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# Polyhedral MV-algebras

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Dedicated to Francesc Esteva, on his 70th birthday

**Abstract**

A *polyhedron* in $\mathbb{R}^n$ is a finite union of simplexes in $\mathbb{R}^n$. An MV-algebra is *polyhedral* if it is isomorphic to the MV-algebra of all continuous [0, 1]-valued piecewise linear functions with integer coefficients, defined on some polyhedron $P$ in $\mathbb{R}^n$. We characterize polyhedral MV-algebras as finitely generated subalgebras of semisimple tensor products $S \otimes F$ with $S$ simple and $F$ finitely presented. We establish a duality between the category of polyhedral MV-algebras and the category of polyhedra with $\mathbb{Z}$-maps. We prove that polyhedral MV-algebras are preserved under various kinds of operations, and have the amalgamation property. Strengthening the Hay–Wójcicki theorem, we prove that every polyhedral MV-algebra is strongly semisimple, in the sense of Dubuc–Poveda.



*Keywords:* Łukasiewicz logic; MV-algebra; Polyhedron; Tensor product; Strongly semisimple; Amalgamation property; Bouligand–Severi tangent; Duality; Finite presentability; Free product; Coproduct; $\mathbb{Z}$-map

## 1. Introduction and preliminary material

This paper is devoted to polyhedral MV-algebras. On the one hand, these algebras constitute a proper subclass of finitely generated strongly semisimple MV-algebras, and are a generalization of finitely presented MV-algebras. On the other hand, polyhedral MV-algebras with homomorphisms are dual to polyhedra in euclidean space, *equipped with* $\mathbb{Z}$*-maps* (Definition 3.1). $\mathbb{Z}$-homeomorphism of two polyhedra $P, Q \subseteq \mathbb{R}^n$ amounts to their continuous $\mathcal{G}_n$-equidissectability, where $\mathcal{G}_n = \mathsf{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ is the $n$-dimensional affine group over the integers, [18]. In the resulting new geometry, already rational polyhedra, with their wealth of combinatorial and numerical invariants, pose challenging algebraic-topological, measure-theoretic and algorithmic problems, [4–6].

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Our paper is organized as follows: Section 2 is devoted to proving the characterization of polyhedral MV-algebras as finitely generated subalgebras of semisimple tensor products $S \otimes F$, with $S$ simple and $F$ finitely presented. In Section 3 we give a virtually self-contained proof of the duality between the category of polyhedral MV-algebras and the category of polyhedra with $\mathbb{Z}$-maps. In Section 4 we prove that polyhedral MV-algebras have the amalgamation property. In Section 5 it is shown that polyhedral MV-algebras are strongly semisimple, in the sense of Dubuc–Poveda [8]. This generalizes the Hay–Wójcicki theorem [10,20].

We refer to [11] and [19] for background on polyhedral topology. A set $Q \subseteq \mathbb{R}^n$ is said to be a *polyhedron* if it is a finite union of simplexes $S_i \subseteq \mathbb{R}^n$. Thus $Q$ need not be convex, nor connected; the simplexes $S_i$ need not have the same dimension. If each $S_i$ can be chosen with rational vertices, then $Q$ is said to be a *rational polyhedron*.

For any integer $n, m > 0$ and polyhedron $P \subseteq \mathbb{R}^n$, a function $f : P \to \mathbb{R}^m$ is *piecewise linear* if it is continuous and there are finitely many linear transformations $L_1, \ldots, L_u : \mathbb{R}^n \to \mathbb{R}^m$ such that for each $x \in P$ there is an index $i \in \{1, \ldots, u\}$ with $f(x) = L_i(x)$. The adjective "linear" is always understood in the affine sense. If in particular the coefficients of $L_1, \ldots, L_u$ are integers, we say that $f$ is *piecewise linear with integer coefficients.*

We refer to [7] and [17] for background on MV-algebras. For any polyhedron $P \subseteq \mathbb{R}^n$ we let $\mathcal{M}(P)$ denote the MV-algebra of piecewise linear functions $f : P \to [0, 1]$ with integer coefficients and the pointwise operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. By [7, 3.6.7], $\mathcal{M}(P)$ is a semisimple MV-algebra. $\mathcal{M}([0, 1]^n)$ is the *free n-generator MV-algebra.* This is McNaughton's theorem, [7, 9.1.5]. By [17, 6.3], an MV-algebra $A$ is finitely presented iff it is isomorphic to $\mathcal{M}(R)$ for some rational polyhedron $R \subseteq [0, 1]^n$. An MV-algebra $A$ is said to be *polyhedral* if, for some $n = 1, 2, \ldots$, it is isomorphic to $\mathcal{M}(P)$ for some polyhedron $P \subseteq \mathbb{R}^n$.

Unless otherwise specified, all polyhedra in this paper are nonempty, and all MV-algebras are nontrivial.

## 2. A characterization of polyhedral MV-algebras

**Lemma 2.1.** *For any polyhedron* $P \subseteq \mathbb{R}^n$ *and function* $f : P \to [0, 1]$, *the following conditions are equivalent:*

(i) $f$ *is piecewise linear. (As specified in the first lines of Section 1, piecewise linearity entails continuity.)*
(ii) *For some triangulation* $\Delta$ *of* $P$, $f$ *is linear on each simplex of* $\Delta$.
(iii) *For any cube* $C = [a, b]^n \subseteq \mathbb{R}^n$ *containing* $P$ *there is a piecewise linear function* $g : C \to [0, 1]$ *such that* $f$ *is the restriction of* $g$ *to* $P$, *in symbols,* $f = g \restriction P$.

**Proof.** (i)⇒(ii) From [19, 2.2.6]. (iii)⇒(i) Is trivial.

(ii)⇒(iii) There is a triangulation $\nabla$ of the cube $C$ such that the set $\nabla_P = \{T \in \nabla \mid T \subseteq P\}$ is a triangulation of $P$ and is a subdivision of $\Delta$. The existence of $\nabla$ is a well-known fact in polyhedral topology [11,19]. A direct proof can be obtained from an adaptation of the De Concini–Procesi theorem in the version of [17, 5.3]. Actually, by a routine adaptation of the affine counterpart of [9, III, 2.8] we may insist that $\nabla_P = \Delta$. Let $g : C \to [0, 1]$ be the continuous function uniquely defined by the following stipulations: $g$ is linear on every simplex of $\nabla$, $g$ coincides with $f$ at each vertex of $\nabla_P$ and $g(v) = 0$ for each vertex $v$ of $\nabla$ not belonging to $P$. Then $f = g \restriction P$. Evidently, $g$ is piecewise linear. □

For any polyhedron $P \subseteq \mathbb{R}^n$, we denote by $\mathcal{M}_{\mathbb{R}}(P)$ the MV-algebra of all functions $f : P \to [0, 1]$ satisfying any (hence all) of the equivalent conditions (i)–(iii) above.

Now suppose the polyhedron $Q$ is contained in $[0, 1]^n$. As in [15, 4.4] or [17, 9.17], the *semisimple tensor product* $[0, 1] \otimes \mathcal{M}(Q)$ can be identified with the MV-algebra of continuous functions from $Q$ into $[0, 1]$ generated by the *pure tensors* $\rho \cdot g = \rho \otimes g$, where $\rho \in [0, 1]$ and $g \in \mathcal{M}(Q)$.

In Theorem 2.4 we will prove that, up to isomorphism, polyhedral MV-algebras coincide with finitely generated subalgebras of a semisimple tensor product $[0, 1] \otimes \mathcal{M}(R)$, for some *rational* polyhedron $R \subseteq [0, 1]^n$, $n = 1, 2, \ldots$. We prepare:

**Lemma 2.2.** *Up to isomorphism,* $[0, 1] \otimes \mathcal{M}([0, 1]^n) = \mathcal{M}_{\mathbb{R}}([0, 1]^n)$.

**Proof.** The inclusion $[0, 1] \otimes \mathcal{M}([0, 1]^n) \subseteq \mathcal{M}_{\mathbb{R}}([0, 1]^n)$ is immediately verified, because the MV-algebra $[0, 1] \otimes \mathcal{M}([0, 1]^n)$ is generated by its pure tensors, each pure tensor $\rho \otimes f = \rho \cdot f$ belongs to $\mathcal{M}_{\mathbb{R}}([0, 1]^n)$, and piecewise linearity is preserved by the MV-algebraic operations.

To prove the converse inclusion $[0, 1] \otimes \mathcal{M}([0, 1]^n) \supseteq \mathcal{M}_{\mathbb{R}}([0, 1]^n)$, we make the following:

**Claim.** *Every* truncated linear *map*

$$t(x) = t(x_1, \ldots, x_n) = 1 \wedge \big(0 \vee (\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n)\big)$$

*defined on* $[0, 1]^n$*, with real coefficients* $\alpha_0, \ldots, \alpha_n$*, belongs to* $[0, 1] \otimes \mathcal{M}([0, 1]^n)$.

The claim is trivially true for every constant function $f(x) = \rho$ ($\rho \in [0, 1]$), because $f$ is the pure tensor $\rho \otimes 1$ of $[0, 1] \otimes \mathcal{M}([0, 1]^n)$. Inductively, we may assume that the function $t$ depends on all its variables, whence each of $\alpha_1, \ldots, \alpha_n$ is nonzero, and

$$0 < t(x) < 1 \quad \text{for some } x \in [0, 1]^n. \tag{1}$$

Now let us agree to say that a function $f : [0, 1]^n \to \mathbb{R}$ is *flat* if it has the form

$$f(x) = f(x_1, \ldots, x_n) = \beta_0 + \sum_{i \in I} \beta_i x_i + \sum_{j \in J} \beta_j (1 - x_j), \tag{2}$$

where $I \cap J = \emptyset$, $I \cup J = \{1, \ldots, n\}$, $\beta_0, \beta_1, \ldots, \beta_n \geq 0$, and $\beta_0 + \beta_1 + \cdots + \beta_n \leq 1$. The graph of $f$ is linear. Let $v = (v_1, \ldots, v_n)$ be the vertex of the $n$-cube $[0, 1]^n$ given by $v_i = 0$ for $i \in I$ and $v_j = 1$ for $j \in J$. Also let $w = (w_1, \ldots, w_n)$ be the vertex of the $n$-cube $[0, 1]^n$ given by $w_i = 1$ for $i \in I$ and $w_j = 0$ for $j \in J$. Then $f(v) = \beta_0$ is the minimum value of $f$, and $f(w) = \beta_0 + \beta_1 + \cdots + \beta_n$ is the maximum. The constant function $\beta_0 = \beta_0 \otimes 1$ is a pure tensor of $[0, 1] \otimes \mathcal{M}([0, 1]^n)$. For each $k \in \{1, \ldots, n\}$, letting $\pi_k : [0, 1]^n \to [0, 1]$ denote the $k$th coordinate projection, define $\pi_k^* = \pi_k$ if $k \in I$, and $\pi_k^* = \neg \pi_k$ if $k \in J$. Then $\beta_k \pi_k^* = \beta_k \otimes \pi_k^* \in [0, 1] \otimes \mathcal{M}([0, 1]^n)$. A direct inspection shows that $f = \beta_0 + \beta_1 \pi_1^* + \cdots + \beta_n \pi_n^* = \beta_0 \oplus \beta_1 \pi_1^* \oplus \cdots \oplus \beta_n \pi_n^*$, whence $f$ belongs to $[0, 1] \otimes \mathcal{M}([0, 1]^n)$.

Next let us say that the function $g : [0, 1]^n \to \mathbb{R}$ is *subflat* if for some flat $f$ as in (2) with $\beta_0 = 0$, and $\sigma$ with $0 < \sigma < 1$, $g$ has the form $g(x) = f(x) \ominus \sigma = f(x) \odot \neg \sigma = f(x) \odot (1 - \sigma)$. Recalling (1), the graph of $g$ consists of two linear pieces. Again, $g$ belongs to $[0, 1] \otimes \mathcal{M}([0, 1]^n)$, because it is obtained from $f \in [0, 1] \otimes \mathcal{M}([0, 1]^n)$ and the pure tensor $\sigma \otimes 1$ via MV-algebraic operations.

To conclude the proof of the claim it is enough to prove that our truncated linear function $t(x) = 1 \wedge (0 \vee (\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n))$ has the form

$$t(x) = m.r(x), \quad \text{for some integer } m \geq 0 \text{ and } r \text{ either flat or subflat.}$$

Following [7, p. 33], we let $m.r$ denote $m$-fold iterated application of the $\oplus$ operation. Letting $l(x) = l(x_1, \ldots, x_n) = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$, there are two possible cases:

*Case 1.* There is no $x \in [0, 1]^n$ such that $l(x) = 0$. Then recalling (1), for all large integers $m > 0$, the range of the function $l(x)/m$ is contained in the open interval $\{\beta \in \mathbb{R} \mid 0 < \beta < 1\}$. The function $l(x)/m$ is flat, whence it belongs to $[0, 1] \otimes \mathcal{M}([0, 1]^n)$, and so does the function $t = m.l/m$.

*Case 2.* There is an $x \in [0, 1]^n$ such that $l(x) = 0$. Then for all large integers $m > 0$, the range of the function $l(x)/m$ is contained in the interval $\{\beta \in \mathbb{R} \mid -1 < \beta < 1\}$. The function $0 \vee l(x)/m$ is subflat, whence it belongs to $[0, 1] \otimes \mathcal{M}([0, 1]^n)$, and so does $t = m.(0 \vee l(x)/m)$.

Having thus settled our claim, we end the proof by recalling that every function $f \in \mathcal{M}_{\mathbb{R}}([0, 1]^n)$ can be written as $f = \bigvee_i \bigwedge_j t_{i,j}$ for suitable truncated linear functions $t_{i,j}$ (the latter belonging to $[0, 1] \otimes \mathcal{M}([0, 1]^n)$ by our claim). The familiar proof follows, e.g., by a routine adaptation of the proof of [7, 9.1.4(ii)]. Since $f$ is obtained from the $t_{i,j}$ via the MV-algebraic operations $\vee, \wedge$ then $f$ belongs to $[0, 1] \otimes \mathcal{M}([0, 1]^n)$. □

Generalizing the above lemma we next prove:

**Theorem 2.3.** *For any polyhedron* $P \subseteq [0, 1]^n$*, the semisimple tensor product* $[0, 1] \otimes \mathcal{M}(P)$ *is* (*isomorphic to*) *the MV-algebra* $\mathcal{M}_{\mathbb{R}}(P)$.

**Proof.** For the inclusion $\mathcal{M}_{\mathbb{R}}(P) \supseteq [0,1] \otimes \mathcal{M}(P)$ one observes that each pure tensor $\sigma \otimes f$ of $[0,1] \otimes \mathcal{M}(P)$ is piecewise linear.

We now prove the converse inclusion $\mathcal{M}_{\mathbb{R}}(P) \subseteq [0,1] \otimes \mathcal{M}(P)$. The restriction to $P$ of any pure tensor $\rho \otimes f = \rho \cdot f : [0,1]^n \to [0,1]$ ($\rho \in [0,1]$, $f \in \mathcal{M}([0,1]^n)$), is a pure tensor of $[0,1] \otimes \mathcal{M}(P)$, because $(\rho \cdot f) \upharpoonright P = \rho \cdot (f \upharpoonright P) = \rho \otimes (f \upharpoonright P)$ and $f \upharpoonright P$ belongs to $\mathcal{M}(P)$ by definition. On the other hand, every pure tensor $\sigma \otimes g : P \to [0,1]$ ($\sigma \in [0,1]$, $g \in \mathcal{M}(P)$), is the restriction to $P$ of some pure tensor $\sigma \otimes h$, $h \in \mathcal{M}([0,1]^n)$. To see this, recalling Lemma 2.1, let $h$ be such that $g = h \upharpoonright P$ and write $(\sigma \otimes h) \upharpoonright P = (\sigma \cdot h) \upharpoonright P = \sigma \cdot (h \upharpoonright P) = \sigma \cdot g = \sigma \otimes g$. Thus the restriction map $\eta : l \in [0,1] \otimes \mathcal{M}([0,1]^n) \mapsto l \upharpoonright P$ is a homomorphism of $[0,1] \otimes \mathcal{M}([0,1]^n)$ onto $[0,1] \otimes \mathcal{M}(P)$ because $\eta$ maps the set of pure tensors of $[0,1] \otimes \mathcal{M}([0,1]^n)$ onto the set of pure tensors of $[0,1] \otimes \mathcal{M}(P)$. By Lemma 2.2, $[0,1] \otimes \mathcal{M}([0,1]^n) = \mathcal{M}_{\mathbb{R}}([0,1]^n)$. So $[0,1] \otimes \mathcal{M}(P)$ contains every function $k \in \mathcal{M}_{\mathbb{R}}(P)$, because any such $k$ is extendible to a function of $\mathcal{M}_{\mathbb{R}}([0,1]^n)$, again by Lemma 2.1. We have proved the inclusion $\mathcal{M}_{\mathbb{R}}(P) \subseteq [0,1] \otimes \mathcal{M}(P)$. □

**Theorem 2.4.** *An MV-algebra $B$ is isomorphic to $\mathcal{M}(Q)$ for some polyhedron $Q \subseteq [0,1]^m$ ($m = 1, 2, \ldots$) iff it is isomorphic to a finitely generated subalgebra of a semisimple tensor product of the form $[0,1] \otimes \mathcal{M}(P)$ for some rational polyhedron $P \subseteq [0,1]^n$, $n = 1, 2, \ldots$.*

**Proof.** ($\Rightarrow$) Let $\Delta$ be a triangulation of $Q$, with its vertices $v_1, \ldots, v_d$. The underlying *abstract* simplicial complex of $\Delta$ has a geometric realization in $[0,1]^d$ sending each $v_i$ to the unit vector $e_i$ along the $i$th axis of $\mathbb{R}^d$, in such a way that $e_1, \ldots, e_d$ are the vertices of a triangulation $\Delta'$ of a *rational* polyhedron $P \subseteq [0,1]^d$, and the map $e_i \mapsto v_i$ determines a piecewise linear homeomorphism $h = (h_1, \ldots, h_m)$ of $P$ onto $Q$, with $h$ linear on each simplex of $\Delta'$. Thus in particular, for every $i = 1, \ldots, m$ the function $h_i$ belongs to $\mathcal{M}_{\mathbb{R}}(P)$. By Theorem 2.3 each $h_i$ belongs to $[0,1] \otimes \mathcal{M}(P)$. Let $A$ be the subalgebra of $[0,1] \otimes \mathcal{M}(P)$ generated by $\{h_1, \ldots, h_m\}$. A routine modification of the argument used for the proof of [17, 3.6] shows that $A$ is isomorphic to $\mathcal{M}(h(P)) = \mathcal{M}(Q) = B$: specifically, letting $\circ$ denote composition, the map $f \in \mathcal{M}(Q) \mapsto f \circ h$ provides an isomorphism of $\mathcal{M}(Q)$ onto $A$.

($\Leftarrow$) Suppose $B \subseteq [0,1] \otimes \mathcal{M}(P)$ is generated by $g_1, \ldots, g_m$. By Theorem 2.3, each generator $g_i$ is a member of $\mathcal{M}_{\mathbb{R}}(P)$. Let the continuous map $g : P \to [0,1]^m$ be defined by $g(x) = (g_1(x), \ldots, g_m(x))$. Then the map $f \in \mathcal{M}(g(P)) \mapsto f \circ g$ is an isomorphism of $\mathcal{M}(g(P))$ onto $B$. Further, the image $Q = g(P) \subseteq [0,1]^m$ of the rational polyhedron $P$ under the map $g$ is a polyhedron (see, e.g., [19, 1.6.8]). We conclude that $B$ is isomorphic to the polyhedral MV-algebra $\mathcal{M}(Q)$. □

**Remark 2.5.** In Theorem 3.6 we will see that the restriction $Q \subseteq [0,1]^n$ is immaterial, and the above characterization holds for every polyhedral MV-algebra.

## 3. $\mathbb{Z}$-maps and the polyhedral duality $\mathcal{M}$

We refer to [1] and [12] for all unexplained notions in category theory. In Corollary 3.5 we will introduce a duality between polyhedral MV-algebras and polyhedra. As a preliminary step, in Theorem 3.3 we give a self-contained proof of the duality [13] between compact sets in euclidean spaces and finitely generated semisimple MV-algebras.

**Definition 3.1.** Given integers $n, m > 0$ together with rational polyhedra $P \subseteq \mathbb{R}^n$ and $Q \in \mathbb{R}^m$, a piecewise linear map with integer coefficients $\xi : P \to Q$ is called a $\mathbb{Z}$*-map*. More generally, given compact sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, a map $\eta : X \to Y$ is called a $\mathbb{Z}$*-map* if there exist rational polyhedra $X \subseteq P$ and $Y \subseteq Q$, and a $\mathbb{Z}$-map $\xi : P \to Q$ such that $\eta = \xi \upharpoonright X$.

We let

$$\mathbb{Z}(X, Y) = \{\eta : X \to Y \mid \eta \text{ is a } \mathbb{Z}\text{-map}\}. \tag{3}$$

By Lemma 2.1, for any polyhedron $P \subseteq \mathbb{R}^n$ we have $\mathcal{M}(P) = \mathbb{Z}(P, [0,1])$.

*Notation* We let $\mathcal{C}$ denote the category whose objects are compact subsets of $\mathbb{R}^n$ ($n = 1, 2, \ldots$) and whose morphisms are $\mathbb{Z}$-maps. We further let $\mathcal{S}$ be the full subcategory of MV-algebras whose objects are finitely generated semisimple MV-algebras.

Let $P$ and $Q$ be rational polyhedra. If (and only if) $P$ and $Q$ are $\mathcal{C}$-isomorphic then there exists an injective surjective $\mathbb{Z}$-map $\eta: P \to Q$ such that $\eta^{-1}$ is also a $\mathbb{Z}$-map. Following [17, 3.1], we then say that $\eta$ is a *$\mathbb{Z}$-homeomorphism*, and that $P$ and $Q$ are *$\mathbb{Z}$-homeomorphic*, in symbols, $P \cong_{\mathbb{Z}} Q$.

In Theorem 3.3 we will see that $\mathcal{C}$ and $\mathcal{S}$ are dually equivalent. We prepare:

**Lemma 3.2.** *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $X, Y \in \mathcal{C}$.*

(i) *The image $\xi(P)$ of $P$ under a $\mathbb{Z}$-map $\xi: P \to \mathbb{R}^m$ is a polyhedron.*
(ii) *For some $n \in \{1, 2, \ldots\}$ there is $W \subseteq [0, 1]^n$ such that $W \in \mathcal{C}$ and $W$ is $\mathcal{C}$-isomorphic to $X$.*
(iii) *If $X \subseteq Y$ then for each $y \in Y \setminus X$ there exists a $\mathbb{Z}$-map $\gamma: Y \to [0, 1]$ such that $\gamma(X) = 0$ and $\gamma(y) = 1$.*

**Proof.** (i) This is [19, 1.6.8]. (ii) Let $R \subseteq \mathbb{R}^n$ be a rational polyhedron containing $X$. By [16, p. 1040] or [14, 3.5], each rational polyhedron $R$ is $\mathcal{C}$-isomorphic (i.e., $\mathbb{Z}$-homeomorphic) to a rational polyhedron $Q \subseteq [0, 1]^n$ for some $n$. Let $\eta: R \to Q$ be a $\mathcal{C}$-isomorphism between $R$ and $Q$ and $W = \eta(X)$. Then $\eta \restriction X: X \to W$ determines a $\mathcal{C}$-isomorphism between $X$ and $W \subseteq Q \subseteq [0, 1]^n$. (iii) Straightforward from [17, 3.7]. This is also a particular case of complete regularity by definable functions, see [13, Lemma 3.5]. $\square$

**Theorem 3.3** *(Duality). Let the functor $\mathcal{M}: \mathcal{C} \to \mathcal{S}$ be defined by:*

Objects: *For any $X \in \mathcal{C}$, $\mathcal{M}(X)$ is the MV-algebra $\mathbb{Z}(X, [0, 1])$ of $\mathbb{Z}$-maps as in (3), with the pointwise MV-algebraic operations of* $[0, 1]$.
Arrows: *For every $X, Y \in \mathcal{C}$ and $\mathbb{Z}$-map $\eta: X \to Y$, $\mathcal{M}(\eta)$ is the map transforming each $f \in \mathcal{M}(Y)$ into the composite function $f \circ \eta$.*

*Then $\mathcal{M}$ is a categorical equivalence between $\mathcal{C}$ and the opposite category $\mathcal{S}^{\mathrm{op}}$ of $\mathcal{S}$. For short, $\mathcal{M}$ is a* duality *between the categories $\mathcal{C}$ and $\mathcal{S}$.*

**Proof.** Once $\mathcal{M}(X)$ is stripped of its MV-structure, $\mathcal{M}$ is just the contravariant hom-functor $\mathbb{Z}(-, [0, 1])$. It is immediately verified that $\mathcal{M}(\eta)$ is an MV-homomorphism of $\mathcal{M}(Y)$ into $\mathcal{M}(X)$ for each $\mathbb{Z}$-map $\eta: X \to Y$.

**Claim 1.** *$\mathcal{M}$ is faithful.*

Let $X, Y \in \mathcal{C}$ and $\eta, \eta' \in \mathbb{Z}(X, Y)$ be such that $\eta \neq \eta'$. Let $x \in X$ be such that $\eta(x) \neq \eta'(x)$. By Lemma 3.2(iii) there exists $f \in \mathcal{M}(Y)$ such that $f(\eta(x)) \neq f(\eta'(x))$. Thus $(\mathcal{M}(\eta))(f) \neq (\mathcal{M}(\eta'))(f)$.

**Claim 2.** *$\mathcal{M}$ is full.*

Let $h: \mathcal{M}(X) \to \mathcal{M}(Y)$ be a homomorphism. By Lemma 3.2(ii) we can assume $X \subseteq [0, 1]^m$. For each $i \in \{1, \ldots, m\}$ let $\pi_i: [0, 1]^m \to [0, 1]$ be the $i$th coordinate function. Since each $\pi_i$ is a $\mathbb{Z}$-map, the restriction $\pi_i \restriction X$ is a member of $\mathcal{M}(X)$. For each $i \in \{1, \ldots, m\}$ there exists a rational polyhedron $P_i \supseteq Y$ together with a $\mathbb{Z}$-map $f_i: P_i \to [0, 1]$ satisfying $h(\pi_i \restriction X) = f_i \restriction Y$. Let the $\mathbb{Z}$-map $\eta: P_1 \cap \cdots \cap P_m \to [0, 1]^m$ be defined by $\eta(x) = (f_1(x), \ldots, f_m(x))$. Since $\{\pi_1, \ldots, \pi_m\}$ is a generating set of $\mathcal{M}([0, 1]^m)$ and $h(\pi_i \restriction X) = f_i = (\pi_i \circ \eta) \restriction Y$, then

$$h(g \restriction X) = (g \circ \eta) \restriction Y, \quad \text{for each } g \in \mathcal{M}\big([0, 1]^m\big).$$

By Lemma 3.2(iii), for each $x \in [0, 1]^m \setminus X$ there is a $\mathbb{Z}$-map, $g: [0, 1]^m \to [0, 1]$ such that $g(X) = 0$ and $g(x) \neq 0$. Thus $h(g \restriction X) = 0 = (g \circ \eta) \restriction Y$ and $x \notin \eta(Y)$. From the inclusion $\eta(Y) \subseteq X$ it follows that $\eta \restriction Y: Y \to X$ is a $\mathbb{Z}$-map and $h = \mathcal{M}(\eta)$.

**Claim 3.** *For each $A \in \mathcal{S}$ there exists $X \in \mathcal{C}$ such that $A \cong \mathcal{M}(X)$.*

This follows from [7, 3.6.7].

Having thus proved Claims 1–3, an application of [12, IV.4.1] yields the desired conclusion. □

**Remark 3.4.** In [13, §3] the authors establish a duality between compact sets $K \subseteq [0, 1]^n$, $n = 1, 2, \ldots$ equipped with "definable maps", and finitely generated semisimple MV-algebras. Theorem 3.3 is an equivalent reformulation of that duality, modulo the following observations:

(i) The $\mathbb{Z}$-maps of Definition 3.1 are a concrete geometric description of the "definable" functions of [13, 1.1].
(ii) In view of Lemma 3.2(ii), polyhedra in $\mathbb{R}^n$ can be safely assumed to be contained in some cube $[0, 1]^m$.

*Notation* We let $\mathcal{P}$ denote the full subcategory of $\mathcal{C}$ whose objects are (not necessarily rational) polyhedra. By $\mathcal{MV}_{\text{poly}}$ we denote the full subcategory of $\mathcal{S}$ of polyhedral MV-algebras.

**Corollary 3.5.** *The restriction to $\mathcal{P}$ of the functor $\mathcal{M}: \mathcal{C} \to \mathcal{S}$ of Theorem 3.3 yields a duality between the categories $\mathcal{P}$ and $\mathcal{MV}_{\text{poly}}$. Moreover, by Lemma 3.2(i), the class of polyhedra is* closed under $\mathcal{C}$-isomorphisms*: whenever a polyhedron $P$ is $\mathcal{C}$-isomorphic to $X \in \mathcal{C}$, then $X$ is a polyhedron.* □

From this corollary together with Lemma 3.2(iii), Theorem 2.4 acquires the following general form:

**Theorem 3.6.** *An MV-algebra is polyhedral iff it is isomorphic to a finitely generated subalgebra of a semisimple tensor product $S \otimes F$, where $S$ is (finitely generated and) simple, and $F$ is finitely presented.*

**Proof.** Up to isomorphism, simple MV-algebras coincide with subalgebras of $[0, 1]$, [7, 3.5.1], and finitely presented MV-algebras coincide with algebras of the form $\mathcal{M}(P)$ as $P$ ranges over rational polyhedra, [17, 6.3]. □

## 4. Amalgamation and coproducts

### *4.1. Amalgamation of polyhedral MV-algebras*

It is well-known that the variety of MV-algebras has the amalgamation property (see [17, §2] and references therein). The same holds for finitely presented MV-algebras [17, 6.7], and for MV-chains [2]. We will prove that both finitely generated semisimple MV-algebras and polyhedral MV-algebras also have the amalgamation property. We prepare:

**Lemma 4.1.** *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be compact sets and $\eta : X \to Y$ a $\mathbb{Z}$-map. Then the following conditions are equivalent:*

(i) *$\eta$ is an epimorphism in $\mathcal{C}$;*
(ii) *$\eta$ is onto $Y$;*
(iii) *$\mathcal{M}(\eta)$ is one–one.*

**Proof.** The equivalence (i) ⇔ (iii) follows directly from Theorem 3.3, upon noting that in any variety of algebras monomorphisms are the same as injective homomorphisms.

The implication (ii)⇒(i) is valid in any concrete category over the category of sets. For the converse implication, by way of contradiction suppose that $\eta$ is an epimorphism in $\mathcal{C}$ but is not onto $Y$. Let $y \in Y \setminus \eta(X)$. By Lemma 3.2(iii) there exists a $\mathbb{Z}$-map $\gamma: Y \to [0, 1]$ such that $\gamma(\eta(X)) = 0$ and $\gamma(y) = 1$. Now let $\gamma'$ be the constant zero map on $Y$. It is easy to see that $\gamma \circ \eta = \gamma' \circ \eta$, but $\gamma \neq \gamma'$, thus contradicting the assumption that $\eta$ is an epimorphism. □

**Theorem 4.2.** *Both finitely generated semisimple MV-algebras and polyhedral MV-algebras have the amalgamation property.*

**Proof.** Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^k$ be objects in $\mathcal{C}$. Let $\eta: X \to Z$ and $\gamma: Y \to Z$ be surjective $\mathbb{Z}$-maps. Then the set

$$X \times_Z Y = \{(x, y) \in X \times Y \mid \eta(x) = \gamma(y)\} \subseteq \mathbb{R}^m$$

is a closed subset of $X \times Y$, and hence is an object of $\mathcal{C}$. From the surjectivity of $\eta$ and $\gamma$, and the definition of $X \times_Z Y$ it follows that the projection maps $\pi_X: X \times_Z Y \to X$ and $\pi_Y: X \times_Z Y \to Y$ are surjective $\mathbb{Z}$-maps. By Theorem 3.3 and Lemma 4.1, the category $\mathcal{S}$ has the amalgamation property.

For any polyhedra $X, Y$ and $Z$, suppose we have surjective $\mathbb{Z}$-maps $\eta: X \to Z$ and $\gamma: Y \to Z$. In view of [19, 2.2.4], let $\Delta$ be a triangulation of $X \times Y$ such that both maps $\eta \circ \pi_X$ and $\gamma \circ \pi_Y$ are linear on each simplex of $\Delta$. The set

$$X \times_Z Y = \{(x, y) \in X \times Y \mid \eta(x) = \gamma(y)\} = \bigcup \{T \in \Delta \mid \eta \circ \pi_X \restriction T = \gamma \circ \pi_Y \restriction T\}$$

is a polyhedron. Then the amalgamation property of polyhedral MV-algebras again follows from Theorem 3.3 and Lemma 4.1. □

### *4.2. Coproducts of polyhedral MV-algebras*

We denote by $\mathcal{SMV}$ the class of semisimple MV-algebras. We will use the notation $\coprod_{\mathcal{S}}$ for $\mathcal{S}$-coproducts, $\coprod_{\mathcal{SMV}}$ for $\mathcal{SMV}$-coproducts, and $\coprod_{\mathcal{MV}}$ for $\mathcal{MV}$-coproducts. A moment's reflection shows that finite $\mathcal{MV}$-coproducts coincide with the finite free products of [17, §7].

The category $\mathcal{C}$ admits finite products, that turn out to coincide with cartesian products. Since a product of two polyhedra is a polyhedron (see [19, p. 29]), then also the category $\mathcal{P}$ has finite products. By Theorem 3.3, for all $P_1, P_2 \in \mathcal{P}$ and $X_1, X_2 \in \mathcal{C}$ we then have

$$\mathcal{M}(P_1 \times P_2) \cong \mathcal{M}(P_1) \coprod_{\mathcal{MV}_{\mathrm{poly}}} \mathcal{M}(P_2) \tag{4}$$

and

$$\mathcal{M}(X_1 \times X_2) \cong \mathcal{M}(X_1) \coprod_{\mathcal{S}} \mathcal{M}(X_2). \tag{5}$$

In [17, 7.3] it is shown that both categories $\mathcal{S}$ (the dual of $\mathcal{C}$) and $\mathcal{MV}_{\mathrm{poly}}$ (the dual of $\mathcal{P}$) *are not closed under finite $\mathcal{MV}$-coproducts.*

**Proposition 4.3.** *Let $\mathfrak{K}$ be a set of algebras in $\mathcal{SMV}$. The $\mathcal{SMV}$-coproduct $\coprod_{\mathcal{SMV}} \mathfrak{K}$ coincides with $\mathcal{MV}$-coproduct $\coprod_{\mathcal{MV}} \mathfrak{K}$ modulo the radical of $\coprod_{\mathcal{MV}} \mathfrak{K}$, in symbols,*

$$\coprod_{\mathcal{SMV}} \mathfrak{K} \cong \Big(\coprod_{\mathcal{MV}} \mathfrak{K}\Big) \Big/ \mathrm{Rad}\Big(\coprod_{\mathcal{MV}} \mathfrak{K}\Big).$$

*Moreover, if $\mathcal{O}$ is any of the two categories $\mathcal{S}$ or $\mathcal{MV}_{\mathrm{poly}}$ and $\mathfrak{K}$ is a finite set of algebras in $\mathcal{O}$, then*

$$\coprod_{\mathcal{O}} \mathfrak{K} = \coprod_{\mathcal{SMV}} \mathfrak{K}.$$

**Proof.** Letting $\mathrm{Rad(A)} = \bigcap\{\text{all maximal ideals of } A\}$, the map $A \mapsto A/\mathrm{Rad}(A)$ determines a functor $R: \mathcal{MV} \to \mathcal{SMV}$. Since $\mathcal{SMV}$ is closed under subalgebras and (finite as well as infinite) cartesian products [7, 3.6.4], the functor $R$ is the left adjoint of the inclusion functor form $\mathcal{SMV}$ to $\mathcal{MV}$. By [12, V.5], for each (finite or infinite) set $\mathfrak{K}$ of semisimple algebras, $\coprod_{\mathcal{SMV}} \mathfrak{K}$ coincides with $R(\coprod_{\mathcal{MV}} \mathfrak{K}) = (\coprod_{\mathcal{MV}} \mathfrak{K})/\mathrm{Rad}(\coprod_{\mathcal{MV}} \mathfrak{K})$.

For the second statement, in case $\mathcal{O} = \mathcal{S}$, suppose $A, B \in \mathcal{S}$. Then $A \coprod_{\mathcal{SMV}} B$ is finitely generated, whence $A \coprod_{\mathcal{S}} B \cong A \coprod_{\mathcal{SMV}} B$. In case $\mathcal{O} = \mathcal{SMV}$, suppose $P_1 \subseteq \mathbb{R}^n$ and $P_2 \subseteq \mathbb{R}^k$ to be polyhedra. By (4)–(5),

$$\mathcal{M}(P_1) \coprod_{\mathcal{MV}_{\text{poly}}} \mathcal{M}(P_2) \cong \mathcal{M}(P_1 \times P_2) \cong \mathcal{M}(P_1) \coprod_{\mathcal{S}} \mathcal{M}(P_2)$$
$$\cong \mathcal{M}(P_1) \coprod_{\mathcal{SMV}} \mathcal{M}(P_2). \qquad \square$$

From [17, 7.9(iv)], it follows that whenever $P$ and $Q$ are rational polyhedra then $\mathcal{M}(P) \coprod_{\mathcal{MV}} \mathcal{M}(Q)$ is the (rational) polyhedral MV-algebra $\mathcal{M}(P \times Q)$. One may now naturally look for more general classes of polyhedral MV-algebras having a polyhedral finite $\mathcal{MV}$-coproduct.

## 5. Polyhedral MV-algebras are strongly semisimple

Following Dubuc and Poveda [8], we say that an MV-algebra $A$ is *strongly semisimple* if for every principal ideal $J \neq A$ of $A$, the quotient $A/J$ is semisimple. Every strongly semisimple MV-algebra is semisimple (because $\{0\}$ is a principal ideal of $A$). Trivially, all hyperarchimedean MV-algebras, whence in particular all boolean algebras, are strongly semisimple. By [7, 3.5 and 3.6.5], all simple and all finite MV-algebras are strongly semisimple. By [10] or [20], every finitely presented MV-algebra is strongly semisimple.

For every set $E$ and real-valued function $f$ on $E$ we denote by $Zf$ the *zeroset* of $f$, in symbols,

$$Zf = \{x \in E \mid f(x) = 0\}.$$

By [7, 3.6.7], every polyhedral MV-algebra $A$ is semisimple. The following stronger result is also a generalization of the Hay–Wójcicki theorem [10,20] (also see [7, 4.6.7] and [17, 1.6]).

**Theorem 5.1.** *Any polyhedral MV-algebra $A$ is strongly semisimple.*

**Proof.** Lemma 3.2(ii) yields a polyhedron $P \subseteq [0, 1]^n$ (for some integer $n > 0$) such that $A \cong \mathcal{M}(P)$. For every $f \in \mathcal{M}([0, 1]^n)$ we will mostly use the abbreviated notation $f^\diamond$ for $f \upharpoonright P$. For any $g \in \mathcal{M}([0, 1]^n)$ we will write $\langle g^\diamond \rangle$ for the principal ideal of $\mathcal{M}(P)$ generated by $g^\diamond$,

$$\langle g^\diamond \rangle = \langle g \upharpoonright P \rangle = \{f^\diamond \in \mathcal{M}(P) \mid f^\diamond \leq m.g^\diamond \text{ for some } m = 0, 1, \ldots\}. \tag{6}$$

We are tacitly assuming $\langle g^\diamond \rangle \neq \mathcal{M}(P)$, whence the quotient $\mathcal{M}(P)/\langle g^\diamond \rangle$ is nontrivial, and the zeroset $Zg^\diamond \subseteq [0, 1]^n$ is nonempty.

**Claim.** *$\langle g^\diamond \rangle$ is an intersection of maximal ideals of $\mathcal{M}(P)$.*

As a matter of fact, let $\langle g \rangle$ be the ideal of $\mathcal{M}([0, 1]^n)$ generated by $g$. Let $\langle g \rangle \upharpoonright P$ be the set of restrictions to $P$ of the elements of $\langle g \rangle$, in symbols,

$$\langle g \rangle \upharpoonright P = \{f \upharpoonright P \mid f \in \mathcal{M}([0, 1]^n) \text{ and } f \leq m.g \text{ for some } m = 0, 1, \ldots\}.$$

From (6) we immediately obtain the identity

$$\langle g^\diamond \rangle = \langle g \upharpoonright P \rangle = \langle g \rangle \upharpoonright P. \tag{7}$$

For all $f \in \mathcal{M}([0, 1]^n)$ we will next prove the equivalence:

$$f^\diamond \in \langle g^\diamond \rangle \quad \Leftrightarrow \quad Zf^\diamond \supseteq Zg^\diamond. \tag{8}$$

The ($\Rightarrow$)-direction is an immediate consequence of (7). For the ($\Leftarrow$)-direction, let $f \in \mathcal{M}([0, 1]^n)$ be such that $Zf^\diamond \supseteq Zg^\diamond$, with the intent of proving

$$\text{there is } m = 0, 1, \ldots \text{ satisfying } m.g \geq f \text{ on } P. \tag{9}$$

To this aim, let $\Delta$ be a triangulation of $[0, 1]^n$ such that $g$ and $f$ are linear on each simplex of $\Delta$, and

$$\bigcup \{T \in \Delta \mid T \subseteq P\} = P. \tag{10}$$

Since $P$ is a polyhedron and $f, g$ are piecewise linear, $\Delta$ is given by an elementary construction in polyhedral topology [19, 2.2.6]. Let $T = \mathrm{conv}(v_0, \ldots, v_r)$ be an arbitrary simplex of $\Delta$. Fix a vertex $v_i$ of $T$. Since $T \subseteq P$ and $Zf^\diamond \supseteq Zg^\diamond$, it is impossible to have $g(v_i) = 0$ and $f(v_i) > 0$ simultaneously. So we consider the following two cases:

(I) $g(v_i) > 0$. Then letting $\mu_i = 1/g(v_i)$ we have $1 = \mu_i g(v_i) \geq f(v_i)$.
(II) $g(v_i) = f(v_i) = 0$. Then, letting $\mu_i = 1$ we have $0 = \mu_i g(v_i) \geq f(v_i) = 0$.

Upon setting

$$m_T = \text{the smallest integer} \geq \max(\mu_0, \ldots, \mu_r),$$

from the linearity of $g$ on $T$ it follows that $m_T.g \geq f$ on $T$. The function $m_T.g$ does belong to $\mathcal{M}([0,1]^n)$. Thus for each $T \in \Delta$ with $T \subseteq P$ there is an integer $m_T \geq 0$ such that $m_T.g \geq f$ on $T$. Letting now $m = \max\{m_T \mid T \in \Delta, T \subseteq P\}$ and recalling (10), we conclude that the McNaughton function $m.g \in \mathcal{M}([0,1]^n)$ satisfies $m.g \geq f$ on $P$. This concludes the proof of (9), as well as of (8). For each $x \in P$, let $J_x$ be the maximal ideal of $\mathcal{M}(P)$ given by of all functions of $\mathcal{M}(P)$ that vanish at $x$. Combining [7, 3.4.3] with (8), for arbitrary $f \in \mathcal{M}([0,1]^n)$ we have: $f^\diamond \in \langle g^\diamond \rangle \Leftrightarrow Zf^\diamond \supseteq Zg^\diamond \Leftrightarrow f^\diamond \in \bigcap\{J_z \mid z \in Zg^\diamond\}$, thus settling our claim.

By [7, 3.6.6], the quotient MV-algebra $\mathcal{M}(P)/\langle g^\diamond \rangle$ is semisimple. We conclude that $A \cong \mathcal{M}(P)$ is strongly semisimple. □

**Remark 5.2.** A much less direct proof of Theorem 5.1 follows from the fact that polyhedra do not have outgoing Bouligand–Severi tangents (see [4, 2.4 and Theorem 3.4]). For $n = 1, 2$ the foregoing theorem is also a consequence of the results of [3].

**Corollary 5.3.** *Let $A$ be a polyhedral MV-algebra, $g \in A$, and $\langle g \rangle$ be the ideal of $A$ generated by $g$. Then the principal quotient $A/\langle g \rangle$ is polyhedral.*

**Proof.** By Lemma 3.2(iii) we can write $A = \mathcal{M}(P)$ for some polyhedron $P \subseteq [0,1]^n$. As proved in Theorem 5.1, $\langle g \rangle$ is an intersection of maximal ideals. By [7, 3.4.5], we have an isomorphism

$$\eta: f/\langle g \rangle \in \mathcal{M}(P)/\langle g \rangle \mapsto f \restriction V_{\langle g \rangle}, \quad \text{where } V_{\langle g \rangle} = \bigcap \left\{ Zl \;\middle|\; l \in \langle g \rangle \right\}.$$

From the proof of Theorem 5.1 we also have the identity $\langle g \rangle = \{l \in \mathcal{M}(P) \mid Zl \supseteq Zg\}$, whence $V_{\langle g \rangle} = Zg$ and $\eta$ is an isomorphism of $\mathcal{M}(P)/\langle g \rangle$ onto $\mathcal{M}(Zg)$. Since $g$ is a piecewise linear map, then $Zg$ is a polyhedron $Q \subseteq [0,1]^n$. We conclude that the principal quotient $\mathcal{M}(P)/\langle g \rangle \cong \mathcal{M}(Q)$ is polyhedral. □

Since polyhedra in the same ambient space $\mathbb{R}^n$ are closed under finite (disjoint) unions, then by duality polyhedral MV-algebras are closed under finite cartesian products. As a final preservation result, from Theorem 3.6 we immediately have:

**Proposition 5.4.** *Let $A$ be a polyhedral MV-algebra. Then any finitely generated MV-subalgebra of $A$ is polyhedral.*

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