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Notes on bounded Hilbert algebras with supremum

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Abstract. In this note we will investigate some particular classes of ideals in Hilbert algebras with supremum. We shall study the relation between α-ideals and annihilator ideals in bounded Hilbert algebras with supremum. We shall introduce the class of σ-ideals and we will see that this class is strongly connected with the deductive systems. We will also characterize the bounded Hilbert algebras with supremum satisfying the Stone identity.

1. Introduction

Hilbert algebras are algebraic models of the implicative fragment of Intuitionistic Propositional Logic. These algebras form a variety as was shown by A. Diego in [13] (see also [18], and [10]). On the other hand, it is well known that in every ordered set with last element $\langle A, \leq, 1\rangle$ the binary operation $\to$ given by $a \to b = 1$, when $a \leq b$, and $a \to b = b$, when $a \nleq b$, define a Hilbert algebra $\langle A, \to, 1\rangle$. This implication is called the *implication given by the order*. This example allows us to define Hilbert algebras on posets, semilattices or lattices. These examples motivate the study of Hilbert algebras with lattice operations. Works in this direction are the papers [4] and [8]. In this note we will consider Hilbert algebras where the order define a join-semilattice. We note that Hilbert algebras are also known as positive implicative BCK-algebras, and Hilbert algebras with lattice operations are a particular case of the BCK-algebras with lattice operations studied by Idziak in [16] and [17].

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An important notion recently considered in the literature on Hilbert algebras is the concept of *order-ideal*. In Hilbert algebras where the order define a join operation the notion of order-ideal is equivalent to the usual notion of ideal of join-semilattice. This notion is used in the topological representation theory for Hilbert algebras given in [7], and the representation theory for Hilbert algebras with supremum recently developed in [8]. In the study of bounded Hilbert algebras with supremum, or $H^{\vee}$-algebras, we can consider other classes of ideals. For example, in [9] we will introduce the notion of *α-ideal*. As was proved in [9] the set of all α-ideals is a Heyting algebra isomorphic to the set of all ideals of the Boolean algebra of the regulars elements. Another interesting class of ideals that can be defined in bounded Hilbert algebras with supremum is the class of *σ-ideals*. It is well-known that a subset I is an ideal of a Boolean algebra $\boldsymbol{A}$ iff the set $(I)^* = \{a^* \mid a \in I\}$ is a filter of $\boldsymbol{A}$, where a^* is the negation of a. For bounded Hilbert algebras with supremum we do not have a similar condition, but we can identify a class of ideals strongly connected with the deductive systems. A *σ-ideal* in a bounded Hilbert algebras with supremum $\boldsymbol{A}$ is an semilattice ideal I such that $I = ((F)^*] = \{a \mid \exists f \in F : (a \leq f^*)\}$, for some deductive system F. In general every σ-ideal is an α-ideal, but the converse is not always true. We prove that the bounded Hilbert algebras with supremum where the converse is valid are exactly those that they satisfies the Stone identity.

The paper is organized as follows. In Section 2 we will recall some notions that will be needed in the sequel. In Section 3 we shall define the class $\mathcal{H}^{\vee}$ of Hilbert algebras with supremum. The class $\mathcal{H}^{\vee}$ is a subclass of BCK-algebras with supremum studied by P. M. Idziak in [16]. In Section 4 we shall study the relation between α-ideals and annihilator ideals in bounded Hilbert algebras with supremum. In Section 5 we shall study the σ-ideals in the class of bounded Hilbert algebras with supremum or $H_0^{\vee}$-algebras. In Section 6 we will give different characterizations of the class of $H_0^{\vee}$-algebras that satisfies the Stone identity.

2. Preliminaries

For basic concepts in Hilbert algebras we refer to [10], [13] and [18], for Hilbert algebras with lattices operations we refer to [4], and for basic concepts in distributive lattices we refer to [1].

Definition 1. A *Hilbert algebra* is an algebra $\boldsymbol{A} = \langle A, \to, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in $\boldsymbol{A}$:

(1) $a \to (b \to a) = 1$.
(2) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$.
(3) $a \to b = 1 = b \to a$ imply $a = b$.

It is easy to see that the binary relation $\leq$ defined in a Hilbert algebra A by $a \leq b$ if and only if $a \to b = 1$, is a partial order on $\boldsymbol{A}$ with greatest element 1. This order is called the *natural ordering* on $\boldsymbol{A}$.

A Hilbert algebra $\boldsymbol{A}$ is *bounded* if there exists an element $0 \in A$ such that $0 \to a = 1$, for all $a \in A$. We shall write a^* for $a \to 0$. We will note by $\mathcal{H}$ and $\mathcal{H}_0$ the varieties of Hilbert algebras and bounded Hilbert algebras, respectively.

Lemma 2. *Let $\boldsymbol{A}$ be a bounded Hilbert algebra. Then the following properties are satisfied for all $a, b, c \in A$:*

(1) $a \to a = 1$.
(2) $1 \to a = a$.
(3) $a \to (b \to c) = b \to (a \to c)$.
(4) $a \to (b \to c) = (a \to b) \to (a \to c)$.
(5) $a \leq ((a \to b) \to b)$.
(6) $a \to (a \to b) = a \to b$.
(7) $(a \to b) \leq (b \to c) \to (a \to c)$.
(8) $(a \to b) \leq (c \to a) \to (c \to b)$.
(9) $a \leq b \to a$.
(10) $a \leq a^{**}$.
(11) $a^* = a^{***}$.
(12) $a \to b \leq b^* \to a^*$.
(13) $a \to b^* = b \to a^*$.
(14) $(a \to b)^{**} \leq a^{**} \to b^{**}$.

Proof. See [3], [4], [10], [13], or [18]. ■

Let us consider a poset $\langle X, \leq \rangle$. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*) if for all $x, y \in X$ such that $x \in U$ $(y \in U)$ and $x \leq y$, we have $y \in U$ $(x \in U)$. For each $Y \subseteq X$, the *increasing* (*decreasing*) set generated by Y is $[Y) = \{x \in X \mid \exists y \in Y \ (y \leq x)\}$ $((Y] = \{x \in X \mid \exists y \in Y \ (x \leq y)\})$. If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. The set of all subsets of X is denoted by $\mathcal{P}(X)$, and the set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. Let $Y \subseteq X$. The complement of Y in X is denoted by $X - Y$, or by $C_X Y$.

Let $\boldsymbol{A} \in \mathcal{H}$. A subset $D \subseteq A$ is a *deductive system of* $\boldsymbol{A}$ if $1 \in D$, and if $a, a \to b \in D$ then $b \in D$. The set of all deductive systems of a Hilbert algebra $\boldsymbol{A}$ is noted $\mathcal{D}s\,(A)$. It is easy to prove that $\mathcal{D}s(A)$ is closed under arbitrary intersections. The deductive system generated by a set X is $\langle X \rangle = \bigcap \{D \in \mathcal{D}s\,(A) \mid X \subseteq D\}$. If $X = \{a\}$, then we will denote $X = \langle a \rangle$. Let us recall that $\langle a \rangle = \{b \in A \mid a \leq b\} = [a)$. Given a sequence $a, a_1, \ldots, a_n \in A$, we define $(a_1, \ldots, a_n; a) = a_1 \to (a_2 \to \ldots (a_n \to a) \ldots)$. So, the deductive system generated by a subset $X \subseteq A$ can be characterized as the set $\langle X \rangle = \{a \in A \mid \exists (a_1, \ldots, a_n) \in A^n$, such that $(a_1, \ldots, a_n; a) = 1\}$.

Let $D \in \mathcal{D}s(A) - \{A\}$. We shall say that D is *irreducible* if and only if for any $D_1, D_2 \in \mathcal{D}s(A)$ such that $D = D_1 \cap D_2$, it follows that $D = D_1$ or $D = D_2$. The set of all irreducible deductive systems of a Hilbert algebra A is denoted by $X(A)$. Let us recall that a deductive system is irreducible iff for every $a, b \in A$ such that $a, b \notin D$ there exists $c \notin D$ such that $a, b \leq c$ (see [5], [13] or [18]). A subset I of A is called an *order-ideal* of $\boldsymbol{A}$ if $b \in I$ and $a \leq b$, then $a \in I$, and for each $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of all order-ideal of $\boldsymbol{A}$ will denoted by $\mathcal{I}(A)$. It is clear that the set $(a] = \{b \in A \mid b \leq a\}$ is an order-ideal, for each $a \in A$.

The following is a Hilbert algebra analogue of Birkhoff's Prime filter Lemma and it is proved in [5].

Theorem 3. *Let $\boldsymbol{A}$ be a Hilbert algebra. Let $D \in \mathcal{D}s(A)$ and let $I \in \mathcal{I}(A)$ be such that $D \cap I = \emptyset$. Then there exists $P \in X(A)$ such that $D \subseteq P$ and $P \cap I = \emptyset$.*

Lemma 4. *Let $\boldsymbol{A} \in \mathcal{H}_0$, $a \in A$ and $P \in X(A)$. Then*

(1) *$a^* \notin P$ if and only if there exists $Q \in X(A)$ such that $P \subseteq Q$ and $a \in Q$.*
(2) *$a^* \notin P$ if and only if there exists a maximal deductive system M such that $P \subseteq M$ and $a \in M$.*

Let $\boldsymbol{A}$ be a Hilbert algebra. Let us consider the poset $\langle X(A), \subseteq \rangle$ and the mapping $\varphi\colon A \to \mathcal{P}_i(X(A))$ defined by $\varphi(a) = \{P \in X(A) \mid a \in P\}$. In [5] was proved that φ is an injective homomorphism of Hilbert algebras. If $\boldsymbol{A} \in \mathcal{H}_0$, then it is easy to prove that $\varphi(0) = \emptyset$, and $\varphi(a^*) = X(A) - (\varphi(a)]$.

Let $\boldsymbol{A} \in \mathcal{H}_0$. The set of *regular elements* of $\boldsymbol{A}$ is the set $R(A) = \{a \in A \mid a^{**} = a\}$. It is know that $R(\boldsymbol{A}) = \langle R(A), \underline{\vee}, \overline{\wedge}, \neg, 0, 1 \rangle$ is a Boolean algebra, where the lattice operations $\underline{\vee}$ and $\overline{\wedge}$ are defined by means of the following conditions:

$$a \overline{\wedge} b = (a \to b^*)^*,$$
$$a \underline{\vee} b = (a^* \to b)^{**},$$

respectively. The set $D(A) = \{a \in A \mid a^* = 0\}$ is a deductive system called the set of *dense elements* of $\boldsymbol{A}$ (for more details see [3] and [14]).

3. Hilbert algebras with supremum

Definition 5. An algebra $\boldsymbol{A} = \langle A, \to, \vee, 1 \rangle$ of type $(2, 2, 0)$ is a *Hilbert algebra with supremum*, or $H^{\vee}$-algebra, if

(1) $\langle A, \to, 1\rangle \in \mathcal{H}$.
(2) for every $a, b \in A$, there exists $a \vee b \in A$ (relative to natural ordering on A.
(3) For all $a, b \in A$, $a \to b = 1$ if and only if $a \vee b = b$.

An important example of Hilbert algebras with supremum are the Tarski algebras, also called Implication algebras. Let us recall that a Tarski algebra is a Hilbert algebra $\langle A, \to, 1\rangle$ such that $(a \to b) \to b = (b \to a) \to a$, for all $a, b \in A$. It is known that $\langle A, \to, 1\rangle$ is a join semilattice under the operation $\vee$ defined by $a \vee b = (a \to b) \to b$ (see [4], [10] or [18]).

The class of $H^{\vee}$-algebras is indeed a variety, denoted by $\mathcal{H}^{\vee}$. An $H_0^{\vee}$-*algebra* is a bounded $H^{\vee}$-algebra. The variety of $H_0^{\vee}$-algebras is denoted by $\mathcal{H}_0^{\vee}$. The fact that $\mathcal{H}^{\vee}$ is a variety follows from the work on BCK-algebras with lattice operations given by P. M. Idziak in [16]. In the subsequent paper [17] Idziak also proved that the lattice operations are compatible with the BCK-congruences. For completeness, we prove these facts.

Theorem 6. *Let us consider an algebra* $\boldsymbol{A} = \langle A, \to, \vee, 1\rangle$ *of type* $(2, 2, 0)$*. Then* $\boldsymbol{A}$ *is an* $H^{\vee}$*-algebra if and only if*

(1) $\langle A, \to, 1\rangle$ *is a Hilbert algebra.*
(2) *For every* $a, b \in A$ *there exists* $a \vee b \in A$*, (relative to the natural ordering of* A*).*
(3) A *satisfies the following equations:*
   (a) $a \to (a \vee b) = 1$,
   (b) $(a \to b) \to ((a \vee b) \to b) = 1$.

Proof. $\Rightarrow$) (a) We note that $a \vee (a \vee b) = a \vee b$ if and only if $a \to (a \vee b) = 1$. We prove (b). We recall that for any $a, b \in A$, we have $a \leq (a \to b) \to b$, and $b \leq (a \to b) \to b$. Then,

$$a \vee (a \to b) \to b = (a \to b) \to b,$$

and

$$b \vee (a \to b) \to b = (a \to b) \to b.$$

So, $(a \vee b) \vee ((a \to b) \to b) = (a \to b) \to b$, and by hypothesis we deduce that $a \vee b \leq (a \to b) \to b$. Therefore, $a \to b \leq (a \vee b) \to b$, i.e., $(a \to b) \to ((a \vee b) \to b) = 1$.

$\Leftarrow$) We prove that $a \to b = 1$ if and only if $a \vee b = b$, for all $a, b \in A$. Suppose that $a \to b = 1$. Hence,

$$1 = (a \to b) \to ((a \vee b) \to b) = 1 \to ((a \vee b) \to b).$$

So, $(a \vee b) \to b = 1$. As $b \to (a \vee b) = 1$, we get from properties of Hilbert algebras, that $a \vee b = b$.

Assume that $a \vee b = b$. Thus, $1 = a \to (a \vee b) = a \to b$. ■

Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. It is well known that if a, b are elements of a BCK-algebra such that $a \vee b$ exists, then for each element c, $(a \to c) \wedge (b \to c)$ also exists and $(a \vee b) \to c = (a \to c) \wedge (b \to c)$. Hence for $a, b \in \boldsymbol{A}$, we have:

$$(a \vee b)^* = a^* \wedge b^*. \tag{3.1}$$

A well-known result given by A. Diego and A. Monteiro (see [13], [10], [18], or [14]) ensures that the lattice of congruences of a Hilbert algebra is isomorphic to the lattice of the deductive systems. This result can be extended to Hilbert algebras with supremum as we shall see below.

Let $\boldsymbol{A} = \langle A, \to, \vee, 1\rangle$ be an $H^{\vee}$-algebra. The lattice of all congruences of $\boldsymbol{A}$ we will denoted by $\mathrm{Con}(A, \to, \vee)$, and the lattice of all congruences of $\langle A, \to, 1\rangle$ we will denoted by $\mathrm{Con}(A, \to)$. Let us recall that for each $D \in \mathcal{D}s(A)$, the relation

$$\theta(D) = \left\{(a, b) \in A^2 \mid a \to b, b \to a \in D\right\},$$

is an element of $\mathrm{Con}(A, \to)$, and for each $\theta \in \mathrm{Con}(A, \to)$, the set

$$1_\theta = \{a \in A \mid (a, 1) \in \theta\},$$

is a deductive system. Moreover, $D = 1_{\theta(D)}$ and $\theta = \theta(1_\theta)$. Now we shall prove that the congruences of an $H^{\vee}$-algebra $\boldsymbol{A}$ are the same that the congruences of the Hilbert algebra $\langle A, \to, 1\rangle$. This result is based in the following lemma.

Lemma 7. *Let $\boldsymbol{A} \in \mathcal{H}^{\vee}$. Let $D \in \mathcal{D}s(A)$. For every $a, b, c \in A$, if $a \to b \in D$, then $(a \vee c) \to (b \vee c) \in D$.*

Proof. Suppose that $a \to b \in D$. As $a \to b \leq a \to (b \vee c) \in D$, we get

$$\begin{aligned}(a \vee c) \to (b \vee c) &= (a \to (b \vee c)) \wedge (c \to (b \vee c)) \\ &= (a \to (b \vee c)) \wedge 1 = a \to (b \vee c) \in D.\end{aligned}$$

■

Theorem 8. *Let $\boldsymbol{A} \in \mathcal{H}^{\vee}$. Then $\mathrm{Con}(A, \to, \vee) = \mathrm{Con}(A, \to)$.*

Proof. It is clear that $\mathrm{Con}(A, \to, \vee) \subseteq \mathrm{Con}(A, \to)$. Let $\theta \in \mathrm{Con}(A, \to)$. Let $(a, b) \in \theta$. Then $a \to b$, $b \to a \in 1_\theta$, and from Lemma 7 we have that $(a \vee c) \to (b \vee c)$, $(b \vee c) \to (a \vee c) \in 1_\theta$, i.e., $(a \vee c, b \vee c) \in \theta$. So, $\mathrm{Con}(A, \to) \subseteq \mathrm{Con}(A, \to, \vee)$. ■

Now we give a characterization of maximal deductive systems in bounded $H^{\vee}$-algebras.

Lemma 9. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$ and $P \in \mathcal{D}s(A)$. Then the following conditions are equivalent:*

(1) *P is maximal.*
(2) *for all $a \in A$ ($a \notin P$, implies that $a^* \in P$).*
(3) *for all $a \in A$ ($a \notin P$, implies that $a^{**} \notin P$).*
(4) *P is irreducible and $D(A) \subseteq P$.*

Proof. The equivalences between the items (1), (2) and (3) follows by the results given in [3] and [18]. The direction (3) $\Rightarrow$ (4) is immediate.

We see (4) $\Rightarrow$ (1). If by contrary P is not maximal, then there is $a \notin P$ such that $a^* \notin P$. Since P is irreducible, there is $b \notin P$ such that $a, a^* \leq b$. Then $b^* \leq a^* \leq b$. So, $b^* \to b = 1$, and consequently $b^{**} = 1$, i.e., $b^* = 0$. Then $b \in D(A) \subseteq P$, a contradiction. ■

Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. We say that a deductive system D is *prime* if and only if $D \neq A$ and for every $a, b \in A$ such that $a \vee b \in D$, we have that $a \in D$ or $b \in D$. It is easy to prove that a deductive system D is irreducible if and only if D is prime.

The next lemma is needed later. The statement and proof of item (1) of the following lemma was suggested by the referee.

Lemma 10. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Then the following properties are satisfied:*

(1) *for $a, b \in A$ there exists $a^{**} \wedge b^*$ and $(a \to b)^* = a^{**} \wedge b^*$.*
(2) *$(a^* \to b)^* = (a \vee b)^*$.*
(3) *If $a \leq b_1^* \vee \cdots \vee b_n^*$, then $(b_1, \ldots, b_n; a^*) = 1$, i.e., $a^* \in \langle b_1, \ldots, b_n \rangle$.*

Proof. (1) Since $b \leq a \to b$ and $a^* \leq a \to b$, then $(a \to b)^* \leq a^{**}$ and $(a \to b)^* \leq b^{**}$.

If consider $t \in A$ such that $t \leq a^{**}$ and $t \leq b^*$, then from $a \to b \leq b^* \to a^*$, we deduce $b^* \leq (a \to b) \to a^*$. Since $t \leq b^*$, we have $t \leq a^{**} \to (a \to b)^*$, i.e., $t \to [a^{**} \to (a \to b)^*] = 1$. Then

$$1 = [t \to a^{**}] \to [t \to (a \to b)^*] = 1 \to [t \to (a \to b)^*] = t \to (a \to b)^*.$$

Thus, $t \leq (a \to b)^*$.

(2) Let $a, b \in A$. Then from (1) above we get that

$$(a^* \to b)^* = a^{***} \wedge b^* = a^* \wedge b^* = (a \vee b)^*.$$

(2) Assume that $a \leq b_1^* \vee \cdots \vee b_n^*$. If $(b_1, \ldots, b_n; a^*) \neq 1$, there exist $P, Q \in X(A)$ such that $b_1, \ldots, b_n \in P$, $P \subseteq Q$, $a^* \notin P$ and $a \in Q$. So, $b_1^* \vee \cdots \vee b_n^*$, and as Q is prime, $b_i^* \in Q$ for some $1 \leq i \leq n$. Then $b_i, b_i^* \in Q$, and this implies that $0 \in Q$, which is impossible. ■

Remark 11. Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. As a consequence of the previous result we have that the join $\underline{\vee}$ in $R(A)$ can be defined by $a \underline{\vee} b = (a \vee b)^{**}$, for each pair $a, b \in R(A)$.

4. α-ideals and annihilators

If $\boldsymbol{A}$ is an $H^{\vee}$-algebra, then the usual notion of ideal in a join-semilattice coincides with the notion of order-ideal, i.e., a subset I of A is an ideal of $\langle A, \vee, 1\rangle$ iff I is an order-ideal of $\boldsymbol{A}$. Moreover, the ideal generated by a set X is the set $I(X) = \{a \in A \mid a \leq x_1 \vee \cdots \vee x_n$, for some $\{x_1, \ldots, x_n\} \subseteq X\}$.

Definition 12. Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Let I be an ideal of A. We shall say that I is an *α-ideal* if $a^{**} \in I$, whenever $a \in I$.

Let $\mathcal{I}_\alpha(\boldsymbol{A})$ be the set of all α-ideals of A. We note that as $a \leq a^{**}$, for all $a \in A$, an ideal I of A is an α-ideal iff $\forall a \in A$ ($a \in I$ iff $a^{**} \in I$).

Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Let $X \subseteq A$. It is easy to see that for any set $X \subseteq A$, the set

$$X^{\perp} = \bigcap \{(x^*] \mid x \in X\},$$

is an ideal of $\boldsymbol{A}$, called the *annihilator* of X. So, $X^{\perp\perp}$ is also an ideal. We shall say that an ideal I is an *annihilator ideal* if $I = I^{\perp\perp}$. When $X = \{a\}$, we write $a^{\perp}$ instead of $\{a\}^{\perp}$. We note that $a^{\perp} = (a^*]$. Let $\mathcal{I}^{\perp}(\boldsymbol{A})$ be the set of all annihilator ideals of A.

Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. An *order-filter* of $\boldsymbol{A}$ is a subset $F \subseteq A$ such that for each pair $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$. A proper ideal I of $\boldsymbol{A}$ is *irreducible* if for all ideals I_1 and I_2 such that $I = I_1 \cap I_2$, then $I = I_1$ or $I = I_2$. It is easy to prove that an ideal I is *irreducible* iff for every $a, b \notin I$ there exists $c \notin I$ and $i \in I$ such that $c \leq a \vee i$ and $c \leq b \vee i$ (see [6]). A proper ideal I is *prime* if $A - I = \{x \in A \mid x \notin I\}$ is an order-filter. We note that an ideal I is prime iff for any $a, b \in A$, if $(a] \cap (b] \subseteq I$ then $a \in I$ or $b \in I$. It is easy to check that every irreducible ideal is a prime ideal.

In the next two results we study the relation between α-ideals and annihilator ideals.

Lemma 13. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$.*

(1) $\mathcal{I}^{\perp}(\boldsymbol{A}) \subseteq \mathcal{I}_{\alpha}(\boldsymbol{A})$.

(2) *Let I be a prime ideal such that $I^{\perp} \neq \{0\}$.*

Then I is an α-ideal.

Proof. (1) Let $I \in \mathcal{I}^{\perp}(\boldsymbol{A})$. Let $a \in I = I^{\perp\perp} = \bigcap\left\{(x^*] \mid x \in I^{\perp}\right\}$. Let $x \in I^{\perp}$. Then $x \leq y^*$ for all $y \in I$. As $a \leq x^*$ implies that $a^{**} \leq x^*$, we have that $a^{**} \in I$, i.e., $I \in \mathcal{I}_{\alpha}(\boldsymbol{A})$.

(2) Since $I^{\perp} \neq \{0\}$, there exists $a \in I^{\perp} = \bigcap\{(x^*] \mid x \in I\}$ with $a \neq 0$. We prove that $I = (a^*]$. As $a \leq x^*$, for all $x \in I$, we have that $x \leq x^{**} \leq a^*$, for all $x \in I$. Thus, $I \subseteq (a^*]$. Let $x \leq a^*$. We prove that $(x] \cap (a] = \{0\}$. Suppose that $z \leq x$ and $z \leq a$. Then $z \leq a^* = a \to 0$. So, $z \leq 0$, i.e., $z = 0$. As $0 \in I$, we have $(x] \cap (a] = \{0\} \subseteq I$, and as I is prime, we obtain that $x \in I$ or $a \in I$. We note that $a \notin I$, because in contrary case, as $a \in I^{\perp}$, we have $a \leq a^*$, and this implies that $a = 0$, a contradiction. So, $x \in I$. Therefore $(a^*] = I$. Now, if $x \in I = (a^*]$, then $x^{**} \leq a^{***} = a^*$. So, $x^{**} \in I$, and thus I is an α-ideal. ∎

In the previous proposition we prove the inclusion $\mathcal{I}^{\perp}(\boldsymbol{A}) \subseteq \mathcal{I}_{\alpha}(\boldsymbol{A})$. In the next result we characterize when is valid the other inclusion.

Proposition 14. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Then the following conditions are equivalent:*

(1) $\mathcal{I}_{\alpha}(\boldsymbol{A}) \subseteq \mathcal{I}^{\perp}(\boldsymbol{A})$.

(2) $I^{\perp} \neq \{0\}$, *for each proper α-ideal I.*

(3) $I \cap D(A) \neq \emptyset$, *for each ideal I such that $I^{\perp} = \{0\}$.*

Proof. (1) $\Rightarrow$ (2). Suppose that there exists a proper α-ideal I such that $I^{\perp} = \{0\}$. As I is an annihilator ideal we have $I = I^{\perp\perp} = \{0\}^{\perp} = A$, a contradiction . Thus, $I^{\perp} \neq \{0\}$.

(2) $\Rightarrow$ (3). Let I be an ideal such that $I^{\perp} = \{0\}$. Suppose that $I \cap D(A) = \emptyset$. As $D(A)$ is a deductive system of $\boldsymbol{A}$, there exists $P \in X(A)$ such that $D(A) \subseteq P$ and $P \cap I = \emptyset$. From Lemma 9 we have that P is maximal. It is easy to see that $H = A - P$ is an α-ideal. As $I \subseteq H$, we have $H^{\perp} \subseteq I^{\perp} = \{0\}$. Then $H^{\perp} = \{0\}$, which is a contradiction because H is a proper ideal. Thus, $I \cap D(A) \neq \emptyset$.

(3) $\Rightarrow$ (1) Let I be an α-ideal. It is clear that $I \subseteq I^{\perp\perp}$. Let $a \in I^{\perp\perp}$. Let us consider the ideal $I \vee I^{\perp} = \left(I \cup I^{\perp}\right]$. Then it is not hard to prove that

$$\left(I \vee I^{\perp}\right)^{\perp} = I^{\perp} \cap I^{\perp\perp} = \{0\}.$$

So, $\left(I \vee I^{\perp}\right) \cap D(A) \neq \emptyset$. Thus there exists $d \in D(A)$, $x \in I$ and $y \in I^{\perp} = \bigcap\{(x^*] \mid x \in I\}$ such that $d \leq x \vee y$. As, $y \leq x^*$, we get $x \leq y^*$. So, $y^{**} \leq x^*$. On the other hand, $(x \vee y)^* = d^* = 0$. By this identity it is easy to see that $y^* \leq x^{**}$. Thus, $y^* = x^{**}$. We note that $x^{**} \in I$ because I is an α-ideal. As $a \in I^{\perp\perp}$, and $y \in I^{\perp}$, we obtain that $a \leq y^* = x^{**} \in I$. Thus $a \in I$. ■

Definition 15. Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. We shall say that a deductive system F is a *deductive order-filter* if it is an order-filter of $\boldsymbol{A}$.

In general, for a deductive system F, the decreasing set $((F)^*]$ is not an ideal. Now we prove that if F is also an order-filter, then $((F)^*]$ is an α-ideal.

Lemma 16. *Let $\boldsymbol{A}$ be an $H_0^{\vee}$-algebra. Let F be a proper deductive order-filter of $\boldsymbol{A}$. Then $((F)^*]$ is an α-ideal such that $((F)^*] \cap F = \emptyset$.*

Proof. Let $a, b \in ((F)^*]$. Then there are elements $x, y \in F$ such that $a \leq x^*$ and $b \leq y^*$. Since F is an order-filter, there exists $c \in F$ such that $c \leq x$ and $c \leq y$. Then $a \vee b \leq c^*$. Thus, $a \vee b \in ((F)^*]$. It is clear that $0 \in ((F)^*]$, because $0 = 1^*$. Let $a \in ((F)^*]$. Then $a \leq f^*$, for some $f \in F$. So, $a^{**} \leq f^*$. This shows that $a^{**} \in ((F)^*]$ and thus $((F)^*]$ is an α-ideal.

We prove that $((F)^*] \cap F = \emptyset$. Suppose that there exists $a \in ((F)^*] \cap F$. Then there exists $f \in F$ such that $a \leq f^*$. So, $f \leq f^{**} \leq a^*$, and this implies that $a^* = a \to 0 \in F$, and as $a \in F$ and it is a deductive system, $0 \in F$ which is a contradiction. ■

5. σ-ideals

Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. For each ideal I of A we consider the following set:

$$\sigma(I) = \{a \in A \mid (a^*] \vee I = A\}.$$

Lemma 17. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Let I be an ideal. Then $\sigma(I)$ is an ideal such that $\sigma(I) \subseteq I$.*

Proof. It is clear that $\sigma(I)$ is a decreasing subset of A such that $0 \in \sigma(I)$. Let $a, b \in \sigma(I)$, i.e., $(a^*] \vee I = A$ and $(b^*] \vee I = A$. As $1 \in A$, there exists $x, y \in I$ such that $a^* \vee x = 1$ and $b^* \vee y = 1$. Suppose that $a \vee b \notin \sigma(I)$, i.e., $((a \vee b)^*] \vee I \neq A$. So, there exists $c \in A$ such that $c \notin ((a \vee b)^*] \vee I$. From Lemma 3 there exists $P \in X(A)$ such that $(((a \vee b)^*] \vee I) \cap P = \emptyset$, and $c \in P$. So, $(a \vee b)^* \notin P$ and $I \cap P = \emptyset$. By Lemma 4 there exists $Q \in X(A)$ such that $P \subseteq Q$ and $a \vee b \in Q$. As $I \cap P = \emptyset$, we have that $x, y \notin P$. Then, $a^*, b^* \in P$. As $a \vee b \in Q$ and Q is prime, $a \in Q$ or $b \in Q$. In the first case we obtain that $a^*, a \in Q$, and as Q is a deductive system, $0 \in Q$, which is impossible. If $b \in Q$ then we obtain also a contradiction. Thus, $((a \vee b)^*] \vee I = A$, i.e., $a \vee b \in \sigma(I)$.

We prove that $\sigma(I) \subseteq I$. If $a \in \sigma(I)$ but $a \notin I$, there exists $P \in X(A)$ such that $P \cap I = \emptyset$ and $a \in P$. As $a^* \vee x = 1$ for some $x \in I$, we get that $a^* \in P$, which is a contradiction. Thus, $\sigma(I) \subseteq I$. ∎

Definition 18. Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Let I be an ideal of A. We shall say that I is a *σ-ideal* if $I = \sigma(I)$.

Now we prove that each σ-ideal is generated by a set $(F)^*$, where F is a deductive system. Recall that $I(X)$ denote the ideal of join-semilattice generated by the set X.

Proposition 19. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. For each ideal I there exists a deductive system F such that $\sigma(I) = I((F)^*)$.*

Proof. Let I be an ideal. Consider the set

$$F = \{a \in A \mid (a^{**}] \vee I = A\}.$$

We prove that F is a deductive system. Let $a, a \to b \in F$. Suppose that $(b^{**}] \vee I \neq A$, i.e., $1 \notin (b^{**}] \vee I$. Then there exists $P \in X(A)$ such that $((b^{**}] \vee I) \cap P = \emptyset$. Then, $b^{**} \notin P$ and $I \cap P = \emptyset$. As $1 = a^{**} \vee x$, and $1 = (a \to b)^{**} \vee y$, for some $x, y \in I$, we have $a^{**} \in P$ and $(a \to b)^{**} \in P$. From Lemma 10 we have that $(a \to b)^{**} \leq a^{**} \to b^{**} \in P$, and thus we obtain that $b^{**} \in P$, which is impossible. Therefore, F is a deductive system.

We prove that $I((F)^*) \subseteq \sigma(I)$. Let $x \in I((F)^*)$. Then there are elements $f_1, f_2, \ldots, f_n \in A$ such that $x \leq f_1^* \vee f_2^* \vee \cdots \vee f_n^*$ and $(f_i^{**}] \vee I = A$, for each $1 \leq i \leq n$. Then there exist $y_1, \ldots, y_n \in I$ such that

$$1 = f_1^{**} \vee y_1 = f_2^{**} \vee y_2 = \cdots = f_n^{**} \vee y_n.$$

We prove that $x^* \vee y_1 \vee \cdots \vee y_n = 1$. Suppose the contrary. Then there exists $P \in X(A)$ such that $x^* \notin P$ and $y_1 \vee \cdots \vee y_n \notin P$. Then there exists $Q \in X(A)$ such that $P \subseteq Q$ and $x \in Q$, and $f_i^{**} \in P$, for all $1 \leq i \leq n$. So, $f_1^* \vee f_2^* \vee \cdots \vee f_n^* \in Q$, and as Q is irreducible, $f_i^* \in Q$ for some $1 \leq i \leq n$. From f_i^*, $f_i^{**} \in Q$ we obtain that $0 \in Q$, which is impossible. Therefore, $x^* \vee y_1 \vee \cdots \vee y_n = 1$. As $y_1 \vee \ldots \vee y_n \in I$, we have that $x \in \sigma(I)$.

We prove that $\sigma(I) \subseteq I((F)^*)$. Let $x \in \sigma(I)$. Then $x^* \vee y = 1$ for some $y \in I$. So, $(x^*)^{**} \vee y = x^* \vee y = 1$. Then $x^* \in F$, and thus $x^{**} \in (F)^*$. Since $x \leq x^{**}$, we get that $x \in I((F)^*)$. ∎

Proposition 20. *Let $\boldsymbol{A} \in \mathcal{H}_0^\vee$. For each σ-ideal I there exists a maximal deductive system U such that $I \cap U = \emptyset$.*

Proof. Let I be a σ-ideal. By Proposition 19 there exists a deductive system F such that $I((F)^*) = I$. We prove that $I((F)^*) \cap F = \emptyset$. Suppose that there exists $a \in I((F)^*) \cap F$. Then there exists $f_1, \dots, f_n \in F$ such that $a \leq f_1^* \vee f_2^* \vee \cdots \vee f_n^*$. By Lemma 10, $(f_1, \dots, f_n; a^*) = 1 \in F$, and this implies that $a^* \in F$. As $a \in F$, we have that $0 \in F$, which is a contradiction. Then, $I((F)^*) \cap F = \emptyset$. By Theorem 3 there exists $P \in X(A)$ such that $F \subseteq P$ and $((F)^*] \cap P = \emptyset$. We prove that P is maximal. Let $a \notin P$. We prove that $a^* \in P$. As $\langle P \cup \{a\}\rangle \cap I((F)^*) \neq \emptyset$, there exists $b \in A$ such that $a \to b \in P$, and there exists $f_1, \dots, f_n \in F$ such that $b \leq f_1^* \vee f_2^* \vee \cdots \vee f_n^*$. From Lemma 10, $(f_1, \dots, f_n; b^*) = 1 \in F$, and this implies that $b^* \in F \subseteq P$. As $a \to b \leq b^* \to a^* \in P$, we have that $a^* \in P$. By Lemma 9 we deduce that P is maximal. ∎

6. Stone $\boldsymbol{H}_0^\vee$-algebras

Definition 21. Let $\boldsymbol{A} \in \mathcal{H}_0^\vee$. We shall say that $\boldsymbol{A}$ is a *Stone $H_0^\vee$-algebra* if $\boldsymbol{A}$ satisfies the identity $a^* \vee a^{**} = 1$.

Now, we characterize the σ-ideals in a Stone $H_0^\vee$-algebra.

Lemma 22. *Let $\boldsymbol{A}$ be a Stone $H_0^\vee$-algebra. Then $I((F)^*)$ is a σ-ideal for any deductive system F.*

Proof. Let F a deductive system. Let $I = I((F)^*)$. By Lemma 17, $\sigma(I) \subseteq I$. We need to prove the inclusion $I \subseteq \sigma(I)$. Let $a \in I$. Then there are elements $f_1, f_2, \dots, f_n \in F$ such that $a \leq f_1^* \vee f_2^* \vee \cdots \vee f_n^*$. From Lemma 10 we have

$$a^* \in \langle f_1, f_2, \dots, f_n\rangle \subseteq F.$$

Then, $a^{**} \in (F)^* \subseteq I((F)^*)$. So, $1 = a^* \vee a^{**} \in (a^*] \vee I$. Thus, $(\neg a] \vee I = A$. ∎

Corollary 23. *Let $\boldsymbol{A}$ be a Stone $H_0^\vee$-algebra. An ideal I is a σ-ideal if and only if there exists a deductive system F such that $I = I((F)^*)$.*

Proof. If I is a σ-ideal, then by Proposition 19 there exists a deductive system F such that $I = \sigma(I) = I((F)^*)$. Conversely, if there exists a deductive system F such that $I = I((F)^*)$, then by Lemma 22, I is a σ-ideal. ∎

Now we give different characterizations of Stone $H_0^\vee$-algebras, but first we need see some concepts.

Lemma 24. *Let $\boldsymbol{A} \in \mathcal{H}_0$. Let I be an order-ideal. Then*

$$I_* = \{a \in A \mid \exists x \in I \ (x^* \leq a)\}$$

is a deductive system of A.

Proof. Let $a, a \to b \in I_*$. Then there exist $x, y \in I$ such that $x^* \leq a$ and $y^* \leq a \to b$. As I is an order-ideal there exists $z \in I$ such that $x, y \leq z$. Then $z^* \leq a$, $z^* \leq a \to b$. As $[z^*)$ is a deductive system, $z^* \leq b$. Thus I_* is a deductive system. ∎

Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Recall that if D is a deductive system, then

$$\theta(D) = \{(a, b) \mid a \to b, b \to a \in D\}$$

is the congruence generated by D. Let I be an ideal of $\boldsymbol{A}$. We define an equivalence relation $\theta_{\vee}(I)$ on A by:

$$(a, b) \in \theta_{\vee}(I) \text{ iff } \exists x \in I \ (a \vee x = b \vee x).$$

It is clear that $\theta_{\vee}(I)$ satisfies the substitution property for the operation $\vee$, i.e. $\theta_{\vee}(I)$ is a $\vee$-congruence, but in general $\theta_{\vee}(I)$ is not a congruence of Hilbert algebras.

Lemma 25. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Then $\theta_{\vee}(I) \subseteq \theta(I_*)$, for any ideal I.*

Proof. Let $(a, b) \in \theta_{\vee}(I)$. Then there exists $x \in I$ such that $a \vee x = b \vee x$. Assume that $(a, b) \notin \theta(I_*)$. Then it is easy to see that there exists $P \in X(A)$ such that $a \in P$, $I_* \subseteq P$, and $b \notin P$. As $x \in I$, we have $x^* \in I_* \subseteq P$. Then $x \notin P$. As $a \leq a \vee x = b \vee x \in P$, we obtain that $b \in P$, which is absurd. Thus, $\theta_{\vee}(I) \subseteq \theta(I_*)$. ∎

In the next theorem we prove that the class of Stone $H_0^{\vee}$-algebras can be characterized as $H_0^{\vee}$-algebras where $\theta(I_*) \subseteq \theta_{\vee}(I)$, for each α-ideal I, and consequently the relation $\theta_{\vee}(I)$ is a congruence of Hilbert algebras.

Theorem 26. *Let $\boldsymbol{A} \in \mathcal{H}_0^{\vee}$. Then the following conditions are equivalent:*

(1) *$\theta(I_*) \subseteq \theta_{\vee}(I)$, for each α-ideal I.*
(2) *$\boldsymbol{A}$ is a Stone $H_0^{\vee}$-algebra.*
(3) *For all $a, b \in A$, and for all $P \in X(A)$, if $\langle\{a, b\}\rangle = A$, then $a^* \in P$ or $b^* \in P$.*
(4) *For all $a, b \in A$, if $\langle\{a, b\}\rangle = A$, then $(a \to b) \vee (b \to a) = 1$.*
(5) *Each irreducible deductive system P is contained in a unique maximal deductive system.*
(6) *For any increasing subsets $U, V \subseteq X(A)$, we have $(U] \cap (V] = (U \cap V]$.*
(7) *$a^* \vee b^* = a \to b^*$, for all $a, b \in A$.*
(8) *$I = \sigma(I)$ for every α-ideal I.*

Proof. (1) $\Rightarrow$ (2). Let $a \in A$. Let $I = (a^{**}]$. It is clear that I is an α-ideal. As $a^* \leq a^*$ and $a \in (a^{**}]$, we have that $(a^*, 1) \in \theta(I_*) = \theta_\vee(I)$. So, there exists $x \in I = (a^{**}]$ such that $a^* \vee x = 1 \vee x = 1$. Since $x \leq a^{**}$, we get $a^* \vee a^{**} = 1$. Thus $\boldsymbol{A}$ satisfies the Stone identity.

(2) $\Rightarrow$ (3) Let $a, b \in A$, and $P \in X(A)$. Suppose that $\langle\{a, b\}\rangle = A$. As $0 \in \langle a, b\rangle$, $a \leq b^*$. If $a^* \notin P$, then $a^{**} \in P$. As $a^{**} \leq b^*$, because $0 \in \langle a, b\rangle$, we get $b^* \in P$.

(3) $\Rightarrow$ (4) Let $a, b \in A$. Suppose that $\langle\{a, b\}\rangle = A$. If $(a \to b) \vee (b \to a) \neq 1$, then there exists $P \in X(A)$ such that $a \to b \notin P$ and $b \to a \notin P$. By hypothesis, $a^* \in P$ or $b^* \in P$, and as $a \leq a \to b$, and $b^* \leq b \to a$, we have $a \to b \in P$ or $b \to a \in P$, which is a contradiction. Thus, $(a \to b) \vee (b \to a) = 1$.

(4) $\Rightarrow$ (5) Let $P \in X(A)$. Suppose that there exist maximal deductive systems U_1, U_2 of A such that $P \subseteq U_1$, $P \subseteq U_2$, and $U_1 \neq U_2$. Then there exists $a \in U_1 - U_2$. As U_1, U_2 are maximal deductive systems, $a^* \notin U_1$ and $a^* \in U_2$. Since $\langle\{a, a^*\}\rangle = A$, $(a \to a^*) \vee (a^* \to a) = 1 \in P$. So, if $a \to a^* \in P$, then $a^* \in U_1$, which is absurd. If $a^* \to a \in P$, then $a \in U_2$, which also is absurd. Therefore, $U_1 = U_2$.

(5) $\Rightarrow$ (6) Let U, V be increasing subsets of $X(A)$. Let $P \in (U] \cap (V]$. Then there exists $Q \in U$ and there exists $D \in V$ such that $P \subseteq Q$ and $P \subseteq D$. Let M_1 and M_2 be maximal deductive systems such that $Q \subseteq M_1$ and $D \subseteq M_2$. As $P \subseteq M_1$ and $P \subseteq M_2$, by hypothesis we get $M_1 = M_2$. Since U and V are increasing, $M_1 \in U \cap V$. Thus, $P \in (U \cap V]$. The inclusion $(U \cap V] \subseteq (U] \cap (V]$ is trivial.

(6) $\Rightarrow$ (7) Suppose that there exist $a, b \in A$ such that $a \to b^* \nleq a^* \vee b^*$. So there exists $P \in X(A)$ such that $a \to b^* \in P$, and $a^* \notin P$, and $b^* \notin P$. So $P \in (\varphi(a)] \cap (\varphi(b)] = (\varphi(a) \cap \varphi(b)]$. Then there exists $Q \in X(A)$ such that $P \subseteq Q$ and $a, b \in Q$. As $a \to b^* \in P \subseteq Q$, we have that $b^* \in Q$, which is a contradiction. Thus, $a \to b^* \leq a^* \vee b^*$. On the other hand, as $a^* = a \to 0 \leq a \to b^*$ and as $b^* \leq a \to b^*$, we get that $a^* \vee b^* \leq a \to b^*$.

(7) $\Rightarrow$ (8). Let I be an α-ideal. We prove that $I \subseteq \{a \in I \mid (a^*] \vee I = A\}$. Let $a \in I$. As I is an α-ideal, $a^{**} \in I$. Then $a^* \vee a^{**} = a \to a^{**} = 1$. So $1 \in (a^*] \vee I$, i.e., $(a^*] \vee I = A$. The other inclusion is immediate.

(8) $\Rightarrow$ (1). We prove the inclusion $\theta(I_*) \subseteq \theta_\vee(I)$. Let $(a, b) \in \theta(I_*)$. As I is an ideal, there exists $z \in I$ such that $z^* \leq a \to b$, $z^* \leq b \to a$. As $(z^*] \vee I = A$, there exists $x \in I$ such that and $x \vee z^* = 1$. We prove that $a \vee x = b \vee x$. Suppose that there exists $P \in X(A)$ such that $a \vee x \in P$ and $b \vee x \notin P$. Then $a \in P$, and $b, x \notin P$. As P is irreducible, and $x \vee z^* = 1 \in P$, $z^* \in P$. So, $a \to b \in P$, and consequently $b \in P$, which is an absurd. Thus $(a, b) \in \theta_\vee(I)$. ∎

Let us recall that a join semilattice $\langle A, \vee\rangle$ is *distributive* if for all $a, b, c \in A$

such that $c \leq a \vee b$ there exist $a_1, b_1 \in A$ such that $a_1 \leq a$, $b_1 \leq b$ and $c = a_1 \vee b_1$. In [15] it was proved that a join semilattice $\langle A, \vee \rangle$ is distributive iff the set of all ideals is a distributive semilattice. Moreover, in [6] it was proved that a join semilattice $\langle A, \vee \rangle$ is distributive iff every irreducible ideal is prime.

Definition 27. We shall say that an $H_0^{\vee}$-algebra $\boldsymbol{A} = \langle A, \to, \vee, 0, 1 \rangle$ is *distributive* if the join semilattice $\langle A, \vee \rangle$ is distributive.

We finish giving a characterization of the Stone $H_0^{\vee}$-algebras when the join semilattice $\langle A, \vee \rangle$ is distributive. First, we recall the following result.

Proposition 28. [15] *Let $\boldsymbol{A}$ be a distributive $H_0^{\vee}$-algebra. Let F be an order-filter and I be an ideal (α-ideal) such that $F \cap I = \emptyset$. Then there exists a prime ideal (prime α-ideal) J such that $F \cap J = \emptyset$ and $I \subseteq J$.*

Lemma 29. *If $\boldsymbol{A}$ is a Stone $H_0^{\vee}$-algebra, then $((A - I)^*]$ is a prime ideal for each prime α-ideal I.*

Proof. Let I be a prime α-ideal. Let $H = A - I$. We prove that H is a deductive system. It is clear that $1 \in H$. Let $a, a \to b \in H$. Since I is prime, there exists $z \notin I$ such that $z \leq a$ and $z \leq a \to b$, i.e., $a, a \to b \in [z)$. Then $b \in [z)$, i.e., $z \leq b$, because $[z)$ is a deductive system. Thus, H is a deductive system. As H is an order-filter, we have H is a deductive order-filter. By Lemma 16 we have that $((H)^*)]$ is an ideal such that $((H)^*] \subseteq I$.

We prove that $((H)^*]$ is prime. Let $a, b \in A$ such that $(a] \cap (b] \subseteq ((H)^*]$. As $((H)^*] \subseteq I$ and I is prime, $a \in I$ or $b \in I$. Assume that $a \in I$. As I is an α-ideal, $a^{**} \in I$, and as $a^{**} \vee a^* = 1 \notin I$, we have that $a^* \notin I$. Then $a^{**} \in (H)^*$. As $a \leq a^{**}$, we obtain that $a \in ((H)^*]$. If we assume that $b \in I$, then we will deduce that $b \in ((H)^*]$. Thus, $((H)^*]$ is a prime ideal. ∎

Theorem 30. *Let $\boldsymbol{A}$ be a distributive $H_0^{\vee}$-algebra. Then the following conditions are equivalent:*

(1) *$\boldsymbol{A}$ is a Stone $H_0^{\vee}$-algebra.*
(2) *$((A - I)^*]$ is a prime ideal for each prime α-ideal I.*

Proof. The direction (1) $\Rightarrow$ (2) follows by the previous lemma. We prove (2) $\Rightarrow$ (1). Suppose that there exists $a \in A$ such that $a^* \vee a^{**} \neq 1$. Let us consider the ideal $(a^* \vee a^{**}]$. As $(a^* \vee a^{**})^{**} = a^* \vee a^{**}$, we have that I is a proper α-ideal. Then by Proposition 28 there exists a prime α-ideal J such that $I \subseteq J$. We note that a^*, $a^{**} \in J$. By hypothesis the ideal $((A - J)^*]$ is a prime ideal. It is clear that $((A - J)^*] \subseteq J$. We prove that $a, a^* \notin ((A - J)^*)]$. If $a \in ((A - J)^*]$, then there exists $d \notin J$ such that $a \leq d^*$. So, $d \leq d^{**} \leq a^*$, and as $a^* \in J$ and J is decreasing, we obtain that $d \in J$, which is a contradiction. If $a^* \in ((A - J)^*]$, then there exists $g \notin J$ such that $a^* \leq g^*$. So, $g \leq g^{**} \leq a^{**}$, and as $a^{**} \in H$, we obtain that $g \in J$, which is an absurd. Thus $a, a^* \notin ((A - J)^*]$. Since $((A - J)^*]$ is prime, there exists $z \notin ((A - J)^*]$ such that $z \leq a$ and $z \leq a^*$. But this implies that $z \leq 0$. Then $z = 0 \notin ((A - J)^*]$, which is impossible, because $((A - J)^*]$ is an ideal. Therefore, $a^* \vee a^{**} = 1$, for all $a \in A$. ■

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