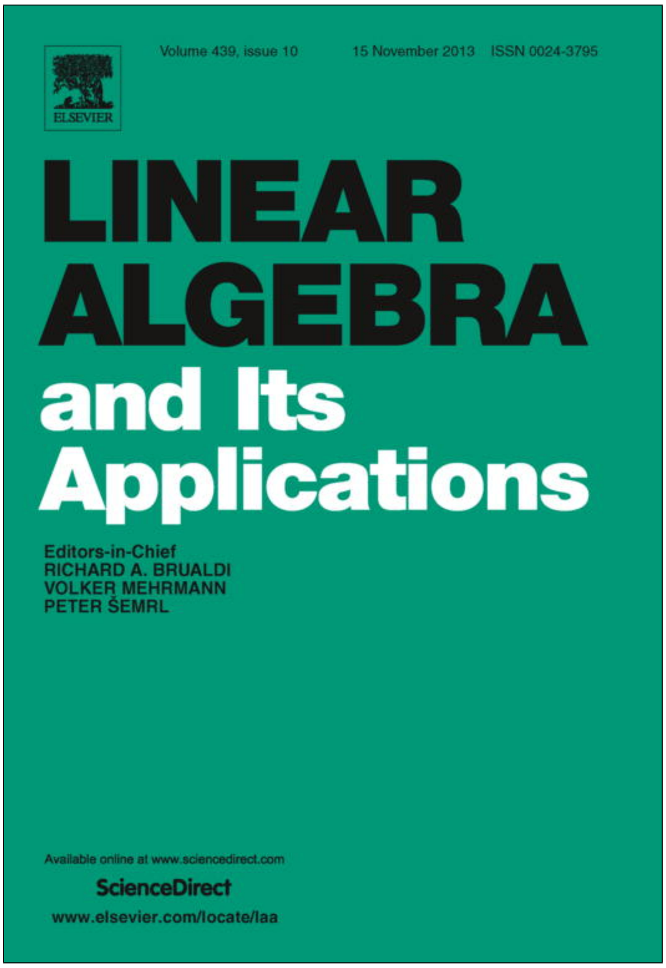
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Best approximation by diagonal compact operators ☆

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ABSTRACT

We study the existence and characterization properties of compact Hermitian operators C on a Hilbert space $\mathcal{H}$ such that

$$\|C\| \leqslant \|C+D\|, \quad \text{for all } D \in \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big)$$

or equivalently

$$\|C\| = \min_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C+D\| = \operatorname{dist}\big(C, \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big)\big)$$

where $\mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ denotes the space of compact self-adjoint diagonal operators in a fixed base of $\mathcal{H}$ and $\|.\|$ is the operator norm. We also exhibit a positive trace class operator that fails to attain the minimum in a compact diagonal.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators and $\mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ the subalgebra of self-adjoint diagonal compact operators (with respect to a fixed orthonormal base). In this paper we study the existence and describe Hermitian compact operators C such that

$$\|C\| \leqslant \|C+D\|, \quad \text{for all } D \in \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big),$$

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or equivalently

$$\|C\| = \mathrm{dist}\big(C, \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big)\big), \tag{1.1}$$

where $\|.\|$ denotes the usual operator norm. These operators C will be called minimal. Our interest in these minimal operators comes from the study of minimal length curves of the orbit manifold of a self-adjoint compact operator A by a particular unitary group (see [1]), that is

$$\mathcal{O}_A = \big\{uAu^*\colon u \text{ unitary in } B(H) \text{ and } u-1 \in \mathcal{K}(H)\big\}.$$

The tangent space at any $b \in \mathcal{O}_A$ is

$$T_b(\mathcal{O}_A) = \big\{zb - bz\colon z \in \mathcal{K}(\mathcal{H})^{ah}\big\}.$$

Where the suffix ah refers to the anti-Hermitian operators (analogously, the suffix h refers to Hermitian operators). If $x \in T_b(\mathcal{O}_A)$, the existence of a (not necessarily unique) minimal element z_0 such that

$$\|x\| = \|z_0\| = \inf\big\{\|z\|\colon z \in \mathcal{K}(\mathcal{H})^{ah},\ zb - bz = x\big\}$$

allows the description of minimal length curves of the manifold by the parametrization

$$\gamma(t) = e^{tz_0} b e^{-tz_0}, \quad t \in [-1, 1].$$

These z_0 can be described as $i(C + D)$, with $C \in \mathcal{K}(\mathcal{H})^h$ and D a real diagonal operator in the orthonormal base of eigenvectors of A.

If we consider a von Neumann algebra $\mathcal{A}$ and a von Neumann subalgebra, named $\mathcal{B}$, of $\mathcal{A}$, it has been proved in [5] that for each $a \in \mathcal{A}$ there always exists a minimal element b_0 in $\mathcal{B}$. It means that $\|a + b_0\| \leqslant \|a + b\|$, for all $b \in \mathcal{B}$. For example, if $M_n^h(\mathbb{C})$ is the algebra of Hermitian matrices of $n \times n$ and $\mathcal{D}(M_n^h(\mathbb{C}))$ is the subalgebra of diagonal Hermitian matrices (or diagonal real matrices), it is easy to prove that, for every $M \in M_n^h(\mathbb{C})$ there always exists a minimal element $D \in \mathcal{D}(M_n^h(\mathbb{C}))$.

However, in the case of $\mathcal{K}(\mathcal{H})^h$, which is only a real Banach algebra, the existence of a best approximant in the general case is not guaranteed. In the particular case that $C \in \mathcal{K}(\mathcal{H})^h$ has finite rank, it was proved in Proposition 5.1 in [1] that there exists a minimal compact diagonal element.

The results we present in this paper are divided in two parts. In the first one we describe a particular case of minimal operators that allow us to prove there is not always a minimal diagonal compact operator. In the second part we present properties and characterizations of minimal compact operators in general.

2. Preliminaries and notation

Let $\mathcal{H}$ be a Hilbert space where we consider a fixed orthonormal base $\{e_i\}_{i\in I}$. Note that since C in (1.1) is compact, then only a countable subset $\{e_{i_n}\}_{n\in\mathbb{N}} \subset \{e_i\}_{i\in I}$ satisfies $C(e_{i_n}) \neq 0$. Therefore, to find $\inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C - D\|$, we can restrict the $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ to those that are diagonal with respect to this countable subset of the basis (and zero elsewhere). Then, in order to describe minimal compact operators, we can suppose without loss of generality that the Hilbert space $\mathcal{H}$ is separable.

We denote by $\langle\, ,\rangle$ the inner product of $\mathcal{H}$ and with $\|x\| = \langle x, x\rangle^{1/2}$ the norm for each $x \in \mathcal{H}$. We designate with $\mathcal{K}(\mathcal{H})$, the two-sided closed ideal of compact operators on $\mathcal{H}$, with $\mathcal{B}_1(\mathcal{H})$, the space of trace class operators, and with $\mathcal{B}(\mathcal{H})$ the set of bounded operators. As usual $\|T\|$ denotes the operator norm of $T \in \mathcal{B}(\mathcal{H})$ and $\|L\|_1 = \mathrm{tr}(|L|) = \mathrm{tr}[(L^*L)^{1/2}]$, the trace norm of $L \in \mathcal{B}_1(\mathcal{H})$. It should cause no confusion the use of the same notation $\|.\|$ to refer to the operator norm or the norm in $\mathcal{H}$, it should be clear from the context.

If $\mathcal{A}$ is any of the previous sets, we denote with $\mathcal{D}(\mathcal{A})$ the set of diagonal operators, that is

$$\mathcal{D}(\mathcal{A}) = \big\{T \in \mathcal{A}\colon \langle Te_i, e_j\rangle = 0, \text{ for all } i \neq j\big\},$$

where $\{e_k\}_{k=1}^{\infty}$ is a fixed orthonormal base of $\mathcal{H}$. We consider an operator $T \in \mathcal{B}(\mathcal{H})$ like an infinite matrix defined for each $i, j \in \mathbb{N}$ as $T_{ij} = \langle Te_i, e_j\rangle$. In this sense, the jth-column and ith-row of T are the vectors in ℓ^2 given by $c_j(T) = (T_{1j}, T_{2j}, \ldots)$ and $f_j(T) = (T_{i1}, T_{i2}, \ldots)$, respectively.

Let $L \in \mathcal{B}(\mathcal{H})^h$, we denote the positive and negative parts of L as:

$$L^+ = \frac{|L| + L}{2} \quad \text{and} \quad L^- = \frac{|L| - L}{2},$$

respectively.

We use $\sigma(T)$ and $R(T)$ to denote the spectrum and range of $T \in \mathcal{B}(\mathcal{H})^h$, respectively.

We define $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{D}(\mathcal{B}(\mathcal{H}))$, $\Phi(X) = \mathrm{Diag}(X)$, which essentially takes the main diagonal (i.e., the elements of the form $\langle Xe_i, e_i\rangle_{i\in\mathbb{N}}$) of an operator X and builds a diagonal operator in the chosen fixed base of $\mathcal{H}$. For a given sequence $(d_n)_{n\in\mathbb{N}}$ we denote with $\mathrm{Diag}((d_n)_{n\in\mathbb{N}})$ the diagonal (infinite) matrix with $(d_n)_{n\in\mathbb{N}}$ in its diagonal and 0 elsewhere.

We define the space $\mathcal{K}(\mathcal{H})^h/\mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ with the usual quotient norm

$$\big\|[C]\big\| = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C + D\| = \mathrm{dist}\big(C, \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big)\big)$$

for each class $[C] = \{C + D\colon D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)\}$.

Given an operator $C \in \mathcal{K}(\mathcal{H})^h$, if there exists an operator D_1 compact and diagonal such that

$$\|C + D_1\| = \mathrm{dist}\big(C, \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big)\big),$$

we say that D_1 is a best approximant of C in $\mathcal{D}(\mathcal{K}(\mathcal{H})^h)$. In other terms, the operator $C + D_1$ verifies the following inequality

$$\|C + D_1\| \leqslant \|C + D\|$$

for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$. In this sense, we call $C + D_1$ a *minimal* operator or similarly we say that D_1 is minimal for C.

3. The existence problem of the best approximant

Some examples of compact Hermitian operators that possess a closest compact diagonal are: i) those constructed with Hermitian square matrices in their main diagonal, ii) tridiagonal operators with zero diagonal, and iii) finite rank compact operators (see [1] for a proof).

In the rest of this section we study some examples of compact Hermitian operators with a unique best diagonal approximant. Then, we use this example to show an operator which has no best compact diagonal approximant. We use frequently the fact that any bounded operator T can be described uniquely as an infinite matrix with the notation T_{ij} that we introduced in Section 2 using the fixed base.

The following statement is about a set of compact symmetric operators ($L = L^t$), which has the following property: every operator has a column (or row) such that every different column (or row) is orthogonal to it (considering L as an infinite matrix). This result has its origins in the finite dimensional result obtained in [6].

Theorem 1. *Let $T \in \mathcal{K}(\mathcal{H})^h$ be described as an infinite matrix by $(T_{ij})_{i,j\in\mathbb{N}}$. Suppose that T satisfies:*

(1) *$T_{ij} \in \mathbb{R}$ for each $i, j \in \mathbb{N}$,*
(2) *there exists $i_0 \in \mathbb{N}$ satisfying $T_{i_0 i_0} = 0$, with $T_{i_0 n} \neq 0$, for all $n \neq i_0$,*
(3) *if $T^{[i_0]}$ is the operator T with zero in its i_0th-column and i_0th-row then*

$$\big\|c_{i_0}(T)\big\| \geqslant \big\|T^{[i_0]}\big\|$$

(where $\|c_{i_0}(T)\|$ denotes the Hilbert norm of the i_0th-column of T), and

(4) *if the T_{nn}'s satisfy that, for each $n \in \mathbb{N}$, $n \neq i_0$:*

$$T_{nn} = -\frac{\langle c_{i_0}(T), c_n(T)\rangle}{T_{i_0 n}}.$$

Then T is minimal, that is

$$\|T\| = \|c_{i_0}(T)\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|T + D\|$$

and moreover, $D = \mathrm{Diag}((T_{nn})_{n\in\mathbb{N}})$ is the unique bounded minimal diagonal operator for T.

Proof. Without loss of generality we can suppose that T is a compact operator with real entries and $i_0 = 1$, therefore it has the matrix form given by

$$T = \begin{pmatrix} 0 & T_{12} & T_{13} & T_{14} & \cdots \\ T_{12} & T_{22} & T_{23} & T_{24} & \cdots \\ T_{13} & T_{23} & T_{33} & T_{34} & \cdots \\ T_{14} & T_{24} & T_{34} & T_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The hypothesis in this case are

- $i_0 = 1$ with $T_{1n} \neq 0$, $\forall n \in \mathbb{N} - \{1\}$.
- $\|c_1(T)\| \geqslant \left\| \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & T_{22} & T_{23} & T_{24} & \cdots \\ 0 & T_{23} & T_{33} & T_{34} & \cdots \\ 0 & T_{24} & T_{34} & T_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{=T^{[1]}} \right\| = \|T^{[1]}\|.$
- Each T_{nn} fulfills:

$$T_{nn} = -\frac{\langle c_1(T), c_n(T)\rangle}{T_{1n}} \quad \text{for every } n \in \mathbb{N} - \{1\}.$$

There are some remarks to be made:

(1) First note that for every $i \in \mathbb{N}$

$$|T_{ii}| = \left|\left\langle T^{[1]}e_i, e_i\right\rangle\right| \leqslant \left\|T^{[1]}e_i\right\| \|e_i\| \leqslant \left\|T^{[1]}\right\| \leqslant \left\|c_1(T)\right\| < \infty$$

namely, $(T_{ii})_{i\in\mathbb{N}}$ is a bounded sequence (each T_{ii} is a diagonal element of $T^{[1]}$ in the fixed base).

(2) A direct computation proves that $\|c_1(T)\|$ and $-\|c_1(T)\|$ are eigenvalues of T with

$$v_+ = \frac{1}{\sqrt{2}\|c_1(T)\|}\left(\left\|c_1(T)\right\|e_1 + c_1(T)\right) \quad \text{and} \quad v_- = \frac{1}{\sqrt{2}\|c_1(T)\|}\left(\left\|c_1(T)\right\|e_1 - c_1(T)\right),$$

which are eigenvectors of $\|c_1(T)\|$ and $-\|c_1(T)\|$, respectively. Let us consider the space $V = \mathrm{Gen}\{v_+, v_-\}$:

- If $w \in V$, then $\|Tw\|^2 = \|c_1(T)\|^2\|w\|^2$.
- If $y \in V^\perp$, then $\|Ty\| = \|T^{[1]}y\| \leqslant \|T^{[1]}\|\|y\|$.

Then, for every $x = w + y \in \mathcal{H}$, with $w \in V$ and $y \in V^\perp$:

$$\left\|T(w+y)\right\|^2 = \|Tw\|^2 + \|Ty\|^2 \leqslant \left\|c_1(T)\right\|^2\|w\|^2 + \left\|T^{[1]}\right\|^2\|y_1\|^2 \leqslant \left\|c_1(T)\right\|^2\|x\|^2.$$

Therefore,

$$\|T\| = \left\|c_1(T)\right\|.$$

(3) Let $D' \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ and define $(\underbrace{T + D'}_{=T'})e_i = T'(e_i) = c_i(T')$ for each $i \in \mathbb{N}$, then the following properties are satisfied:

- If $D'_{11} \neq 0$ then
$$\left\|T'(e_1)\right\|^2 = \left\|c_1(T')\right\|^2 = D'_{11} + \left\|c_1(T)\right\|^2 > \left\|c_1(T)\right\|^2 = \|T\|^2 \quad \Rightarrow \quad \left\|T'\right\| > \|T\|.$$
Therefore, we can assume that if $T + D'$ is minimal then $D'_{11} = 0$.
- Now suppose that there exists $i \in \mathbb{N}$, $i > 1$, such that D' does not have its ith-column orthogonal to the first one, that is:
$$\langle T'e_1, T'e_i\rangle = \langle c_1(T'), c_i(T')\rangle = a \neq 0.$$
Then,
$$T'\left(\frac{c_1(T)}{\|c_1(T)\|}\right) = \left(\left\|c_1(T)\right\|, \frac{a_2}{\|c_1(T)\|}, \dots, \frac{a_i}{\|c_1(T)\|}, \dots\right)$$
$$\Rightarrow \quad \left\|T'(c_1(T))\right\|^2 > \left\|c_1(T)\right\|^2 = \|T\|.$$
Hence, $\|T'\| > \|T\|$.

Therefore, $D = \mathrm{Diag}((T_{nn})_{n\in\mathbb{N}})$ is the unique minimal diagonal for T and it is bounded. $\square$

Note that the minimal diagonal obtained in Theorem 1 is clearly bounded but we do not know if it is compact. An interesting question is if there exist an operator T which fulfills the hypothesis of Theorem 1 and it has an only minimal bounded diagonal non-compact. To answer this question we analyzed several examples, we show the most relevant among them.

Let $\gamma \in \mathbb{R}$ be such that $|\gamma| < 1$ and take an operator $T \in \mathcal{B}(\mathcal{H})^h$ defined as $(T_{ij})_{i,j\in\mathbb{N}}$ where

$$T_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \gamma^{\max\{i,j\}-2} & \text{if } i \neq j \text{ and } j, i \neq 1, \\ \gamma^{|i-j|} & \text{if } j = 1 \text{ or } i = 1. \end{cases}$$

Writing T as an infinite matrix

$$T = \begin{pmatrix} 0 & \gamma & \gamma^2 & \gamma^3 & \gamma^4 & \cdots \\ \gamma & 0 & \gamma & \gamma^2 & \gamma^3 & \cdots \\ \gamma^2 & \gamma & 0 & \gamma^2 & \gamma^3 & \cdots \\ \gamma^3 & \gamma^2 & \gamma^2 & 0 & \gamma^3 & \cdots \\ \gamma^4 & \gamma^3 & \gamma^3 & \gamma^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

T is symmetric and $c_n(T)$ is the nth-column. Since, $(T^*T)_{nn} = \langle f_n(T), c_n(T)\rangle = \langle c_n(T), c_n(T)\rangle$ for each $n \in \mathbb{N}$, then

- $(T^*T)_{11} = \langle c_1(T), c_1(T)\rangle = 0 + \gamma\gamma + \gamma^2\gamma^2 + \cdots = \sum_{k=1}^{\infty} \gamma^{2k} = \frac{\gamma^2}{1-\gamma^2}$,
- $(T^*T)_{22} = \langle c_2(T), c_2(T)\rangle = \gamma\gamma + 0 + \gamma\gamma + \gamma^2\gamma^2 + \cdots = \gamma^2 + \sum_{k=1}^{\infty} \gamma^{2k} = \gamma^2 + \frac{\gamma^2}{1-\gamma^2}$,
- and for any $n \geqslant 3$:

$$\begin{aligned}(T^*T)_{nn} &= \langle c_n(T), c_n(T)\rangle \\ &= \gamma^{n-1}\gamma^{n-1} + \underbrace{\gamma^{n-2}\gamma^{n-2} + \cdots + \gamma^{n-2}\gamma^{n-2}}_{n-2 \text{ times}} + 0 \\ &\quad + \gamma^{n-1}\gamma^{n-1} + \gamma^n\gamma^n + \gamma^{n+1}\gamma^{n+1} + \gamma^{n+2}\gamma^{n+2} + \cdots \\ &= \gamma^{2(n-1)} + (n-2)\gamma^{2(n-2)} + \sum_{k=n-1}^{\infty} \gamma^{2k} \\ &= \gamma^{2(n-1)} + (n-2)\gamma^{2(n-2)} + \frac{\gamma^{-2+2n}}{1-\gamma^2} < \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{tr}(T^*T) &= \sum_{n=1}^{\infty}(T^*T)_{nn}\\
&= \frac{\gamma^2}{1-\gamma^2} + \gamma^2 + \frac{\gamma^2}{1-\gamma^2} + \sum_{n=3}^{\infty}\left[\gamma^{2(n-1)} + (n-2)\gamma^{2(n-2)} + \frac{\gamma^{2n-2}}{1-\gamma^2}\right]\\
&= \frac{\gamma^2}{1-\gamma^2} + \gamma^2 + \frac{\gamma^2}{1-\gamma^2} + \frac{1}{1-\gamma^2} + \frac{-\gamma^2+2\gamma^4}{\gamma^4(1-\gamma^2)^2} + \frac{1}{(1-\gamma^2)^2}\\
&= \frac{-1+4\gamma^2+2\gamma^4-4\gamma^6+\gamma^8}{\gamma^2(-1+\gamma^2)^2} < \infty.
\end{aligned}$$

Then, T is a Hilbert–Schmidt operator. Consider a diagonal operator D, given by $D = \operatorname{Diag}((d_n)_{n\in\mathbb{N}})$, with the sequence $(d_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

(1) $d_1 = 0$.
(2) $\langle c_1(T), c_n(T+D)\rangle = 0$, for every $n \in \mathbb{N}$, $n > 1$.

Indeed, for every $n \geqslant 2$ each d_n is uniquely determined by

$$d_n = -\frac{1-\gamma^{n-2}}{1-\gamma} - \frac{\gamma^n}{1-\gamma^2}.$$

We can also note that $d_n \to \frac{1}{\gamma-1}$ when $n \to \infty$, so the diagonal operator $D = \operatorname{Diag}((d_n)_{n\in\mathbb{N}})$ is bounded but non-compact.

On the other hand, if we consider $T^{[1]}$, the operator given by

$$T^{[1]} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & \gamma & \gamma^2 & \gamma^3 & \cdots\\ 0 & \gamma & 0 & \gamma^2 & \gamma^3 & \cdots\\ 0 & \gamma^2 & \gamma^2 & 0 & \gamma^3 & \cdots\\ 0 & \gamma^3 & \gamma^3 & \gamma^3 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then $T^{[1]}$ is also a Hilbert–Schmidt operator. Then $T^{[1]} + D \in \mathcal{B}(\mathcal{H})$. Now consider the operator T_r, given by

$$T_r = \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & r\gamma^4 & \cdots\\ r\gamma & 0 & \gamma & \gamma^2 & \gamma^3 & \cdots\\ r\gamma^2 & \gamma & 0 & \gamma^2 & \gamma^3 & \cdots\\ r\gamma^3 & \gamma^2 & \gamma^2 & 0 & \gamma^3 & \cdots\\ r\gamma^4 & \gamma^3 & \gamma^3 & \gamma^3 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.1}$$

with $r = \frac{\|T^{[1]}+D\|}{\|c_1(T)\|}$. Then, we claim that the following operator

$$T_r + D = \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & r\gamma^4 & \cdots\\ r\gamma & d_2 & \gamma & \gamma^2 & \gamma^3 & \cdots\\ r\gamma^2 & \gamma & d_3 & \gamma^2 & \gamma^3 & \cdots\\ r\gamma^3 & \gamma^2 & \gamma^2 & d_4 & \gamma^3 & \cdots\\ r\gamma^4 & \gamma^3 & \gamma^3 & \gamma^3 & d_5 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is minimal and unique, which means:

$$\|[T_r]\| = \inf_{D' \in \mathcal{D}(\mathcal{B}_h(H))} \|T + D'\| = \inf_{D' \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|T + D'\| = \|T_r + D\|.$$

This is true because T_r is an operator which clearly satisfies the hypothesis of Theorem 1. It follows from the non-compacity of D that there is no best compact diagonal approximation of T_r.

The operator T_r is also a positive trace class operator. In effect, if we consider the lower triangular operator $C_a \in \mathcal{B}(\mathcal{H})$, given by $(C_a)_{ij} = a^i$, for $i \geqslant j$, and take $a = \sqrt{\gamma}$, then

$$C^*_{\sqrt{\gamma}} C_{\sqrt{\gamma}} = \frac{1}{1-\gamma} \begin{pmatrix} \gamma & \gamma^2 & \gamma^3 & \gamma^4 & \gamma^5 & \cdots \\ \gamma^2 & \gamma^2 & \gamma^3 & \gamma^4 & \gamma^5 & \cdots \\ \gamma^3 & \gamma^3 & \gamma^3 & \gamma^4 & \gamma^5 & \cdots \\ \gamma^4 & \gamma^4 & \gamma^4 & \gamma^4 & \gamma^5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \frac{1}{1-\gamma} Q.$$

Therefore,

$$\operatorname{tr}(|Q|) = (1-\gamma)\operatorname{tr}\big(\big|C^*_{\sqrt{\gamma}} C_{\sqrt{\gamma}}\big|\big) = (1-\gamma)\operatorname{tr}\big(C^*_{\sqrt{\gamma}} C_{\sqrt{\gamma}}\big) = \operatorname{tr}(Q),$$

which shows that $Q \in \mathcal{B}_1(\mathcal{H})$. On the other hand, the operator

$$R = \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & \cdots \\ r\gamma & 0 & 0 & 0 & \cdots \\ r\gamma^2 & 0 & 0 & 0 & \cdots \\ r\gamma^3 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has finite rank, thus $\left(\begin{smallmatrix} 0 & \cdots \\ \vdots & Q \end{smallmatrix}\right) + R \in \mathcal{B}_1(\mathcal{H})$. But also $\left(\begin{smallmatrix} 0 & \cdots \\ \vdots & Q \end{smallmatrix}\right) + R - \operatorname{Diag}(Q) = T_r$, which is equivalent to say that $\left(\begin{smallmatrix} 0 & \cdots \\ \vdots & Q \end{smallmatrix}\right) + R$ is in the same class that T_r. As $\operatorname{Diag}(Q) \in \mathcal{B}_1(\mathcal{H})$, it follows that $T_r \in \mathcal{B}_1(\mathcal{H})$. Moreover, since Q and R are positive then T_r is also positive.

Remark 2 *(About the implications of the uniqueness condition on the existence of minimal diagonal operators).* For a given Hermitian compact operator C the existence of a unique bounded real diagonal operator D_0 minimal for C does not imply that D_0 is not compact. On the other hand, if there exist infinite bounded real diagonal operators that are minimal for C, this does not imply that there exists a compact minimal diagonal.

The next examples of operators show that the existence of a unique (respectively non-unique) minimal diagonal does not necessarily imply that there does not exist (respectively that there exists) a minimal compact diagonal.

(1) Let $L \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$, $L \neq 0$, then $-L$ is the only minimal diagonal compact operator. In this case, we can observe that there is uniqueness for the minimal, but the best approximant is also compact.

(2) Let us consider the example T_r defined in (3.1) and the block operator $S = \left(\begin{smallmatrix} S_n & 0 \\ 0 & T_r \end{smallmatrix}\right)$, where $S_n \in M_n^h(\mathbb{C})$ is a matrix whose quotient norm is $\|[T_r]\|$ and has infinite minimal diagonals of $n \times n$ (consider matrices like those in [3,4] or [6]). Then, all minimal diagonal bounded operators for S are of the form $D' = \left(\begin{smallmatrix} D_n & 0 \\ 0 & D \end{smallmatrix}\right)$, with any of the infinite D_n minimals for S_n and D the unique minimal bounded diagonal operator for T_r. Thus, none of these D' is compact. This case shows that if uniqueness of a minimal diagonal does not hold this does not necessarily imply the existence of a minimal compact diagonal operator.

4. A characterization of minimal compact operators

In the previous section we showed an example of a compact operator T_r that has no compact diagonal best approximant. The main property that allowed us to prove the non-existence of a minimal compact diagonal is the uniqueness of the best approximant for T_r.

Nevertheless, there are a lot of compact operators which have at least one best compact diagonal approximation, for example the operators of finite rank. The spirit of this part follows the main ideas in [2]. The main purpose of this subsection is to study properties and equivalences that characterize minimal compact operators.

The next two propositions are closely related with the Hahn–Banach theorem for Banach spaces and they relate the space $\mathcal{K}(\mathcal{H})^h$ with $\mathcal{B}_1(\mathcal{H})^h$.

Proposition 3. *Let $C \in \mathcal{K}(\mathcal{H})^h$ and consider the set*

$$\mathcal{N} = \left\{Y \in \mathcal{B}_1(\mathcal{H})^h \colon \|Y\|_1 = 1,\ \operatorname{tr}(YD) = 0,\ \forall D \in \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big)\right\}.$$

Then, there exists $Y_0 \in \mathcal{N}$ such that

$$\big\|[C]\big\| = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C+D\| = \operatorname{tr}(Y_0C). \tag{4.1}$$

Proof. It is an immediate consequence from the Hahn–Banach theorem that since $\mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ is a closed subspace of $\mathcal{K}(\mathcal{H})^h$ and $C \in \mathcal{K}(\mathcal{H})^h$, then there exists a functional $\rho : \mathcal{K}(\mathcal{H})^h \to \mathbb{R}$ such that $\|\rho\| = 1$, $\rho(D) = 0$, $\forall D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$, and

$$\rho(C) = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C+D\| = \operatorname{dist}\big(C, \mathcal{D}\big(\mathcal{K}(\mathcal{H})^h\big)\big).$$

But, since any functional ρ can be written as $\rho(.) = \operatorname{tr}(Y_0.)$, with $Y_0 \in \mathcal{B}_1(\mathcal{H})$, the result follows. $\square$

Proposition 4 *(Banach duality formula). Let $C \in \mathcal{K}(\mathcal{H})^h$, then*

$$\inf_{D\in\mathcal{D}(\mathcal{K}(H))} \|C+D\| = \max_{Y\in\mathcal{N}} \big|\operatorname{tr}(CY)\big|. \tag{4.2}$$

Proof. Let $C \in \mathcal{K}(\mathcal{H})^h$. By Proposition 3, there exists $Y_0 \in \mathcal{N}$ such that

$$\inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C+D\| = \operatorname{tr}(Y_0C).$$

Then

$$\inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C+D\| = \operatorname{tr}(Y_0C) \leqslant \max_{Y\in\mathcal{N}} \big|\operatorname{tr}(CY)\big| = \max_{Y\in\mathcal{N}} \big|\operatorname{tr}\big((C+D)Y\big)\big| \leqslant \underbrace{\|Y\|_1}_{=1} \|C+D\|,$$

for any $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$. Therefore, the equality (4.2) can be proved as a consequence of this fact. $\square$

Note that if $Y \in \mathcal{B}_1(\mathcal{H})$ is such that $\operatorname{tr}(YD) = 0$ for every $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ then $\operatorname{tr}(YD) = 0$ for every $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$. Moreover, it is easy to prove that if $\operatorname{tr}(YD) = 0$ for every $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$, then $\operatorname{Diag}(Y) = 0$.

It is apparent that

$$\inf_{D\in\mathcal{D}(\mathcal{B}(\mathcal{H})^h)} \|C+D\| \leqslant \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C+D\|.$$

Observe that there always exists $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$ such that $\|C+D_0\| = \inf_{D\in\mathcal{D}(\mathcal{B}(H))^h} \|C+D\|$, since $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $\mathcal{D}(\mathcal{B}(\mathcal{H}))$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ (see [5]).

With the above properties we can prove the reverse inequality, as we show in the following proposition.

Proposition 5. *Let $C \in \mathcal{K}(\mathcal{H})^h$, then*

$$\inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)} \|C + D\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C + D\|.$$

Proof. Let D_0 a minimal bounded diagonal operator such that

$$\inf_{D \in \mathcal{D}(\mathcal{B}(H))^h} \|C + D\| = \|C + D_0\|.$$

Then, using Proposition 3, there exists $Y_0 \in \mathcal{B}_1(\mathcal{H})$, with $\|Y_0\|_1 = 1$ such that

$$\inf_{D \in \mathcal{D}(\mathcal{K}(H))^h} \|C + D\| = \left|\operatorname{tr}(Y_0 C)\right| = \left|\operatorname{tr}\big(Y_0(C + D_0)\big)\right| \leqslant \|C + D_0\|$$

which completes the proof. □

A natural fact that has been proved for minimal Hermitian matrices is a balanced spectrum property: if $M \in M_n^h(\mathbb{C})$ and M is minimal then $\|M\|$ and $-\|M\|$ are in the spectrum of M. This property holds for minimal compact operators.

Proposition 6 *(Balanced spectrum property). Let $C \in \mathcal{K}(H)^h$, $C \neq 0$. Suppose that there exists $D_1 \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ such that $C + D_1$ is minimal, then*

$$\pm\|C + D_1\| \in \sigma(C + D_1).$$

Proof. The proof is a routine application of functional calculus to the Hermitian operator $C + D_1$. □

Theorem 7. *Let $C \in \mathcal{K}(\mathcal{H})^h$ and $D_1 \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$. Consider E_+ and E_-, the spectral projections of the eigenvalues $\lambda_{\max}(C + D_1)$ and $\lambda_{\min}(C + D_1)$, respectively. The following statements are equivalent*:

(1) *$C + D_1$ is minimal.*
(2) *There exists $X \in \mathcal{B}_1(\mathcal{H})^h$, $X \neq 0$, such that*
- $\langle Xe_i, e_i\rangle = 0$, $\forall i \in \mathbb{N}$;
- $|\operatorname{tr}(X(C + D_1))| = \|C + D_1\|\|X\|_1$;
- $E_+X^+ = X^+$, $E_-X^- = X^-$.

(3) *$\lambda_{\min}(C + D_1) + \lambda_{\max}(C + D_1) = 0$ and for each $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ there exist $y \in R(E_+)$, $z \in R(E_-)$ such that*:
- $\|y\| = \|z\| = 1$;
- $\langle Dy, y\rangle \leqslant \langle Dz, z\rangle$.

Proof. (2) ⇒ (1). Let $C \in \mathcal{K}(\mathcal{H})^h$ and $D_1 \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$. If there exists $X \in \mathcal{B}_1(\mathcal{H})^h$ which fulfills the properties in (2), then:

$$\|C + D_1\| = \frac{\operatorname{tr}(X(C + D_1))}{\|X\|_1} = \operatorname{tr}\left(\frac{X}{\|X\|_1}C\right) \leqslant \max_{Y \in \mathcal{N}}\left|\operatorname{tr}(YC)\right| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|C + D\|,$$

where the last equality holds for the Banach duality formula (see Proposition 4). Then, $C + D_1$ is minimal.

(1) ⇒ (2). Without loss of generality, we can suppose that $\|C + D_1\| = 1$. The proof of this part follows the same techniques used in Theorem 2 in [2] for matrices and we include it for the sake of completeness. The Banach duality formula implies that there exists $X \in \mathcal{B}_1(H)^h$ such that

$$\langle Xe_i, e_i\rangle = 0, \quad \forall i \in \mathbb{N}, \qquad \|X\|_1 = 1, \qquad \operatorname{tr}\big(X(C + D_1)\big) = \operatorname{tr}(XC) = 1.$$

Let us prove that $X(C + D_1) = (C + D_1)X$. Since $C + D_1$ is minimal Proposition 6 implies that $-1, 1 \in \sigma(C + D_1)$. Consider the spectral projections E_+, E_- and $E_3 = I - E_+ - E_-$. The operators $C + D_1$ and

X can be written matricially, in terms of the orthogonal decomposition $\mathcal{H} = R(E_+) \oplus R(E_-) \oplus R(E_3)$, as

$$C + D_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & (C+D_1)_{3,3} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{pmatrix}.$$

It is enough to prove that $X_{1,2} = X_{1,3} = X_{2,3} = X_{3,3} = 0$. To this end, if we consider Theorem 1.19 in [8], the following inequalities hold

$$\left\| \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} \right\|_1 + \|X_{3,3}\|_1 \leqslant \|X\|_1$$

and

$$\|X_{1,1}\|_1 + \|X_{2,2}\|_1 \leqslant \left\| \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} \right\|_1 .$$

Suppose that $\|X_{3,3}\|_1 \neq 0$, then

$$\begin{aligned} 1 &= \operatorname{tr}\big(X(C+D_1)\big) = \operatorname{tr}(X_{1,1}) - \operatorname{tr}(X_{2,2}) + \operatorname{tr}\big(X_{3,3}(C+D_1)_{3,3}\big) \\ &< \|X_{1,1}\|_1 + \|X_{2,2}\|_1 + \|X_{3,3}\|_1 \leqslant \left\| \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} \right\|_1 + \|X_{3,3}\|_1 \leqslant \|X\|_1 \leqslant 1, \end{aligned}$$

which is a contradiction. Then, $X_{3,3} = 0$.

It also follows that

$$\left. \begin{aligned} \operatorname{tr}(X_{1,1}) &= \|X_{1,1}\|_1 \\ \operatorname{tr}(-X_{2,2}) &= \|-X_{2,2}\|_1 \end{aligned} \right\} \quad \Rightarrow \quad X_{1,1} \geqslant 0 \wedge -X_{2,2} \geqslant 0.$$

On the other hand,

$$1 = \operatorname{tr}\big(X(C+D_1)\big) = \|X_{1,1}\|_1 + \|-X_{2,2}\|_1 \leqslant \big\|X(C+D_1)\big\|_1 \leqslant \|X\|_1 \|C+D_1\| \leqslant 1.$$

Therefore,

$$\operatorname{tr}\big(X(C+D_1)\big) = \big\|X(C+D_1)\big\|_1.$$

Then $X(C+D_1) \geqslant 0$, which implies that

$$\begin{cases} X_{3,1}(C+D_1)_{3,3} = X_{1,3}^*(C+D_1)_{3,3} = X_{3,1} & \Leftrightarrow \quad X_{3,1} = X_{1,3}^* = 0, \\ X_{3,2}(C+D_1)_{3,3} = X_{2,3}^*(C+D_1)_{3,3} = X_{3,2} & \Leftrightarrow \quad X_{3,2} = X_{2,3}^* = 0. \end{cases}$$

Analogously, we can deduce that

$$\operatorname{tr} \begin{pmatrix} X_{1,1} & X_{1,2} \\ -X_{2,1} & -X_{2,2} \end{pmatrix} = \left\| \begin{pmatrix} X_{1,1} & X_{1,2} \\ -X_{2,1} & -X_{2,2} \end{pmatrix} \right\|_1 .$$

Then $\left(\begin{smallmatrix} X_{1,1} & X_{1,2} \\ -X_{2,1} & -X_{2,2} \end{smallmatrix}\right) \geqslant 0$ and $-X_{2,1} = X_{1,2}^* = X_{2,1} = 0$. Therefore,

$$X = \begin{pmatrix} X_{1,1} & 0 & 0 \\ 0 & X_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and this operator commutes with $C + D_1$. Also,

$$X^+ = E_+ X_{1,1} E_+ \quad \Longrightarrow \quad E_+ X^+ = X^+ \quad \text{and} \quad X^- = E_- X_{2,3} E_- \quad \Longrightarrow \quad E_- X^- = X^-.$$

(2) $\Rightarrow$ (3). Let $X \in \mathcal{B}_1(\mathcal{H})^h$, $X \neq 0$ such that $\operatorname{Diag}(X) = 0$, $\operatorname{tr}(CX) = \|X\|_1$ and $E_+ X^+ = X^+$, $E_- X^- = X^-$. Let $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ and define numbers m and M as

$$m = \min_{y \in R(E_+)} \frac{\langle Dy, y\rangle}{\|y\|^2}, \qquad M = \max_{z \in R(E_-)} \frac{\langle Dz, z\rangle}{\|z\|^2}. \tag{4.3}$$

Observe that $\dim(R(E_+))$, $\dim(R(E_-)) < \infty$, so the minimum and maximum, respectively, are always attained. We claim that

$$\operatorname{tr}\left(\frac{X^+}{\|X^+\|_1} D\right) \geqslant m. \tag{4.4}$$

In order to prove it observe that $X^+ = E_+ X^+$ and note that

$$\operatorname{tr}\left(\frac{X^+}{\|X^+\|_1} D\right) = \operatorname{tr}\left(\frac{E_+ X^+ E_+}{\|X^+\|_1} D\right) = \operatorname{tr}\left(\frac{X^+}{\|X^+\|_1} E_+ D E_+\right).$$

Therefore, inequality (4.4) is equivalent to

$$\operatorname{tr}\left[\frac{X^+}{\|X^+\|_1}(E_+ D E_+ - mE_+)\right] \geqslant 0,$$

since $\frac{X^+}{\|X^+\|_1} \geqslant 0$. Then, if we prove that $E_+ D E_+ - mE_+ \geqslant 0$ we obtain (4.4). Let $h \in \mathcal{H}$:

$$\langle E_+ D E_+ h, h\rangle = \langle D E_+ h, E_+ h\rangle = \langle Dy, y\rangle \geqslant m\|y\|^2,$$

with $E_+ h = y \in R(E_+)$. Then, $\underbrace{\langle Dy, y\rangle}_{<\infty} - m\underbrace{\langle y, y\rangle}_{<\infty} \geqslant 0$, for all $y \in R(E_+)$. Finally, since $y = E_+ h$, we have

$$\big\langle (DE_+ - mE_+)h, E_+ h\big\rangle \geqslant 0 \quad \Leftrightarrow \quad \big\langle (E_+ D E_+ - mE_+)h, h\big\rangle \geqslant 0.$$

Analogously, it can be proved that $\operatorname{tr}(\frac{X^-}{\|X^-\|_1} D) \leqslant M$.

On the other hand, the condition $\operatorname{Diag}(X) = 0$ with $X \neq 0$ forces that $\operatorname{Diag}(X^+) = \operatorname{Diag}(X^-) \neq 0$, and since $X^+, X^- \geqslant 0$ we have

$$\|X^+\|_1 = \|\operatorname{Diag}(X^+)\|_1 = \|\operatorname{Diag}(X^-)\|_1 = \|X^-\|_1$$

and

$$\operatorname{tr}(X^+ D) = \operatorname{tr}(X^- D).$$

Therefore, there exist $y_0 \in R(E_+)$ and $z_0 \in R(E_-)$ such that $\|y_0\| = \|z_0\| = 1$ and

$$\langle Dy_0, y_0\rangle = m \leqslant \operatorname{tr}\left(\frac{X^+}{\|X^+\|_1} D\right) = \operatorname{tr}\left(\frac{X^-}{\|X^-\|_1} D\right) \leqslant M = \langle Dz_0, z_0\rangle.$$

(3) $\Rightarrow$ (2). For this part we follow the main ideas used in the proof of Theorem 2 in [2]: take the function $\Phi(X) = \operatorname{Diag}(X)$ defined in Section 2 and the following sets

$$\mathcal{A} = \left\{Y \in \mathcal{B}_1(\mathcal{H})^h\colon E_+ Y = Y \geqslant 0,\ \operatorname{tr}(Y) = 1\right\} \quad \text{and}$$
$$\mathcal{B} = \left\{Z \in \mathcal{B}_1(\mathcal{H})^h\colon E_- Z = Z \geqslant 0,\ \operatorname{tr}(Z) = 1\right\}.$$

Since $\dim(R(E_+)) < \infty$ (and $\dim(R(E_-)) < \infty$), every $Y \in \mathcal{A}$ (and every $Z \in \mathcal{B}$) is a Hermitian operator between finite fixed dimensional spaces. Then, all norms restricted to those spaces are equivalent. Thus, we can consider that $\Phi(\mathcal{A})$ and $\Phi(\mathcal{B})$ are compact subsets of $\ell^2(\mathbb{R})$ for every norm (and of course, they are convex also).

Assume the non-existence of X satisfying (2). This implies that $\Phi(\mathcal{A}) \cap \Phi(\mathcal{B}) = \emptyset$. Since $\Phi(\mathcal{A})$ and $\Phi(\mathcal{B})$ are compact and convex sets of $\ell^2(\mathbb{R})$ considering the Euclidean norm, there exist $a, b \in \mathbb{R}$ and a functional ρ defined for every $x \in \mathcal{H}$ such that $\rho(x) = \sum_{i=1}^{\infty} x_i d_i$, with $d = (d_i)_{i \in \mathbb{N}} \in c_0$, such that

$$\rho(y) \geqslant a > b \geqslant \rho(z),$$

for each $y \in \Phi(\mathcal{A})$ and $z \in \Phi(\mathcal{B})$. Then

$$\langle \Phi(Y), d \rangle \geqslant a > b \geqslant \langle \Phi(Z), d \rangle \quad \Rightarrow \quad \min_{Y \in \mathcal{A}} \langle \Phi(Y), d \rangle > \max_{Z \in \mathcal{B}} \langle \Phi(Z), d \rangle,$$

and this cannot occur because if $D = \mathrm{Diag}(d) \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$, then

$$\min_{Y \in \mathcal{A}} \langle \Phi(Y), d \rangle = m \quad \text{and} \quad \max_{Z \in \mathcal{B}} \langle \Phi(Z), d \rangle = M,$$

with m and M defined in (4.3). Therefore $M < m$ and this fact contradicts condition (3). $\square$

Remark 8. The operator X in statement (2) of Theorem 7 has finite rank. Moreover, X can be described as a finite diagonal block operator in the base of eigenvectors of the minimal compact operator $C + D_1$.

Let $\mathcal{A}$ a Banach space, $\mathcal{B}$ a proper closed subspace of $\mathcal{A}$, and $Z \in \mathcal{A}$ a minimal element, that is $\|Z\| = \inf_{B \in \mathcal{B}} \|Z + B\|$. Then, a functional $\psi : \mathcal{A} \to \mathbb{C}$ is called a witness of the $\mathcal{B}$-minimality of Z if $\|\psi\| = 1$, $\psi|_{\mathcal{B}} \equiv 0$ and $\psi(Z) = \|Z\|$ (see [7]).

Remark 9. Let $Z = C + D_1 \in \mathcal{K}(\mathcal{H})^h$ and suppose that there exists an operator X which satisfies the conditions of statement (2) of Theorem 7. Then, we can define $\Psi : \mathcal{K}(\mathcal{H})^h \to \mathbb{R}$, given by $\Psi(\cdot) = \mathrm{tr}(X\cdot)$, such that

(1) $\|\Psi\| = 1$,
(2) $\Psi(C + D_1) = \mathrm{tr}(X(C + D_1)) = \|[C]\|$,
(3) $\Psi(D) = 0 \ \forall D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$.

Observe that Ψ acts as a functional witness of the $\mathcal{D}(\mathcal{K}(\mathcal{H})^h)$-minimality of $C + D_1$.

If we take $v, w \in \mathcal{H}$, we can write $v = \sum_{i=1}^{\infty} v^i e_i$ and $w = \sum_{i=1}^{\infty} w^i e_i$ with $v^i, w^i \in \mathbb{C}$ for all $i \in \mathbb{N}$. Then, we denote with $v \circ w$ the vector in $\mathcal{H}$ defined by

$$v \circ w = \left(v^1 w^1, v^2 w^2, v^3 w^3, \ldots\right) \in \mathcal{H}.$$

The proof of the following corollary is the analogue of that of Corollary 3 in [2], considering the special treatment for compact operators instead of matrices.

Corollary 10. *Let $C \in \mathcal{K}(\mathcal{H})^h$, $C \neq 0$, such that $\lambda_{\max}(C) + \lambda_{\min}(C) = 0$. Then, the following statements are equivalent*:

(1) *C is minimal* (*as defined in Section 2*).
(2) *There exist $\{v_i\}_{i=1}^r \subset R(E_+)$ and $\{v_j\}_{j=r+1}^{r+s} \subset R(E_-)$, orthonormal sets such that*

$$\mathrm{co}\left(\{v_i \circ \overline{v_i}\}_{i=1}^r\right) \cap \mathrm{co}\left(\{v_j \circ \overline{v_j}\}_{i=r+1}^{r+s}\right) \neq 0.$$

Here $\mathrm{co}(\{w_k\}_{k=n_0}^{n_1})$ denotes the convex hull of the space generated by the finite family of vectors $\{w_k\}_{k=n_0}^{n_1} \subset \mathcal{H}$, and if $w_k = (w_k^1, w_k^2, w_k^3, \ldots)$ in the fixed base chosen in $\mathcal{H}$ (see Section 2), then we denote with $\overline{w_k} = (\overline{w_k^1}, \overline{w_k^2}, \overline{w_k^3}, \ldots) \in \mathcal{H}$.

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