Linear Algebra and its Applications 439 (2013) 1730–1741

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Products of projections and positive operators

M. Laura Arias [a,1], Gustavo Corach [a,b,*,2], M. Celeste Gonzalez [a,c,3]

[a] Instituto Argentino de Matemática "Alberto P. Calderón", Saavedra 15, Piso 3, 1083 Buenos Aires, Argentina
[b] Dpto. de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires, Argentina
[c] Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina

ARTICLE INFO

Article history:
Received 17 April 2013
Accepted 11 May 2013
Available online 5 June 2013
Submitted by A. Böttcher

Keywords:
Projections
Positive operators
Polar decomposition
Partial isometries
Compatibility

ABSTRACT

This article is devoted to the study of the set $\mathcal{T}$ of all products PA with P an orthogonal projection and A a positive (semidefinite) operator. We describe this set and study optimal factorizations. We also relate this factorization with the notion of compatibility and explore the polar decomposition of the operators in $\mathcal{T}$.

1. Introduction

Given two classes of operators $\mathcal{M}$ and $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space), a problem which naturally arises is that of characterizing the set $\mathcal{M} \cdot \mathcal{B}$ of all products AB, $A \in \mathcal{M}$, $B \in \mathcal{B}$. These problems are as old as matrix theory and they form now an interesting part of factorization theory for matrices and operators. In 1958 Chandler Davis [8, Theorem 6.3] proved that, if $\mathcal{I}$ denotes the set of Hermitian involutions (i.e., $T = T^* = T^{-1}$) then $\mathcal{I} \cdot \mathcal{I}$ coincides with all unitaries T such that T is similar to T^{-1}. H. Radjavi and J.P. Williams [21] proved later that $\mathcal{I} \cdot \mathcal{L}^h$, where $\mathcal{L}^h$ denotes the set of Hermitian operators on $\mathcal{H}$, is the set of all $T \in \mathcal{L}(\mathcal{H})$ such that T is unitarily equivalent to T^{-1}. Their paper

* Corresponding author at: IAM-CONICET, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina.
E-mail addresses: lauraarias@conicet.gov.ar (M.L. Arias), gcorach@fi.uba.ar (G. Corach), celeste.gonzalez@conicet.gov.ar (M.C. Gonzalez).
1 Partially supported by UBACYT 20020100100250; 2011–2014.
2 Partially supported by PICT 808-2008 (FONACYT) and UBACYT 20020100100250; 2011–2014.
3 Partially supported by PIP 11220090100757 (CONICET).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.05.008

also contains a characterization of $\mathcal{P}\cdot\mathcal{P}$ due to T. Crimmins and a characterization of $\mathcal{P}\cdot\mathcal{L}^h$ (here, $\mathcal{P}$ denotes the set of all orthogonal projectors of $\mathcal{L}(\mathcal{H})$). Other characterizations of $\mathcal{P}\cdot\mathcal{P}$ have been found by S. Nelson and M. Neumann [17], A. Arias and S. Gudder [1], T. Oikhberg [18] and the second author and A. Maestripieri [6]. In a series of papers, J.R. Holub [14–16] (see also Fujii and Furuta [12]) studied, as an approach to general Wiener–Hopf or Toeplitz operators, some properties of the class $\mathcal{P}\cdot\mathcal{G}^+ = \{PA\colon\ P\in\mathcal{P}$ and $A\in\mathcal{L}^+$ is invertible$\}$, where $\mathcal{L}^+$ denotes the cone of positive semidefinite operators in $\mathcal{L}(\mathcal{H})$. They observed that the set $\mathcal{Q}$ of oblique (i.e., not necessarily orthogonal) projections in $\mathcal{L}(\mathcal{H})$ is contained in $\mathcal{P}\cdot\mathcal{G}^+$.

In this paper, we characterize operators in $\mathcal{T} := \mathcal{P}\cdot\mathcal{L}^+$. We extend several results on $\mathcal{P}\cdot\mathcal{P}$ and Holub's theorem that $\mathcal{Q}$ is contained in $\mathcal{P}\cdot\mathcal{G}^+$. It should be noticed that $\mathcal{Q}$ is not contained in $\mathcal{P}\cdot\mathcal{P}$, but it is contained in $(\mathcal{P}\cdot\mathcal{P})^\dagger$, the set of all Moore–Penrose inverses of products PQ, $P, Q\in\mathcal{P}$. This is an old result by Penrose [20] and Greville [13] which has been extended to the infinite dimensional case in [5] and [4]. The paper [21] by H. Radjavi and J. Williams and the survey [25] by P.Y. Wu contain many characterizations of classes of the type $\mathcal{M}\cdot\mathcal{B}$.

One of the main features of the class $\mathcal{P}\cdot\mathcal{L}^+$ is that their elements admit a particular polar decomposition where the partial isometry is an orthogonal projection. In fact, for $T\in\mathcal{T}$, any factorization $T = PA$, with $P\in\mathcal{P}$ and $A\in\mathcal{L}^+$ provides one such polar decomposition. Among all these expressions, we find one (the optimal factorization) with some relevant minimal properties. The main characterization of $\mathcal{T}$ is based on a result of Z. Sebestyén [22]. We include a proof, which is completely different from the original one, because it illustrates how the classical majorization theorem of R.G. Douglas [10,11] can be used to provide special solutions of some operator equations. In fact, if $T\in\mathcal{T}$ and P is the orthogonal projection onto the closure of the image of T, then the positive solutions of the equation $PX = T$ play a natural role in this paper.

The contents of the paper are the following. Section 2 contains notations and the statements of some theorems by Crimmins [11, Theorem 2.2], Douglas [10, Theorem 1] and Sebestyén [22]. We include a proof of the last one based on Douglas' theorem. Section 3 is devoted to several properties of the set $\mathcal{T}$ and different characterizations of its elements. Just to mention two of them, $T\in\mathcal{L}(\mathcal{H})$ belongs to $\mathcal{T}$ if and only if there exists $\lambda\geqslant 0$ such that $(T^*T)^2\leqslant\lambda T^*T^2$ (Theorem 3.2). If $R(T)$ is closed then $T\in\mathcal{T}$ if and only if $R(T)\dotplus N(T) = \mathcal{H}$ and $TP\in\mathcal{L}^+$, where $P = P_{R(T)}$ (Theorem 3.3). A formula for the oblique projection onto $R(T)$ with nullspace $N(T)$ is exhibited at Section 4, where a particular factorization of $T\in\mathcal{T}$ is shown to have several optimal properties. For instance, if $T\in\mathcal{T}$ then there exist $P_T\in\mathcal{P}$ and $A_T\in\mathcal{L}^+$ such that $T = P_TA_T$ and $P_T\leqslant P$ and $A_T\leqslant A$ for all $P\in\mathcal{P}$, $A\in\mathcal{L}^+$ such that $T = PA$. The last result of Section 4 is the characterization of the fiber of $T\in\mathcal{T}$ by the map $(P, A)\to PA$, i.e., we find all pairs $(P, A)\in\mathcal{P}\times\mathcal{L}^+$ such that $PA = T$. In Section 5 we relate the different factorizations of $T\in\mathcal{T}$ with the notions of compatibility and quasi-compatibility between positive operators and closed subspaces. It turns out that, if $T\in\mathcal{T}$ and $T = PA$ for some $P = P_{\mathcal{S}}\in\mathcal{P}$ and $A\in\mathcal{L}^+$, then the pair $(A, \mathcal{S})$ is compatible if and only if $\mathcal{H} = \overline{R(T)}\dotplus N(T)$. The last section studies some properties of the standard polar decomposition of $T\in\mathcal{T}$.

2. Preliminaries

Throughout $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{K}$ denote separable complex Hilbert spaces. By $\mathcal{L}(\mathcal{H},\mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The algebra $\mathcal{L}(\mathcal{H},\mathcal{H})$ is abbreviated by $\mathcal{L}(\mathcal{H})$. By $\mathcal{L}(H)^+$ we denote the cone of positive (semidefinite) operators of $\mathcal{L}(\mathcal{H})$ i.e., $T\in\mathcal{L}(\mathcal{H})^+$ if and only if $\langle Tx, x\rangle\geqslant 0$ for all $x\in\mathcal{H}$. Furthermore, $\mathcal{G}(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$ and $\mathcal{CR}(\mathcal{H})$ the set of closed range operators on $\mathcal{H}$. When no confusion can arise, we omit the Hilbert space and we write it simply $\mathcal{L}^+, \mathcal{G}$ and $\mathcal{CR}$ respectively. Moreover, we denote $\mathcal{G}^+ = \mathcal{G}\cap\mathcal{L}^+$. Given $T\in\mathcal{L}(\mathcal{H},\mathcal{K})$, $R(T)$ denotes the range or image of T, $N(T)$ the nullspace of T, T^* the adjoint of T and $T^\dagger$ the Moore–Penrose inverse of T. Recall that $T^\dagger\in\mathcal{L}(\mathcal{K},\mathcal{H})$ if and only if $R(T)$ is closed. We shall denote by $\mathcal{Q} = \{Q\in\mathcal{L}(\mathcal{H})\colon\ Q = Q^2\}$ and $\mathcal{P} = \{P\in\mathcal{Q}\colon\ P = P^*\}$. Moreover, fixed a closed subspace $\mathcal{S}$, $P_{\mathcal{S}}$ stands for the orthogonal projection onto $\mathcal{S}$. In the sequel we denote by $\mathcal{S}\dotplus\mathcal{W}$ the direct sum of the subspaces $\mathcal{S}$ and $\mathcal{W}$. In particular, if $\mathcal{S}\subseteq\mathcal{W}^\perp$ we denote $\mathcal{S}\oplus\mathcal{W}$.

We end this section by stating three important results that we will frequently use along this article.

Theorem 2.1. *(See [11, Theorem 2.2].) If $A, B \in \mathcal{L}(\mathcal{H})$ then $R(A) + R(B) = R((AA^* + BB^*)^{1/2})$.*

Theorem 2.2. *(See Douglas, [10].) Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{L}(\mathcal{F}, \mathcal{K})$. The following conditions are equivalent:*

1. *$R(B) \subseteq R(A)$.*
2. *There is a positive number λ such that $BB^* \leqslant \lambda AA^*$.*
3. *There exists $C \in \mathcal{L}(\mathcal{F}, \mathcal{H})$ such that $AC = B$.*

*If one of these conditions holds then there is a unique operator $D \in \mathcal{L}(\mathcal{F}, \mathcal{H})$ such that $AD = B$ and $R(D) \subseteq N(A)^{\perp}$. We shall call D the **reduced solution** of $AX = B$. Moreover, $N(D) = N(B)$.*

The following result due to Sebestyén will be crucial along this article. Here, we present a different proof by means of Douglas' theorem.

Theorem 2.3. *(See [22].) Let $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. The equation $AX = B$ has a positive solution if and only if $BB^* \leqslant \lambda AB^*$ for some $\lambda \geqslant 0$.*

Proof. Let Y be a positive solution of $AX = B$. Since $R(AY) \subseteq R(AY^{1/2})$ we obtain, by Douglas' theorem, that $BB^* = AYYA^* \leqslant \lambda AY^{1/2}Y^{1/2}A^* = \lambda AYA^* = \lambda AB^*$ for some $\lambda \geqslant 0$.

Conversely, if $BB^* \leqslant \lambda AB^*$ for some $\lambda \geqslant 0$ then, by Douglas' theorem, there exists $D \in \mathcal{L}(\mathcal{H})$ such that $(AB^*)^{1/2}D = B$, $R(D) \subseteq N((AB^*)^{1/2})^{\perp}$ and $N(D) = N(B)$. Then,

$$\left(AB^*\right)^{1/2} DA^* = BA^* = \left(AB^*\right)^{1/2}\left(AB^*\right)^{1/2}. \tag{1}$$

Therefore, DA^* and $(AB^*)^{1/2}$ are both the reduced solution of $(AB^*)^{1/2}X = BA^*$. Thus, by the uniqueness of the reduced solution, we get that $DA^* = (AB^*)^{1/2}$. So $AD^*D = (AB^*)^{1/2}D = B$, i.e., $Y = D^*D \in \mathcal{L}^+$ is solution of $AX = B$ and the result is obtained. □

Corollary 2.4. *If the operator equation $AX = B$ has a positive solution then there exists $Y \in \mathcal{L}(\mathcal{H})^+$ such that $AY = B$ and $N(Y) = N(B)$.*

Proof. Let D be the reduced solution of $(AB^*)^{1/2}X = B$. Then, by the proof of Theorem 2.3, $Y = D^*D$ is a positive solution of $AX = B$ with $N(Y) = N(B)$. □

3. The set $\mathcal{T}$

This section is devoted to the study of the set defined as

$$\mathcal{T} := \mathcal{P} \cdot \mathcal{L}^+ = \left\{T \in \mathcal{L}(\mathcal{H})\colon\ T = PA \text{ with } P \in \mathcal{P} \text{ and } A \in \mathcal{L}^+\right\}.$$

As we mentioned, the subclass $\mathcal{P}\cdot\mathcal{P}$ has been studied in [6] where several properties of this set have been provided. However, it must be noted that many properties of $\mathcal{P}\cdot\mathcal{P}$ are not longer valid in $\mathcal{T}$. For instance, given $T \in \mathcal{P}\cdot\mathcal{P}$, it holds that $T \in \mathcal{CR}$ if and only if $\mathcal{H} = \overline{R(T)} + N(T)$ (see [6, Theorem 3.2]). Now, this characterization is not true if $T \in \mathcal{T}$. Indeed, consider $T \in \mathcal{L}^+$ with non-closed range then $T \in \mathcal{T}$ and $\mathcal{H} = \overline{R(T)} + N(T)$. Moreover, both sets have different topological properties. For example, $\mathcal{P}\cdot\mathcal{P}$ is closed but $\mathcal{T}$ is not. In fact, $T_n = \left[\begin{smallmatrix} 1/n & 1 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1/n & 1 \\ 1 & n \end{smallmatrix}\right] \in \mathcal{T}$. However, $\lim_{n\to\infty} T_n = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \notin \mathcal{T}$ (see Theorem 3.2).

In what follows, given $T \in \mathcal{T}$ we denote

$$\mathcal{T}_T^+ := \left\{A \in \mathcal{L}(\mathcal{H})^+\colon\ \exists P \in \mathcal{P} \text{ such that } T = PA\right\}$$

and

$$\mathcal{T}_T^{\mathcal{P}} := \left\{P \in \mathcal{P}\colon\ \exists A \in \mathcal{L}(\mathcal{H})^+ \text{ such that } T = PA\right\}.$$

In the following lemma we collect some properties of $\mathcal{T}$.

Lemma 3.1. *Let $T \in \mathcal{T}$. Then, the following conditions hold*:

1. $P_{\overline{R(T)}} \in \mathcal{T}_T^{\mathcal{P}}$.
2. *The spectrum of T, $\sigma(T)$, is positive.*
3. $T^n \in \mathcal{T}$ *for all* $n \in \mathbb{N}$.
4. $T \in \mathcal{G}$ *if and only if* $T \in \mathcal{G}^+$.
5. *if* $TT^* = T^*T$ *then* $T \in \mathcal{L}^+$.
6. $\overline{R(T)} \cap N(T) = \{0\}$, *i.e.,* $N(T^*) + \overline{R(T^*)}$ *is a dense subspace of* $\mathcal{H}$.
7. $R(T^*) \cap N(T^*) = \{0\}$ *but, in general,* $\overline{R(T^*)} \cap N(T^*) \neq \{0\}$ *(i.e.,* $\overline{R(T)} + N(T)$ *is not dense, in general). As a consequence, in general,* $T^* \notin \mathcal{T}$.

Proof. 1. Since $T \in \mathcal{T}$, then $T = P_{\mathcal{S}}A$ for some $P_{\mathcal{S}} \in \mathcal{P}$ and $A \in \mathcal{L}^+$. Therefore, $R(T) \subseteq \mathcal{S}$ and as $\mathcal{S}$ is a closed subspace, then $\overline{R(T)} \subseteq \mathcal{S}$. Hence, $T = P_{\overline{R(T)}}T = P_{\overline{R(T)}}P_{\mathcal{S}}A = P_{\overline{R(T)}}A$ and so $P_{\overline{R(T)}} \in \mathcal{T}_T^{\mathcal{P}}$.

2. Let $T = PA$ then $\sigma(T) = \sigma(PA) = \sigma(A^{1/2}PA^{1/2}) \geqslant 0$.

3. Let $T = PA \in \mathcal{T}$ and $k \in \mathbb{N}$. Then, $T^{2k} = (PA)^{2k} = P(AP)^k(PA)^k = P(T^*)^kT^k \in \mathcal{T}$. On the other side, $T^{2k+1} = TT^{2k} = PAP(T^*)^kT^k = P(T^*)^{k+1}T^k = P(T^*)^kAPT^k = P((T^*)^kAT^k) \in \mathcal{T}$. Then the assertion follows.

4. If $T \in \mathcal{G}$ then, by item 1, $I \in \mathcal{T}_T^{\mathcal{P}}$ and so $T \in \mathcal{G}^+$.

5. Applying item 2, we have that T is a normal operator with $\sigma(T) \geqslant 0$, then $T \in \mathcal{L}^+$.

6. Let $T = PA$ and $x \in \overline{R(T)} \cap N(T)$. Since $R(T) \subseteq R(P)$ then $PAPx = 0$, i.e., $A^{1/2}Px = 0$ and so $APx = T^*x = 0$. Thus, $x \in \overline{R(T)} \cap N(T^*) = \{0\}$ and the result is obtained.

7. Let $T = PA$ and $z \in R(T^*) \cap N(T^*)$. Hence, $z = APx$ for some $x \in \mathcal{H}$ and $APz = 0$. Thus, $APAPx = 0$, and so $PAPAPx = 0$. Hence, $PAPx = 0$ and so $A^{1/2}Px = 0$. Therefore, $APx = z = 0$.

For the second part, consider $A \in \mathcal{L}^+$ with non-closed range and $x \in \overline{R(A)} \setminus R(A)$. Define $\mathcal{S} = \operatorname{span}\{x\}^{\perp}$ and $T = P_{\mathcal{S}}A$. Clearly, $T \in \mathcal{T}$ and $N(T) = N(A)$. Thus, $\overline{R(T^*)} = \overline{R(A)}$ and $\{0\} \neq \operatorname{span}\{x\} = \mathcal{S}^{\perp} \cap \overline{R(A)} = \mathcal{S}^{\perp} \cap \overline{R(T^*)} \subseteq N(T^*) \cap \overline{R(T^*)}$. □

In [21], it is proven that $T \in \mathcal{P} \cdot \mathcal{L}^h$ if and only if T^*T^2 is selfadjoint. In particular, this shows that $\mathcal{P} \cdot \mathcal{L}^h$ is closed; recall that $\mathcal{T} = \mathcal{P} \cdot \mathcal{L}^+$ is not. It is natural to ask if a necessary and sufficient condition for $T \in \mathcal{T} = \mathcal{P} \cdot \mathcal{L}^+$ is that T^*T^2 be positive. The next result proves that the answer is negative, and that a stronger condition is needed.

Theorem 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ and $P = P_{\overline{R(T)}}$. The following conditions are equivalent*:

1. $T \in \mathcal{T}$;
2. $TT^* \leqslant \lambda TP$ *for some* $\lambda \geqslant 0$;
3. $TP \in \mathcal{L}^+$ *and* $R(T(I-P)) \subseteq R((TP)^{1/2})$;
4. $(T^*T)^2 \leqslant \lambda T^*T^2$ *for some* $\lambda \geqslant 0$.

Proof. $1 \Rightarrow 2$. If $T \in \mathcal{T}$ then the equation $T = PX$ has a positive solution and so, by Theorem 2.3, $TT^* \leqslant \lambda TP$ for some $\lambda \geqslant 0$.

$2 \Rightarrow 3$. If $TT^* \leqslant \lambda TP$ for some $\lambda \geqslant 0$ then $TP \in \mathcal{L}^+$. In addition, $T(I-P)T^* = TT^* - TPT^* \leqslant TT^* \leqslant \lambda TP$. Therefore, by Douglas' theorem, $R(T(I-P)) \subseteq R((TP)^{1/2})$.

$3 \Rightarrow 4$. Suppose $TP \in \mathcal{L}^+$ and $R(T(I-P)) \subseteq R((TP)^{1/2})$. Then, by Douglas' theorem, $TT^* - TPT^* \leqslant \alpha TP$ for some $\alpha \geqslant 0$. As $R(TP) \subseteq R((TP)^{1/2})$, using again Douglas' theorem, we get that $(TP)^2 \leqslant \beta TP$ for some $\beta \geqslant 0$. Hence $TT^* \leqslant (\alpha + \beta)TP$. Now, the assertion follows multiplying with T^*, T.

$4 \Rightarrow 1$. Assume that item 4 holds. By Theorem 2.3, there exists $X_0 \in \mathcal{L}^+$ such that $T^*T = T^*X_0$, and so $T^*T = T^*X_0 = T^*PX_0$. Thus, T and PX_0 are both solutions of the operator equation $T^*X = T^*T$. Moreover, $R(T), R(PX_0) \subseteq N(T^*)^{\perp} = \overline{R(T)}$, i.e., T and PX_0 are both the reduced solution of $T^*X = T^*T$. Hence, by the uniqueness of this solution, we obtain that $T = PX_0$ and so $T \in \mathcal{T}$. □

Theorem 3.3. *Let $T \in \mathcal{CR}$ and $P = P_{R(T)}$. The following conditions are equivalent:*

1. $T \in \mathcal{T}$;
2. *there exists* $A \in \mathcal{G}^+$ *such that* $T = PA$;
3. $R(T) \dot{+} N(T) = \mathcal{H}$ *and* $TP \in \mathcal{L}^+$.

Proof. $1 \Rightarrow 2$. Let $T \in \mathcal{T}$. Hence, there exists $B \in \mathcal{L}^+$ such that $PB = T$. Therefore, $R(B) + R(T)^\perp = \mathcal{H}$. Define $A := B + P_{R(T)^\perp}$. Hence, $R(A^{1/2}) = R(B^{1/2}) + R(T)^\perp \supset \mathcal{H}$ and so $A \in \mathcal{G}^+$. Now, as $PA = PB = T$, the result is obtained.

$2 \Rightarrow 3$. Suppose that $T = PA$ with $A \in \mathcal{G}^+$. Since $A \in \mathcal{G}^+$, then $\langle x, y\rangle_A := \langle Ax, y\rangle$ defines a inner product equivalent to $\langle , \rangle$. Now, since $N(T) = A^{-1}(R(T)^\perp) = R(T)^{\perp_A}$ then $R(T) \dot{+} N(T) = \mathcal{H}$. In addition, $TP = PAP \in \mathcal{L}^+$.

$3 \Rightarrow 1$. Assume that $R(T) \dot{+} N(T) = \mathcal{H}$ and $TP \in \mathcal{L}^+$. Let us define $A := T^*(TP)^\dagger T$. Note that since $R(TP) = R(T)$ (because $R(T) \dot{+} N(T) = \mathcal{H}$) then TP has closed range and so $(TP)^\dagger \in \mathcal{L}^+$. Thus, $A \in \mathcal{L}^+$ and $T = PA$, i.e., $T \in \mathcal{T}$. □

Observe that as an immediate consequence of Theorem 3.3 we obtain that $\mathcal{Q} \subseteq \mathcal{T}$.

Remark 3.4. Taking into account Lemma 3.1 and Theorem 3.3, a natural question is if $\overline{R(T)} \cap N(T) = \{0\}$ and $TP_{\overline{R(T)}} \in \mathcal{L}^+$ imply $T \in \mathcal{T}$. However, this is false in general. In fact, consider a Hilbert space decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$ and define $T = \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$ with a a positive injective operator with non-closed range and b such that $R(b) \not\subseteq R(a^{1/2})$. Then, $\overline{R(T)} = \overline{R(a)} = \mathcal{S}$ and so, by the injectivity of a, we have that $\overline{R(T)} \cap N(T) = \{0\}$. Moreover, $TP_{\overline{R(T)}} = \left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{L}^+$. However, since $R(b) \not\subseteq R(a^{1/2})$, then there does not exist $A \in \mathcal{L}^+$ such that $T = P_{\overline{R(T)}}A$, i.e., $T \notin \mathcal{T}$ (see [23] and [19]).

In the sequel, we abbreviate

$$\mathcal{T}_{cr} = \mathcal{T} \cap \mathcal{CR}.$$

Note that, by Theorem 3.3, $\mathcal{T}_{cr} = \mathcal{P} \cdot \mathcal{G}^+$.

Proposition 3.5. *It holds* $\mathcal{T}_{cr}^\dagger = \mathcal{T}_{cr}$.

Proof. Let $T \in \mathcal{T}_{cr}$, then by Theorem 3.3, $T = PA$ with $A \in \mathcal{G}^+$ and $P = P_{R(T)}$. Now, define $C = P_{R(AP)}A^{-1}$. Observe that $TC = P$ and $R(C) = N(T)^\perp$ then, by Theorem 3.1 in [2], $T^\dagger = C = P_{R(AP)}A^{-1}$ and so $T^\dagger \in \mathcal{T}_{cr}$. The converse follows from the fact that $(T^\dagger)^\dagger = T$. □

4. Optimal factorization

In this section, given $T \in \mathcal{T}$, we describe all factors $A \in \mathcal{T}_T^+$ and $P \in \mathcal{T}_T^{\mathcal{P}}$ such that $T = PA$. In particular we show that T admits an optimal factorization.

Proposition 4.1. *Let $T \in \mathcal{T}$. Then, there exists $A \in \mathcal{T}_T^+$ with $N(A) = N(T)$. Moreover, there exists a unique $A \in \mathcal{T}_T^+$ with $N(A) = N(T)$ if and only if $\overline{R(T^*)} \cap N(T^*) = \{0\}$.*

Proof. Let $P = P_{\overline{R(T)}}$. As $T \in \mathcal{T}$ then $PX = T$ has a positive solution. Now, by Corollary 2.4, there exists $A \in \mathcal{L}^+$ such that $PA = T$ and $N(A) = N(T)$.

On the other hand, suppose that $\mathcal{S} = \overline{R(T^*)} \cap N(T^*) \neq \{0\}$ and let $A \in \mathcal{T}_T^+$ with $N(A) = N(T)$. Define $Y := A + P_{\mathcal{S}}$. Observe that $Y \in \mathcal{T}_T^+$ and so $N(Y) \subseteq N(T)$. Now, let $x \in N(T) \subseteq \mathcal{S}^\perp$. Then $Yx = Ax + P_{\mathcal{S}}x = 0$. Thus, $N(T) \subseteq N(Y)$, i.e., $N(T) = N(Y)$ and so the uniqueness does not hold. Conversely, suppose that there exist $A_1, A_2 \in \mathcal{T}_T^+$ with $N(A_1) = N(A_2) = N(T)$. Then $A_1 - A_2$ is a selfadjoint solution of the equation $PX = 0$. Then, by Lemma 2.8 in [3], $A_1 - A_2 = (I - P)(A_1 - A_2)(I - P)$. Therefore $R(A_1 - A_2) \subseteq \overline{R(T^*)} \cap R(T)^{\perp} = \overline{R(T^*)} \cap N(T^*) = \{0\}$. So that $A_1 = A_2$. □

Remark 4.2. In the sequel, given $T \in \mathcal{T}$ we shall denote by

$$A_T := \left(\left((TP)^{1/2}\right)^{\dagger} T\right)^* \left((TP)^{1/2}\right)^{\dagger} T,$$

where $P = P_{\overline{R(T)}}$. Note that:

1. $A_T \in \mathcal{T}_T^+$.
2. $N(A_T) = N(T)$.
3. $T = PA_T$.
4. If $T \in \mathcal{T}_{cr}$ then $R(A_T) = R(T^*)$.

Indeed, as $T \in \mathcal{T}$, then the equation $PX = T$ has a positive solution. Therefore, items 1 and 2 follow by the proof of Corollary 2.4. Moreover, since $A_T \in \mathcal{T}_T^+$ then there exists $P_{\mathcal{S}} \in \mathcal{P}$ such that $T = P_{\mathcal{S}} A_T$. Then, as $\overline{R(T)} \subseteq \mathcal{S}$, it holds that $T = P_{\overline{R(T)}} P_{\mathcal{S}} A_T = P_{\overline{R(T)}} A_T$. On the other hand, if $T \in \mathcal{T}_{cr}$ then, by Theorem 3.3, $\mathcal{H} = R(T) \dotplus N(T)$. Hence, $R(TP) = R(T)$, and $R(((TP)^{1/2})^{\dagger} T) = R(((TP)^{1/2})^{\dagger} TP) = R((TP)^{1/2}) = R(TP) = R(T)$. Then $R(A_T)$ is closed and so, by item 2, $R(A_T) = R(T^*)$.

Observe that, by Theorem 3.3, given $T \in \mathcal{T}_{cr}$, it holds that $\mathcal{H} = R(T) \dotplus N(T)$. Thus, the projection $Q_{R(T)//N(T)}$ with range $R(T)$ and nullspace $N(T)$ is well-defined. In the next proposition we show that this projection can also be factorized in terms of the factors of $T \in \mathcal{T}$.

Proposition 4.3. *Let $T \in \mathcal{T}_{cr}$ and $P = P_{R(T)}$. Then,*

$$Q_{R(T)//N(T)} = P(A_T P)^{\dagger} A_T.$$

Proof. It is easy to check that $P(A_T P)^{\dagger} A_T$ is an idempotent operator with $R(P(A_T P)^{\dagger} A_T) \subseteq R(T)$. Thus, let us show that $N(P(A_T P)^{\dagger} A_T) = N(T)$. Now, let $x \in N(P(A_T P)^{\dagger} A_T)$. Then $(A_T P)^{\dagger} A_T x \in R(T)^{\perp} \cap N(A_T P)^{\perp} = R(T)^{\perp} \cap R(T) = \{0\}$. So that $A_T x \in R(T^*) \cap N((A_T P)^{\dagger}) = R(T^*) \cap N(PA_T) = R(T^*) \cap N(T) = \{0\}$. Hence $x \in N(A_T) = N(T)$ and so $N(P(A_T P)^{\dagger} A_T) \subseteq N(T)$. The other inclusion is trivial. In consequence $P(A_T P)^{\dagger} A_T = Q_{R(T)//N(T)}$. □

Our next result fully describes the set $\mathcal{T}_T^+$.

Proposition 4.4. *Let $T \in \mathcal{T}$ and $P = P_{\overline{R(T)}}$. Then*

$$\mathcal{T}_T^+ = \left\{A_T + (I - P)C(I - P)\colon C \in \mathcal{L}^+\right\}.$$

In particular, $\mathcal{T}_T^+$ is a closed convex set.

Proof. Let us consider the orthogonal decomposition $\mathcal{H} = \overline{R(T)} \oplus R(T)^{\perp}$. Then, under this decomposition, $T = \begin{bmatrix} t_1 & t_2 \\ 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, if $A \in \mathcal{T}_T^+$ then $T = PA$ and so, by [23] (see also [19]), $A = \begin{bmatrix} t_1 & t_2 \\ t_2^* & d^*d + f \end{bmatrix}$ with $d = (t_1^{1/2})^{\dagger} t_2$ and $f \in \mathcal{L}(R(T)^{\perp})^+$. Now, $A = \begin{bmatrix} t_1 & t_2 \\ t_2^* & d^*d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} = A_T + (I - P)C(I - P)$ with $C \in \mathcal{L}^+$.

The other inclusion follows from Remark 4.2. □

Corollary 4.5. *Let $T \in \mathcal{T}_{cr}$ and $P = P_{R(T)}$. Then,*

$$\mathcal{T}_T^+ \cap \mathcal{G} = \left\{A_T + (I - P)C(I - P)\colon C \in \mathcal{G}^+\right\}.$$

Proof. Let $A \in \mathcal{T}_T^+$ be an invertible operator. Then, by Proposition 4.4, $A = A_T + (I-P)S(I-P)$ with $S \in \mathcal{L}^+$. Now, let us define $C = S + TT^* \in \mathcal{L}^+$. Note that $R(C) = \mathcal{H}$. In fact, by Theorem 2.1 and Remark 4.2, $\mathcal{H} = R(A) = R(T^*) + R((I-P)S^{1/2})$. Now, by Theorem 3.3, $R(I-P) \dot{+} R(T^*) = N(T^*) \dot{+} R(T^*) = \mathcal{H}$. Hence, $R((I-P)S^{1/2}) = R(I-P)$ and so $\mathcal{H} = R(S^{1/2}) + R(T) = R(C)$. Thus, $C \in \mathcal{G}(\mathcal{H})^+$ and $A = A_T + (I-P)S(I-P) = A_T + (I-P)C(I-P)$.

For the converse, it is sufficient to note that given $C \in \mathcal{G}(\mathcal{H})^+$ then $R((A_T + (I-P)C(I-P))^{1/2}) = R((A_T)^{1/2}) + R((I-P)C^{1/2}) = R(T^*) + N(T^*) = \mathcal{H}$, where the last equality is consequence of Theorem 3.3. Therefore, $A_T + (I-P)C(I-P) \in \mathcal{G}^+$ and the result is proved. □

In the next proposition we describe the elements of $\mathcal{G}^+$ that factorize $\mathcal{CR}^+$ and $\mathcal{Q}$.

Proposition 4.6. *Let $A \in \mathcal{G}^+$. The following equivalence holds*:

1. $PA \in \mathcal{CR}^+$ *for some* $P \in \mathcal{P}$ *if and only if* $A(\mathcal{S}) = \mathcal{S}$ *for some closed subspace* $\mathcal{S}$.
2. $PA \in \mathcal{Q}$ *for some* $P \in \mathcal{P}$ *if and only if* $A^{1/2}|_{\mathcal{S}}$ *is an isometry for some closed subspace* $\mathcal{S}$.

Proof. 1. If $PA \in \mathcal{CR}^+$ for some $P \in \mathcal{P}$ with $R(P) = \mathcal{S}$ then $PA = AP$ and so $A\mathcal{S} = \mathcal{S}$. Conversely, if $A\mathcal{S} = \mathcal{S}$ then $P_{\mathcal{S}}AP_{\mathcal{S}} = AP_{\mathcal{S}}$ and so $AP_{\mathcal{S}} \in \mathcal{CR}^+$.

2. See Theorem 2 in [15]. □

In the next proposition we show that A_T is optimal in $\mathcal{T}_T^+$ in two senses.

Proposition 4.7. *Let $T \in \mathcal{T}$. Then, $A_T = \min \mathcal{T}_T^+$. Moreover, $\|A_T\| = \min\{\|A\|\colon A \in \mathcal{T}_T^+\}$.*

Proof. If $A \in \mathcal{T}_T^+$ then, by Proposition 4.4, $A = A_T + C$ with $C \in \mathcal{L}^+$ and so $A_T \leqslant A$. Thus, the first equality is proved. For the second equality, as $0 \leqslant A_T \leqslant A$, then for all $x \in \mathcal{H}$ with $\|x\| = 1$, we have that $\langle A_T x, x\rangle \leqslant \langle Ax, x\rangle \leqslant \|A\|$. Thus, $\|A_T\| = \sup_{\|x\|=1}\langle A_T x, x\rangle \leqslant \|A\|$. □

We now study the set $\mathcal{T}_T^{\mathcal{P}}$.

Proposition 4.8. *Let $T \in \mathcal{T}$. Then,*

$$\mathcal{T}_T^{\mathcal{P}} = \left\{P_{\mathcal{S}} \in \mathcal{P}\colon \mathcal{S} = \overline{R(T)} \oplus \mathcal{M} \text{ for some } \mathcal{M} \subseteq N(T)\right\}.$$

Moreover, fixed $A \in \mathcal{T}_T^+$ then

$$\{P_{\mathcal{S}} \in \mathcal{P}\colon T = P_{\mathcal{S}}A\} = \left\{P_{\mathcal{S}} \in \mathcal{P}\colon \mathcal{S} = \overline{R(T)} \oplus \mathcal{M} \text{ for some } \mathcal{M} \subseteq N(A)\right\}.$$

On the other hand, fixed $P \in \mathcal{T}_T^{\mathcal{P}}$,

$$\left\{A \in \mathcal{L}^+\colon T = PA\right\} = \left\{A_T + (I-P)C(I-P)\colon C \in \mathcal{L}^+\right\}.$$

Proof. Let us prove the first equality. For this, if $T = P_{\mathcal{S}}A$ with $A \in \mathcal{L}^+$ then $\overline{R(T)} \subseteq \mathcal{S}$ and so $T = P_{\overline{R(T)}}A$. Furthermore, $\mathcal{M} := \mathcal{S} \ominus \overline{R(T)}$ is well-defined and $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$. Therefore, $P_{\mathcal{S}} = P_{\overline{R(T)}} + P_{\mathcal{M}}$ and $P_{\overline{R(T)}}A = T = P_{\mathcal{S}}A = P_{\overline{R(T)}}A + P_{\mathcal{M}}A$. After cancellation, we get $P_{\mathcal{M}}A = 0$, i.e., $\mathcal{M} \subseteq N(A)$. Now, since $N(A) \subseteq N(T)$ we obtain the desired inclusion.

Conversely, let $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(T)$. Since $T \in \mathcal{T}$, there exists $A \in \mathcal{L}^+$ with $N(A) = N(T)$ such that $T = P_{\overline{R(T)}}A$. Now, as $\mathcal{M} \subseteq N(T) = N(A)$ we obtain that $P_{\mathcal{S}}A = P_{\overline{R(T)}}A + P_{\mathcal{M}}A = T$. The equality is proved.

The second equality can be proved similarly.

Now, given $P \in \mathcal{T}_T^{\mathcal{P}}$, we know that $R(P) = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(T)$. Thus, $P = P_{\overline{R(T)}} + P_{\mathcal{M}}$. Note that as $N(T) = N(A_T)$ then we get that $PA_T = T$. Now, let $A \in \mathcal{L}^+$ such that $PA = T$. Hence, by Proposition 4.4, $A = A_T + (I - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}})$ for some $C \in \mathcal{L}^+$. Hence, $T = PA = PA_T +$

$(P - PP_{\overline{R(T)}})C(I - P_{\overline{R(T)}}) = T + (P - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}})$. Thus, $(P - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}}) = 0$ and so $(P - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}})P = (P - P_{\overline{R(T)}})C(P - P_{\overline{R(T)}}) = 0$ Now, as $P - P_{\overline{R(T)}} = P_{\mathcal{M}}$, then $P_{\mathcal{M}}CP_{\mathcal{M}} = 0$, i.e., $R(C^{1/2}) \subseteq \mathcal{M}^{\perp}$. Finally, $(I - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}}) = (I - (P - P_{\mathcal{M}}))C(I - (P - P_{\mathcal{M}})) = (I - P)C(I - P)$ and so $A = A_T + (I - P)C(I - P)$. The other inclusion follows from $PA_T = T$. □

As consequence of the previous results we obtain a characterization of the set $\{(P, A): PA = T\}$ for a given $T \in \mathcal{T}$. Observe that Proposition 4.8 gives partial answers of this problem.

Theorem 4.9. *Let $T \in \mathcal{T}$, $P \in \mathcal{P}$ and $A \in \mathcal{L}^+$. Then, $T = PA$ if and only if there exists a closed subspace, $\mathcal{M}$, of $\mathcal{H}$ and $C \in \mathcal{L}^+$ such that*

1. $R(P) = \overline{R(T)} \oplus \mathcal{M}$;
2. $\mathcal{M} \subseteq N(T)$;
3. $A = A_T + (I - P)C(I - P)$.

Proof. It follows from Proposition 4.8. □

We prove now the minimality of $P_{\overline{R(T)}}$ in $\mathcal{T}_T^{\mathcal{P}}$.

Proposition 4.10. *Let $T \in \mathcal{T}$. Then, $P_{\overline{R(T)}} = \min \mathcal{T}_T^{\mathcal{P}}$.*

Proof. Let $P_{\mathcal{S}} \in \mathcal{T}_T^{\mathcal{P}}$. Then, $\overline{R(T)} \subseteq \mathcal{S}$, i.e., $P_{\overline{R(T)}} \leqslant P_{\mathcal{S}}$. □

Definition 4.11. For $T \in \mathcal{T}$ the identity $T = P_{\overline{R(T)}}A_T$ is called the **optimal factorization** of T.

Remark 4.12. In [6] it is proven that, for $T \in \mathcal{P} \cdot \mathcal{P}$, the identity $T = P_{\overline{R(T)}}P_{N(T)^{\perp}}$, found by T. Crimmins (see [21, Theorem 8]) has several minimality properties. We show now that it coincides with the optimal factorization of T, i.e., for $T \in \mathcal{P} \cdot \mathcal{P}$ it holds $A_T = P_{N(T)^{\perp}}$. In fact, $P_{N(T)^{\perp}}$ is a positive operator with nullspace $N(T)$, so, by Crimmins' result, $P_{N(T)^{\perp}} \in \mathcal{T}_T^{+}$. On the other hand, by [6, Theorem 3.2], $\overline{R(T^*)} \cap N(T^*) = \{0\}$. Then, by Proposition 4.1, we get $A_T = P_{N(T)^{\perp}}$.

5. Compatibility

The aim of this section is to relate the factors P, A of a given $T \in \mathcal{T}$ with compatibility. The notion of compatibility relates a closed subspace $\mathcal{S}$ of $\mathcal{H}$ and a positive operator $A \in \mathcal{L}(\mathcal{H})$. More precisely, the pair $(A, \mathcal{S})$ is called **compatible** if there exists $Q \in \mathcal{Q}$ with $R(Q) = \mathcal{S}$ such that $AQ = Q^*A$ (this means that Q is Hermitian respect to the semi-inner product induced by A). This notion can be also extended to unbounded projections, in which case the pair $(A, \mathcal{S})$ is called **quasi-compatible** if there exists a densely defined closed projection Q onto $\mathcal{S}$ such that AQ is symmetric. The quasi-compatibility (resp., compatibility) of a pair $(A, \mathcal{S})$ is equivalent to $\overline{\mathcal{S} + (A\mathcal{S})^{\perp}} = \mathcal{H}$ (resp. $\mathcal{S} + (A\mathcal{S})^{\perp} = \mathcal{H}$). In particular, the notion of compatibility is also equivalent to certain angle condition, more precisely, the Dixmier angle between $\mathcal{S}^{\perp}$ and $\overline{A\mathcal{S}}$ is non-zero. Recall that the Dixmier angle between two closed subspaces $\mathcal{S}_1, \mathcal{S}_2$ is that whose cosine is $c_0(\mathcal{S}_1, \mathcal{S}_2) = \sup\{|\langle x, y\rangle|: x \in \mathcal{S}_1, \ y \in \mathcal{S}_2, \|x\|, \|y\| \leqslant 1\}$. Therefore, it holds that $(A, \mathcal{S})$ is compatible if and only if $c_0(\mathcal{S}^{\perp}, \overline{A\mathcal{S}}) < 1$. For more results on the theory of compatibility see [7] and references therein. For details on quasi-compatibility see [4].

Given $T \in \mathcal{T}$, we study the quasi-compatibility (resp., compatibility) of the pairs $(A, \mathcal{S})$ such that $T = P_{\mathcal{S}}A$. We begin by showing that such compatibility is independent of the factors chosen, that is, it depends only on T and not on the particular $P_{\mathcal{S}}$, A.

Proposition 5.1. *Let $T = P_{\mathcal{S}}A \in \mathcal{T}$.*

1. *The following conditions are equivalent:*
   (a) $(A, \mathcal{S})$ *is quasi-compatible;*
   (b) $\overline{R(T)} + N(T)$ *is a dense subspace of* $\mathcal{H}$;
   (c) $\overline{R(T^*)} \cap N(T^*) = \{0\}$.
2. *The following conditions are equivalent:*
   (a) $(A, \mathcal{S})$ *is compatible;*
   (b) $c_0(\mathcal{S}^\perp, \overline{A\mathcal{S}}) < 1$;
   (c) $\overline{R(T)} \dot{+} N(T) = \mathcal{H}$;
   (d) $\overline{R(T^*)} \dot{+} N(T^*) = \mathcal{H}$;
   (e) $c_0(\overline{R(T)}, N(T)) < 1$;
   (f) $c_0(\overline{R(T^*)}, N(T^*)) < 1$.

Proof. By Proposition 4.8, $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(A)$. Then, $(A\mathcal{S})^\perp = (A\overline{R(T)})^\perp = R(AP_{\overline{R(T)}})^\perp = N(P_{\overline{R(T)}}A) = N(T)$.

1. If $(A, \mathcal{S})$ is quasi-compatible then $\mathcal{H} = \overline{\mathcal{S} + (A\mathcal{S})^\perp} = \overline{\mathcal{S} + N(T)} \subseteq \overline{\overline{R(T)} + \mathcal{M} + N(T)} = \overline{\overline{R(T)} + N(T)}$ because $\mathcal{M} \subseteq N(A) \subseteq N(T)$. The converse is similar. (b) $\Leftrightarrow$ (c) follows by taking orthogonal complements.

2. (a) $\Leftrightarrow$ (b) follows from Theorem 2.15 in [7]. Now, if item (a) holds then $\mathcal{H} = \mathcal{S} + (A\mathcal{S})^\perp$. That is, $\mathcal{H} = \mathcal{S} + N(T) = \overline{R(T)} \oplus \mathcal{M} + N(T) = \overline{R(T)} + N(T)$, where the last equality follows from $\mathcal{M} \subseteq N(A) \subseteq N(T)$. Moreover, by Lemma 3.1, $\mathcal{H} = \overline{R(T)} \dot{+} N(T)$. Thus, (a) $\Rightarrow$ (c). (c) $\Leftrightarrow$ (d) is consequence of Lemma 11 in [9]. (d) $\Rightarrow$ (e) follows from Theorems 12 and 15 in [9]. (e) $\Rightarrow$ (f) is also consequence of Theorem 12 in [9].

Finally, if item (f) holds, as $\overline{R(T^*)} = \overline{A\mathcal{S}}$ and $\mathcal{S}^\perp \subseteq N(T^*)$, then $c_0(\overline{A\mathcal{S}}, \mathcal{S}^\perp) < c_0(\overline{R(T^*)}, N(T^*))$. Therefore, (f) $\Rightarrow$ (a) because of Theorem 2.15 in [7]. □

In the next result, given a positive operator A and a closed subspace $\mathcal{S}$, we characterize the quasi-compatibility of $(A, \mathcal{S})$ in terms of the existence of certain operator in $\mathcal{T}$.

Proposition 5.2. *Let $A \in \mathcal{L}^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. The pair $(A, \mathcal{S})$ is quasi-compatible if and only if there exists $T \in \mathcal{T}$ such that $\overline{R(T)} = \overline{A\mathcal{S}}$ and $N(T) = (\mathcal{S} \ominus (A\mathcal{S})^\perp)^\perp$.*

Proof. If $(A, \mathcal{S})$ is quasi-compatible then, by [4, Proposition 2.15] there exists $T \in \mathcal{L}(\mathcal{H})$ such that $TT^*T = T^2$, $\overline{R(T)} = \overline{A\mathcal{S}}$ and $N(T) = (\mathcal{S} \ominus (A\mathcal{S})^\perp)^\perp$. Then, $(T^*T)^2 = T^*T^2$ and so, by Theorem 3.2, $T \in \mathcal{T}$. Conversely, if there exists $T \in \mathcal{T}$ such that $\overline{R(T)} = \overline{A\mathcal{S}}$ and $N(T) = (\mathcal{S} \ominus (A\mathcal{S})^\perp)^\perp$ then, by Lemma 3.1, $\overline{A\mathcal{S}} \cap (\mathcal{S} \ominus (A\mathcal{S})^\perp)^\perp = \{0\}$. So that $\mathcal{S} + (A\mathcal{S})^\perp$ is dense in $\mathcal{H}$. Therefore, $(A, \mathcal{S})$ is quasi-compatible. □

Given a quasi-compatible pair $(A, \mathcal{S})$ there exists a distinguished element with optimal properties among all densely defined idempotents Q with domain $\mathcal{S} + (A\mathcal{S})^\perp$, $R(Q) = \mathcal{S}$ and AQ symmetric, namely, $P_{A,\mathcal{S}} := Q_{\mathcal{S}//(A\mathcal{S})^\perp \ominus \mathcal{S}}$ (see [4]). If the pair $(A, \mathcal{S})$ is compatible then $P_{A,\mathcal{S}}$ is bounded.

Proposition 5.3. *Let $T \in \mathcal{T}$ be such that $\overline{R(T)} \dot{+} N(T) = \mathcal{H}$. Therefore, if $T = P_{\mathcal{S}}A$ then $(A, \mathcal{S})$ is compatible and*

$$P_{A,\mathcal{S}} = Q_{\overline{R(T)}//N(T)} + P_{\mathcal{M}},$$

where $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$ and $\mathcal{M} \subseteq N(A)$.

Proof. The compatibility of the pair $(A, \mathcal{S})$ follows from Proposition 5.1. Moreover, by Proposition 4.8, $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(A)$. Now, define $E = Q_{\overline{R(T)}//N(T)} + P_{\mathcal{M}}$. Since $\mathcal{M} \subseteq N(T)$ and

$\mathcal{M} \perp \overline{R(T)}$ then $E^2 = E$. Furthermore, $AE = E^*A$. Indeed, since $N(T) = (A\overline{R(T)})^\perp$ then $AQ_{\overline{R(T)}//N(T)} = Q^*_{\overline{R(T)}//N(T)}A$. Now, since $AE = AQ_{\overline{R(T)}//N(T)}$ we get that $AE = E^*A$. In addition, it is clear that $R(E) \subseteq \mathcal{S}$. Hence, it remains to show that $N(E) = N(P_{A,\mathcal{S}})$. Observe that $N(P_{A,\mathcal{S}}) = (A\mathcal{S})^\perp \cap ((A\mathcal{S})^\perp \cap \mathcal{S})^\perp = N(T) \cap (N(T) \cap (\overline{R(T)} + \mathcal{M}))^\perp = N(T) \cap (N(T) \cap \mathcal{M})^\perp = N(T) \cap \mathcal{M}^\perp$. Now, we prove the equality $N(E) = N(T) \cap \mathcal{M}^\perp$. Clearly, $N(T) \cap \mathcal{M}^\perp \subseteq N(E)$. For the other inclusion, if $x \in N(E)$ then $Q_{\overline{R(T)}//N(T)}x = -P_{\mathcal{M}}x \in \overline{R(T)} \cap \mathcal{M} = \{0\}$. So that $x \in N(T) \cap \mathcal{M}^\perp$. Then $N(E) = N(P_{A,\mathcal{S}})$ and so $E = P_{A,\mathcal{S}}$. □

Remark 5.4. Given $T \in \mathcal{T}$ such that $\overline{R(T)} + N(T)$ is a dense subspace of $\mathcal{H}$ then, by Proposition 5.1, the pair $(A, \mathcal{S})$ is quasi-compatible for all A, $\mathcal{S}$ such that $T = P_{\mathcal{S}}A$. In this case,

$$P_{A,\mathcal{S}} = Q_{\overline{R(T)}//N(T)} + P_{\mathcal{M}}|_{\overline{R(T)}+N(T)}.$$

Corollary 5.5. *Let $T \in \mathcal{T}_{cr}$. Then*:

1. *if $T = P_{\mathcal{S}}A$ then $(A, \mathcal{S})$ is compatible*;
2. *$(A, R(T))$ is compatible for all $A \in \mathcal{T}_T^+$ and $P_{A,R(T)} = P(A_T P)^\dagger A_T$ where $P = P_{R(T)}$.*

Proof. It follows from Theorem 3.3 and Propositions 4.3 and 5.3. □

6. Polar decomposition

This section is devoted to the study of the polar decomposition of the operators in $\mathcal{T}$. For this, given $T \in \mathcal{L}(\mathcal{H})$ we shall denote by V_T the partial isometry of the polar decomposition of T, i.e., $V_T \in \mathcal{J} = \{V \in \mathcal{L}(\mathcal{H}):\ VV^*V = V\}$ and $T = V_T|T|$ with $N(V_T) = N(T)$ and $|T| = (T^*T)^{1/2}$. In addition, given a class of operators $\mathcal{M}$, we denote

$$\mathcal{J}_{\mathcal{M}} = \{V \in \mathcal{J}\colon \text{there exists } T \in \mathcal{M} \text{ such that } V = V_T\}.$$

The sets $\mathcal{J}_{\mathcal{Q}}$ and $\mathcal{J}_{\mathcal{P}\cdot\mathcal{P}}$ have been studied in [5] and [6], respectively. Our goal in this section is to describe the set $\mathcal{J}_{\mathcal{T}}$.

Proposition 6.1. *The following relations hold*:

1. $\mathcal{T} \cap \mathcal{J} = \mathcal{J}_{\mathcal{Q}}$;
2. $\mathcal{J}_{\mathcal{Q}} \subseteq \mathcal{J}_{\mathcal{P}\cdot\mathcal{P}} \subseteq \mathcal{J}_{\mathcal{T}}$.

Proof. 1. If follows from [5, Theorem 5.1].

2. The first inclusion can be deduced from [5,6]. In fact, if $E \in \mathcal{Q}$ then $E^\dagger = P_{N(E)^\perp}P_{R(E)}$ by a result of Penrose [20] (see also Greville [13] or Vidav [24]): the reader can easily check that $X = P_{N(E)^\perp}P_{R(E)}$ satisfies the four Penrose conditions $EXE = E$, $XEX = X$, $(XE)^* = XE$, $(EX)^* = EX$. This shows that $\mathcal{Q}^\dagger \subseteq \mathcal{P}\cdot\mathcal{P}$ and, therefore, $\mathcal{J}_{\mathcal{Q}^\dagger} \subseteq \mathcal{J}_{\mathcal{P}\cdot\mathcal{P}}$. On the other side, for any class $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$, the properties of the standard polar decomposition show that $\mathcal{J}_{\mathcal{M}^\dagger} = \mathcal{J}_{\mathcal{M}^*}$. Since Q is closed by the adjoint operation, we get $\mathcal{J}_{\mathcal{Q}} = \mathcal{J}_{\mathcal{Q}^\dagger} \subseteq \mathcal{J}_{\mathcal{P}\cdot\mathcal{P}}$ as claimed.

The second inclusion is consequence of $\mathcal{P}\cdot\mathcal{P} \subset \mathcal{T}$. □

Proposition 6.2. *Let $A \in \mathcal{G}^+$. Then, $PA \in \mathcal{J}_{\mathcal{Q}}$ for some $P \in \mathcal{P}$ if and only if $A|_{\mathcal{S}}$ is an isometry for some closed subspace $\mathcal{S}$.*

Proof. Suppose that $PA \in \mathcal{J}_{\mathcal{Q}}$ for some $P \in \mathcal{P}$. Then, $P = PA(PA)^* = PA^2P$ and so, $(PA^2)^2 = PA^2PA^2 = PA^2$, i.e., $PA^2 \in \mathcal{Q}$ and the result follows by Proposition 4.6. Conversely, if $A|_{\mathcal{S}}$ is an isometry for some closed subspace $\mathcal{S}$ then, by Proposition 4.6, $PA^2 \in \mathcal{Q}$. Moreover, $R(PA^2) = R(P)$ because $A \in \mathcal{G}$. Hence, $PA(PA)^*PA = PA^2PA = PA$, i.e., $PA \in \mathcal{J} \cap \mathcal{T} = \mathcal{J}_{\mathcal{Q}}$. □

Proposition 6.3. *Let $T \in \mathcal{T}$. Then $V_T^* A = |T|$ for all $A \in \mathcal{T}_T^+$ and $|T| V_T \in \mathcal{L}^+$.*

Proof. Let $T = P_{\overline{R(T)}} A = V_T |T|$, i.e., $V_T V_T^* A = V_T |T|$. Hence, $V_T^* A = V_T^* V_T V_T^* A = V_T^* V_T |T| = |T|$ and so $|T| V_T = V_T^* A V_T \in \mathcal{L}^+$. □

Proposition 6.4.

$$\mathcal{J}_{\mathcal{T}} = \left\{ V \in \mathcal{J} \colon \exists A \in \mathcal{L}^+ \text{ such that } AV \in \mathcal{L}^+ \text{ and } R(V) \cap N(A) = \{0\} \right\}.$$

Proof. Consider $T = V_T |T| \in \mathcal{T}$. Therefore, by Proposition 6.3, $A V_T = |T| \in \mathcal{L}^+$ for all $A \in \mathcal{T}_T^+$. In addition, $N(AV_T) = N(|T|) = N(T)$. Therefore, if $y \in R(V_T) \cap N(A)$ then $y = V_T x$ for some $x \in \mathcal{H}$ and $Ay = AV_T x = 0$, i.e., $x \in N(AV_T) = N(T) = N(V_T)$. So, $y = V_T x = 0$ and the first inclusion is proved.

Conversely, let $V \in \mathcal{J}$ such that $AV \in \mathcal{L}^+$ for some $A \in \mathcal{L}^+$ with $R(V) \cap N(A) = \{0\}$. Define $T := VV^*A \in \mathcal{T}$. Then, $T^*T = AVV^*VV^*A = AVV^*A = (V^*A)^2$, i.e., $|T| = V^*A$. Moreover, $N(T) = N(|T|) = N(V^*A) = N(AV) = N(V)$ where the last equality holds because $R(V) \cap N(A) = \{0\}$. Therefore VV^*A is the polar decomposition of $T \in \mathcal{T}$ and so $V \in \mathcal{J}_{\mathcal{T}}$. □

Given two operators $T, S \in \mathcal{L}(\mathcal{H})$ we write $T \sim_+ S$ if there exists $A \in \mathcal{G}^+$ such that $T = A^{-1} S A$.

Corollary 6.5.

$$\begin{aligned} \mathcal{J}_{\mathcal{T}_{cr}} &= \left\{ V \in \mathcal{J} \colon \exists A \in \mathcal{G}^+ \text{ such that } AV \in \mathcal{L}^+ \right\} \\ &= \left\{ V \in \mathcal{J} \colon V \sim_+ B \text{ for some } B \in L(\mathcal{H})^+ \right\}. \end{aligned}$$

Proof. Let us prove the first equality. Consider $T = V_T |T| \in \mathcal{T}_{cr}$. Then, by Theorem 3.3, there exists $A \in \mathcal{G}^+$ such that $T = V_T V_T^* A = V_T |T|$. So $V_T^* A = |T| \in L(\mathcal{H})^+$ with $A \in \mathcal{G}^+$.

For the other inclusion, let $V \in \mathcal{J}$ such that $AV \in L(\mathcal{H})^+$ for some $A \in \mathcal{G}^+$. Define $T := VV^*A$. Clearly, $T \in \mathcal{T}_{cr}$. Moreover, it is straightforward that $|T| = V^*A$ and $N(V) = N(T)$.

For the second equality, note that $AV \geqslant 0$ for some $A \in \mathcal{G}^+$ if and only if $A^{1/2} V A^{-1/2} = B \geqslant 0$, i.e., $V \sim_+ B$ with $B \geqslant 0$. □

References

[1] A. Arias, S. Gudder, Almost sharp quantum effects, J. Math. Phys. 45 (2004) 4196–4206.
[2] M.L. Arias, G. Corach, M.C. Gonzalez, Generalized inverses and Douglas equations, Proc. Am. Math. Soc. 136 (2008) 3177–3183.
[3] M.L. Arias, M.C. Gonzalez, Reduced solutions of Douglas type equations and angles between subspaces, J. Math. Anal. Appl. 355 (2009) 426–433.
[4] G. Corach, M.C. Gonzalez, A. Maestripieri, Unbounded symmetrizable idempotents, Linear Algebra Appl. 437 (2012) 659–674.
[5] G. Corach, A.L. Maestripieri, Polar decomposition of oblique projections, Linear Algebra Appl. 433 (2010) 511–519.
[6] G. Corach, A.L. Maestripieri, Products of orthogonal projections and polar decompositions, Linear Algebra Appl. 434 (2011) 1594–1609.
[7] G. Corach, A. Maestripieri, D. Stojanoff, A classification of projectors, Banach Cent. Publ. 67 (2005) 145–160.
[8] C. Davis, Separation of two linear subspaces, Acta Sci. Math. (Szeged) 19 (1958) 172–187.
[9] F. Deutsch, The angles between subspaces of a Hilbert space, in: S.P. Singh (Ed.), Approximation Theory, Wavelets and Applications, Kluwer, Netherlands, 1995, pp. 107–130.
[10] R.G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert spaces, Proc. Am. Math. Soc. 17 (1966) 413–416.
[11] P.A. Fillmore, J.P. Williams, On operator ranges, Adv. Math. 7 (1971) 254–281.
[12] J.I. Fujii, T. Furuta, Holub's factorization and normal approximants of idempotent operators, Math. Jpn. 25 (1980) 143–145.
[13] T.N.E. Greville, Solutions of the matrix equation $XAX = X$, and relations between oblique and orthogonal projectors, SIAM J. Appl. Math. 26 (1974) 828–832.
[14] J.R. Holub, A note on Wiener–Hopf operators and projections, Math. Jpn. 25 (1980) 31–32.
[15] J.R. Holub, Wiener–Hopf operators and projections II, Math. Jpn. 25 (1980) 251–253.
[16] J.R. Holub, On invertibility of Wiener–Hopf operators, Math. Jpn. 25 (3) (1980) 341–343.

[17] S. Nelson, M. Neumann, Generalizations of the projection method with applications to SOR theory for Hermitian positive semidefinite linear systems, Numer. Math. 51 (1987) 123–141.
[18] T. Oikhberg, Products of orthogonal projections, Proc. Am. Math. Soc. 127 (1999) 3659–3669.
[19] E.L. Pekarev, J.L. Šmul'jan, Parallel addition and parallel substraction of operators, Math. USSR, Izv. 10 (1976) 351–370.
[20] R. Penrose, A generalized inverse for matrices, Proc. Camb. Philos. Soc. 51 (1955) 406–413.
[21] H. Radjavi, J.P. Williams, Products of self-adjoint operators, Mich. Math. J. 16 (1969) 177–185.
[22] Z. Sebestyén, Restrictions of positive operators, Acta Sci. Math. 46 (1983) 299–301.
[23] J.L. Šmul'jan, A Hellinger operator integral, Mat. Sb. 49 (91) (1959) 381–430 (Russian), English transl. AMS Transl. 22 (1962), 289–337.
[24] I. Vidav, On idempotent operators in a Hilbert space, Publ. Inst. Math. (Belgr.) 4 (18) (1964) 157–163.
[25] P.Y. Wu, The operator factorization problems, Linear Algebra Appl. 117 (1989) 35–63.