Many real-world networks depend on other networks, often in nontrivial ways, to maintain their functionality. These interdependent “networks of networks” are often extremely fragile. When a fraction $1 - p$ of nodes in one network randomly fails, the damage propagates to nodes in networks that are interdependent and a dynamic failure cascade occurs that affects the entire system. We present dynamic equations for two interdependent networks that allow us to reproduce the failure cascade for an arbitrary pattern of interdependency. We study the “rich club” effect found in many real interdependent network systems in which the high-degree nodes are extremely interdependent, correlating a fraction $\alpha$ of the higher-degree nodes on each network. We find a rich phase diagram in the plane $p-\alpha$, with a triple point reminiscent of the triple point of liquids that separates a nonfunctional phase from two functional phases.

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Real-world infrastructures that provide essential services such as energy supply, transportation, and communications [1] can be understood as interdependent networks. Although this interdependency enhances the functionality of each network, it also increases the vulnerability of the entire system to attack or random failure [2]. In these interdependent infrastructures, the disruption of a small fraction of nodes in one network can generate a failure cascade that disconnects the entire system.

Failure cascades in real-world interdependent systems, such as the 2003 electrical blackout in Italy caused by failures in the telecommunications network [3], are physically explainable as abrupt percolating transitions [4–6]. In Ref. [4], the authors study the simplest case of two networks $A$ and $B$ of the same size $N$ with random interdependent nodes. Within each network the nodes are randomly connected through connectivity links, and pairs of nodes of different networks are randomly connected via one-to-one bidirectional interdependent links, enabling the failures to propagate through the links in either direction. The random failure of a fraction $1 - p$ of nodes in one network produces a failure cascade in both networks. As a consequence, the size of the giant component (GC) of each network, i.e., the still-functioning network within each network, dynamically decreases until the system reaches a steady state. Reference [4] describes the existence of a critical threshold $p_c$, which is a measure of the robustness of the entire network, below which the size of the functioning network within each network abruptly collapses as a first-order percolating transition and above which these functioning networks are preserved.

In many real systems, however, this interdependency is not fully random [7,8]. Instead, nodes of different networks connect to form a “rich club” in which a portion of high-degree nodes in one network depends on corresponding high-degree nodes in other networks. This occurs in trading and finance networks in which a well-integrated country in the global trade market is also well integrated in the financial system.

Another example of the nontrivial patterns of interdependency can be found in telecommunication networks in which important nodes often acquire a battery backup system in order to decrease their dependence on the electrical supply network. To understand the effect of these realistic features on failure cascades, some studies have focused separately on the correlation between the degrees of interdependent nodes [5,7] and the random or targeted autonomization [9–12]. In these studies, the original theoretical formalism [4] is reformulated to take into account these features.

In this Rapid Communication, we present a simple, unified theoretical framework that allows us to describe the dynamics of failure cascades in interdependent networks for an arbitrary interdependency between networks. We apply our framework to interdependent heterogeneous networks when a fraction $\alpha$ of the higher-degree nodes is interdependent, and a fraction $1 - \alpha$ is randomly dependent. Here $\alpha$ is a parameter that controls the level of correlation and allows us to explore its effect on system robustness.

We consider for simplicity, but without loss of generality, two networks $A$ and $B$ in which the degree distribution of the connectivity links is given by $P[k_A]$ and $P[k_B]$, where $k_A$ and $k_B$ are the connectivity links of nodes in $A$ and $B$, respectively. We define $q_A[k_A,k_B]$ and $q_B[k_A,k_B]$ as the fraction of nodes in network $A$ ($B$) that depends on network $B$ ($A$). When $q_i[k_A,k_B] = 1$ (with $i = A, B$) the system is one to one and all the interdependent links are bidirectional, and for $q_i[k_A,k_B] < 1$ a node in network $A$ ($B$) with degree $k_A$ ($k_B$) is independent of the other network with a probability $1 - q_i[k_A,k_B]$, i.e., the link cannot transmit the failure to that node. After a random failure of $1 - p$ nodes in network $A$ that triggers the process, at each stage $n$ of the failure cascade that goes from $A$ to $B$, a node is considered functional if it belongs to the GC of its own network and the others become dysfunctional because they lose support. As $f_{An}(f_{Bn})$ is the probability that, transversing a link, a node of the giant connected component is reached in network $A$ ($B$) at stage $n$ [13–15], a node on network $A$ with degree $k_A$ is functional if it can be reached on its own network with a probability $p[1 - (1 - p f_{An})^{k_A}]$. This node will not be affected by the failure cascade (a) if it is independent
of network B with a probability \(1 - q_A[k_A,k_B]\), or (b) if it depends on network B, but its interdependent node in B is connected to the GC at the previous stage with a probability \(q_A[k_A,k_B](1 - (1 - f_{Bn-1}^{k_B}))\). The relative size \(\Psi_n\) of the GC of network A at stage \(n\) is then given by

\[
\Psi_n = p \left( \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} P[k_A,k_B] \left[ (1 - q_A[k_A,k_B])(1 - f_{An})^{k_A-1} \right] + \right. \\
\left. \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} P[k_A,k_B] \left[ q_A[k_A,k_B](1 - (1 - f_{Bn})^{k_B}) \right] \right),
\]

where \(P[k_A,k_B]\) is the joint degree distribution for the interdependent links. The first term in Eq. (1) takes into account the functional nodes in A with degree \(k_A\) which do not depend on network B, and the second term corresponds to the case where functional nodes in network A with degree \(k_A\) depend on functional nodes of network B with degree \(k_B\) at step \(n - 1\). Here \(f_{An}\) fulfills the self-consistent equation

\[
f_{An} = \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} \frac{k_A P[k_A,k_B]}{\langle k_A \rangle} (1 - q_A[k_A,k_B])(1 - f_{An})^{k_A-1} + \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} \frac{k_A P[k_A,k_B]}{\langle k_B \rangle} q_A[k_A,k_B](1 - (1 - f_{Bn})^{k_B})[1 - (1 - f_{Bn})^{k_B}].
\]

Similarly, at stage \(n\) the relative size \(\phi_n\) of the GC of network B is given by

\[
\phi_n = \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} P[k_A,k_B] \left[ (1 - q_B[k_A,k_B])(1 - f_{Bn})^{k_B} \right] + p \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} P[k_A,k_B] q_B[k_A,k_B](1 - (1 - f_{An})^{k_A})[1 - (1 - f_{An})^{k_A}].
\]

where \(f_{Bn}\) satisfies the self-consistent equation

\[
f_{Bn} = \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} \frac{k_B P[k_A,k_B]}{\langle k_B \rangle} (1 - q_B[k_A,k_B])(1 - f_{Bn})^{k_B-1} + \sum_{k_A = k_{\text{min}}}^{k_{\text{max}}} \sum_{k_B = k_{\text{min}}}^{k_{\text{max}}} \frac{k_B P[k_A,k_B]}{\langle k_B \rangle} q_B[k_A,k_B][1 - (1 - f_{An})^{k_A}]1 - (1 - f_{Bn})^{k_B-1}.
\]

Note that in the right-hand side of Eq. (4) \(f_{Bn}\) is not multiplied by \(p\), since we assume that the initial failure of 1 – \(p\) nodes occurs only in network \(A\).

In the steady state, i.e., for \(n \to \infty\), \(\Psi_n \approx \Psi_\infty\) and \(\phi_n \approx \phi_\infty\), thus \(\Psi_n\) and \(\phi_n\) converge to \(\Psi_\infty\) and \(\phi_\infty\), respectively. Our equations for the steady state were obtained by Son et al. [16] for uncorrelated interdependent networks and used by Baxter et al. [17] to explain the origin of the avalanche collapse.

We introduce here a correlated interdependency model, in which interdependent links are connected bidirectionally and one to one \((q_A[k_A,k_B] = q_B[k_A,k_B] = 1)\), and a fraction \(\alpha\) of the higher-degree nodes are fully correlated. This extends the “rich club” concept [18, 19] to interdependent networks. Assuming that the degree distribution of both networks is the same, the joint degree distribution \(P[k_A,k_B]\) is given by

\[
P[k_A,k_B] = \begin{cases} 
P[k_A]P[k_B]/(1 - \alpha), & k_A < k_S, k_B < k_S, \\
(1 - w)P[k_S]P[k_B]/(1 - \alpha), & k_A = k_S, k_B < k_S, \\
(1 - w)P[k_A]P[k_S]/(1 - \alpha), & k_B = k_S, k_A < k_S, \\
(1 - w)^2P[k_S]P[k_S]/(1 - \alpha) + wP[k_S], & k_A = k_B = k_S, \\
P[k_A]\delta_{k_A,k_S}, & k_S < k_A, k_S < k_B.
\end{cases}
\]

Here \(k_S\) is the degree above which a fraction \(\alpha\) of interdependent nodes are correlated, and \(w\) is the fraction of correlated nodes with degree \(k_S\) such that \(wP[k_S] + \sum_{k=k_S+1}^{k_{\text{max}}} P[k] = \alpha\).
In order to demonstrate how correlation improves the robustness of the networks, in Fig. 3(a) we show the NOI of these systems. For very low values of $\alpha$ there is only one peak at the critical threshold $p_c$ that is related to a first-order percolating transition. Surprisingly, for increasing $\alpha$ (see the case of $\alpha = 0.01\%$ in the figure) there is another peak around the threshold $p_c^\ast > p_c \equiv p_c^\ast$ at which the sizes of the GCs decrease abruptly but, because the hubs support each other, the functional networks are not destroyed, and the decay slows down, indicating that there is time to intervene and prevent the collapse of the GC. This slow behavior around critical points is shown as peaks in the number of iteration (NOI) steps needed to reach the steady state, as we will show below. Figure 2(b) shows that, as $\alpha$ increases, the system is still functional for high initial failure values. The critical threshold $p_c$ at which the system is completely destroyed decreases and thus the networks are more robust.

Note that, because we are using a finite degree cutoff when $\alpha \rightarrow 1$, the threshold does not go to zero, but when $k_{\text{max}} \rightarrow \infty$ in SF networks with $\lambda \leq 3$ and $\alpha = 1$, $p_c \rightarrow 0$ in this limit [5].
robustness of the system against failure cascades is enhanced. For higher values of $\alpha$ we also find that there is a sharp peak that corresponds to a first-order phase transition at $p = p^c_\alpha$, and a rounded peak at $p = p^*_\alpha$ around which the size of the GC decreases continuously with an increasing value of its derivative with respect to $p$, $d\Psi_\infty/dp$ close to $p^c_\alpha$. These findings suggest that finite correlations generate a crossover between an abrupt and a continuous, sharp decrease in the sizes of the GCs.

Figure 3(b) shows the rich phase diagram in the $p-\alpha$ plane. Note that as $\alpha$ increases, the line of the first-order transition that separates a functional GC phase from a nonfunctional phase forks into two branches, generating a new phase characterized by a small GC ($\lesssim 10^{-3}$). Around this point small fluctuations in the temporal evolution—or in the steady state—can induce an abrupt change in the size of the GC, which is reminiscent of the instability of the triple point of liquids where three phases coexist [23]. The lower branch that emerges from the triple point corresponds to the first-order transition that separates functional from nonfunctional phases. The upper one corresponds to the second threshold where the dynamics slows down and, at $\alpha = \alpha_c = 0.0218\%$ [24], the transition changes from an abrupt variation to a rapid but continuous variation of $\Psi_\infty(p)$. The small value of $\alpha_c$ indicates that a small correlation of the highest-degree nodes can avoid the abrupt change in the size of the GC. We found the same qualitative behavior for other SF networks with $2 < \lambda \leq 3$ [25], indicating that the triple point is characteristic of nontrivial patterns of interdependency.

In summary, we have used a general framework to describe the temporal behavior of failure cascades with any pattern of interdependency links, and we have found a rich phase diagram for degree-degree correlated interdependency with a triple point at which a first-order transition line splits into two first-order lines with an abrupt collapse of the sizes of the functional networks. The agreement between theory and simulations is excellent. Our framework can be extended to study the dynamics of failure cascades and the robustness of networks with degree-degree correlation in their connectivity links and in their multiple interdependent links, where we expect to find a rich phase diagram [26].

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[22] We assume that the system reaches a steady state when $\Psi_n - \Psi_{n+1} < 10^{-18}$.
[24] In order to calculate the critical point, we solve geometrically Eq. (6) using the intersection between the identity function and the right-hand side of Eq. (6). A critical point $p_c$ corresponds to the value of $p$ at which the angle in the intersection point between the identity function and the right-hand side of Eq. (6) is minimum.
[25] Close to $\lambda = 2$ the robustness of the system is dominated by the divergence of the average degree, and the correlations have little effect on robustness of the systems.