sqrt{2}}$, then trivially

$$\|P_a - E_+\| = \|\Delta_J(P) - E_+\| \leq \frac{1}{\sqrt{2}} < \|P - E_+\|.$$

Otherwise, if $\|P - E_+\| \leq \frac{1}{\sqrt{2}}$, since Δ_J is the identity in this closed ball, then $\Delta_J(P) = P$. $\square$

Let us study the conditions (7). Note that one may reduce this study to the case when $N(A^2 - 1) = \{0\}$:

Remark 5.4. First note that $N(A^2 - 1)$ is trivial if and only if $1_{\mathcal{H}_+} - aa^*$ and $1_{\mathcal{H}_-} - a^*a$ have trivial nullspaces. If $P \in \Delta_J^{-1}(E)$, $A = E - JEJ = P - JPJ$. As remarked above, this implies that $N(A^2 - 1)$ reduces, simultaneously, P, JPJ, E and JEJ. Thus our completion problem can be considered separately in $N(A^2 - 1)$ and $N(A^2 - 1)^\perp$.

In the first space, aa^* and a^*a are the identity operators (of $\mathcal{H}_+ \cap N(A^2 - 1)$ and $\mathcal{H}_- \cap N(A^2 - 1)$, respectively). The relations (7) in this case imply $(x - 1)^2 = 0$ and $(y - 1)^2 = 0$, i.e. there is a unique solution $x = 1$ and $y = 1$.

We may suppose therefore that $N(A^2 - 1) = 0$.

Proposition 5.5. *Given a contraction $a \in \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$, with $N(aa^* - 1) = N(a^*a - 1) = \{0\}$, the set of pairs $x \in \mathcal{B}(\mathcal{H}_+)$ and $y \in \mathcal{B}(\mathcal{H}_-)$ satisfying the relations (7), is given by*

$$x = 1_{\mathcal{H}_+} + \epsilon\big(1 - aa^*\big)^{1/2}, \qquad y = 1_{\mathcal{H}_-} + \delta\big(1 - a^*a\big)^{1/2},$$

where ϵ and δ are symmetries of $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively, which satisfy

$$\big[\epsilon, aa^*\big] = 0, \qquad \big[\delta, a^*a\big] = 0 \quad \text{and} \quad \epsilon a + a\delta = 0. \tag{8}$$

Proof. Let x, y be a pair of operators satisfying (7). Equation $aa^* = 2x - x^2$ is equivalent to $1 - aa^* = (x - 1)^2$, or $(1 - aa^*)^{1/2} = |x - 1|$. Let

$$x-1=\epsilon|x-1|=\epsilon\big(1-aa^*\big)^{1/2}$$

be the polar decomposition of $x-1$ in $\mathcal{H}_+$. Since $x-1$ is selfadjoint and $1-aa^*$ has trivial nullspace, ϵ is a symmetry in $\mathcal{H}_+$, which commutes with $1-aa^*$. Similarly, in $\mathcal{H}_-$,

$$y-1=\delta\big(1-a^*a\big)^{1/2},$$

with δ a symmetry which commutes with $1-a^*a$. Note that $(aa^*)a=a(a^*a)$, thus, for any continuous function f defined in the spectra of a^*a and aa^* (which are contained in $[0,1]$), $f(aa^*)a=af(a^*a)$. Thus

$$a\delta\big(1-a^*a\big)^{1/2}=a\big(1-a^*a\big)^{1/2}\delta=\big(1-aa^*\big)^{1/2}a\delta,$$

and therefore

$$\big(1-aa^*\big)^{1/2}(\epsilon a+a\delta)=\epsilon\big(1-aa^*\big)^{1/2}a+a\delta\big(1-a^*a\big)^{1/2}=(x-1)a+a(y-1)=0.$$

Since $(1-aa^*)^{1/2}$ has trivial nullspace, it follows that $\epsilon a+a\delta=0$.

Conversely, straightforward computations show that if ϵ and δ are symmetries of $\mathcal{H}_+$ and $\mathcal{H}_-$, satisfying (8), then x and y defined as above, satisfy (7). □

Remark 5.6. In the above description of $\Delta_J^{-1}(E)$, one obtains E by considering $\epsilon=1_{\mathcal{H}_+}$ and $\delta=-1_{\mathcal{H}_-}$. Note also that if $aa^*=1_{\mathcal{H}_+}$ and $a^*a=1_{\mathcal{H}_-}$, E is the unique element in $\Delta_J^{-1}(E)$. Note that such E lies at the border of the range of Δ_J: $\|E-E_+\|=\frac{1}{\sqrt{2}}$.

The fiber over elements in the restricted Grassmannian can be characterized:

Proposition 5.7. *Let $1\le p\le\infty$. The following are equivalent:*

1. $\Delta_J(P)\in\mathcal{P}_p$.
2. $[P,E_+]=2[P,J]\in\mathcal{B}_p(\mathcal{H})$.
3. $a\in\mathcal{B}_p(\mathcal{H}_-,\mathcal{H}_+)$.

Proof. Suppose first that $\Delta_J(P)\in\mathcal{P}_p$, i.e. $\Delta_J(P)-E_+\in\mathcal{B}_p(\mathcal{H})$. Then

$$\begin{aligned}\Delta_J(P)-E_+-J\big(\Delta_J(P)-E_+\big)J&=\Delta_J(P)-J\Delta_J(P)J=a_J\big(p_J\big(a_J(P)\big)\big)\\&=a_J(P)\in\mathcal{B}_p(\mathcal{H}).\end{aligned}$$

Thus

$$PJ-JP=(P-JPJ)J=a_J(P)J\in\mathcal{B}_p(\mathcal{H}).$$

Conversely, $[P, J] \in \mathcal{B}_p(\mathcal{H})$ implies that $a_J(P) \in \mathcal{B}_p(\mathcal{H})$ by the same calculations above. By the results in Section 2, this implies that

$$\Delta_J(P) = p_J\big(a_J(P)\big) \in \mathcal{P}_p.$$

Apparently

$$[P, J] = \begin{pmatrix} 0 & -a \\ a^* & 0 \end{pmatrix}.$$

Thus $[P, J] \in \mathcal{B}_p(\mathcal{H})$ if and only if $a \in \mathcal{B}_p(\mathcal{H}_-, \mathcal{H}_+)$. □

5.1. The retraction in the restricted Grassmannian

Let us consider now the map Δ_J restricted to the full restricted Grassmannian $\mathcal{P}_{res}$, induced by the composition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, for the case $p = \infty$ (recall that $\mathcal{B}_\infty(\mathcal{H})$ denotes the compact operators). In previous sections we only considered the connected component of E_+ (or zero index component) in $\mathcal{P}_{res}$. We recall the definition [11] of the restricted Grassmannian:

A projection P belongs to $\mathcal{P}_{res}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ if and only if

1.
$$E_+P|_{R(P)} : R(P) \to \mathcal{H}_+ \in \mathcal{B}\big(R(P), \mathcal{H}_+\big)$$
is a Fredholm operator in $\mathcal{B}(R(P), \mathcal{H}_+)$, and
2.
$$E_-P|_{R(P)} : R(P) \to \mathcal{H}_- \in \mathcal{B}_\infty\big(R(P), \mathcal{H}_-\big).$$

The index of the first operator is usually called the index of P [11]. The index characterizes the connected components of $\mathcal{P}_{res}$: $P_1, P_2 \in \mathcal{P}_{res}$ belong to the same connected component if and only if they have the same index [11].

The following result is elementary:

Lemma 5.8. *Let $P \in \mathcal{P}$ with matrix*

$$P = \begin{pmatrix} x/2 & a/2 \\ a^*/2 & y/2 \end{pmatrix}.$$

Then $P \in \mathcal{P}_{res}$ if and only if x is Fredholm in $\mathcal{B}(\mathcal{H}_+)$, $y \in \mathcal{B}_\infty(\mathcal{H}_-)$ and $a \in \mathcal{B}_\infty(\mathcal{H}_-, \mathcal{H}_+)$.

Proof. The proof is based on the following elementary facts:

- $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Fredholm operator if and only if AA^* is a Fredholm operator in $\mathcal{H}_1$ and $N(A)$ is finite dimensional.
- $A \in \mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $A^*A \in \mathcal{B}_\infty(\mathcal{H}_1)$.

Suppose first that $P \in \mathcal{P}_{res}$. Then $E_+P \in \mathcal{B}(R(P), \mathcal{H}_+)$ is Fredholm, and thus

$$E_+P(E_+P)^*\big|_{\mathcal{H}_+} = E_+PE_+|_{\mathcal{H}_+} = x/2$$

is Fredholm in $\mathcal{H}_+$. Also $E_-P \in \mathcal{B}_\infty(R(P), \mathcal{H}_-)$, and thus

$$E_-P(E_-P)^*\big|_{\mathcal{H}_-} = E_-PE_-|_{\mathcal{H}_-} = y/2$$

is compact in $\mathcal{H}_-$. The fact that P is a projection, implies the relation $2y - y^2 = a^*a$, and thus $a \in \mathcal{B}_\infty(\mathcal{H}_-, \mathcal{H}_+)$.

Conversely, by the last computations, if a and y are compact, then $E_-P \in \mathcal{B}_\infty(R(P), \mathcal{H}_-)$. Similarly, $E_+P(E_+P)^*|_{\mathcal{H}_+} = x/2$ is Fredholm. Thus E_+P, as an operator in $\mathcal{B}(R(P), \mathcal{H}_+)$, has closed range (equal to the range of x) with finite codimension. Let us prove that its nullspace is finite dimensional. Let $\xi = \xi_+ + \xi_- = P\xi$ such that $E_+\xi = 0$ ($\xi_+ \in \mathcal{H}_+$, $\xi_- \in \mathcal{H}_-$). This implies that

$$\begin{cases} 2\xi_+ = x\xi_+ + a\xi_- \\ 2\xi_- = a^*\xi_+ + y\xi_- \end{cases}$$

and $\xi_+ = 0$. The second equation then reduces to $2\xi_- = y\xi_-$, i.e. ξ_- lies in the 2-eigenspace of the compact operator y. Thus ξ_- lies in a finite dimensional space. It follows that $N(E_+P|_{R(P)})$ is finite dimensional. $\square$

Corollary 5.9. *The restriction of Δ_J to $\mathcal{P}_{res}$ is a retraction onto $\{P \in \mathcal{P}_{res} : \|P - E_+\| < \frac{1}{\sqrt{2}}\}$.*

Proof. If $P \in \mathcal{P}_{res}$, then $a \in \mathcal{B}_\infty(\mathcal{H}_-, \mathcal{H}_+)$. By Proposition 5.7, this means that $\Delta_J(P) \in \mathcal{P}_J^\infty$, the connected component of E_+ in $\mathcal{P}_{res}$. $\square$

Remark 5.10. It follows that Δ_J solves an analogous completion problem, in the context of compact operators and restricted Grassmannians. Let $a : \mathcal{H}_- \to \mathcal{H}_+$ be a compact operator, with $\|a\| \leq 1$. Then there is a unique way to complete the matrix

$$\begin{pmatrix} * & a/2 \\ a^*/2 & * \end{pmatrix}$$

in order to obtain an orthogonal projection E in the restricted Grassmannian, which is closest to E_+. Pick $P_a = p_J(A)$, where as before,

$$A = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}.$$

Clearly $P_a \in \mathcal{P}_J^\infty$. For any other completion P, $\Delta_J(P) = P_a$, and

$$\|P_a - E_+\| \leq \|P - E_+\|,$$

with equality only if $P = P_a$.

References

[1] W.O. Amrein, K.B. Sinha, On pairs of projections in a Hilbert space, Linear Algebra Appl. 208/209 (1994) 425–435.
[2] E. Andruchow, Pairs of projections: geodesics, Fredholm and compact pairs, Complex Anal. Oper. Theory 8 (7) (2014) 1435–1453.
[3] E. Andruchow, Operators which are the difference of two projections, J. Math. Anal. Appl. 420 (2) (2014) 1634–1653.
[4] J. Avron, R. Seiler, B. Simon, The index of a pair of projections, J. Funct. Anal. 120 (1) (1994) 220–237.
[5] D. Beltita, T.S. Ratiu, A.B. Tumpach, The restricted Grassmannian, Banach Lie–Poisson spaces, and coadjoint orbits, J. Funct. Anal. 247 (1) (2007) 138–168.
[6] K.Y. Chung, Subspaces and graphs, Proc. Amer. Math. Soc. 119 (1) (1993) 141–146.
[7] G. Corach, H. Porta, L. Recht, The geometry of spaces of projections in C^*-algebras, Adv. Math. 101 (1) (1993) 59–77.
[8] C. Davis, Separation of two linear subspaces, Acta Sci. Math. (Szeged) 19 (1958) 172–187.
[9] P.R. Halmos, Two subspaces, Trans. Amer. Math. Soc. 144 (1969) 381–389.
[10] H. Porta, L. Recht, Minimality of geodesics in Grassmann manifolds, Proc. Amer. Math. Soc. 100 (3) (1987) 464–466.
[11] G. Segal, G. Wilson, Loop groups and equations of KdV type, Publ. Math. Inst. Hautes Etudes Sci. 61 (1985) 5–65.