Central European Journal of Mathematics

A note on the differentiable structure of generalized idempotents.

--Manuscript Draft--

Manuscript Number:	
Full Title:	A note on the differentiable structure of generalized idempotents.
Article Type:	Research Article
Section/Category:	Functional Analysis
Keywords:	generalized idempotents; generalized projections
Corresponding Author:	Esteban Andruchow Universidad Nacional de General Sarmiento Los Polvorines, Buenos Aires ARGENTINA
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Universidad Nacional de General Sarmiento
Corresponding Author's Secondary Institution:	
First Author:	Esteban Andruchow
First Author Secondary Information:	
Order of Authors:	Esteban Andruchow
	Gustavo Corach, Dr.
	Mostafa Mbekhta, Dr.
Order of Authors Secondary Information:	
Abstract:	For a fixed n > 2, we study the set of generalized idempotents, which are operators satisfying T^{n+1} = T. Also the subsets of operators such that T^{n-1} is the Moore-Penrose pseudo-inverse of T, and the subset of operators such that T^{n-1} = T^* (known as generalized projections) are studied. The local smooth structure of these sets is examined.
Suggested Reviewers:	Ilya Spitkovsky College of William and Mary ilya@math.wm.edu We would value his opinion.
Opposed Reviewers:	

A note on the differentiable structure of generalized idempotents

E. Andruchow, G. Corach, M. Mbekhta

April 24, 2012

Abstract

For a fixed $n > 2$, we study the set Λ of generalized idempotents, which are operators satisfying $T^{n+1} = T$. Also the subsets $\Lambda_\dagger$, of operators such that T^{n-1} is the Moore-Penrose pseudo-inverse of T, and Λ_*, of operators such that $T^{n-1} = T^*$ (known as generalized projections) are studied. The local smooth structure of these sets is examined.

2010 MSC: 47A05, 47B15.
Keywords: generalized idempotents, generalized projections.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. For a fixed integer $n \geq 3$, we consider the following subsets of $\mathcal{B}(\mathcal{H})$:

$$\Lambda = \{T \in \mathcal{B}(\mathcal{H}) : T^{n+1} = T\},$$

$$\Lambda_* = \{T \in \mathcal{B}(\mathcal{H}) : T^{n-1} = T^*\},$$

and

$$\Lambda_\dagger = \{T \in \mathcal{B}(\mathcal{H}) : T^{n-1} = T^\dagger\},$$

where $T^\dagger$ denotes the Moore-Penrose pseudo-inverse of T; observe that $T^\dagger$ is bounded if and only if $R(T)$, the range of T, is closed; therefore operators in $\Lambda_\dagger$ have closed ranges. This paper is devoted to a topological and geometrical study of the sets Λ, Λ_* and $\Lambda_\dagger$, which are all smooth submanifolds of $\mathcal{B}(\mathcal{H})$. We show that $\Lambda_* \subset \Lambda_\dagger \subset \Lambda$ and that all inclusions are proper and smooth.

These submanifolds have interesting characterizations, which relate them to the sets of idempotents, operators with ascent and descent not greater than 1, and their intersections with the sets of self-adjoints, normal, or quasi-nilpotent operators are relevant.

The group $Gl(\mathcal{H})$ of invertible operators acts locally transitively by similarity on Λ, and the orbits of the action are the connected components of Λ. Analogously, the group $U(\mathcal{H})$ of unitary operators acts locally transitively on Λ_*. However, though $U(\mathcal{H})$ acts on $\Lambda_\dagger$, the action is not locally transitive there. These facts allow the determination of the (arc) connected components of Λ, Λ_*, $\Lambda_\dagger$.

The operator norm defines Finsler metrics on these manifolds, and the range map (see definition below)

$$\Lambda \to \mathcal{P}(\mathcal{H})$$

is a smooth submersion. When restricted to $\Lambda_\dagger$, the range map decreases norms between the tangent spaces. We also study the geodesic distance in $\Lambda_\dagger$ and Λ_*, and compare it to the distance given by the usual norm.

The sets Λ, Λ_* and $\Lambda_\dagger$ have been studied before, under different names, by several authors. Kovarik and Sherif [15], [16], [17] studied, for a Banach space $\mathcal{X}$, the geometry of the set

$$\mathcal{E} = \{(E_1, \ldots, E_{n+1}) : E_k \in \mathcal{B}(\mathcal{X}), E_k E_i = \delta_{ki} E_i, \sum_{k=1}^{n+1} E_k = 1\}.$$

Corach, Porta and Recht [8] observed, in a Banach algebra setting, that $\mathcal{E}$ is diffeomorphic to the submanifold $\{T : p(T) = 0\}$, where p is a complex polynomial of degree $n+1$ with simple roots. Thus, for $p(T) = T^{n+1} - T$, this set is precisely Λ. In 1997, Gross and Trenkler [13] initiated the study of matrices $A \in \mathbb{C}^{n\times n}$ such that $A^2 = A^*$ (or $A^2 = A^\dagger$). J.K. Baksalary, O.M. Baksalary and X. Liu [4], [5], [6] extended their results. Benitez and Thome [7] started the study of the set of matrices $\{A : A^k = A^\dagger\}$ and $\{A : A^k = A^*\}$. Du and Li [11] found a spectral characterization of operators $A \in \mathcal{B}(\mathcal{H})$ such that $A^2 = A^*$ and G.W. Stewart [20], independently, extended, for matrices, the spectral characterization of A such that $A^k = A^*$ or $A^k = A^\dagger$. Lebtahi and Thome [18] generalized these spectral descriptions to operators. In [12] Du, Wang and Duan proved some connectedness results for $\{A : A^k = A^*\}$.

The contents of the paper are the following. Section 2 contains notations, preliminaries and a short description of the spectral properties of $T \in \Lambda$. Section 3 is devoted to prove the inclusions $\Lambda_* \subset \Lambda_\dagger \subset \Lambda$, and several characteristic properties of elements of Λ_* and $\Lambda_\dagger$. The intersections

$$\mathcal{Q}(\mathcal{H}) \cap \Lambda_\dagger = \mathcal{P}(\mathcal{H}), \Lambda \cap Gl(\mathcal{H}) = \Lambda_\dagger \cap Gl(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T^n = 1\},$$

$$\Lambda \cap U(\mathcal{H}) = \Lambda_\dagger \cap Gl(\mathcal{H}) = \{T \in \mathcal{N}(\mathcal{H}) : T^n = 1\},$$

$$\Lambda_\dagger \cap \mathcal{B}_h(\mathcal{H}) = \Lambda \cap \mathcal{B}_h(\mathcal{H}) = \Lambda \cap \mathcal{B}^+(\mathcal{H}) = \mathcal{P}(\mathcal{H}),$$

$$\Lambda \cap \mathcal{N}(\mathcal{H}) = \Lambda_\dagger \cap \mathcal{N}(\mathcal{H}) = \Lambda_*,$$

$$\Lambda \cap QN(\mathcal{H}) = \{0\} \text{ and } \Lambda \cap \mathcal{C}(\mathcal{H}) = \Lambda \cap \mathcal{I}(\mathcal{H}) = \Lambda_*$$

are determined, where $\mathcal{Q}(\mathcal{H})$ denotes the idempotents, $\mathcal{P}(\mathcal{H})$ the orthogonal projectors, $\mathcal{B}_h(\mathcal{H})$ the self-adjoint operators, $\mathcal{B}^+(\mathcal{H})$ the positive operators, $\mathcal{N}(\mathcal{H})$ the normals, $QN(\mathcal{H})$ the quasi-nilpotents, $\mathcal{C}(\mathcal{H})$ the contractions and $\mathcal{I}(\mathcal{H})$ the partial isometries. Section 4 deals with the action of $Gl(\mathcal{H})$ (resp. $U(\mathcal{H})$) over Λ (resp. $\Lambda_\dagger$ and Λ_*), and the description of the submanifold structures. It is proved that for $T \in \Lambda$, T^n is an idempotent with the same range as T; T^n is the orthogonal projection onto the range of T if and only if $T \in \Lambda_\dagger$. The map

$$\nu : \Lambda_\dagger \to \mathcal{P}(\mathcal{H}),\ \nu(T) = T^n$$

is called the range map. Section 5 is devoted to the proof that the tangent map of ν is norm-decreasing. This result is applied to show the existence of curves of minimal length in $\Lambda_\dagger$. Finally, in section 6 the geodesic metric and the norm metric in Λ_* are shown to be equivalent. Clearly, if d denotes the geodesic metric (computed as the infimum of the lengths of the curves joining the given points), then $d(T_0, T_1) \geq \|T_0 - T_1\|$. We compute a constant for the reverse inequality, depending on n.

2 Preliminary facts

If $T \in \Lambda$, T is a root of the polynomial $p(t) = t^{n+1} - t$, therefore its spectrum is contained in the set $\Omega_n = \{0\} \cup \{z : z^n = 1\}$. Fix a primitive root of 1, for instance $w = e^{i\frac{2\pi}{n}}$. There exist polynomials $p_0, p_1, \dots, p_n$ of degree $n+1$, such that $p_k(T)$ are idempotent operators, and

$$T = \sum_{k=1}^{n} w^k p_k(T) \quad \text{and } p_0(T) = 1 - \sum_{k=1}^{n} p_k(T).$$

That is, $p_k(T)$ are the spectral idempotents corresponding to the eigenvalues 0, w^k (by convention $p_k(T) = 0$ if w^k is not in $\sigma(T)$) . Note that $p_0(t) = 1 - t^n$ (and thus $\sum_{k=1}^{n} p_k(t) = t^n$), and that for $k = 1, \dots, n$,

$$p_k(t) = t \prod_{j:j\neq k} \frac{t - w^j}{w^k - w^j}.$$

Moreover, a straightforward verification shows that $p_n(t) = \frac{1}{n}\sum_{j=1}^{n} t^j$. Also it is clear that for $k \geq 1$,

$$p_k(t) = p_n(w^{-k}t) = \sum_{j=1}^{n} w^{j-k} t^j.$$

In this paper, if $T \in \mathcal{B}(\mathcal{H})$, $R(T)$ denotes the range of T, and for a given subspace $\mathcal{S} \subset \mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$.

3 Characterizations of Λ, $\Lambda_\dagger$ and Λ_*

Let us prove first the following inclusions

Proposition 3.1. $\Lambda_* \subset \Lambda_\dagger \subset \Lambda$.

Proof. If $T^{n-1} = T^*$, then T is normal and $(T^*)^{n-1} = T$. Then

$$(T^*T)^{n-1} = (T^*)^{n-1}T^{n-1} = TT^* = T^*T.$$

Since T^*T is apositive operator and a root of the polynomial $t^{n-1} - t$, therefore $\sigma(T^*T) \subseteq \{0, 1\}$. Note that

$$T^n = T^{n-1}T = T^*T,$$

and therefore $\sigma(T) \subseteq \{0\} \cup \{z : z^n = 1\}$. Then $T = \sum_{k=1}^{n} w^k p_k(T)$ and $T^{n+1} = T$.

Now, $TT^*T = TT^{n-1}T = T^{n+1} = T$, and also $T^*TT^* = (TT^*T)^* = T^*$. Thus T^* is a pseudo-inverse for T. Clearly the idempotent $T^*T = TT^*$ is self-adjoint. Then $T^{n-1} = T^* = T^\dagger$.

If $T^{n-1} = T^\dagger$, then $P_{R(T)} = TT^\dagger = TT^{n-1} = T^n$. Thus $T = P_{R(T)}T = T^nT = T^{n+1}$. □

Let us collect in the next remark several elementary observations. Recall that an operator T has *finite ascent* if there exists $k \in \mathbb{N}$ such that $N(T^k) = N(T^{k+1})$. The smallest k with this property is called $a(T)$, the ascent of T. An operator T has finite descent if there exists k such that $R(T^k) = R(T^{k+1})$, an the smallest such k is called de descent $d(T)$ of T.

Remark 3.2.

1. As remarked above,

   (i) if $T \in \Lambda$, T^n and $1 - T^n$ are idempotents onto $R(T)$ and $N(T)$, respectively. It follows that $R(T) \dot{+} N(T) = \mathcal{H}$.

   (ii) If $a(T)$, $d(T)$ denote the ascent the descent of T, then

$$\Lambda = \{T; T^n \in \mathcal{Q}(\mathcal{H}) \text{ and } a(T) \leq 1\} = \{T; T^n \in \mathcal{Q}(\mathcal{H}) \text{ and } d(T) \leq 1\}$$

2. Apparently $\mathcal{Q}(\mathcal{H}) \subset \Lambda$ and $\mathcal{P}(\mathcal{H}) \subset \Lambda_*$ (in both cases, all the eigenspaces corrresponding to the eigenvalues w^j for $1 \leq j \leq n-1$ are trivial). Moreover $\mathcal{Q}(\mathcal{H}) \cap \Lambda_\dagger = \mathcal{P}(\mathcal{H})$.

3. $\Lambda_\dagger = \{T \in \Lambda;\ R(T) \subseteq R(T^*)\} = \{T \in \Lambda;\ R(T^*) \subseteq R(T)\} = \{T \in \Lambda;\ R(T) = R(T^*)\} = \{T \in \Lambda;\ N(T) \subseteq N(T^*)\} = \{T \in \Lambda;\ N(T^*) \subseteq N(T)\} = \{T \in \Lambda;\ N(T) = N(T^*)\}$

4. When the kernel is trivial, we have the following straightforward identities:

$$\Lambda \cap Gl(\mathcal{H}) = \Lambda_\dagger \cap Gl(\mathcal{H}) = \{\ n\text{-roots of } 1\},$$

   and

$$\Lambda \cap U(\mathcal{H}) = \Lambda_* \cap Gl(\mathcal{H}) = \{\text{ normal } n\text{-roots of } 1\}.$$

5. Recall that $\mathcal{B}_h(\mathcal{H})$ denotes the set of self-adjoint operators, and $\mathcal{B}^+(\mathcal{H})$ denotes the set of positive operators. Then

$$\mathcal{P}(\mathcal{H}) = \Lambda_\dagger \cap \mathcal{B}_h(\mathcal{H}) = \Lambda \cap \mathcal{B}_h(\mathcal{H}) = \Lambda \cap \mathcal{B}^+(\mathcal{H}).$$

   Since $\mathcal{P}(\mathcal{H})$ is a subset of all other sets involved, it suffices to show that $\Lambda \cap \mathcal{B}_h(\mathcal{H}) \subset \mathcal{P}(\mathcal{H})$. If $T \in \Lambda$ is self-adjoint, then in particular it is normal, and thus all eigenspaces are orthogonal. Therefore, the fact that it is self-adjoint implies that the projections onto the eigenspaces $p_j(T)$, $1 \leq j \leq n-1$, corresponding to the non-real eigenvalues, are trivial. Thus $T = p_1(T) \in \mathcal{P}(\mathcal{H})$.

6. If $\mathcal{N}(\mathcal{H})$ denotes the set of normal operators, then

$$\Lambda \cap \mathcal{N}(\mathcal{H}) = \Lambda_\dagger \cap \mathcal{N}(\mathcal{H}) = \Lambda_*$$

7. If $\mathcal{QN}(\mathcal{H})$ denotes the set of quasi-nilpotent operators, then

$$\Lambda \cap \mathcal{QN}(\mathcal{H}) = \{0\}$$

Proposition 3.3. *Let $T \in \Lambda$. Then:*

1. *$T \in \Lambda_\dagger$ if and only if $R(T)$ and $N(T)$ are orthogonal. In that case, $T^n = P_{R(T)}$.*
2. *$T \in \Lambda_*$ if and only if all the eigenspaces of T are mutually orthogonal.*

Proof. If $T \in \Lambda_\dagger$, $T^n = TT^\dagger = P_{R(T)}$, and then $1 - T^n$ is a self-adjoint idempotent onto $N(T)$, and thus $p_0(T) = 1 - T^n = P_{N(T)}$. Conversely, if $R(T) \perp N(T)$, then T^n is an idempotent whose range and null-space are orthogonal. Therefore it is a self-adjoint projection. Then the pseudo-inverse T^{n-1} of T verifies $T^{n-1}T = TT^{n-1} = P_{R(T)}$, i.e. $T^{n-1} = T^\dagger$.

If $T \in \Lambda_*$, then T is normal, and therefore all $n+1$ idempotents $p_j(T)$, $0 \leq j \leq n$ are normal, thus self-adjoint. Conversely, if the eigenspaces are orthogonal, these idempotents are self-adjoint. Then

$$T^* = (\sum_{j=1}^{n} w^j p_j(T))^* = \sum_{j=1}^{n} \bar{w}^j p_j(T) = \sum_{j=1}^{n} (w^j)^{n-1} p_j(T) = T^{n-1}.$$

□

If $\mathcal{C}(\mathcal{H})$ denotes the set of contractive operators in $\mathcal{H}$, then

Proposition 3.4.

$$\Lambda \cap \mathcal{C}(\mathcal{H}) = \Lambda_*.$$

Proof. The elements of Λ_* are clearly contractive (in fact, they are partial isometries). Conversely, if $T \in \Lambda$ is contractive, then $p_k(T)$ are contractive idempotents, thus self-adjoint projections. Indeed, for $1 \leq k \leq n$

$$\|p_k(T)\| = \frac{1}{n} \| \sum_{j=1}^{n} w^{-j} T^j \| \leq \frac{1}{n} \{ \sum_{j=1}^{n} \|T\|^j) \leq 1.$$

Then also $p_0(T) = 1 - \sum_{j=1}^{n} p_j(T)$ is self-adjoint. It follows that $T \in \Lambda_*$. □

Let us examine how these sets relate to partial isometries. We denote by $\mathcal{I}$ the set of partial isometries. As noted above, $\Lambda_* \subset \mathcal{I}$. Then apparently

Corollary 3.5.

$$\Lambda \cap \mathcal{I} = \Lambda_*.$$

4 Actions of the unitary and invertible groups

As remarked above, if $T \in \Lambda$, it can be diagonalized,

$$T = \sum_{j=1}^{n} w^j p_j(T).$$

If $G \in Gl(\mathcal{H})$, then clearly $GTG^{-1} \in \Lambda$, and

$$GTG^{-1} = \sum_{j=1}^{n} w^j G p_j(T) G^{-1} = \sum_{j=1}^{n} w^j p_j(GTG^{-1})$$

Fix $T \in \Lambda$, and consider

$$\pi_T : Gl(\mathcal{H}) \to \Lambda, \pi_T(G) = GTG^{-1}.$$

Let us recall some facts from [9] and [3], concerning similarity orbits of operators.

Remark 4.1.

Let $(Q_0, \ldots, Q_n)$ be an $n+1$-tuple of idempotents such that $Q_iQ_j = 0$ if $i \neq j$ and $\sum_{j=0}^{n} Q_j = 1$. Then there exists an invertible operator G such that GQ_iG^{-1} are orthogonal projections. This fact is well known. It was proved in [9]. In [2] it was proved by a procedure similar to the Gram-Schmidt orthogonalization process.

This implies that $T \in \Lambda$ is similar to an element $S \in \Lambda_*$. Indeed, consider the $n+1$-tuple of idempotents $(p_0(T), \ldots, p_n(T))$ (which verifies the above hypothesis), and pick $G \in Gl(\mathcal{H})$ such that $Gp_i(T)G^{-1}$ are self-adjoint. Then clearly $S = GTG^{-1} \in \Lambda_*$. Since S is normal and has finite spectrum, it is what D. Herrero [14] called a nice Jordan operator. In [3] it was proved that this implies several properties for the map π_T:

1. The map π_T has local cross sections: for each $T_0 \in \Lambda$ there exists a neighbourhood $\mathcal{V}$ of T_0 in Λ and a continuous map $\sigma_{T_0} : \mathcal{V} \to Gl(\mathcal{H})$ such that $\pi_T(\sigma_{T_0}(S)) = S$ for all $S \in \mathcal{V}$.
2. In particular, the action of $Gl(\mathcal{H})$ is locally trivial (close elements in Λ are conjugate by the action). Therefore, the connected comoponent of T in Λ coincides with the orbit $\{GTG^{-1} : G \in Gl(\mathcal{H})\}$.
3. Each connected component of Λ is an analytic submanifold of $\mathcal{B}(\mathcal{H})$, and the map π_T onto the component containing T is an analytic submersion.
4. Pick now $T \in \Lambda_*$, and denote by π_T^u the restriction of π_T to the unitary group $U(\mathcal{H})$. Apparently it takes values on the unitary orbit of T, which lies inside Λ_*. Then also π_T^u has (unitary) continuous local cross sections. The connected component of T in Λ_* coincides with the unitary orbit of T, and is a C^∞ submanifold of $\mathcal{B}(\mathcal{H})$. The corresponding map $\pi_T^u : U(\mathcal{H}) \to \Lambda_*$ (or rather, the connected component of T in Λ_*) is a C^∞ submersion.
5. We remark that $U(\mathcal{H})$ acts on $\Lambda_\dagger$, but the action is not locally transitive (as with Λ_*). Indeed, the unitary action fixes the angles between the eigenspaces, and this is not a local property of $\Lambda_\dagger$

In particular, the similarity orbit of a single idempotent $Q \in \mathcal{Q}(\mathcal{H})$, verifies these conditions. In this case the analytic cross sections can be explicitely computed [9]. Namely, if $R \in \mathcal{Q}(\mathcal{H})$ such that $\|R - Q\| < 1$, then $\sigma(R) = RQ + (1-R)(1-Q)$ is invertible, and verifies $\sigma(R)Q = R\sigma(R)$, thus providing a local cross section for π_Q on a neighbourhood of Q. It is apparently analytic. A cross section near $Q' = GQG^{-1}$ is obtained by translating σ_Q: put $\sigma_{Q'}(R) = G\sigma_Q(G^{-1}RG)$, defined on the set $\{R \in \mathcal{Q}(\mathcal{H}) : \|G^{-1}RG - Q'\| < 1\}$, which is clearly open in $\mathcal{Q}(\mathcal{H})$.

Recall that if $T \in \Lambda$, then T^n is an idempotent onto the range of T, and that T^n is the orthogonal projection onto the range of T if and only if $T \in \Lambda_\dagger$.

Proposition 4.2. *The map*

$$\nu : \Lambda \to \mathcal{Q}(\mathcal{H}) \ , \quad \nu(T) = T^n$$

is a analytic submersion.

Proof. It suffices to show that this map has analytic local cross sections. Fix $T_0 \in \Lambda$, and let $\mathcal{W}$ be a neighbourhood of $Q_0 = T_0^n$ in $\mathcal{Q}(\mathcal{H})$ on which the cross section for π_{Q_0} is defined. Let $\mathcal{V} = \{T \in \Lambda : T^n \in \mathcal{W}\}$, and define

$$s_{T_0} : \mathcal{V} \to \Lambda, \quad s_{T_0}(T) = \sigma_{Q_0}(T^n) T_0 \sigma_{Q_0}^{-1}(T^n).$$

It is clearly well defined and analytic. Moreover

$$(s_{T_0}(T))^n = \sigma_{Q_0}(T^n)T_0^n\sigma_{Q_0}^{-1}(T^n) = \sigma_{Q_0}(T^n)Q_0\sigma_{Q_0}^{-1}(T^n) = T^n,$$

i.e. it is a local cross section for the map $T \mapsto T^n$. □

Corollary 4.3. *The set $\Lambda_\dagger$ is a C^∞ submanifold of Λ (and of $\mathcal{B}(\mathcal{H})$).*

Proof. As seen in the previous section, $T \in \Lambda$ belongs to $\Lambda_\dagger$ if and only if the idempotent T^n is a self-adjoint projection. Then

$$\Lambda_\dagger = \nu^{-1}(\mathcal{P}(\mathcal{H})).$$

Since $\mathcal{P}(\mathcal{H})$ is a C^∞ submanifold of $\mathcal{Q}(\mathcal{H})$ and ν is a submersion, it follows that $\Lambda_\dagger$ is a C^∞ submanifold of Λ. □

Since the map $\pi_T : Gl(\mathcal{H}) \to \Lambda$, $\pi_T(G) = GTG^{-1}$ is a submersion, it follows that continuous curves in Λ can be lifted to continuous curves in $Gl(\mathcal{H})$. Let us describe a natural procedure to lift smooth curves, borrowed essentially form [9]. Suppose that $T(t) \in \Lambda$ varies smoothly for $t \in I$, where smooth means C^k, $1 \le k \le \infty$. It follows that $p_j(T(t))$, $0 \le j \le n$, are smooth curves in $\mathcal{Q}(\mathcal{H})$. Then the curve

$$\Sigma_t = -\sum_{j=0}^{n} p_j(T(t))\frac{d}{dt}p_j(T(t)) \qquad (1)$$

is continuous (in fact C^{k-1}). Consider the following linear differential equation in $\mathcal{B}(\mathcal{H})$, which we shall call the *transport equation*: fix $t_0 \in I$

$$\begin{cases} \dot\Gamma = \Sigma_t\Gamma \\ \Gamma(t_0) = 1 \end{cases} \qquad (2)$$

It is a standard fact that an operator linear equation as above, with invertible initial condition, remains invertible for all $t \in I$. The solutions of this equation, which are smooth curves in $Gl(\mathcal{H})$, lift the curve $T(t)$:

Proposition 4.4. *Fix $t_0 \in I$ and let Γ be the solution of (2). Then*

$$\Gamma(t)T(t_0)\Gamma^{-1}(t) = T(t),$$

for all $t \in I$. Moreover, if $T(t) \in \Lambda_*$, then $\Gamma(t) \in U(\mathcal{H})$.

Proof. Fix $0 \le j \le n$, denote $p_j = p_j(T(t))$, and differentiate $\Gamma^{-1}p_j\Gamma$,

$$-\Gamma^{-1}\dot\Gamma\Gamma^{-1}p_j\Gamma + \Gamma^{-1}\dot p_j\Gamma + \Gamma^{-1}p_j\dot\Gamma$$

$$= \Gamma^{-1}\left(-\Sigma_t p_j + \dot p_j + p_j\Sigma_t\right)\Gamma.$$

Note that since $p_ip_j = 0$ if $i \ne j$, then $p_j\Sigma_t = -p_j\dot p_j$. Also differentiating $p_ip_j = 0$, gives $\dot p_ip_j = -p_i\dot p_j$ if $i \ne j$. If $i = j$, differentiating $p_j^2 = p_j$ one obtains $\dot p_jp_j + p_j\dot p_j = 0$ and thus $p_j\dot p_jp_j = 0$. Then

$$\Sigma_tp_j = -\sum_{i\ne j} p_i\dot p_j = (1-p_j)\dot p_j = -\dot p_j + p_j\dot p_j.$$

Therefore $-\Sigma_t p_j + \dot{p}_j + p_j\Sigma_t = 0$, and thus $\Gamma^{-1}p_j\Gamma$ is constant, and thus

$$\Gamma^{-1}p_j\Gamma = \Gamma^{-1}(t_0)p_j(t_0)\Gamma(t_0) = p_j(t_0).$$

Then

$$T(t) = \Gamma\left(\sum_{j=1}^{n} w^j p_j(T)\right)\Gamma^{-1} = \Gamma T(t_0)\Gamma^{-1}.$$

If $T(t) \in \Lambda_*$, then Σ_t is anti-self-adjoint. Indeed, p_j is self-adoint, and using again that $\dot{p}_j p_j + p_j\dot{p}_j$, one has

$$\Sigma_t^* = -\sum_{i=0}^{n} \dot{p}_i p_i = \sum_{i=0}^{n} p_i\dot{p}_i - \sum_{=0}^{n} \dot{p}_i = -\Sigma_t,$$

because $\sum_{=0}^{n} \dot{p}_i = 0$ (since $\sum_{=0}^{n} p_i = 1$). Therefore in this case the solution Γ consists of unitary operators. □

The actions of the unitary and invertible groups allow the easy characterization of the connected components of these sets. The case for Λ and Λ_* is apparent. We state them in the following result. We use the following notation. If $T \in \Lambda$, $0 \le i \le n$,

$$\mu_i(T) = \dim(R(p_i(T))).$$

Proposition 4.5. *The $n+1$-tuple $\vec{\mu}(T) = (\mu_0(T), \mu_1(T), \ldots, \mu_n(T))$ characterizes the connected components of Λ (resp. $\Lambda_\dagger$, resp. Λ_*). That is, $T, T' \in \Lambda$ (resp. $\Lambda_\dagger$, resp. Λ_*) lie in the same connected component of this set if and only if $\vec{\mu}(T) = \vec{\mu}(T')$.*

Proof. Only the assertion on $\Lambda_\dagger$ needs a proof. If T and T' lie in the same component of $\Lambda_\dagger$, then in particular they can be connected in Λ, and thus $\vec{\mu}(T) = \vec{\mu}(T')$. Conversely, suppose that $T, T' \in \Lambda_\dagger$ verify $\vec{\mu}(T) = \vec{\mu}(T')$. Then T^n and $(T')^n$ are self-adjoint projections whose range and kernels have the same dimensions. Therefore they are unitarily equivalent. Pick $U \in U(\mathcal{H})$ such that $UT^nU^* = (T')^n$. Note that $U^*T'U \in \Lambda_\dagger$ and $\vec{\mu}(U^*T'U) = \vec{\mu}(T)$. Thus we are reduced to the case when T and T' have the same range and kernel. This case is trivial: they can be joined via a continuous curve of invertibles $G(t)$, which leave invariant the fixed kernels and ranges, by means of $T(t) = G(t)TG^{-1}(t)$. Apparently this curve lies inside $\Lambda_\dagger$. □

Remark 4.6. Denote $\Lambda^k = \{T \in \mathcal{B}(\mathcal{H}) : T^{k+1} = T\}$. Apparently $\Lambda^k \subset \Lambda^m$ if and only if k divides m. Moreover, from the above result characterizing the connected components, it is also apparent that if $m = kl$, then Λ^k consists of the connected components of Λ^m such that $\mu_j(T) = 0$ if j is not a power of l. In particular, Λ^k is a submanifold of Λ^m.

5 The range projections decrease distances

In this section we endow the tangent spaces of Λ, $\Lambda_\dagger$ and Λ_* with the metric induced by the usual norm of operators. First let us describe these tangent spaces. Since the maps π_T and π_T^u are submersions, their differentials are onto. Therefore if $T_0 \in \Lambda$,

$$(T\Lambda)_{T_0} = \{XT_0 - T_0X : X \in \mathcal{B}(\mathcal{H})\}.$$

Indeed, a typical smooth curve $T(t)$ in Λ with $T(0) = T_0$ is of the form $T(t) = G(t)T_0G^{-1}(t)$, for $G(t)$ a smooth curve in $Gl(\mathcal{H})$, with $G(0) = 1$ and $\dot{G}(0) = X$. Since $\dot{T} = \dot{G}T_0G - GT_0G^{-1}\dot{G}G^{-1}$, it follows that $\dot{T}(0) = XT_0 - T_0X$. Analogously, reasoning with the action of the unitary group, if $T_0 \in \Lambda_*$,

$$(T\Lambda_*)_{T_0} = \{XT_0 - T_0X : X \in \mathcal{B}_{ah}(\mathcal{H})\}.$$

Finally:

Proposition 5.1. *If $T_0 \in \Lambda_\dagger$, a tangent vector $XT_0 - T_0X \in (T\Lambda)_{T_0}$ is tangent to $\Lambda_\dagger$ if and only if $XT_0^n - T_0^nX$ is self-adjoint, i.e.*

$$(T\Lambda_\dagger)_{T_0} = \{XT_0 - T_0X : X \in \mathcal{B}(\mathcal{H}) \text{ such that } XT_0^n - T_0^nX \in \mathcal{B}_h(\mathcal{H})\}.$$

Proof. Consider the smooth submersion

$$\kappa = \nu \circ \pi_{T_0} : Gl(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}), \kappa(G) = GT_0^nG^{-1}.$$

Then $\mathcal{S} = \kappa^{-1}(\mathcal{P}(\mathcal{H}))$ is a closed submanifold of $Gl(\mathcal{H})$, such that $\pi_{T_0}(\mathcal{S}) = \Lambda_\dagger$. Therefore,

$$(T\Lambda_\dagger)_{T_0} = (d\pi_{T_0})_{T_0}((T\mathcal{S})_1).$$

Note that $(T\mathcal{S})_1 = \{X \in \mathcal{B}(\mathcal{H}) : XT_0^n - T_0^nX \in \mathcal{B}_h(\mathcal{H})\}$ and that $(d\pi_{T_0})_{T_0}(X) = XT_0 - T_0X$, and therefore our claim follows. ☐

We shall need the following lemma, which states that the most efficient way to complete a co-diagonal 2×2 self-adjoint block matrix order that its norm remains minimal, is by putting zeros in the diagonal. It is the simple case in the theory of operator extensions ([10]). We include the proof of this fact.

Lemma 5.2. *Let P be an orthogonal projection, and $A \in \mathcal{B}(\mathcal{H})$ self-adjoint. Then*

$$\|PA(1-P) - (1-P)AP\| \leq \|A\|.$$

Proof. Given P, one can write operators in $\mathcal{B}(\mathcal{H})$ as 2×2 matrices in terms of P,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix},$$

with A_{ii} self-adjoint. Then

$$PA(1-P) - (1-P)AP = \begin{pmatrix} 0 & A_{12} \\ A_{12}^* & 0 \end{pmatrix}.$$

Note that

$$(PA(1-P) - (1-P)AP)^2 = \begin{pmatrix} A_{12}A_{12}^* & 0 \\ 0 & A_{12}^*A_{12} \end{pmatrix},$$

and that

$$A^2 = \begin{pmatrix} A_{11}^2 + A_{12}A_{12}^* & B \\ B^* & A_{22}^2 + A_{12}^*A_{12} \end{pmatrix}.$$

The linear map $X \mapsto PXP + (1-P)X(1-P)$ is contractive. Thus

$$\|PA^2P + (1-P)A^2(1-P)\| \leq \|A\|^2.$$

On the other hand, it is clear that

$$PA^2P + (1-P)A^2(1-P) = \begin{pmatrix} A_{11}^2 + A_{12}A_{12}^* & 0 \\ 0 & A_{22}^2 + A_{12}^*A_{12} \end{pmatrix} \geq \begin{pmatrix} A_{12}A_{12}^* & 0 \\ 0 & A_{12}^*A_{12} \end{pmatrix}$$

$$= (PA(1-P) - (1-P)AP)^2$$

Therefore

$$\|(PA(1-P)-(1-P)AP)\|^2 = \|(PA(1-P)-(1-P)AP)^2\| \leq \|PA^2P+(1-P)A^2(1-P)\| \leq \|A\|^2.$$

□

Theorem 5.3. *The differential of the map $\nu : \Lambda_\dagger \to \mathcal{P}(\mathcal{H})$, $\nu(T) = T^n = P_{R(T)}$ is norm-decreasing between the tangent spaces, i.e. for any $Z \in (T\Lambda_\dagger)_{T_0}$*

$$\|(d\nu)_{T_0}(Z)\| \leq \|Z\|.$$

Proof. Pick $Z = XT_0 - T_0X \in (T\Lambda_\dagger)_{T_0}$. Then there exists a smooth curve $G_t \in Gl(\mathcal{H})$ such that $G_0 = 1$, $\dot{G}(0) = X$ and $G_tT_0G_t^{-1} \in \Lambda_\dagger$. Then $\nu(G_tT_0G_t^{-1}) = G_tT_0^nG_t^{-1} \in \mathcal{P}(\mathcal{H})$. Thus

$$((d\nu)_{T_0}d\nu)_{T_0}(Z) = Y = \frac{d}{dt}|_{t=0}\nu(G_tT_0G_t^{-1}) = XT_0^n - T_0^nX.$$

Note that $Y^* = Y$. Let $P_j = p_j(T_0)$, $j = 0, \ldots, n$. Then for $i \geq 1$,

$$P_jZP_k = P_jXT_0P_k - P_jT_0XP_k = (w^k - w^j)P_jXP_k,$$

and, analogously using that $T_0^n = \sum_{i=1}^n P_i$,

$$P_jYP_k = P_jX(\sum_{i=1}^n P_i)P_k - P_j(\sum_{i=1}^n P_i)XP_k = P_jXP_k - P_jXP_k = 0.$$

Also it is apparent that $P_0ZP_0 = P_0YP_0 = 0$. Moreover, for $j \geq 1$, by similar computations

$$P_0ZP_j = w^jP_0XP_j, \quad P_jZP_0 = -w^jP_jXP_0$$

and

$$P_0YP_j = P_0XP_j, \quad P_jYP_0 = -P_jXP_0.$$

Thus Y is self-adjoint and P_0 co-diagonal. Using these computations let us write Y and Z in matrix form in terms of the decomposition $P_0, P_1, \ldots, P_n$. Then

$$Y = \begin{pmatrix} 0 & P_0XP_1 & P_0XP_2 & \ldots & P_0XP_n \\ P_1XP_0 & 0 & 0 & \ldots & 0 \\ P_2XP_0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ P_nXP_0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \qquad (3)$$

Put $\Delta_{jk} = w^j - w^k$. Then

$$Z = \begin{pmatrix} 0 & wP_0XP_1 & w^2P_0XP_2 & \dots & w^nP_0XP_n \\ -wP_1XP_0 & 0 & \Delta_{12}P_1XP_2 & \dots & \Delta_{1n}P_1XP_n \\ -w^2P_2XP_0 & \Delta_{21}P_2XP_1 & 0 & \dots & \Delta_{2n}P_2XP_n \\ \dots & \dots & \dots & \dots & \dots \\ -w^nP_nXP_0 & \Delta_{n1}P_nXP_1 & \dots & \Delta_{n(n-1)}P_nXP_{n-1} & 0 \end{pmatrix}. \tag{4}$$

Consider the unitary (diagonal) operator $W = \sum_{i=0}^n w^{-i}P_i$. Then clearly $\|Z\| = \|WZW\|$. A straightforward matrix computation shows that the first row and the first column of WZW coincide (respectively) with the first row and column of Y. Since Y is self-adjoint, this implies that these coincide with the first row and column of $\frac{1}{2}(WZW + (WZW)^*)$. Clearly

$$\|Z\| = \|WZW\| \geq \|\frac{1}{2}(WZW + (WZW)^*)\|.$$

On the other hand, this last (self-adjoint) operator is a completion of the matrix Y (which as a 2×2 matrix in terms of P_0, has zeros in the diagonal). It follows that, by the above lemma, that

$$\|\frac{1}{2}(WZW + (WZ^*W))\| \geq \|Y\|,$$

which completes the proof. □

Corollary 5.4. *The tangent map of* $\rho_0 : \Lambda_\dagger \to \mathcal{P}(\mathcal{H})$, $\rho_0(T) = P_{N(T)} = 1 - T^n$ *is norm-decreasing at any point.*

Proof. The proof follows using that $\rho_0(T) = 1 - \nu(T)$, and thus $(d\rho_0)_T = -(d\nu_T)$. □

If $T(t) \in \Lambda$, $t \in I$ is a smooth curve, one computes the length $\ell(T)$ of $T(t)$ (with the Finsler metric considered here) as

$$\ell(T) = \int_I \|\dot{T}(t)\| dt.$$

Corollary 5.5. *If* $T(t) \in \Lambda_\dagger$ *is a smooth curve, then*

$$\ell(\nu(T)) \leq \ell(T) \quad \text{and } \ell(\rho_0(T)) \leq \ell(T).$$

In [19] it was shown that if $X^* - X$ is co-diagonal with respect to a self-adjoint projection P, i.e. $PXP = (1 - P)X(1 - P) = 0$, then the curve $e^{tX}Pe^{-tX}$ in $\mathcal{P}$ has minimal length along its path, in any interval such that $|t|\|X\| \leq \pi/2$ (by this we mean that this path has minimal length among all possible smooth curves joining any given pair of points in the path). A straightforward consequence of this fact is the following.

Proposition 5.6. *Let* $T \in \Lambda_\dagger$ *and* $P = \nu(T) = P_{R(T)}$. *Let* $X \in \mathcal{B}_{ah}(\mathcal{H})$ *such that* $PXP = (1 - P)X(1 - P) = 0$. *Then the curve* $\tau(t) = e^{tX}Te^{-tX}$ *has minimal length in* $\Lambda_\dagger$ *along its path on any interval* I *such that* $|I| \leq \frac{\pi}{2\|X\|}$.

Proof. Let $\gamma(t) \in \Lambda_\dagger$, be a smooth curve, which is parametrized in the interval $I = [t_0, t_1]$, and verifies $\gamma(t_0) = \tau(t_0)$ and $\gamma(t_1) = \tau(t_1)$. By the above corollary (measuring the lengths of both curves in the common interval I),

$$\ell(\nu(\gamma)) \leq \ell(\gamma).$$

On the other hand, $\nu(\tau(t)) = (e^{tX}Te^{-tX})^n = e^{tX}Pe^{-tX}$, i.e. $\nu(\tau)$ is a minimal geodesic in I. Then, by the result from [19],

$$\ell(\nu(\tau)) \leq \ell(\nu(\gamma)).$$

We claim that $\ell(\tau) = \ell(\nu(\tau))$, a fact which would conclude the proof. Indeed, note that

$$\ell(\nu(\tau)) = \int_I \|\dot\tau(t)\| dt = \int_I \|e^{tZ}XPe^{-tX} - e^{tX}Pe^{-tX}X\| dt = \|XP - PX\| \; |I|.$$

Similarly $\ell(\tau) = \|XT - TX\| \; |I|$. Thus we need to show that $\|XP - PX\| = \|XT - TX\|$. Recall from the proof of Theorem 5.3, the matrix forms (3) and (4) (in terms of $P_0, P_1, \ldots, P_n$) of the conmutators $XP - XP$ and $XT - TX$.

$$XP - PX = \begin{pmatrix} 0 & P_0XP_1 & P_0XP_2 & \ldots & P_0XP_n \\ P_1XP_0 & 0 & 0 & \ldots & 0 \\ P_2XP_0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ P_nXP_0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

In the case of $XT - TX$, note that since $P = \sum_{j=1}^n P_j$, for $1 \leq j \leq n$,

$$P_jXP_k = P_jPXPP_k = 0.$$

Therefore

$$XT - TX = \begin{pmatrix} 0 & wP_0XP_1 & w^2P_0XP_2 & \ldots & w^nP_0XP_n \\ -wP_1XP_0 & 0 & 0 & \ldots & 0 \\ -w^2P_2XP_0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -w^nP_nXP_0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Using the unitary operator W as in the proof of Theorem 5.3, it is apparent that $\|XP - PX\| = \|XT - TX\|$. □

If $\mathcal{S} \subset \mathcal{H}$ is a closed subspace, denote by $\Lambda^{\mathcal{S}}$ (resp. $\Lambda_\dagger^{\mathcal{S}}$, $\Lambda_*^{\mathcal{S}}$) the set of elements T in Λ (resp. $\Lambda_\dagger$, Λ_*) such that $R(T) = \mathcal{S}$. In other words, $\Lambda^{\mathcal{S}} = \nu^{-1}(p_{\mathcal{S}})$.

Corollary 5.7. *Let $T_0, T_1 \in \Lambda_\dagger$, such that $\|P_{R(T_0)} - P_{R(T_1)}\| < 1$. Then there exists a curve $\tau(t) \in \Lambda_\dagger$, $t \in [0, 1]$, of the form $\tau(t) = e^{tX}T_0e^{-tX}$, with $X^* = -X$ and $\|X\| < \pi/2$, such that*

1. *τ has minimal length in $\Lambda_\dagger$ along its path.*
2. *τ has minimal length among all smooth curves in $\Lambda_\dagger$ joining T_0 and $\Lambda_\dagger^{\mathcal{S}}$.*

Proof. If $\|P_{R(T_0)} - P_{R(T_1)}\| < 1$, then $P_{R(T_0)}$ and $P_{R(T_1)}$ can be joined with a minimal geodesic of $\mathcal{P}$, which is given by a $P_{R(T_0)}$-co-diagonal anti-hermitic operator X with $\|X\| < \pi/2$. Then, by the above result, $\tau(t) = e^{tX}T_0e^{-tX}$ has minimal length along its path in $\Lambda_\dagger$. Suppose that γ is another smooth curve in $\Lambda_\dagger$ with $\gamma(0) = T_0$, $\gamma(1) \in \Lambda_\dagger^{\mathcal{S}}$. Then $\nu(\gamma)$ joins $P_{R(T_0)}$ and $P_{R(T_1)}$. Thus

$$\ell(\nu(\gamma)) \geq \ell(\nu(\tau)) = \ell(\tau)$$

by the computation in the preceding Proposition. By Corollary 5.5,

$$\ell(\nu(\gamma)) \leq \ell(\gamma),$$

and the result follows. □

Remark 5.8.

1. Let us denote by $d(A, B)$ the rectifiable distance, obtained as the infimum of the lengths of curves joining A and B (either in Λ or $\mathcal{P}$). Then $\|P_{R(T_0)} - P_{R(T_1)}\| < 1$ is equivalent to $d(P_{R(T_0)}, P_{R(T_1)}) < \pi/2$ (see [19] or [1]). By Corollary 5.5, if $T_0, T_1 \in \Lambda_\dagger$,

$$d(P_{R(T_0)}, P_{R(T_1)}) \leq d(T_0, T_1).$$

   Thus the hypothesis $\|P_{R(T_0)} - P_{R(T_1)}\| < 1$ of the above Corollary, could be replaced by: there exists a smooth curve in $\Lambda_\dagger$ joining T_0 and T_1, of length less than $\pi/2$.

2. Let $\mathcal{S}_0, \mathcal{S}_1$ be closed subspaces of $\mathcal{H}$ such that $\|P_{\mathcal{S}_0} - P_{\mathcal{S}_1}\| < 1$, then

$$d(\Lambda_\dagger^{\mathcal{S}_0}, \Lambda_\dagger^{\mathcal{S}_1}) = d(P_{\mathcal{S}_0}, P_{\mathcal{S}_1}) = \arcsin(\|P_{\mathcal{S}_0} - P_{\mathcal{S}_1}\|).$$

   For the last equality, see for instance [1]. Pick X such that $X^* = -X$, $\|X\| < \pi/2$ and X is $P_{\mathcal{S}_0}$-co-diagonal, such that $e^XP_{\mathcal{S}_0}e^{-X} = P_{\mathcal{S}_1}$. Pick any $T_0 \in \Lambda_\dagger$ such that $R(T_0) = \mathcal{S}_0$. Then $\tau(t) = e^{tX}T_0e^{-tX}$, $t \in [0, 1]$, is minimal in $\Lambda_\dagger$, and its length is $\|XT_0 - T_0X\| = \|XP_{\mathcal{S}_0} - P_{\mathcal{S}_0}X\| = d(P_{\mathcal{S}_0}, P_{\mathcal{S}_1})$. This number does not depend on the choice of $T_0 \in \Lambda_\dagger^{\mathcal{S}_0}$.

3. Otherwise, if $\|P_{R(T_0)} - P_{R(T_1)}\| = 1$, an easy approximation argument shows that

$$d(\Lambda_\dagger^{\mathcal{S}_0}, \Lambda_\dagger^{\mathcal{S}_1}) = \pi/2.$$

6 Comparison between the norm and the geodesic metric in Λ_*

In this section we examine the metric d in Λ, given by the infima of lengths of curves in Λ_*. The metric at the tangent spaces is given by the usual operator norm, therefore any curve joining T_0 and T_1 in Λ_* will be longer than the line segment. Therefore

$$d(T_0, T_1) \geq \|T_0 - T_1\|.$$

In this section, we shall estimate a constant for the reverse inequality. We shall use the local cross section of the action of the unitary group in Λ_*. First note that

Lemma 6.1. *Let $T_0, T_1 \in \Lambda$, and $p_j(T_i)$, $i = 0, 1$, $j = 0, \ldots, n$ the spectral projections given in section 1.*

1. *In general,*
$$\|p_j(T_0) - p_j(T_1)\| \leq \frac{n+1}{2}\|T_0 - T_1\|.$$

2. *If, additionally, $T_0, T_1 \in \Lambda_*$, then, for $1 \leq j \leq n$,*
$$\|p_j(T_0) - p_j(T_1)\| \leq \kappa(n)\|T_0 - T_1\|,$$
where $\kappa(n) = \max\{2\sin(\frac{\pi}{n}), \frac{1}{\sin(\frac{\pi}{n})}\}$. For $j = 0$,
$$\|p_0(T_0) - P_0(T)\| \leq 2\|T_0 - T\|.$$

Proof. If $T_0, T_1 \in \Lambda$, then

$$T_0^j - T_1^j = T_0^j - T_0^{j-1}T_1 + T_0^{j-1}T_1 - \ldots + T_0T_1^{j-1} - T_1^j$$
$$= \sum_{k=0}^{j-1} T_0^k(T_0 - T_1).$$

Therefore $\|T_0^j - T_1^j\| \leq j\|T_0 - T_1\|$. Using this inequality in the formula

$$p_k(t) = p_n(w^{-k}t) = \sum_{j=1}^{n} w^{j-k}t^j$$

one obtains

$$\|p_k(T_0) - P_k(T_1)\| \leq \frac{1}{n}\sum_{j=1}^{n} j\|T_0 - T_1\| = \frac{n+1}{2}\|T_0 - T_1\|.$$

If $T_0, T_1 \in \Lambda_*$, then the resolvent operators of T_0 and T_1 are normal. Using the Riesz integral form of the spectral projection,

$$p_j(T_0) - p_j(T_1) = \frac{1}{2\pi i}\int_{C_j} (z1 - T_0)^{-1} - (z1 - T_1)^{-1} dz = \frac{1}{2\pi i}\int_{C_j} (z1 - T_0)^{-1}(T_1 - T_0)(z1 - T_1)^{-1} dz,$$

where C_j is a circle centered at w^j if $j \neq 0$, with radius equal to the minimum between $1/2$ and $\sin(\frac{\pi}{n})$ (which is half the distance between w^j and the nearest eigenvalue). For $j = 0$, C_0 is centered at 0, with radius $1/2$. Therefore, for $j \geq 1$

$$\|p_j(T_0) - p_j(T_1)\| \leq \frac{1}{2\pi}\int_{C_j} \|(z1 - T_0)^{-1}\|\|(z1 - T_0)^{-1}\|dz\|T_0 - T_1\|.$$

Using that the resolvents are normal,

$$\|(z1 - T_i)^{-1}\| = \frac{1}{d(z, \Omega_n)} = \frac{1}{\min\{1/2, \sin(\frac{\pi}{n})\}}.$$

Thus

$$\|p_j(T_0) - p_j(T_1)\| \leq \frac{1}{2\pi}(\max\{2, \frac{1}{\sin(\frac{\pi}{n})}\})^2 2\pi \sin(\frac{\pi}{n})\|T_0 - T_1\| = \kappa(n)\|T_0 - T_1\|.$$

For $j = 0$, if $z \in C_0$ (i.e. $|z| = 1/2$), $d(z, \Omega_n) \geq 1/2$, and therefore

$$\|p_0(T_0) - p_0(T)\| \leq 2\|T_0 - T\|.$$

□

Let us recall the formula for the local cross section of the unitary action on systems of self-adjoint projections [9], which serves as the local cross sections for action on Λ_*. Fix $T_0 \in \Lambda_*$, and let $T \in \Lambda_*$ such that $\|p_k(T_0) - p_k(T)\| < 1$ (for instance, if $\|T_0 - T\| < \frac{1}{\kappa(n)}$). Then

$$G = \sum_{j=0}^{n} p_j(T)p_j(T_0)$$

is invertible. Indeed, put $P_j = p_j(T_0)$ and $Q_j = p_j(T)$. Then $G^*G = \sum_{j=0}^{n} P_jQ_jP_j$. Note that each $P_jQ_jP_j$ is an invertible operator acting in $R(P_j)$, beacause $\|P_jQ_jP_j - P_j\| \leq \|P_j - Q_j\| < 1$. Therefore G^*G is invertible in $\mathcal{H}$, which is the direct sum of the ranges $R(P_j)$. Analogously GG^* is invertible, and thus G is invertible.

Let $G = U|G|$ be the polar decomposition of G, i.e. $U_{T_0}(T) = U = G|G|^{-1}$. Then this unitary operator $U_{T_0}(T)$ verifies

$$U_{T_0}(T)T_0U^*_{T_0}(T) = T.$$

We shall need the following elementary estimate:

Lemma 6.2. *Let $T, Z \in \mathcal{B}(\mathcal{H})$. Then*

$$\|[T, e^Z]\| \geq \|[T, Z]\|(1 - \|Z\|e^{\|Z\|}).$$

Proof. Note that

$$[T, e^Z] = [T, Z] + \frac{1}{2}[T, Z^2] + \frac{1}{6}[T, Z^3] + \ldots$$

On the other hand

$$[T, Z^k] = TZ^k - ZTZ^{k-1} + ZTZ^{k-1} - Z^2TZ^{k-2} + \ldots + Z^{k-1}TZ - Z^kT = \sum_{j=1}^{k} Z^{k-j}[T, Z]Z^j,$$

and thus

$$\|[T, Z^k]\| \leq k\|Z\|^k\|[T, Z]\|.$$

Therefore

$$\|[T, e^Z]\| \geq \|[T, Z]\| - \|[T, Z]\|\sum_{k=1}^{\infty}\frac{1}{k!}k\|Z\|^k = \|[T, Z]\|(1 - \|Z\|e^{\|Z\|}).$$

□

Let T, T_0 in Λ_* as above. We estimate now $\|U-1\|$, where $U=U_{T_0}(T)$.

Lemma 6.3. *With the current notations,*

$$\|U-1\| < \frac{2r}{1-r},$$

where $r=\max_{0\le j\le n}\|p_j(T_0)-p_j(T)\|$.

Proof. Denote by $P_j=p_j(T_0)$ and $Q_j=p_j(T)$. Then $U=G(G^*G)^{-1/2}$, where $G=\sum_{j=0}^n Q_jP_j$. Thus

$$U=(\sum_{j=0}^n Q_jP_j)(\sum_{l=0}^n P_lQ_lP_l)^{-1/2}.$$

Note that since $\|P_l-Q_l\|<1$, $P_lQ_lP_l$ is a positive invertible operator in $\mathcal{B}(R(P_l))$ (we shall denote by $(P_lQ_lP_l)^{-1}$ its inverse there, note that also the square root is computed there). In particular, note that $Q_lP_l(P_lQ_lP_l)^{-1/2}=Q_l(P_lQ_lP_l)^{-1/2}$. Thus

$$U-1=(G-(G^*G)^{1/2})(G^*G)^{-1/2}$$

Then

$$\|U-1\|\le\|G-(G^*G)^{1/2}\|\|(G^*G)^{-1/2}\|.$$

Note that

$$\|G-(G^*G)^{1/2}\|=\|\sum_{j=0}^n Q_jP_j-(P_jQ_jP_j)^{1/2}\|\le\|\sum_{j=0}^n Q_jP_j-P_j\|+\|\sum_{j=0}^n P_j-(P_jQ_jP_j)^{1/2}\|.$$

The first term is bounded by r. The second term is bounded by $\max_{0\le j\le n}\|P_j-(P_jQ_jP_j)^{1/2}\|$. Note that if an operator A verifies $0\le A\le 1$, then $\|1-A^{1/2}\|\le\|1-A\|$. Using this fact with $P_jQ_jP_j$ in $\mathcal{B}(R(P_j))$, one has that the second term is also bounded by r.

It remains to consider

$$\|(G^*G)^{-1/2}\|=\|(G^*G)^{-1}\|^{1/2}=\|\sum_{j=0}^n(P_jQ_jP_j)^{-1}\|^{1/2}=\max_{0\le j\le n}\|(P_jQ_jP_j)^{-1}\|^{1/2}.$$

Since $\|P_jQ_jP_j-P_j\|\le\|Q_j-P_j\|<1$, it follows that $(P_jQ_jP_j)^{-1}=\sum_{l\ge0}(P_j-P_jQ_jP_j)^l$. Thus

$$\|(P_jQ_jP_j)^{-1}\|\le\sum_{l\ge0}\|P_j-P_jQ_jP_j\|^l\le\sum_{l\ge0}\|P_j-Q_j\|^l\le\frac{1}{1-r}.$$

□

If $U=e^Z$, with $Z^*=-Z$, and U is close to 1 in order that Z can be chosen (unique) with $\|Z\|<\pi$, then it is a straightforward fact that

$$\|U-1\|=2\sin(\frac{\|Z\|}{2}).$$

Putting these facts together, we have the following

Lemma 6.4. *Let $T_0, T \in \Lambda_*$ such that $\|p_j(T_0) - p_j(T)\| < r < 1$. Put $U_{T_0}(T) = e^Z$, with $Z^* = -Z$ and $\|Z\| < \pi$. Then*

$$\|[T_0, U_{T_0}(T)]\| \geq \|[T_0, Z]\| \left(1 - 2\arcsin(\frac{r}{1-r})e^{2\arcsin(\frac{r}{1-r})}\right).$$

With the above notations, note that

$$\|T - T_0\| = \|e^Z T_0 e^{-Z} - T_0\| = \|[T_0, e^Z]\| \geq \|[T_0, Z]\| C(r),$$

where $C(r) = 1 - 2\arcsin(\frac{r}{1-r})e^{2\arcsin(\frac{r}{1-r})}$. On the other hand, the curve $\gamma(t) = e^{tZ}T_0e^{-tZ}$ is a smooth curve in Λ_* joining $\gamma(0) = T_0$ and $\gamma(1) = T$. Therefore

$$d(T, T_0) \leq \ell(\gamma) = \int_0^1 \|\dot\gamma(t)\| dt = \int_0^1 \|e^{tZ}ZT_0e^{-tZ} - e^{tZ}T_0Ze^{-tZ}\| = \|[T_0, Z]\|.$$

Thus we have proved the following corollary:

Corollary 6.5. *Let $T, T_0 \in \Lambda_*$ such that $\|T - T_0\| < \frac{r}{\kappa(n)}$. Then*

$$d(T, T_0) \leq C(r)\|T - T_0\|.$$

References

[1] E. Andruchow, G. Corach, D. Stojanoff, Projective spaces of a C^*-algebra. Integral Equations Operator Theory 37 (2000), no. 2, 143–168.

[2] E. Andruchow, D. Stojanoff, Nilpotent operators and systems of projections. J. Operator Theory 20 (1988), no. 2, 359–374.

[3] E. Andruchow, D. Stojanoff, Differentiable structure of similarity orbits. J. Operator Theory 21 (1989), no. 2, 349–366.

[4] J.K. Baksalary, X. Liu, Xiaoji, An alternative characterization of generalized projectors. Linear Algebra Appl. 388 (2004), 61–65.

[5] J.K. Baksalary, O.M. Baksalary, X. Liu, Further properties of generalized and hypergeneralized projectors. Linear Algebra Appl. 389 (2004), 295–303.

[6] J.K. Baksalary, O.M. Baksalary, X. Liu, G. Trenkler, Further results on generalized and hypergeneralized projectors. Linear Algebra Appl. 429 (2008), no. 5-6, 1038–1050.

[7] J. Benítez, N. Thome, Characterizations and linear combinations of k-generalized projectors. Linear Algebra Appl. 410 (2005), 150–159.

[8] G. Corach, H. Porta, L. Recht, Differential geometry of systems of projections in Banach algebras. Pacific J. Math. 143 (1990), no. 2, 209–228.

[9] G. Corach, H. Porta, L. Recht, The geometry of spaces of projections in C^*-algebras, Adv. Math. 101 (1993), 59–77.

[10] C. Davis, W.M. Kahan, H.F. Weinberger, NORM PRESERVING DILATIONS AND THEIR APPLICATIONS TO OPTIMAL ERROR BOUNDS. SIAM J. Numer. Anal. 19 (1982), 445–469.

[11] H.-K. Du, Y. Li, Yuan, THE SPECTRAL CHARACTERIZATION OF GENERALIZED PROJECTIONS. Linear Algebra Appl. 400 (2005), 313–318.

[12] H.-K. Du, W.-F. Wang, Y.-T. Duan, Ying-Tao, PATH CONNECTIVITY OF k-GENERALIZED PROJECTORS. Linear Algebra Appl. 422 (2007), no. 2-3, 712–720.

[13] J. Gross, G. Trenkler, Gtz GENERALIZED AND HYPERGENERALIZED PROJECTORS. Linear Algebra Appl. 264 (1997), 463–474.

[14] D.A. Herrero. Approximation of Hilbert space operators. Vol. 1. Second edition. Pitman Research Notes in Mathematics Series, 224. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. xii+332 pp.

[15] Z.V. Kovárik, MANIFOLDS OF FRAMES OF PROJECTORS. Linear Algebra Appl. 31 (1980), 151–158

[16] Z.V. Kovárik, N. Sherif, CHARACTERIZATION OF SIMILARITIES BETWEEN TWO n-FRAMES OF PROJECTORS. Linear Algebra Appl. 57 (1984), 57–69.

[17] Z.V. Kovárk, N. Sherif, GEODESICS AND NEAR-GEODESICS IN THE MANIFOLDS OF PROJECTOR FRAMES. Linear Algebra Appl. 99 (1988), 259–277.

[18] L. Lebtahi, N.A. Thome, A NOTE ON k-GENERALIZED PROJECTIONS. Linear Algebra Appl. 420 (2007), no. 2-3, 572–575.

[19] H. Porta, L. Recht, MINIMALITY OF GEODESICS IN GRASSMANN MANIFOLDS, Proc. Amer. Math. Soc. 100 (1987), 464–466.

[20] G. W. Stewart, A NOTE ON GENERALIZED AND HYPERGENERALIZED PROJECTORS. Linear Algebra Appl. 412 (2006), no. 2-3, 408–411.

E. Andruchow
Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento
J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina
and
Instituto Argentino de Matemática
Saavedra 15, 3er. piso, (1083) Buenos Aires, Argentina.
e-mail: eandruch@ungs.edu.ar

G. Corach
Instituto Argentino de Matemática
Saavedra 15, 3er. piso, (1083) Buenos Aires, Argentina.
e-mail: gcorach@fi.uba.ar

M. Mbekhta
UFR de Mathematiques, F-59655 Villeneuve d'Ascq, France
e-mail: mostafa.mbekhta@math.univ-lille1.fr