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Stone style duality for distributive nearlattices

Sergio Celani and Ismael Calomino

Abstract. The aim of this paper is to study the variety of distributive nearlattices with greatest element. We will define the class of N-spaces as sober-like topological spaces with a basis of open, compact, and dually compact subsets satisfying an additional condition. We will show that the category of distributive nearlattices with greatest element whose morphisms are semi-homomorphisms is dually equivalent to the category of N-spaces with certain relations, called N-relations. In particular, we give a duality for the category of distributive nearlattices with homomorphisms. Finally, we apply these results to characterize topologically the one-to-one and onto homomorphisms, the subalgebras, and the lattice of the congruences of a distributive nearlattice.

1. Introduction and preliminaries

Implication algebras, also called Tarski algebras, were introduced by J. C. Abbott in [1]. It is well known that this class of algebras is the algebraic semantic of the $\{\to\}$-fragment of the classical propositional logic. Abbott [1] established a bijective correspondence between the variety of Tarski algebras and the class of all upper-bounded join-semilattices for which every principal filter is a Boolean lattice. The implication algebras are an example of a more general case, i.e., upper-bounded join-semilattices where each principal filter is only a lattice. They are called *nearlattices*. These structures have been investigated by W. H. Cornish and R. C. Hickman in [11] and [14], and recently by I. Chajda, R. Halaš, J. Kühr and M. Kolařík in [7], [8], [9] and [10]. The class of nearlattices is a variety. This fact was proved first by Hickman in [14], and subsequently by Chajda and Kolařík in [10]. In this latter paper, they show that the class of distributive nearlattices is a variety of a certain type.

Topological dualities are very useful in the study of various types of algebras. In [12], G. Grätzer gave a topological representation for distributive semilattices extending the known topological representation due to Stone for bounded distributive lattices and Boolean algebras [15]. Grätzer's representation was extended in [5] to a full duality. Similarly, a full duality between Tarski algebras and certain topological spaces with a distinguished topological basis of compact and open subsets was developed in [6]. In this paper, we will present a Stone style duality for distributive nearlattices with greatest element

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that extends the ones developed in [6]. We will introduce the notion of N-space and we will prove that there is a dual equivalence between the category of distributive nearlattices with greatest element, whose morphisms are semi-homomorphisms, and the category of N-spaces with certain relations, called N-relations. As a particular case, if the distributive nearlattice has a least element, we obtain a bounded distributive lattice and the well-known representation of Stone. Later, this duality is a generalization of the Stone duality for bounded distributive lattices. Moreover, if every prime ideal is maximal, then the distributive nearlattice is a Tarski algebra. Thus, we obtain the representation of Tarski algebras developed in [6].

The paper is organized as follows. In Section 2, we will recall the definitions and some basic properties of distributive nearlattices. Also, we prove that every prime ideal is maximal if and only if the distributive nearlattice is a Tarski algebra. In Section 3, we will introduce N-spaces and we will prove that any distributive nearlattice A is isomorphic to the dual distributive nearlattice of some N-space, and conversely that for any N-space, there exists a distributive nearlattice A that is homeomorphic to the dual space of A. In Section 4, we shall define the category of N-spaces with N-relations and we will apply the results of Section 3 to prove that there exists a correspondence between semi-homomorphisms of distributive nearlattices and N-relations. Later, we will extend these results to homomorphisms and N-functional relations. In Section 5, we shall give several applications of duality developed in the previous sections to describe some algebraic concepts. First, we give a dual description of 1-1 and onto homomorphisms. We will show a topological representation of lattices of subalgebras and congruences of distributive nearlattices.

Let us consider a poset $\langle X, \leq \rangle$. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*) if for all $x, y \in X$ such that $x \in U$ ($y \in U$) and $x \leq y$, we have $y \in U$ ($x \in U$). The set of all decreasing subsets of X is denoted by $\mathcal{P}_d(X)$. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \exists y \in Y\, y \leq x\}$ ($(Y] = \{x \in X : \exists y \in Y\, x \leq y\}$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. The set complement of a subset $Y \subseteq X$ will be denoted by Y^c or $X \setminus Y$.

A *join-semilattice with greatest element* is an algebra $\langle A, \vee, 1\rangle$ of type $(2, 0)$ such that the operation $\vee$ is idempotent, commutative, associative, and $a \vee 1 = 1$ for all $a \in A$. As usual, the binary relation $\leq$ defined by $x \leq y$ if and only if $x \vee y = y$ is a partial order. In what follows, we shall write simply *semilattice*.

A *filter* of a semilattice A is a non-empty subset $F \subseteq A$ with $1 \in F$, such that if $x \leq y$ and $x \in F$, then $y \in F$, and if $x, y \in F$, then $x \wedge y \in F$ whenever $x \wedge y$ exists. The set of all filters of A is denoted by $\mathrm{Fi}(A)$. The intersection of any collection of filters is again a filter. For any non-empty subset $X \subseteq A$, the set $F(X) = \{a \in A : \exists x_1, \ldots, x_n \in X, \exists x_1 \wedge \cdots \wedge x_n \text{ and } x_1 \wedge \cdots \wedge x_n \leq a\}$ is the *filter generated by* X. A filter F is said to be *finitely generated* if $F = F(X)$ for some finite non-empty subset X of A. The set of all finitely generated filters of A will be denoted by $\mathrm{Fi}_f(A)$.

A subset I of a semilattice A is called an *ideal* if for every $x, y \in A$, if $x \leq y$ and $y \in I$, then $x \in I$, and if $x, y \in I$, then $x \vee y \in I$. The set of all ideals of A is denoted by $\mathrm{Id}(A)$. The least ideal containing X is called *ideal generated by* X and will be denoted by $I(X)$. We shall say that a non-empty proper ideal P is *prime* if for all $x, y \in A$, if $x \wedge y$ exists and is in P, then $x \in P$ or $y \in P$. The set of all prime ideals of A will be denoted by $X(A)$.

2. Nearlattices

In this section, we will recall the definitions and basic properties of distributive nearlattices with greatest element.

Definition 2.1. A *nearlattice* is a semilattice A where for each $a \in A$, the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded lattice with respect to the induced order $\leq$ of A.

In [14], R. C. Hickman proves that the class of nearlattices forms a variety. Since the operation meet is defined only in a corresponding principal filter, we will indicate this fact by indices, i.e., $\wedge_a$ denotes the meet in $[a)$. Note that if $x, y \in [a)$ and $b \leq a$, then $x, y \in [b)$ and $x \wedge_a y = x \wedge_b y$. The operation $\wedge$ is not everywhere defined, and so nearlattices are partial algebras only. However, they can be treated as total algebras via the ternary operation m on A defined by

$$m(x, y, a) = (x \vee a) \wedge_a (y \vee a). \tag{$*$}$$

Lemma 2.2. *Let A be a nearlattice, and let m be defined by $(*)$. The following identities are satisfied:*

(1) $m(x, y, x) = x$,
(2) $m(x, x, y) = m(y, y, x)$,
(3) $m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z))$,
(4) $m(x, y, z) = m(y, x, z)$,
(5) $m(m(x, y, z), w, z) = m(x, m(y, w, z), z)$,
(6) $m(x, m(y, y, x), z) = m(x, x, z)$,
(7) $m(m(x, x, z), m(x, x, z), m(x, y, z)) = m(x, x, z)$,
(8) $m(m(x, x, z), m(y, y, z), z) = m(x, y, z)$,
(9) $m(x, x, 1) = 1$.

Let $\langle A, m, 1\rangle$ be an algebra of type $(3, 0)$ satisfying the identities (1), (2), and (3) of Lemma 2.2. If we define $x \vee y = m(x, x, y)$, then $\langle A, \vee, 1\rangle$ is a semilattice with greatest element. We can introduce the *induced order* $\leq$ by $x \leq y$ if and only if $m(x, x, y) = y$. It is clear that $\leq$ is an order on the set A which coincides with the induced order of the assigned semilattice $\langle A, \vee, 1\rangle$. The following theorem shows that nearlattices can be regarded as pure algebras.

Theorem 2.3. *Let $\langle A, m, 1\rangle$ be an algebra of type $(3, 0)$ satisfying the identities (1)–(9) of Lemma 2.2. Then the assigned semilattice $\mathcal{S}(A) = \langle A, \vee, 1\rangle$ is a nearlattice, where for every $a \in A$ and $x, y \in [a)$,*

$$x \wedge_a y = m(x, y, a).$$

Let $\langle S, \vee, 1\rangle$ be a nearlattice and $\mathcal{A}(S) = \langle S, m, 1\rangle$ be an algebra with the ternary operation m given by $(*)$. Then $\mathcal{S}(\mathcal{A}(S)) = S$. On the other hand, if $\langle A, m, 1\rangle$ is an algebra of type $(3, 0)$ satisfying the identities (1)–(9) of Lemma 2.2, then $\mathcal{A}(\mathcal{S}(A)) = A$.

By Lemma 2.2 and Theorem 2.3, there is a one-to-one correspondence between nearlattices and ternary algebras satisfying the above conditions. So, we shall alternate between these two faces of nearlattices and use that one which will be more convenient. The class of all nearlattices, considered as ternary algebras, is a variety. We denote by $\mathcal{N}$ the variety of nearlattices.

As in lattice theory, the class of distributive nearlattices play a special role.

Definition 2.4. Let $A \in \mathcal{N}$. Then A is *distributive* if for each $a \in A$, the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded distributive lattice.

Example 2.5. Let $\langle X, \leq\rangle$ be a poset. Then $\langle \mathcal{P}_d(X), m, X\rangle$ is a distributive nearlattice where $m(A, B, C) = (A \cup C) \cap (B \cup C)$ for every $A, B, C \in \mathcal{P}_d(X)$. The triple $\langle \mathcal{P}_d(X), m, X\rangle$ is of great importance because any distributive nearlattice can be embedded into a distributive nearlattice of this form, as we will prove later (see also [8]).

The distributivity of a nearlattice A can be characterized in terms of the ternary operation m or the set $\mathrm{Fi}(A)$. The following result can be found in [8], [10] and [11].

Theorem 2.6. *Let $A \in \mathcal{N}$. Then A is distributive if and only if satisfies either of the following identities:*

(1) $m(x, m(y, y, z), w) = m(m(x, y, w), m(x, y, w), m(x, z, w))$,
(2) $m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w)$.

We will denote by $\mathcal{DN}$ the variety of distributive nearlattices.

Theorem 2.7. *Let $A \in \mathcal{N}$. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *$\langle \mathrm{Fi}(A) \cup \{\emptyset\}, \subseteq\rangle$ is a distributive lattice.*
(3) *$\langle \mathrm{Fi}_f(A), \subseteq\rangle$ is a distributive lattice.*

One of the most important results in the theory of distributive lattices is Birkhoff's Prime Ideal Theorem. We have a theorem analogous for the variety of distributive nearlattices. See [13] or [8].

Theorem 2.8. *Let $A \in \mathcal{DN}$. Let $I \in \mathrm{Id}(A)$ and let $F \in \mathrm{Fi}(A)$ such that $I \cap F = \emptyset$. Then there exists $P \in X(A)$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Corollary 2.9. *Let $A \in \mathcal{DN}$. Then every proper ideal of A is the intersection of prime ideals.*

Proof. Let I be a proper ideal of A. For each $a \notin I$, we have $I \cap [a) = \emptyset$. Since $[a) \in \mathrm{Fi}(A)$, by Theorem 2.8 there exists $P_a \in X(A)$ such that $I \subseteq P_a$ and $a \notin P_a$. Thus, $I = \bigcap\{P_a \in X(A) : a \notin I\}$. □

Let $A \in \mathcal{DN}$; consider the poset $\langle X(A), \subseteq\rangle$ and $\varphi\colon A \to \mathcal{P}_d(X(A))$, defined by $\varphi(a) = \{P \in X(A) : a \notin P\}$. We have the following result.

Theorem 2.10 (**Representation theorem**)**.** *Let $A \in \mathcal{DN}$. Then A is isomorphic to the subalgebra $\varphi(A) = \{\varphi(a) : a \in A\}$ of $\mathcal{P}_d(X(A))$.*

Proof. It is clear that $\varphi(a) \in \mathcal{P}_d(X(A))$ for all $a \in A$. It is also easy to check that $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$, $\varphi(1) = X(A)$, and if there exists $a \wedge b$, then $\varphi(a \wedge b) = \varphi(a) \cap \varphi(b)$. So, $\varphi(m(a, b, c)) = m(\varphi(a), \varphi(b), \varphi(c))$. It follows that φ is 1-1 by Theorem 2.8. Thus, $A \cong \varphi(A)$. □

Definition 2.11. Let $A \in \mathcal{DN}$ and I a non-empty ideal of A.

(1) We say that I is *irreducible* if for every $I_1, I_2 \in \mathrm{Id}(A)$ such that $I_1 \cap I_2 = I$, then $I_1 = I$ or $I_2 = I$.
(2) We say that I is *maximal* if it is proper and for every $J \in \mathrm{Id}(A)$, if $I \subseteq J$, then $J = I$ or $J = A$.

Similar to the theory of distributive lattices, we have the following result.

Lemma 2.12. *Let $A \in \mathcal{DN}$. Let $P \in \mathrm{Id}(A)$.*

(1) *If P is irreducible, then P is prime.*
(2) *If P is maximal, then P is prime.*
(3) *P is maximal if and only if for all $a \in A$, if $a \notin P$, then $I(P \cup \{a\}) = A$.*

Proof. (1): Let P be a irreducible ideal. Let $a, b \in A$ be such that $a \wedge b$ exists and $a \wedge b \in P$. Then $(a \wedge b] = (a] \cap (b] \subseteq P$. We prove that $(P \vee (a]) \cap (P \vee (b]) \subseteq P \vee ((a] \cap (b])$. Let $x \in (P \vee (a]) \cap (P \vee (b])$. Then there exist $p_1, p_2 \in P$ such that $x \leq p_1 \vee a$ and $x \leq p_2 \vee b$. Since P is a ideal, $p = p_1 \vee p_2 \in P$ and $p \vee a, p \vee b \in [x)$. As $[x)$ is a distributive lattice, $x \leq (p \vee a) \wedge_x (p \vee b) = p \vee (a \wedge b)$. Hence, $x \in (P \cup \{a \wedge b\}] = P \vee ((a] \cap (b])$. The other inclusion it is immediate. So, $P = (P \vee (a]) \cap (P \vee (b])$ and consequently, $a \in P$ or $b \in P$. Thus, P is prime.

(2): Clearly, every maximal ideal is irreducible, so (2) follows from (1).

(3): If P is maximal, then it is clear that $I(P \cup \{a\}) = A$, for all $a \notin A$.

Conversely. Suppose that there exists $Q \in \mathrm{Id}(A)$ such that $P \subset Q$, i.e., there exists $a \in Q \setminus P$. We prove that $Q = A$. Let $b \in A$. So, $b \in I(P \cup \{a\})$, i.e., there exists $p \in P$ such that $b \leq p \vee a$. As $p \vee a \in Q$ and Q is an ideal, $b \in Q$. Thus, $Q = A$. □

Let $A \in \mathcal{DN}$ and $a, b \in A$. Suppose that $b \in [a)$. We define the sets

$$b^{\top} = \{x \in A : x \vee b = 1\} \text{ and } b_a^{\perp} = \{x \in A : \exists\,(x \wedge b) \text{ and } x \wedge b = a\},$$

where the set $b_a^{\perp}$ depends of a.

Lemma 2.13. *Let $A \in \mathcal{DN}$ and $a \in A$.*

(1) $b^{\top}$ *is a filter.*
(2) $b_a^{\perp}$ *is closed under join.*

Proof. (1): We prove that $b^{\top}$ is a filter. Let $x, y \in A$ such that $x \leq y$ and $x \in b^{\top}$. Then $x \vee b \leq y \vee b$ and $x \vee b = 1$. So, $y \vee b = 1$ and $y \in b^{\top}$. Let $x, y \in b^{\top}$ such that $x \wedge y$ exists. Since $[b)$ is a distributive lattice, $(x \wedge y) \vee b = (x \vee b) \wedge_b (y \vee b) = 1$. Thus, $x \wedge y \in b^{\top}$ and $b^{\top}$ is a filter.

(2): Let $x, y \in b_a^{\perp}$. Then there exist $x \wedge b$ and $y \wedge b$ such that $x \wedge b = a$ and $y \wedge b = a$. Thus, $a \leq x \wedge b$ and $a \leq y \wedge b$. As $[a)$ is a distributive lattice, $(x \wedge b) \vee (y \wedge b) = (x \vee y) \wedge_a b = a$. So, $x \vee y \in b_a^{\perp}$. □

If every prime ideal of a distributive nearlattice is maximal, then we have a Tarski algebra or implication algebra introduced by Abbott [1].

Theorem 2.14. *Let $A \in \mathcal{DN}$. The following conditions are equivalent:*

(1) *For all $a \in A$, $[a)$ is a Boolean lattice.*
(2) *Every prime ideal is maximal.*

Proof. (1) $\Rightarrow$ (2): Let $P \in X(A)$ and $a \notin P$. Let us consider $I(P \cup \{a\})$; we prove that $I(P \cup \{a\}) = A$. Suppose that $I(P \cup \{a\}) \subset A$. Then there exists $x \in A$ such that $x \notin I(P \cup \{a\})$. So, by Theorem 2.8, there exists $Q \in X(A)$ such that $a \in Q$, $P \subseteq Q$ and $x \notin Q$. Let $p \in P$. Since $p \leq p \vee a$ and $[p)$ is a Boolean lattice, there exists $z \in [p)$ such that $(p \vee a) \vee z = 1$ and $(p \vee a) \wedge z = p$. As $(p \vee a) \wedge z \in P$ and P is prime, we have $p \vee a \in P$ or $z \in P$. If $p \vee a \in P$, then $a \in P$, which is a contradiction. If $z \in P$, then $z \in Q$. Thus, we have $a \vee z = (p \vee a) \vee z = 1 \in Q$, which is a contradiction because Q is prime. Therefore, $I(P \cup \{a\}) = A$ and P is maximal.

(2) $\Rightarrow$ (1): Let $a \in A$. We prove that $[a)$ is a Boolean lattice, i.e., that every $b \in [a)$ has a complement. Let $b \in [a)$ such that $b \neq 1$ and $b \neq a$. Suppose that b has no complement. Let us consider the sets $b^{\top}$ and $b_a^{\perp}$. It follows that $b \notin b^{\top}$ and $b \notin b_a^{\perp}$. We prove that $I(b_a^{\perp} \cup \{b\})$ is a proper ideal of A. In effect, if $1 \in I(b_a^{\perp} \cup \{b\})$, then there exists $x \in b_a^{\perp}$ such that $x \vee b = 1$. So, $x \wedge b = a$ exists, which is a contradiction because we assumed that b has no complement. Then $1 \notin I(b_a^{\perp} \cup \{b\})$ and there exists $P \in X(A)$ such that $b \in P$ and $b_a^{\perp} \subseteq P$. Now, we prove that $a \notin F(P^c \cup \{b\})$. If $a \in F(P^c \cup \{b\})$, then there exists $p \notin P$ such that $p \wedge b$ exists and $p \wedge b \leq a$. Since $p \vee a, b \in [a)$ and $[a)$ is a distributive lattice, we have

$$(p \vee a) \wedge_a b = (p \wedge b) \vee (a \wedge b) = (p \wedge b) \vee a = a.$$

So, $p \vee a \in b_a^{\perp}$ and $p \vee a \in P$. As P is an ideal, $p \in P$, which is a contradiction. Then $a \notin F(P^c \cup \{b\})$ and by Theorem 2.8, there exists $Q \in X(A)$ such that $a \in Q$, $Q \cap P^c = \emptyset$, and $b \notin Q$. So, $Q \subseteq P$. Since every prime ideal is maximal, we have $P = Q$. Therefore, $b \in P$ and $b \notin P$, which is a contradiction. Then b has a complement and $[a)$ is a Boolean lattice. □

A filter P of a distributive nearlattice A is *prime* if for all $x, y \in A$, if $x \vee y \in P$, then $x \in P$ or $y \in P$. It is easy to see that an ideal P is prime if and only if P^c is a prime filter. Moreover, in the case of Tarski algebras, the concepts of filter and deductive system coincide.

3. Topological representation

In this section, we will define the dual topological space of a distributive nearlattice, called N-space, and we will prove that any distributive nearlattice can be represented by means of an N-space.

3.1. N-spaces. We recall some topological notions. A topological space $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ with a base $\mathcal{K}$ will be denoted by $\langle X, \mathcal{K}\rangle$. A subset $Y \subseteq X$ is *basic saturated* if $Y = \bigcap\{U_i : U_i \in \mathcal{K} \text{ and } Y \subseteq U_i\}$, i.e., it is an intersection of basic open sets. The basic saturation $\mathrm{Sb}(Y)$ of a subset Y is the smallest basic saturated set containing Y. If $Y = \{y\}$, we write $\mathrm{Sb}(\{y\}) = \mathrm{Sb}(y)$.

Given a topological space $\langle X, \mathcal{K}\rangle$ we consider the following family of subsets of $\mathcal{P}(X)$: $D_{\mathcal{K}}(X) = \{U : U^c \in \mathcal{K}\}$, i.e., $D_{\mathcal{K}}(X)$ is the set of complements of elements of $\mathcal{K}$.

Definition 3.1. Let $\langle X, \mathcal{K}\rangle$ be a topological space. Let Y be a non-empty subset of X.

(1) We say that Y is *irreducible* if for every $U, V \in D_{\mathcal{K}}(X)$, we have that $U \cap V \in D_{\mathcal{K}}(X)$, and $Y \cap (U \cap V) = \emptyset$ implies $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$.
(2) We say that Y is *dually compact* if for every family $\mathcal{F} = \{U_i : i \in I\} \subseteq \mathcal{K}$ such that $\bigcap\{U_i : i \in I\} \subseteq Y$, there exists a finite family $\{U_1, \ldots, U_n\}$ of $\mathcal{F}$ such that $U_1 \cap \cdots \cap U_n \subseteq Y$.

It is easy to see that $\mathrm{Sb}(x)$ is irreducible for all $x \in X$. We will introduce on X the following relation: $x \leq y$ iff $y \in \mathrm{Sb}(x)$.

We note that $\mathrm{Sb}(x) = [x)$. The relation $\leq$ is reflexive and transitive, but not necessarily antisymmetric. The following result is well known, but we include it for the reader's convenience.

Lemma 3.2. *Let $\langle X, \mathcal{K}\rangle$ be a topological space.*

(1) *If each irreducible basic saturated subset is the saturation of a unique single point, then $\leq$ is an order relation.*
(2) *The relation $\leq$ is an order if and only if $\langle X, \mathcal{K}\rangle$ is T_0.*

Proof. (1): It is easy to check that $\leq$ is reflexive and transitive. Finally, to show that is antisymmetric, suppose that $x \leq y$ and $y \leq x$. Then $\mathrm{Sb}(x) = \mathrm{Sb}(y)$. By uniqueness, $x = y$ holds.

(2): Let $x, y \in X$ such that $x \neq y$. Since $\leq$ is an order, $x \nleq y$ or $y \nleq x$. Suppose, for example, that $x \nleq y$. Then $y \notin \mathrm{Sb}(x)$, i.e., there exists $U \in \mathcal{K}$ such that $x \in U$ and $y \notin U$. Thus, $\langle X, \mathcal{K}\rangle$ is T_0.

Conversely, we prove that $\leq$ is antisymmetric. Let $x, y \in X$ such that $x \leq y$ and $y \leq x$, i.e., $y \in \mathrm{Sb}(x)$ and $x \in \mathrm{Sb}(y)$. Suppose that $x \neq y$. Since $\langle X, \mathcal{K}\rangle$ is T_0, there exists $U \in D_{\mathcal{K}}(X)$ such that $x \in U^c$ and $y \notin U^c$. But $y \in \mathrm{Sb}(x)$ and $y \in U^c$, which is a contradiction. $\square$

Now, we define the topological spaces that are dual to distributive nearlattices.

Definition 3.3. An *N-space* is a structure $\langle X, \mathcal{K}\rangle$ such that

(1) $\mathcal{K}$ is a basis of open, compact, and dually compact subsets for a topology $\mathcal{T}_{\mathcal{K}}$ on X.
(2) For every $U, V, W \in \mathcal{K}$, we have $(U \cap W) \cup (V \cap W) \in \mathcal{K}$.
(3) For every irreducible basic saturated subset Y of X, there exists a unique $x \in X$ such that $\mathrm{Sb}(x) = Y$.

Remark 3.4. (1) By Lemma 3.2, the relation $\leq$ is an order in an N-space.

(2) It is clear that an N-space is automatically T_0 and every $U \in D_{\mathcal{K}}(X)$ is decreasing.

(3) By item (2) of the Definition 3.3, we have that for every $U, V \in \mathcal{K}$, $(U \cap V) \cup (U \cap V) = U \cap V \in \mathcal{K}$. Therefore, $\mathcal{K}$ is closed under finite intersections and $\langle D_{\mathcal{K}}(X), \cup, X\rangle$ is a semilattice.

(4) We note that N-spaces are a generalization of topological spaces associated with Tarski algebras introduced in [6].

Let us prove that the triple $\langle D_{\mathcal{K}}(X), \cup, X\rangle$ has the structure of a distributive nearlattice.

Theorem 3.5. *Let $\langle X, \mathcal{K}\rangle$ be an N-space. Then $\langle D_{\mathcal{K}}(X), \cup, X\rangle$ is a distributive nearlattice.*

Proof. Let $C \in D_{\mathcal{K}}(X)$. We consider $[C) = \{U \in D_{\mathcal{K}}(X) : C \subseteq U\}$ and show that $\langle [C), \cap_C, \cup, C, X\rangle$ is a bounded distributive lattice. Let $A, B \in [C)$. Then $C \subseteq A$ and $C \subseteq B$. Since $D_{\mathcal{K}}(X)$ is a semilattice, $A \cup B \in [C)$. On the other hand, by condition (2) of the Definition 3.3, we have

$$(A \cup C) \cap_C (B \cup C) = A \cap_C B \in D_{\mathcal{K}}(X).$$

Then $A \cap_C B \in [C)$. Further, $(A \cup C) \cap_C (B \cup C) = (A \cap_C B) \cup C$ and $[C)$ is a bounded distributive lattice. Thus, $\langle D_{\mathcal{K}}(X), \cup, X\rangle$ is a distributive nearlattice. $\square$

The structure $\langle D_{\mathcal{K}}(X), \cup, X\rangle$ will be called the *dual distributive nearlattice* of X.

We will give some equivalences of item (3) of Definition 3.3.

Proposition 3.6. *Let $\langle X, \mathcal{K}\rangle$ be a topological space where $\mathcal{K}$ is a basis of open and compact subsets for a topology $\mathcal{T}_{\mathcal{K}}$ on X. Suppose $(U \cap W) \cup (V \cap W) \in \mathcal{K}$ for every $U, V, W \in \mathcal{K}$. The following conditions are equivalent:*

(1) $\langle X, \mathcal{K}\rangle$ *is* T_0*, and if* $\{U_i : i \in I\}$ *and* $\{V_j : j \in J\}$ *are non-empty families of* $D_{\mathcal{K}}(X)$ *such that* $\bigcap\{U_i : i \in I\} \subseteq \bigcup\{V_j : j \in J\}$*, then there exist* $U_1, \ldots, U_n$ *and* $V_1, \ldots, V_k$ *such that* $U_1 \cap \cdots \cap U_n \subseteq V_1 \cup \cdots \cup V_k$ *and* $U_1 \cap \cdots \cap U_n \in D_{\mathcal{K}}(X)$*.*

(2) $\langle X, \mathcal{K}\rangle$ *is* T_0*, every* $U \in \mathcal{K}$ *is dually compact, and* $H \colon X \to X(D_{\mathcal{K}}(X))$ *defined by* $H(x) = \{U \in D_{\mathcal{K}}(X) : x \notin U\}$ *for each* $x \in X$*, is onto.*

(3) *Every* $U \in \mathcal{K}$ *is dually compact and for every irreducible basic saturated subset* Y *of* X*, there exists a unique* $x \in X$ *such that* $\mathrm{Sb}(x) = Y$*.*

Proof. (1) $\Rightarrow$ (2): It is clear that every $U \in \mathcal{K}$ is dually compact and H is well defined. Let $P \in X(D_{\mathcal{K}}(X))$. We prove that

$$\mathcal{F} = \bigcap \{U_i : U_i \notin P\} \cap \bigcap \left\{V_j^c : V_j \in P\right\} \neq \emptyset.$$

If $\mathcal{F} = \emptyset$, then $\bigcap \{U_i : U_i \notin P\} \subseteq \bigcup \{V_j : V_j \in P\}$. Thus, there are $U_1, \ldots, U_n$ and $V_1, \ldots, V_k$ such that $U_1 \cap \cdots \cap U_n \subseteq V_1 \cup \cdots \cup V_k$ and $U_1 \cap \cdots \cap U_n \in D_{\mathcal{K}}(X)$. Since $V_1 \cup \cdots \cup V_k \in P$ and P is an ideal, $U_1 \cap \cdots \cap U_n \in P$. As P is prime, we have that $U_i \in P$ for some $1 \leq i \leq n$, which is a contradiction. Then $\mathcal{F} \neq \emptyset$, i.e., there exists $x \in \bigcap \{U_i : U_i \notin P\} \cap \bigcap \left\{V_j^c : V_j \in P\right\}$, which implies that $P = H(x)$.

(2) $\Rightarrow$ (3): Let Y be an irreducible basic saturated subset of X. Let us consider the set $P_Y = \{U \in D_{\mathcal{K}}(X) : Y \cap U = \emptyset\}$. It is easy to see that P_Y is an ideal of $D_{\mathcal{K}}(X)$. We prove that P_Y is prime. Suppose that there exists $U_1 \cap U_2 \in D_{\mathcal{K}}(X)$ such that $U_1 \cap U_2 \in P_Y$. Then $Y \cap (U_1 \cap U_2) = \emptyset$. Since Y is irreducible, $Y \cap U_1 = \emptyset$ or $Y \cap U_2 = \emptyset$, i.e., $U_1 \in P_Y$ or $U_2 \in P_Y$. Thus, P_Y is a prime ideal of $D_{\mathcal{K}}(X)$. Since X is T_0, the map H is injective, and as H is onto, there exists a unique $y \in X$ such that $H(y) = P_Y$. Now it is easy to check that $Y = \mathrm{Sb}(y)$.

(3) $\Rightarrow$ (1): By Lemma 3.2, X is T_0. Let $A = \{U_i : i \in I\}$ and $B = \{V_j : j \in J\}$ be non-empty families of $D_{\mathcal{K}}(X)$ such that $\bigcap\{U_i : i \in I\} \subseteq \bigcup\{V_j : j \in J\}$. If $I(B) \cap F(A) = \emptyset$, then by Theorem 2.8 there exists $P \in X(D_{\mathcal{K}}(X))$ such that $I(B) \subseteq P$ and $P \cap F(A) = \emptyset$. Let us consider the set $Y = \bigcap \{W^c : W \in P\}$. It follows that Y is a basic saturated. We see that Y is irreducible. Let $U, V \in D_{\mathcal{K}}(X)$ such that $U \cap V \in D_{\mathcal{K}}(X)$ and $Y \cap (U \cap V) = \emptyset$. Then $Y \subseteq U^c \cup V^c$. Since $U^c \cup V^c$ is dually compact, there exist $W_1, \ldots, W_n \in P$ such that $W_1^c \cap \cdots \cap W_n^c \subseteq U^c \cup V^c$, i.e., $U \cap V \subseteq W_1 \cup \cdots \cup W_n$. Thus, $U \cap V \in P$ and by the primality of P, $U \in P$ or $V \in P$. It follows that $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$. So, Y is irreducible. By hypothesis, there exists a unique $y \in X$ such that $\mathrm{Sb}(y) = Y$. It is easy to see that $H(y) = P$. Then $B \subseteq H(y)$ and $H(y) \cap A = \emptyset$. Thus, $y \in \bigcap\{U_i : i \in I\}$ and $y \notin \bigcup\{V_j : j \in J\}$, which is a contradiction. So, there exists $Q \in F(A) \cap I(B)$, i.e., there exist $U_1, \ldots, U_n \in A$ and $V_1, \ldots, V_k \in B$ such that $U_1 \cap \cdots \cap U_n \in D_{\mathcal{K}}(X)$ and $U_1 \cap \cdots \cap U_n \subseteq Q \subseteq V_1 \cup \cdots \cup V_k$. Therefore, we have $U_1 \cap \cdots \cap U_n \subseteq V_1 \cup \cdots \cup V_k$. $\square$

Following the definition given in [3], we recall that a *Stone space* (also called *spectral space*) is a topological space $\langle X, \mathcal{K}\rangle$ such that the following hold:

(1) $\langle X, \mathcal{K}\rangle$ is T_0.
(2) The family $\mathcal{K}$ of all compact and open subsets is a ring of sets and a basis for a topology $\mathcal{T}_{\mathcal{K}}$ on $\langle X, \mathcal{K}\rangle$.
(3) If $\{U_i : i \in I\}$ and $\{V_j : j \in J\}$ are non-empty families of non-empty compact and open subsets and $\bigcap\{U_i : i \in I\} \subseteq \bigcup\{V_j : j \in J\}$, then there exist $U_1, \ldots, U_n$ and $V_1, \ldots, V_k$ such that $U_1 \cap \cdots \cap U_n \subseteq V_1 \cup \cdots \cup V_k$.

By Proposition 3.6, we see that Stone spaces are a particular class of N-spaces.

Remark 3.7. We note that if $\langle X, \mathcal{K}\rangle$ is an N-space, then $X \in \mathcal{K}$ iff $D_{\mathcal{K}}(X)$ is a bounded distributive lattice iff $\mathcal{K}$ is a ring of sets. Moreover, by item (2) of the Definition 3.3, we have that $\mathcal{K}$ is a ring of sets iff $\mathcal{K}$ is the set of all compact and open subsets of X. So, we obtain the well-known topological representation for bounded distributive lattices given by M. H. Stone in [15].

3.2. The dual space of a distributive nearlattice. We will provide a construction which shows that any distributive nearlattice A is isomorphic to the dual distributive nearlattice of some N-space. In other words, we will prove that for any distributive nearlattice A, there exists an N-space $\langle X, \mathcal{K}\rangle$ such that $A \cong D_{\mathcal{K}}(X)$.

Let $A \in \mathcal{DN}$. Let us consider the set $X(A)$ and the family of sets

$$\mathcal{K}_A = \{X(A) \setminus \varphi(a) = \varphi(a)^c : a \in A\},$$

where we recall that $\varphi(a) = \{P \in X(A) : a \notin P\}$ for $a \in A$. We note that $X(A) = \bigcup\{\varphi(a)^c : a \in A\}$ because any prime ideal is non-empty. Moreover, for any $a, b \in A$ and $P \in X(A)$ such that $P \in \varphi(a)^c \cap \varphi(b)^c$, there exists $c = a \vee b \in A$ such that $P \in \varphi(c)^c = \varphi(a)^c \cap \varphi(b)^c$. Thus, the family $\mathcal{K}_A$ is a basis for a topology $\mathcal{T}_A$ on $X(A)$. Let us denote by $\mathcal{F}(A) = \langle X(A), \mathcal{K}_A\rangle$ the topological space associated with A, called the *dual space* of A.

Remark 3.8. It is immediate to see that $\mathcal{F}(A)$ is T_0.

Proposition 3.9. *Let $A \in \mathcal{DN}$ and let $\mathcal{F}(A)$ be the dual space of A. If $\{\varphi(b_i) : b_i \in B\}$ and $\{\varphi(c_j) : c_j \in C\}$ are non-empty families of $D_{\mathcal{K}_A}(X(A))$ such that*

$$\bigcap\{\varphi(c_j) : c_j \in C\} \subseteq \bigcup\{\varphi(b_i) : b_i \in B\},$$

then there are $b_1, \ldots, b_n \in B$ and $c_1, \ldots, c_k \in C$ with

$$\varphi(c_1) \cap \cdots \cap \varphi(c_k) \subseteq \varphi(b_1) \cup \cdots \cup \varphi(b_n)$$

such that $c_1 \wedge \cdots \wedge c_k$ exists.

Proof. Let $I(B)$ be the ideal generated by B, and let $F(C)$ be the filter generated by C. If $I(B) \cap F(C) = \emptyset$, then by Theorem 2.8, there exists $P \in X(A)$ such that $I(B) \subseteq P$ and $P \cap F(C) = \emptyset$. Moreover, $P \notin \varphi(b_i)$ for every $b_i \in B$.

So, $P \notin \bigcup\{\varphi(b_i) : b_i \in B\}$. On the other hand, $P \in \varphi(c_j)$ for every $c_j \in C$, i.e., $P \in \bigcap\{\varphi(c_j) : c_j \in C\}$, which is a contradiction. Thus, $I(B) \cap F(C) \neq \emptyset$. Then there exist $b_1, \dots, b_n \in B$ and $c_1, \dots, c_k \in C$ such that $c_1 \wedge \cdots \wedge c_k$ exists and $c_1 \wedge \cdots \wedge c_k \leq b_1 \vee \cdots \vee b_n$. Therefore, we have $\varphi(c_1 \wedge \cdots \wedge c_k) \subseteq \varphi(b_1 \vee \cdots \vee b_n)$ and $\varphi(c_1) \cap \cdots \cap \varphi(c_k) \subseteq \varphi(b_1) \cup \cdots \cup \varphi(b_n)$. $\square$

For each $I \in \mathrm{Id}(A)$ and each $F \in \mathrm{Fi}(A)$, consider the sets

$$\alpha(I) = \{P \in X(A) : I \nsubseteq P\} \quad \text{and} \quad \beta(F) = \{P \in X(A) : P \cap F = \emptyset\}.$$

It is easy to prove that $\alpha(I) = \bigcup\{\varphi(a) : a \in I\}$ and $\beta(F) = \bigcap\{\varphi(b) : b \in F\}$ for each $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$, respectively. In particular, we have the following result for finitely generated filters.

Lemma 3.10. *Let $A \in \mathcal{DN}$. Let $F = F(\{b_1, \dots, b_k\})$ be a finitely generated filter. Then $\beta(F) = \varphi(b_1) \cap \cdots \cap \varphi(b_k)$.*

Proof. Let $P \in \beta(F)$. Then $P \cap F = \emptyset$ and $\{b_1, \dots, b_k\} \subseteq P^c$. Thus, $b_i \notin P$ for every b_i. Therefore, $P \in \varphi(b_1) \cap \cdots \cap \varphi(b_k)$. Conversely, let $P \in \varphi(b_1) \cap \cdots \cap \varphi(b_k)$. Then $\{b_1, \dots, b_k\} \subseteq P^c$. Since P is a prime ideal, P^c is a filter and $F(\{b_1, \dots, b_k\}) \subseteq P^c$. Thus, $P \cap F = \emptyset$ and $P \in \beta(F)$. $\square$

In the following proposition, we characterize certain special subsets of the dual space of a distributive nearlattice.

Proposition 3.11. *Let $A \in \mathcal{DN}$ and let $\mathcal{F}(A)$ be the dual space of A.*

(1) *A subset $Y \subseteq X(A)$ is basic saturated in $\mathcal{F}(A)$ if and only if there exists an ideal I of A such that $Y = \alpha(I)^c$.*
(2) *A subset $U \subseteq X(A)$ is open in $\mathcal{F}(A)$ if and only if there exists a filter F of A such that $U = \beta(F)^c$.*
(3) *A subset $U \subseteq X(A)$ is open and compact in $\mathcal{F}(A)$ if and only if there exist $a_1, \dots, a_n \in A$ such that $U = \beta(F(\{a_1, \dots, a_n\}))^c$.*
(4) *Every element of $\mathcal{K}_A$ is an open, compact, and dually compact subset of $\mathcal{F}(A)$.*
(5) *For every $a, b, c \in A$, $[\varphi(a)^c \cap \varphi(c)^c] \cup [\varphi(b)^c \cap \varphi(c)^c] \in \mathcal{K}_A$.*

Proof. (1): Let $Y \subseteq \mathcal{F}(A)$ be basic saturated. Then $Y = \bigcap\{\varphi(b)^c : b \in B\}$ for some $B \subseteq A$. Let us consider the ideal $I = I(B)$. So, we have $\alpha(I)^c = \bigcap\{\varphi(a)^c : a \in I\}$. We prove that $Y = \alpha(I)^c$. It is evident that $\alpha(I)^c \subseteq Y$. On the other hand, let $P \in \bigcap\{\varphi(b)^c : b \in B\}$ and let $a \in I$. Then there exist $b_1, \dots, b_n \in B$ such that $a \leq b_1 \vee \cdots \vee b_n$. Thus, $\varphi(a) \subseteq \varphi(b_1) \cup \cdots \cup \varphi(b_n)$, or equivalently, $\varphi(b_1)^c \cap \cdots \cap \varphi(b_n)^c \subseteq \varphi(a)^c$. Since $\bigcap\{\varphi(b)^c : b \in B\} \subseteq \varphi(b_1)^c \cap \cdots \cap \varphi(b_n)^c$, we have $P \in \varphi(a)^c$. As this holds for $a \in I$, then $P \in \bigcap\{\varphi(a)^c : a \in I\} = \alpha(I)^c$.

(2): Let U be an open subset of $\mathcal{F}(A)$. Since $\mathcal{K}_A$ is a base for a topology $\mathcal{T}_A$ on $X(A)$, $U = \bigcup\{\varphi(b)^c : b \in B\}$ for some $B \subseteq A$. Let us consider the filter $F = F(B)$. We prove that $U^c = \beta(F)$. Let $P \in U^c$; then $b \notin P$ for every $b \in B$. We prove that $b \notin P$ for every $b \in F$. In the contrary case, if $b \in P$ for some $b \in F$, then there exist $b_1, \dots, b_n \in B$ such that $b_1 \wedge \cdots \wedge b_n$ exists and $b_1 \wedge \cdots \wedge b_n \leq b$. So, $b_1 \wedge \cdots \wedge b_n \in P$ and as P is prime, we have $b_i \in P$ for some b_i, which is a contradiction. Therefore , $P \cap F = \emptyset$ and $P \in \beta(F)$.

(3): Let U be an open and compact subset of $\mathcal{F}(A)$. By item (2) above, we have $U = \beta(F)^c = \bigcup\{\varphi(a)^c : a \in F\}$ for some filter F on A. Since U is compact, there exists $\{a_1, \dots, a_n\} \subseteq F$ such that

$$U = \varphi(a_1)^c \cup \cdots \cup \varphi(a_n)^c = [\varphi(a_1) \cap \cdots \cap \varphi(a_n)]^c = \beta(F(\{a_1, \dots, a_n\}))^c.$$

The converse follows from Lemma 3.10.

(4): For every $a \in A$, $\varphi(a)^c = \beta([a))^c$. By (3), we have that $\varphi(a)^c$ is an open and compact subset of $\mathcal{F}(A)$. It follows from Proposition 3.9 that each $\varphi(a)^c$ is dually compact.

(5): Let $a, b, c \in A$. Then

$$\begin{aligned}[][\varphi(a)^c \cap \varphi(c)^c] \cup [\varphi(b)^c \cap \varphi(c)^c] &= \varphi(a \vee c)^c \cup \varphi(b \vee c)^c \\ &= \varphi((a \vee c) \wedge_c (b \vee c))^c,\end{aligned}$$

where $(a \vee c) \wedge_c (b \vee c)$ exists in $[c)$ and $\varphi((a \vee c) \wedge_c (b \vee c))^c \in \mathcal{K}_A$. □

Remark 3.12. In distributive semilattices (see [5]), the set of all open and compact subsets forms a basis for a topology. In the case of distributive nearlattices, not all open and compact subsets of the topology $\mathcal{T}_A$ are of the form $\varphi(a)^c$. Indeed, if $U \subseteq X(A)$ is open then $U = \bigcup\{\varphi(b)^c : b \in B\}$, for some subset $B \subseteq A$. If U is compact, there exist $b_1, \dots, b_n \in B$ such that

$$U = \varphi(b_1)^c \cup \cdots \cup \varphi(b_n)^c = [\varphi(b_1) \cap \cdots \cap \varphi(b_n)]^c.$$

But we have $\varphi(b_1) \cap \cdots \cap \varphi(b_n) = \varphi(b_1 \wedge \cdots \wedge b_n)$ only in the case that the infimum $b_1 \wedge \cdots \wedge b_n$ exists.

Theorem 3.13. *Let $A \in \mathcal{DN}$. Then $\mathcal{F}(A)$ is an N-space and the mapping $\varphi\colon A \to D_{\mathcal{K}_A}(X(A))$ is an isomorphism of distributive nearlattices.*

Proof. By Propositions 3.6, 3.9, 3.11, and by definition of $D_{\mathcal{K}_A}(X(A))$, we have $A \cong D_{\mathcal{K}_A}(X(A))$, where φ is the isomorphism. □

Let $\langle X, \mathcal{T}\rangle$ be a topological space. We will denote by $\mathcal{O}(X)$ the set of all open subsets of X. Let us denote by $\mathcal{KO}(X)$ the set of all compact and open subsets of X. Note that $\mathcal{O}(X)$ is a lattice and $\mathcal{KO}(X)$ is a join-semilattice, under set inclusion.

Remark 3.14. Let $A \in \mathcal{DN}$. Then $\mathcal{KO}(X(A))$ is a distributive lattice.

Lemma 3.15. *Let $A \in \mathcal{DN}$ and let $\mathcal{F}(A)$ be the dual space of A.*

(1) *The lattices* $\mathrm{Fi}(A)$ *and* $\mathcal{O}(X(A))$ *are isomorphic under the mapping* $\Psi\colon \mathrm{Fi}(A) \to \mathcal{O}(X(A))$ *defined by* $\Psi(F) = \beta(F)^c$.

(2) *The isomorphism Ψ induces an isomorphism between the lattices* $\mathrm{Fi}_f(A)$ *and* $\mathcal{KO}(X(A))$.

Proof. This follows from Proposition 3.11 (2) and (3), respectively. □

There is a natural question when an N-space is homeomorphic to the dual space of a distributive nearlattice. Given an N-space, we will prove that there exists a distributive nearlattice A such that the dual space $\mathcal{F}(A)$ is homeomorphic to the initial N-space.

Theorem 3.16. *Let $\langle X, \mathcal{K}\rangle$ be an N-space. The mapping $H\colon X \to X(D_{\mathcal{K}}(X))$ is a homeomorphism between the topological spaces X and $X(D_{\mathcal{K}}(X))$.*

Proof. By condition (3) of the Definition 3.3 and by Proposition 3.6, it follows that H is well defined, 1-1, and onto. Now we will prove that H is continuous. By Proposition 3.11, given an open subset U of $X(D_{\mathcal{K}}(X))$, there exists a filter F of $D_{\mathcal{K}}(X)$ such that $U = \beta(F)^c$. Let $V = \bigcap\{O : O \in F\}$. Then V is closed in X. Let us prove that $H^{-1}(U) = V^c$. Let $x \in X$. Then

$$\begin{aligned} x \notin V \quad &\text{iff} \quad \exists O \in F(x \notin O) &&\text{iff} \quad \exists O \in F(O \in H(x)) \\ &\text{iff} \quad H(x) \cap F \neq \emptyset &&\text{iff} \quad H(x) \notin \beta(F) \\ &\text{iff} \quad H(x) \in U &&\text{iff} \quad x \in H^{-1}(U). \end{aligned}$$

Thus, H is continuous.

Let us prove that for all $U \in \mathcal{K}$, $H(U) \in \mathcal{K}_{D_{\mathcal{K}}(X)}$. Let $U \in \mathcal{K}$, then

$$\begin{aligned} x \notin U \quad &\text{iff} \quad x \in U^c &&\text{iff} \quad U^c \notin H(x) \\ &\text{iff} \quad H(x) \in \varphi(U^c) &&\text{iff} \quad H(x) \notin \varphi(U^c)^c, \end{aligned}$$

where $\varphi(U^c)^c \in \mathcal{K}_{D_{\mathcal{K}}(X)}$. Therefore, $H(U) = \varphi(U^c)^c$. $\square$

4. Topological duality

In the previous section, we have seen that distributive nearlattices are related to N-spaces. In this section, we will consider the algebraic category whose objects are distributive nearlattices with semi-homomorphisms as arrows, and we will prove that it is dually equivalent to the category whose objects are N-spaces with certain binary relations as arrows.

Recall the definition of semi-homomorphism of distributive nearlattices.

Definition 4.1. Let A,$B \in \mathcal{DN}$. We say that a map $h\colon A \to B$ is a *semi-homomorphism* if for every $a, b \in A$,

(1) $h(a \vee b) = h(a) \vee h(b)$,
(2) $h(1) = 1$.

Note that a semi-homomorphism $h\colon A \to B$ preserves the natural order, i.e., if $a \leq b$, then $h(a) \leq h(b)$. Moreover, if $a \wedge b$ exists, then $h(a) \wedge h(b)$ exists. Indeed, as $a \wedge b \leq a, b$, then $h(a), h(b) \in [h(a \wedge b))$. Since B is a nearlattice, $h(a) \wedge h(b)$ exists.

A *homomorphism* from the distributive nearlattice A into the distributive nearlattice B is a semi-homomorphism h such that for all $a, b \in A$, if $a \wedge b$ exists, then $h(a \wedge b) = h(a) \wedge h(b)$.

Remarks 4.2. Let $A, B \in \mathcal{DN}$ and $h\colon A \to B$ a semi-homomorphism. Then h is a homomorphism if and only if $[b) \subseteq [a_1) \vee [a_2)$ implies $[h(b)) \subseteq [h(a_1)) \vee [h(a_2))$, for all $a_1, a_2, b \in A$. Indeed, suppose that h is a homomorphism. Let $a_1, a_2, b \in A$ such that $[b) \subseteq [a_1) \vee [a_2)$. Then, by the distributivity of $\mathrm{Fi}(A)$, we have

$$[b) = [b) \wedge ([a_1) \vee [a_2)) = ([b) \wedge [a_1)) \vee ([b) \wedge [a_2)) = [b \vee a_1) \vee [b \vee a_2).$$

Since $(b \vee a_1) \wedge (b \vee a_2)$ exists, we have $b = (b \vee a_1) \wedge (b \vee a_2)$. Then, as h is a homomorphism and B is a distributive nearlattice, $h(b) = h(b) \vee (h(a_1) \wedge h(a_2))$ and $[h(b)) \subseteq [h(a_1) \wedge h(a_2))$, i.e., $[h(b)) \subseteq [h(a_1)) \vee [h(a_2))$.

Conversely, let $a_1, a_2 \in A$ such that $a_1 \wedge a_2$ exists. Since h preserves the natural order, $h(a_1 \wedge a_2) \leq h(a_1) \wedge h(a_2)$. Let $z \in B$ such that $z \leq h(a_1)$ and $z \leq h(a_2)$. Then $[h(a_1)) \vee [h(a_2)) \subseteq [z)$. Moreover, as $a_1 \wedge a_2$ exists, then $[a_1 \wedge a_2) = [a_1) \vee [a_2)$. By hypothesis, $[h(a_1 \wedge a_2)) \subseteq [h(a_1)) \vee [h(a_2))$ and $[h(a_1 \wedge a_2)) \subseteq [z)$, i.e., $z \leq h(a_1 \wedge a_2)$. Therefore, $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2)$.

The following lemma gives a characterization of homomorphisms.

Lemma 4.3. *Let $A, B \in \mathcal{DN}$. The following conditions are equivalent:*

(1) *h is a homomorphism.*
(2) *$h^{-1}(P) \in X(A)$ for every $P \in X(B)$.*

Proof. (1) $\Rightarrow$ (2): Let $P \in X(B)$. If $h^{-1}(P) = A$, then $1 \in h^{-1}(P)$ and $h(1) = 1 \in P$, which is a contradiction because P is a proper ideal. Since h preserves the natural order and it is a homomorphism, it follows that $h^{-1}(P)$ is an ideal. Let $a, b \in A$ be such that $a \wedge b$ exists and $a \wedge b \in h^{-1}(P)$. Then $h(a \wedge b) = h(a) \wedge h(b) \in P$. Since P is prime, $h(a) \in P$ or $h(b) \in P$, i.e., $a \in h^{-1}(P)$ or $b \in h^{-1}(P)$. Therefore, $h^{-1}(P) \in X(A)$.

(2) $\Rightarrow$ (1): We prove that h is monotone. Let $a, b \in A$ such that $a \leq b$. Suppose that $h(a) \nleq h(b)$. Then there exists $P \in X(B)$ such that $h(b) \in P$ and $h(a) \notin P$; thus, $b \in h^{-1}(P)$ and $a \notin h^{-1}(P)$, which is in contradiction with $h^{-1}(P)$ being an ideal. Now we prove that h is a homomorphism. If $h(1) < 1$, then there exists $P \in X(B)$ such that $h(1) \in P$, that is, $1 \in h^{-1}(P)$, which contradicts (2). Thus, $h(1) = 1$. Since h is monotone, $h(a) \vee h(b) \leq h(a \vee b)$ for all $a, b \in A$. Suppose that $h(a \vee b) \nleq h(a) \vee h(b)$. Then there exists $Q \in X(B)$ such that $h(a) \vee h(b) \in Q$ and $h(a \vee b) \notin Q$. So, $h(a), h(b) \in Q$ and $a, b \in h^{-1}(Q)$. Since $h^{-1}(Q) \in X(A)$, we have $a \vee b \in h^{-1}(Q)$. Thus, $h(a \vee b) \in Q$, which is a contradiction. Therefore, $h(a \vee b) = h(a) \vee h(b)$. By a similar argument, we obtain that if $a \wedge b$ exists, then $h(a \wedge b) = h(a) \wedge h(b)$. So, h is a homomorphism. $\square$

We will denote by $\mathcal{SDN}(A, B)$ the set of all semi-homomorphisms from A into B. Let us consider the following algebraic categories whose objects are distributive nearlattices:

$$\mathcal{SDN} = \text{Distributive nearlattices} \ + \ \text{semi-homomorphisms},$$
$$\mathcal{HDN} = \text{Distributive nearlattices} \ + \ \text{homomorphisms}.$$

We will prove that these categories are dually equivalent, respectively, to the following categories, which will be defined later:

$$\mathcal{NR} = N\text{-spaces} \ + \ N\text{-relations},$$
$$\mathcal{NF} = N\text{-spaces} \ + \ N\text{-functional relations}.$$

4.1. Duality for $\mathcal{SDN}$. Let X_1 and X_2 be two sets and let $R \subseteq X_1 \times X_2$ be a binary relation. For each $x \in X_1$, let $R(x) = \{y \in X_2 : (x,y) \in R\}$. Recall that R is *serial* if for all $x \in X_1$, we have that $R(x) \neq \emptyset$.

Before studying the topological counterparts of semi-homomorphisms, we consider the next example.

Example 4.4. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be a binary relation. Suppose that R is serial. We define the mapping $h_R\colon \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ by $h_R(U) = \{x \in X_1 : R(x) \cap U \neq \emptyset\}$. It is easy to prove that $h_R \in \mathcal{SDN}(\mathcal{P}(X_2), \mathcal{P}(X_1))$.

Definition 4.5. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let us consider a binary relation $R \subseteq X_1 \times X_2$. We say that R is an *N-relation* if it satisfies the following properties:

(1) $h_R(U) \in D_{\mathcal{K}_1}(X_1)$, for every $U \in D_{\mathcal{K}_2}(X_2)$.
(2) $R(x)$ is a basic saturated subset of X_2, for each $x \in X_1$.
(3) R is serial.

We will denote by $\mathcal{NR}(X_1, X_2)$ the set of all N-relations between X_1 and X_2. The following lemma characterizes condition (2) of Definition 4.5 by means of the concepts developed in the previous section.

Lemma 4.6. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be a binary relation. Suppose that $h_R(U) \in D_{\mathcal{K}_1}(X_1)$, for every $U \in D_{\mathcal{K}_2}(X_2)$. Then the following conditions are equivalent:*

(1) *$R(x)$ is a basic saturated subset of X_2, for all $x \in X_1$.*
(2) *For any $(x,y) \in X_1 \times X_2$,*

$$(x,y) \in R \textit{ iff } h_R^{-1}(H_{X_1}(x)) \subseteq H_{X_2}(y).$$

Proof. (1) $\Rightarrow$ (2): Let $x \in X_1$ and $y \in X_2$. If $(x,y) \in R$, then it is easy to see that $h_R^{-1}(H_{X_1}(x)) \subseteq H_{X_2}(y)$.

Suppose that $(x,y) \notin R$. Since $R(x)$ is basic saturated, we have $R(x) = \bigcap\{V^c : V \in D_{\mathcal{K}_2}(X_2) \text{ and } R(x) \subseteq V^c\}$. Then there exists $V \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq V^c$ and $y \notin V^c$. Thus, $R(x) \cap V = \emptyset$ and $y \in V$, i.e., $x \notin h_R(V) \in D_{\mathcal{K}_1}(X_1)$ and $V \notin H_{X_2}(y)$. So, $h_R^{-1}(H_{X_1}(x)) \nsubseteq H_{X_2}(y)$.

(2) $\Rightarrow$ (1): Let $x \in X_1$. We prove that

$$R(x) = \bigcap\{V^c : V \in D_{\mathcal{K}_2}(X_2) \text{ and } R(x) \subseteq V^c\}.$$

Clearly, $R(x) \subseteq \bigcap\{V^c : V \in D_{\mathcal{K}_2}(X_2) \text{ and } R(x) \subseteq V^c\}$.

Let $y \in \bigcap\{V^c : V \in D_{\mathcal{K}_2}(X_2) \text{ and } R(x) \subseteq V^c\}$. Suppose that $y \notin R(x)$. Then $h_R^{-1}(H_{X_1}(x)) \nsubseteq H_{X_2}(y)$, i.e., there exists $V \in D_{\mathcal{K}_2}(X_2)$ such that $h_R(V) \in H_{X_1}(x)$ and $V \notin H_{X_2}(y)$. Thus, $x \notin h_R(V)$ and $y \in V$. It follows that $R(x) \subseteq V^c$ and $y \notin V^c$, which is a contradiction. □

Let $\langle X_1, \mathcal{K}_1\rangle, \langle X_2, \mathcal{K}_2\rangle$, and $\langle X_3, \mathcal{K}_3\rangle$ be three N-spaces, $R \in \mathcal{NR}(X_1, X_2)$, and $S \in \mathcal{NR}(X_2, X_3)$. Similar to the case of distributive semilattices developed in [4], the usual set-theoretic composition of two N-relations may not be an N-relation. This motivates us to define a new composition of two N-relations. Define $S * R \subseteq X_1 \times X_3$ by

$$(x, z) \in (S * R) \text{ iff } (\forall V \in D_{\mathcal{K}_3}(X_3))((S \circ R)(x) \cap V = \emptyset \Rightarrow z \notin V),$$

where $S \circ R$ is the usual set-theoretic composition of R and S.

Remark 4.7. Note that $S \circ R \subseteq S * R$, and if $S \circ R$ is an N-relation, then $S * R = S \circ R$.

We have the following result.

Lemma 4.8. *Let $\langle X_1, \mathcal{K}_1\rangle, \langle X_2, \mathcal{K}_2\rangle$, and $\langle X_3, \mathcal{K}_3\rangle$ be three N-spaces. Let $R \in \mathcal{NR}(X_1, X_2)$ and $S \in \mathcal{NR}(X_2, X_3)$. Then*

$$(x, z) \in (S * R) \textit{ iff } (h_R \circ h_S)^{-1}(H_{X_1}(x)) \subseteq H_{X_3}(z).$$

Proof. Let $(x, z) \in (S * R)$. For $V \in D_{\mathcal{K}_3}(X_3)$, if $(S \circ R)(x) \cap V = \emptyset$, then $z \notin V$. So, $x \notin (h_R \circ h_S)(V)$. It follows that $(h_R \circ h_S)(V) \in H_{X_1}(x)$, which means that $V \in (h_R \circ h_S)^{-1}(H_{X_1}(x))$. Thus, for $V \in D_{\mathcal{K}_3}(X_3)$, if $V \in (h_R \circ h_S)^{-1}(H_{X_1}(x))$, then $V \in H_{X_3}(z)$, i.e., $(h_R \circ h_S)^{-1}(H_{X_1}(x)) \subseteq H_{X_3}(z)$. Conversely, we also obtain that if $(h_R \circ h_S)^{-1}(H_{X_1}(x)) \subseteq H_{X_3}(z)$, then $(x, z) \in (S * R)$. □

Remark 4.9. By Lemma 4.8, it is easy to see that $(S * R)(x) = \mathrm{Sb}((S \circ R)(x))$ for every $x \in X_1$.

Corollary 4.10. *Let $\langle X_1, \mathcal{K}_1\rangle, \langle X_2, \mathcal{K}_2\rangle$, and $\langle X_3, \mathcal{K}_3\rangle$ be three N-spaces. Let $R \in \mathcal{NR}(X_1, X_2)$ and $S \in \mathcal{NR}(X_2, X_3)$. Then $h_{(S*R)}(U) = (h_R \circ h_S)(U)$.*

Proof. Let $U \in D_{\mathcal{K}_3}(X_3)$ and $x \in (h_R \circ h_S)(U)$; then $(h_R \circ h_S)(U) \notin H_{X_1}(x)$, and so $U \notin (h_R \circ h_S)^{-1}(H_{X_1}(x))$. Since $D_{\mathcal{K}_3}(X_3)$ is a distributive nearlattice, by Theorem 2.8 there exists $P \in X(D_{\mathcal{K}_3}(X_3))$ with $(h_R \circ h_S)^{-1}(H_{X_1}(x)) \subseteq P$ and $U \notin P$. By Proposition 3.6, there exists $z \in X_3$ such that $P = H_{X_3}(z)$. So, $(h_R \circ h_S)^{-1}(H_{X_1}(x)) \subseteq H_{X_3}(z)$ and $U \notin H_{X_3}(z)$. It follows by Lemma 4.8 that $(x, z) \in (S * R)$ and $z \in U$, i.e., $(S * R)(x) \cap U \neq \emptyset$. Therefore, $x \in h_{(S*R)}(U)$.

Conversely, let $x \in h_{(S*R)}(U)$. Then $(S * R)(x) \cap U \neq \emptyset$, i.e., there exists $z \in X_3$ such that $(x, z) \in (S * R)$ and $z \in U$. By Lemma 4.8, we have $(h_R \circ h_S)^{-1}(H_{X_1}(x)) \subseteq H_{X_3}(z)$. Since $U \notin H_{X_3}(z)$, so $U \notin (h_R \circ h_S)^{-1}(H_{X_1}(x))$. Thus, $(h_R \circ h_S)(U) \notin H_{X_1}(x)$. Therefore, $x \in (h_R \circ h_S)(U)$. $\square$

The following technical result is needed to affirm that $\mathcal{NR}$, the N-spaces with N-relations as arrows, is a category.

Theorem 4.11. *Let $\langle X_1, \mathcal{K}_1\rangle, \langle X_2, \mathcal{K}_2\rangle$, and $\langle X_3, \mathcal{K}_3\rangle$ be three N-spaces. Let $R \in \mathcal{NR}(X_1, X_2)$ and $S \in \mathcal{NR}(X_2, X_3)$.*

(1) $\leq_1 \in \mathcal{NR}(X_1, X_1)$.
(2) $R* \leq_1 = R = \leq_2 *R$.
(3) $S * R \in \mathcal{NR}(X_1, X_3)$.

Proof. (1): It is easy to see that $\leq_1$ is serial and that $\leq_1 (x)$ is a basic saturated subset of X_2 for all $x \in X_1$. We prove that $h_{\leq_1}(U) = U$ for all $U \in D_{\mathcal{K}_1}(X_1)$. By reflexivity of $\leq_1$, we have $U \subseteq h_{\leq_1}(U)$. Conversely, suppose that $h_{\leq_1}(U) \nsubseteq U$. Thus, there exists $x \in h_{\leq_1}(U)$ such that $x \in U^c$. So, $\leq_1 (x) \cap U \neq \emptyset$, i.e., there is $y \in \leq_1 (x)$ and $y \in U$. Then $x \leq_1 y$. By (2) of Remark 3.4, U is decreasing and $x \in U$, which is a contradiction. Therefore, $h_{\leq_1}(U) = U$ and $\leq_1$ is an N-relation.

(2): By Lemmas 4.6, 4.8 and (1) above, we have

$$\begin{aligned}(x, z) \in (R* \leq_1) &\text{ iff } (h_{\leq_1} \circ h_R)^{-1}(H_{X_1}(x)) \subseteq H_{X_3}(z)\\ &\text{ iff } {h_R}^{-1}(H_{X_1}(x)) \subseteq H_{X_3}(z) \text{ iff } (x, z) \in R.\end{aligned}$$

Analogously, $\leq_2 *R = R$.

(3): Let $U \in D_{\mathcal{K}_3}(X_3)$. By Corollary 4.10, it follows that

$$h_{(S*R)}(U) = (h_R \circ h_S)(U) \in D_{\mathcal{K}_1}(X_1)$$

because S and R are N-relations. By Lemma 4.8, we have $(S * R)(x) = \mathrm{Sb}((S \circ R)(x))$ for all $x \in X_1$. Finally, since $S \circ R$ is serial, we have that $S * R$ is serial. So, $S * R \subseteq X_1 \times X_3$ is an N-relation. $\square$

In Section 3, we studied the relationship between distributive nearlattices and N-spaces. We complete the duality by studying the correspondence between semi-homomorphisms and N-relations.

Let $A, B \in \mathcal{DN}$ and $h \in \mathcal{SDN}(A, B)$. Let us define the following binary relation $R_h \subseteq X(B) \times X(A)$ by $(P, Q) \in R_h$ iff $h^{-1}(P) \subseteq Q$.

The following Proposition is needed to show later that there exists a contravariant functor from the category $\mathcal{SDN}$ into $\mathcal{NR}$.

Proposition 4.12. *Let $A, B \in \mathcal{DN}$ and $h \in \mathcal{SDN}(A, B)$.*

(1) *For every $P \in X(B)$ and for every $a \in A$, $h(a) \notin P$ if and only if there exists $Q \in X(A)$ such that $(P, Q) \in R_h$ and $a \notin Q$.*
(2) $R_h \in \mathcal{NR}(X(B), X(A))$.
(3) *If $C \in \mathcal{DN}$ and $k \in \mathcal{SDN}(B, C)$, then $R_{k \circ h} = R_h * R_k$.*

(4) *The mapping* $h_{R_h}\colon D_{\mathcal{K}_A}(X(A)) \to D_{\mathcal{K}_B}(X(B))$ *satisfies*

$$\varphi_B \circ h = h_{R_h} \circ \varphi_A.$$

Proof. (1): Let $P \in X(B)$ and $a \in A$. If $h(a) \notin P$, then $a \notin h^{-1}(P)$. Since h is a semi-homomorphism, it is easy to see that $h^{-1}(P)$ is an ideal of A. Thus, $h^{-1}(P) \cap [a) = \emptyset$. By Theorem 2.8, there exists $Q \in X(A)$ such that $h^{-1}(P) \subseteq Q$ and $Q \cap [a) = \emptyset$. Therefore, $(P, Q) \in R_h$ and $a \notin Q$. Conversely, by hypothesis, there exists $Q \in X(A)$ such that $(P, Q) \in R_h$ and $a \notin Q$. Then $h^{-1}(P) \subseteq Q$ and $a \notin Q$. It follows that $h(a) \notin P$.

(2): Let $P \in X(B)$. So, $h^{-1}(P)$ is an ideal of A. We prove that $1 \notin h^{-1}(P)$. If $1 \in h^{-1}(P)$, then $h(1) = 1 \in P$, which is a contradiction because P is proper. So, $1 \notin h^{-1}(P)$. Then there exists $Q \in X(A)$ such that $h^{-1}(P) \subseteq Q$. Hence, $(P, Q) \in R_h$ and $R_h(P)$ is serial. We prove $R_h(P) = \bigcap\{\varphi_A(a)^c : h(a) \in P\}$. If $Q \in R_h(P)$, then $h^{-1}(P) \subseteq Q$. For each $h(a) \in P$, $a \in h^{-1}(P) \subseteq Q$. So, $a \in Q$ and $Q \in \varphi_A(a)^c$. Therefore, $Q \in \bigcap\{\varphi_A(a)^c : h(a) \in P\}$. To see the converse, suppose that $Q \in \bigcap\{\varphi_A(a)^c : h(a) \in P\}$ and $Q \notin R_h(P)$. Then $h^{-1}(P) \nsubseteq Q$, i.e., there exists $a \in h^{-1}(P)$ such that $a \notin Q$. Thus, $h(a) \in P$ and $Q \notin \varphi_A(a)^c$, which is a contradiction. Finally, by (1), it follows that $\varphi_B(h(a)) = h_{R_h}(\varphi_A(a))$ for all $a \in A$. Thus, $h_{R_h}(\varphi_A(a)) \in \varphi_B(B)$ for each $\varphi_A(a) \in \varphi_A(A)$. Therefore, R_h is an N-relation.

(3): It suffices to prove that for all $P \in X(C)$, we have

$$(R_{k\circ h})(P) = \mathrm{Sb}((R_h \circ R_k)(P)) = \bigcap\left\{\varphi(a)^c \in \mathcal{K}_A : (R_h \circ R_k)(P) \subseteq \varphi(a)^c\right\}.$$

If $Q \in (R_{k\circ h})(P)$, then $h^{-1}(k^{-1}(P)) \subseteq Q$. Let $\varphi(a)^c \in \mathcal{K}_A$ be such that $(R_h \circ R_k)(P) \subseteq \varphi(a)^c$. We prove that $Q \in \varphi(a)^c$, i.e., $a \in Q$. Suppose, on the contrary, that $a \notin Q$; then $a \notin h^{-1}(k^{-1}(P))$. Since $h(a) \notin k^{-1}(P)$, there exists $R \in X(B)$ such that $k^{-1}(P) \subseteq R$ and $h(a) \notin R$. Again, since $a \notin h^{-1}(R)$, there exists $S \in X(A)$ such that $h^{-1}(R) \subseteq S$ and $a \notin S$. Thus, $(P, R) \in R_k$ and $(R, S) \in R_h$. So, $(P, S) \in R_h \circ R_k$ and $S \in (R_h \circ R_k)(P)$. Then $S \in \varphi(a)^c$, or equivalently, $a \in S$, which is a contradiction. Therefore, $a \in Q$ and $Q \in \mathrm{Sb}((R_h \circ R_k)(P))$.

Conversely, let $Q \in \mathrm{Sb}((R_h \circ R_k)(P))$. We prove that $h^{-1}(k^{-1}(P)) \subseteq Q$. Let $a \in h^{-1}(k^{-1}(P))$. It is easy to prove that $(R_h \circ R_k)(P) \subseteq \varphi(a)^c$. So, by hypothesis, $Q \in \varphi(a)^c$ and $a \in Q$.

(4): This is an immediate consequence of (1). □

Remark 4.13. Let $A \in \mathcal{DN}$. If $Id\colon A \to A$ denotes the identity map, then

$$R_{Id} = \{(P, Q) \in X(A) \times X(A) : P \subseteq Q\} = \subseteq.$$

By Theorem 3.13, Proposition 4.12 and the previous remark we can define a contravariant functor $\mathbf{X}\colon \mathcal{SDN} \to \mathcal{NR}$ as follows: If A is a distributive nearlattice, then $\mathbf{X}(A) = \langle X(A), \mathcal{K}_A\rangle$ and if h is a semi-homomorphism, then $\mathbf{X}(h) = R_h$.

To complete the duality, we prove that there exists a contravariant functor from $\mathcal{NR}$ into $\mathcal{SDN}$. We have the following result.

Theorem 4.14. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be N-spaces and let R belong to $\mathcal{NR}(X_1, X_2)$.*

(1) *The map $h_R \colon D_{\mathcal{K}_2}(X_2) \to D_{\mathcal{K}_1}(X_1)$ defined as in Example 4.4 is a semi-homomorphism.*
(2) *The binary relation $R \subseteq X_1 \times X_2$ satisfies $R_{h_R} \circ H_{X_1} = H_{X_2} \circ R$.*

Proof. (1): Since R is an N-relation, we have that $h_R(U) \in D_{\mathcal{K}_1}(X_1)$ for all $U \in D_{\mathcal{K}_2}(X_2)$. Thus, h_R is well defined. If $U, V \in D_{\mathcal{K}_2}(X_2)$, then clearly $h_R(U \cup V) = h_R(U) \cup h_R(V)$. On the other hand, since R is serial, we have $h_R(X_2) = X_1$. So, h_R is a semi-homomorphism.

(2): Let $(x, z) \in R_{h_R} \circ H_{X_1}$. Then there exists $y \in X(D_{\mathcal{K}_1}(X_1))$ such that $(x, y) \in H_{X_1}$ and $(y, z) \in R_{h_R}$. By Theorem 3.16, H_{X_1} and H_{X_2} are bijections; thus, $H_{X_1}(x) = y$ and there exists $t \in X_2$ such that $H_{X_2}(t) = z$. It follows that $(H_{X_1}(x), H_{X_2}(t)) \in R_{h_R}$ and by Lemma 4.6, we have that $(x, t) \in R$. So, $(x, z) \in H_{X_2} \circ R$. The converse is similar. □

Remark 4.15. Let $\langle X, \mathcal{K}\rangle$ be an N-space and let $\leq\, \subseteq X \times X$ be the N-relation identity. By Theorem 4.11(1), we have $h_{\leq}(U) = \{x \in X : {\leq}(x) \cap U \neq \emptyset\} = U$. Therefore, $h_{\leq} \colon D_{\mathcal{K}}(X) \to D_{\mathcal{K}}(X)$ is the identity map.

By using Theorems 3.5 and 4.14, we can define a contravariant functor $\mathbf{D} \colon \mathcal{NR} \to \mathcal{SDN}$ as follows: If $\langle X, \mathcal{K}\rangle$ is an N-space, then $\mathbf{D}(\langle X, \mathcal{K}\rangle) = D_{\mathcal{K}}(X)$, and if R is an N-relation, then $\mathbf{D}(R) = h_R$.

So, by Theorems 3.16, 4.14, and Lemma 4.6, H is a natural equivalence between the identity functor of $\mathcal{NR}$ and the composition functor $\mathbf{X} \circ \mathbf{D}$.

Analogously, by Theorem 3.13 and Proposition 4.12, we have that φ is a natural equivalence between the identity functor of $\mathcal{SDN}$ and the composition functor $\mathbf{D} \circ \mathbf{X}$.

We summarize the above results in the following theorem.

Theorem 4.16. *The contravariant functors $\mathbf{X}$ and $\mathbf{D}$ define a dual equivalence between the algebraic category of distributive nearlattices with semi-homomorphisms and the category of N-spaces with N-relations.*

4.2. Duality for $\mathcal{HDN}$. We present a dual description of homomorphisms between distributive nearlattices.

Lemma 4.17. *Let $A, B \in \mathcal{DN}$ and $h \colon A \to B$ be a homomorphism. Then for each $P \in X(B)$ and $Q \in X(A)$, we have $R_h(P) = \mathrm{Sb}(Q)$ iff $h^{-1}(P) = Q$.*

Proof. Let $R_h(P) = \mathrm{Sb}(Q)$ and $h^{-1}(P) \neq Q$. Since $Q \in \mathrm{Sb}(Q) = R_h(P)$, $h^{-1}(P) \subseteq Q$. If $Q \nsubseteq h^{-1}(P)$, since $h^{-1}(P) \in X(A)$ and $h^{-1}(P) \subseteq h^{-1}(P)$, then $h^{-1}(P) \in R_h(P) = \mathrm{Sb}(Q)$, i.e., $h^{-1}(P) \in \bigcap\{\varphi_A(a)^c : Q \in \varphi_A(a)^c\}$. So, $a \in h^{-1}(P)$ for all $a \in Q$, or equivalently, $Q \subseteq h^{-1}(P)$, which is a contradiction.

Reciprocally, suppose that $h^{-1}(P) = Q$. Then

$$H \in R_h(P) \text{ iff } Q = h^{-1}(P) \subseteq H \text{ iff } \forall a \in A(a \in Q \Rightarrow a \in H)$$
$$\text{iff } \forall \varphi_A(a)^c \in \mathcal{K}_A \ (Q \in \varphi_A(a)^c \Rightarrow H \in \varphi_{\mathbf{A}}(a)^c) \text{ iff } H \in \mathrm{Sb}(Q).$$

Therefore, $R_h(P) = \mathrm{Sb}(Q)$. □

By Lemmas 4.3 and 4.17, we have a dual description of homomorphisms. The above lemma leads to the following definition.

Definition 4.18. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let us consider a binary relation $R \subseteq X_1 \times X_2$. We say that R is an *N-functional relation* if R is an N-relation satisfying that for each $x \in X_1$, there exists $y \in X_2$ such that $R(x) = \mathrm{Sb}(y)$.

Using Theorem 4.16, we obtain the following result.

Theorem 4.19. *The contravariant functors $\mathbf{X} \mid_{\mathcal{HDN}}$ and $\mathbf{D} \mid_{\mathcal{NF}}$ define a dual equivalence between the algebraic category of distributive nearlattices with homomorphisms and the category of N-spaces with N-functional relations.*

We will show that N-functional relations can be characterized by means of special functions between N-spaces. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be Stone spaces. We recall (see [3]) that a map $f\colon X_1 \to X_2$ is a *Stone morphism* if $f^{-1}(U)$ is compact and open set of X_1 for each compact and open set U of X_2. Equivalently, if $U \in D_{\mathcal{K}_2}(X_2)$ implies $f^{-1}(U) \in D_{\mathcal{K}_1}(X_1)$. In what follows, we generalize Stone morphisms.

Definition 4.20. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be N-spaces. A map $f\colon X_1 \to X_2$ is an *N-morphism* if $f^{-1}(U) \in D_{\mathcal{K}_1}(X_1)$ for every $U \in D_{\mathcal{K}_2}(X_2)$.

As N-spaces are a generalization of Stone spaces, it follows that Stone morphisms are a special case of N-morphisms. We will denote by $\mathcal{NS}$ the category of N-spaces with N-morphisms.

Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be N-spaces and $R \subseteq X_1 \times X_2$ an N-functional relation. We define $f_R\colon X_1 \to X_2$ by $f_R(x) = y$ iff $R(x) = \mathrm{Sb}(y)$.

Lemma 4.21. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be an N-functional relation. Then f_R is an N-morphism.*

Proof. We prove that $f_R^{-1}(U) = h_R(U)$, for all $U \in D_{\mathcal{K}_2}(X_2)$. Let $x \in f_R^{-1}(U)$. Then $f_R(x) = y \in U$ and $\mathrm{Sb}(y) \cap U \neq \emptyset$. So, $R(x) \cap U \neq \emptyset$, and therefore $x \in h_R(U)$. Conversely, if $x \in h_R(U)$, then $\mathrm{Sb}(y) \cap U \neq \emptyset$. Thus, there exists $z \in \mathrm{Sb}(y) = [y)$ such that $z \in U$. Since $y \leq z$ and U is decreasing, we have $y = f_R(x) \in U$. So, $x \in f_R^{-1}(U)$. Finally, as $h_R(U) \in D_{\mathcal{K}_1}(X_1)$, it follows that f_R is an N-morphism. □

Conversely, let $f\colon X_1 \to X_2$ be an N-morphism. Consider the relation $R_f \subseteq X_1 \times X_2$ defined as follows: $(x, y) \in R_f$ iff $f(x) \leq_2 y$.

Lemma 4.22. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $f\colon X_1 \to X_2$ an N-morphism. Then R_f is an N-functional relation.*

Proof. Since $f(x) \leq_2 f(x)$ for all $x \in X_1$, R_f is serial. Also, by definition, it follows that $R_f(x) = \mathrm{Sb}(f(x)) = [f(x))$. We prove that $h_{R_f}(U) = f^{-1}(U)$, for all $U \in D_{\mathcal{K}_2}(X_2)$. Let $x \in h_{R_f}(U)$. Then $R_f(x) \cap U \neq \emptyset$, i.e., there exists $y \in [f(x))$ and $y \in U$. Since U is decreasing, $f(x) \in U$. So, $x \in f^{-1}(U)$. Conversely, let $x \in f^{-1}(U)$. Thus, $f(x) \in U$ and since $f(x) \in R_f(x)$, we have $R_f(x) \cap U \neq \emptyset$. Then $x \in h_{R_f}(U)$. Therefore, R_f is an N-functional relation. □

Finally, we have the following theorem.

Theorem 4.23. *The categories $\mathcal{NS}$ and $\mathcal{NF}$ are isomorphic.*

Proof. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $f\colon X_1 \to X_2$ be a Stone morphism and $R \subseteq X_1 \times X_2$ an N-functional relation. We prove that $R_{f_R} = R$ and $f_{R_f} = f$. Indeed, we have $(x, y) \in R_{f_R}$ iff $f_R(x) \leq_2 y$ iff $y \in [f_R(x)) = R(x)$ iff $(x, y) \in R$. Similarly, we have $f_{R_f}(x) = y$ iff $R_f(x) = [y)$ iff $f(x) = y$. □

It is immediately seen that Theorem 4.23 is an extension of Stone duality.

5. Application of the duality

In this section, we present several applications of the above isomorphism for a dual description of some algebraic concepts of the theory of distributive nearlattices.

5.1. Description of 1-1 and onto homomorphisms. Our next aim is to give a dual description of 1-1 and onto homomorphisms. We define the notion of strong 1-1 homomorphisms, which is a special case of 1-1 homomorphisms, and show that strong 1-1 homomorphisms and onto homomorphisms of distributive nearlattices correspond to onto N-functional relations and 1-1 N-functional relations, respectively.

Definition 5.1. Let $A, B \in \mathcal{DN}$ and $h\colon A \to B$ a homomorphism. We say that h is *strong* 1-1 if for all $n \geq 0$ and $a, b_1, \ldots, b_n \in A$,

$$[h(a)) \subseteq [h(b_1)) \vee \cdots \vee [h(b_n)) \text{ yields } [a) \subseteq [b_1) \vee \cdots \vee [b_n).$$

As an immediate consequence, we have the following result.

Remark 5.2. Let $A, B \in \mathcal{DN}$ and $h\colon A \to B$ a homomorphism. If h is strong 1-1, then h is 1-1.

Remark 5.3. Note that if A and B are distributive lattices, the notions of strong 1-1 and 1-1 homomorphisms coincide.

Definition 5.4. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be an N-functional relation.

(1) We say that R is *onto* if for each $y \in X_2$, there exists $x \in X_1$ such that $R(x) = \mathrm{Sb}(y)$.
(2) We say that R is 1-1 if for each $x \in X_1$ and $U \in D_{\mathcal{K}_1}(X_1)$ with $x \notin U$, there exists $V \in D_{\mathcal{K}_2}(X_2)$ such that $U \subseteq h_R(V)$ and $x \notin h_R(V)$.

Theorem 5.5. *Let $A, B \in \mathcal{DN}$ and $h\colon A \to B$ a homomorphism. Then*

(1) *h is strong* 1-1 *iff R_h is onto,*
(2) *h is onto iff R_h is* 1-1*.*

Proof. (1): Suppose that h is strong 1-1. Let $P \in X(A)$. We prove that $I(h(P)) \cap F(h(P^c)) = \emptyset$. Suppose the contrary. Then there are $a \in P$ and $p_1, \ldots, p_n \in P^c$ such that $h(p_1)\wedge\cdots\wedge h(p_n)$ exists and $h(p_1)\wedge\cdots\wedge h(p_n) \leq h(a)$. Thus, $[h(a)) \subseteq [h(p_1)) \vee \cdots \vee [h(p_n))$ and since h is strong 1-1, we have that $[a) \subseteq [p_1) \vee \cdots \vee [p_n)$. As P^c is a filter, $[p_1) \vee \cdots \vee [p_n) \subseteq P^c$. So, $a \in P^c$, which is a contradiction. Thus, $I(h(P)) \cap F(h(P^c)) = \emptyset$ and by Theorem 2.8, there exists $Q \in X(B)$ such that $h(P) \subseteq Q$ and $Q \cap h(P^c) = \emptyset$. Therefore, $h(P) \subseteq Q$ and $Q \subseteq h(P)$, i.e., $h(P) = Q$. By Lemma 4.17, R_h is onto.

Conversely, let $a, b_1, \ldots, b_n \in A$ be such that $[h(a)) \subseteq [h(b_1)) \vee \cdots \vee [h(b_n))$. We prove that $[a) \subseteq [b_1) \vee \cdots \vee [b_n)$. Suppose that $a \notin [b_1) \vee \cdots \vee [b_n) = [\{b_1, \ldots, b_n\})$. Then by Theorem 2.8, there exists $Q \in X(A)$ such that $a \in Q$ and $Q \cap [\{b_1, \ldots, b_n\}) = \emptyset$. By hypothesis, there exists $P \in X(B)$ such that $R_h(P) = \mathrm{Sb}(Q)$ and by Lemma 4.17, we have $h^{-1}(P) = Q$. Thus, $h(a) \in P$ and $h(b_1), \ldots, h(b_n) \notin P$. But since $[h(a)) \subseteq [h(b_1)) \vee \cdots \vee [h(b_n))$, there is a subset $\{b_{k_1}, \ldots, b_{k_m}\} \subseteq \{b_1, \ldots, b_n\}$ such that $h(b_{k_1}) \wedge \cdots \wedge h(b_{k_m})$ exists and as P is prime, there is $b_{k_j} \in \{b_{k_1}, \ldots, b_{k_m}\}$ such that $h(b_{k_j}) \in P$, which is a contradiction. Therefore, $[a) \subseteq [b_1) \vee \cdots \vee [b_n)$ and h is strong 1-1.

(2): Suppose that h is onto. Let $P \in X(B)$ and $\varphi_B(b) \in D_{\mathcal{K}_B}(X(B))$ such that $P \notin \varphi_B(b)$. Since h is onto, there exists $a \in A$ such that $h(a) = b$. So, by Proposition 4.12, $\varphi_B(b) = \varphi_B(h(a)) = h_{R_h}(\varphi_A(a))$. Thus, $\varphi_B(b) \subseteq h_{R_h}(\varphi_A(a))$ and $P \notin h_{R_h}(\varphi_A(a))$. We have proved that R_h is 1-1.

Now suppose that R_h is 1-1. Let $b \in B$. For each $P \in X(B)$ such that $b \in P$, we have $P \notin \varphi_B(b)$. As R_h is 1-1, there exists $\varphi_A(a_P) \in D_{\mathcal{K}_A}(X(A))$ such that $\varphi_B(b) \subseteq h_{R_h}(\varphi_A(a_P))$ and $P \notin h_{R_h}(\varphi_A(a_P))$. Thus,

$$\varphi_B(b)^c = \bigcap\{h_{R_h}(\varphi_A(a_P))^c : P \notin \varphi_B(b)\}.$$

Since $\varphi_B(b)^c$ is dually compact, there are $a_1, \ldots, a_n \in A$ such that $\varphi_B(b)^c = h_{R_h}(\varphi_A(a_1))^c \cap \cdots \cap h_{R_h}(\varphi_A(a_n))^c$. So, $\varphi_B(b) = h_{R_h}(\varphi_A(a_1 \vee \cdots \vee a_n))$ and by Proposition 4.12, we have $\varphi_B(b) = h_{R_h}(\varphi_A(a)) = \varphi_B(h(a))$. Therefore, $\varphi_B(b) = \varphi_B(h(a))$. By injectivity of φ_B, it follows that h is onto. $\square$

Theorem 5.6. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces and $R \subseteq X_1 \times X_2$ be an N-functional relation. Then*

(1) *R is* 1-1 *iff h_R is onto,*

(2) *R is onto iff h_R is strong* 1-1.

Proof. This follows by Theorems 4.14 and 5.5. □

5.2. Congruences. Further, we focus on congruences of distributive nearlattices. In [11], the authors have shown that congruence lattices of distributive nearlattices are isomorphic to congruence lattices of certain lattices. Using the representation from Section 3, we present a different characterization of these lattices.

Clearly, by a congruence on a distributive nearlattice A is meant any equivalence on A compatible with the ternary operation m. The corresponding congruence lattice will be denoted by $\mathrm{Con}(A)$.

Recall that if $\langle X, \mathcal{T}\rangle$ is a topological space and Y is a subset of X, then the family $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ of subsets of Y is a topology for Y called the *relative topology* and the topological space $\langle Y, \mathcal{T}_Y\rangle$ is a *subspace* of $\langle X, \mathcal{T}\rangle$.

Lemma 5.7. *Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be a topological space where $\mathcal{K}$ is a basis of the topology $\mathcal{T}_{\mathcal{K}}$ and let $Y \subseteq X$. Then the family $\mathcal{K}_Y = \{U \cap Y : U \in \mathcal{K}\}$ is a basis for a topology $\mathcal{T}_{\mathcal{K}_Y}$ on Y such that $\mathcal{T}_Y \subseteq \mathcal{T}_{\mathcal{K}_Y}$.*

Definition 5.8. Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be a topological space with a basis $\mathcal{K}$ of open and compact subsets. Let $Y \subseteq X$. We shall say that Y is a $\mathcal{K}$-*subset* of X if $U \cap Y$ is a compact set in the topology $\mathcal{T}_Y$ on Y, for each $U \in \mathcal{K}$.

Lemma 5.9. *Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be a topological space with a basis $\mathcal{K}$ of open and compact subsets. Let Y be a $\mathcal{K}$-subset of X. Then $\mathcal{K}_Y = \{U \cap Y : U \in \mathcal{K}\}$ is a basis of open and compact subsets for a topology $\mathcal{T}_{\mathcal{K}_Y}$ on Y such that $\mathcal{T}_Y = \mathcal{T}_{\mathcal{K}_Y}$.*

Proof. By Lemma 5.7, $\mathcal{K}_Y = \{U \cap Y : U \in \mathcal{K}\}$ is a basis for a topology $\mathcal{T}_{\mathcal{K}_Y}$ on Y and $\mathcal{T}_Y \subseteq \mathcal{T}_{\mathcal{K}_Y}$. We prove that $\mathcal{T}_{\mathcal{K}_Y} \subseteq \mathcal{T}_Y$. Let $O' \in \mathcal{T}_{\mathcal{K}_Y}$. So, there exists a family $\{U_i \cap Y : U_i \in \mathcal{K}\} \subseteq \mathcal{K}_Y$ such that $O' = \bigcup\{U_i \cap Y : U_i \in \mathcal{K}\}$. Since Y is a $\mathcal{K}$-subset of X, we have that $U_i \cap Y$ is an open and compact subset in the topology $\mathcal{T}_Y$ on Y. Thus, $O' \in \mathcal{T}_Y$. □

The following result gives necessary and sufficient conditions under which the pair $\langle Y, \mathcal{K}_Y\rangle$ is an N-space.

Theorem 5.10. *Let $\langle X, \mathcal{K}\rangle$ be an N-space and let $Y \subseteq X$. The following conditions are equivalent:*

(1) *$\langle Y, \mathcal{K}_Y\rangle$ is an N-space.*
(2) *Y is a $\mathcal{K}$-subset and if $\{U_i \cap Y : i \in I\}$ and $\{V_j \cap Y : j \in J\}$ are non-empty families of $D_{\mathcal{K}_Y}(Y)$ such that $\bigcap\{U_i \cap Y : i \in I\} \subseteq \bigcup\{V_j \cap Y : j \in J\}$, then there exist $U_1, \ldots, U_n$ and $V_1, \ldots, V_k$ such that $(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) \in D_{\mathcal{K}_Y}(Y)$ and $(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) \subseteq (V_1 \cap Y) \cup \cdots \cup (V_k \cap Y)$.*

Proof. (1) $\Rightarrow$ (2): We prove that Y is a $\mathcal{K}$-subset of X, i.e., $W \cap Y$ is a compact set in the topology $\mathcal{T}_Y$ on Y, for each $W \in \mathcal{K}$. Since $\mathcal{K}$ is a basis of $\mathcal{T}_{\mathcal{K}}$, it suffices to take a family $\{V_i : i \in I\} \subseteq \mathcal{K}$ such that $W \cap Y \subseteq \bigcup\{V_i \cap Y : V_i \in \mathcal{K}\}$. Let $D = \{V_i \cap Y : V_i \in \mathcal{K}\}$. We denote $\bar{D} = \{V_i^c \cap Y : V_i \in \mathcal{K}\}$. As $\langle Y, \mathcal{K}_Y\rangle$ is an N-space, we have that $D_{\mathcal{K}_Y}(Y) = \{U^c \cap Y : U \in \mathcal{K}\}$ is a distributive nearlattice. We prove that $I(W^c \cap Y) \cap F(\bar{D}) \neq \emptyset$. Assume on the contrary, i.e., $I(W^c \cap Y) \cap F(\bar{D}) = \emptyset$. Then there exists $P \in X(D_{\mathcal{K}_Y}(Y))$ with $I(W^c \cap Y) \subseteq P$ and $P \cap F(\bar{D}) = \emptyset$. On the other hand, by Proposition 3.6, we have $H\colon Y \to X(D_{\mathcal{K}_Y}(Y))$ is onto. So, there exists $y \in Y$ such that $P = H(y)$. Thus, $W^c \cap Y \in H(y)$ and $V_i^c \cap Y \notin H(y)$ for all $V_i^c \cap Y \in \bar{D}$. Then $y \in W \cap Y$ and $y \notin \bigcup\{V_i \cap Y : V_i \in \mathcal{K}\}$, which is a contradiction. So, $I(W^c \cap Y) \cap F(\bar{D}) \neq \emptyset$ and there exist $V_1^c, \ldots, V_n^c$ such that $(V_1^c \cap Y) \cap \cdots \cap (V_n^c \cap Y) \in D_{\mathcal{K}_Y}(Y)$ and $(V_1^c \cap Y) \cap \cdots \cap (V_n^c \cap Y) \subseteq W^c \cap Y$, i.e., $W \cap Y \subseteq (V_1 \cap Y) \cup \cdots \cup (V_n \cap Y)$. Therefore, $W \cap Y$ is a compact set of $\mathcal{T}_Y$ and Y is a $\mathcal{K}$-subset of X.

(2) $\Rightarrow$ (1): Since Y is a $\mathcal{K}$-subset, by Lemma 5.9, $\mathcal{K}_Y = \{U \cap Y : U \in \mathcal{K}\}$ is a basis of open and compact subsets of $\mathcal{T}_{\mathcal{K}_Y}$. It is easy to see that for every $(U \cap Y), (V \cap Y), (W \cap Y) \in \mathcal{K}_Y$,

$$[(U \cap Y) \cap (W \cap Y)] \cup [(V \cap Y) \cap (W \cap Y)] \in \mathcal{K}_Y.$$

So, by Proposition 3.6, $\langle Y, \mathcal{K}_Y\rangle$ is an N-space. □

Given $A \in \mathcal{DN}$ and $\theta \in \mathrm{Con}(A)$, the natural homomorphism $q_\theta\colon A \to A/\theta$ assigns to $a \in A$ the equivalence class $q_\theta(a) = a/\theta$. Consider the set

$$Y_\theta = \{q_\theta^{-1}(P) : P \in X(A/\theta)\}.$$

By Lemma 4.3, $q_\theta^{-1}(P) \in X(A)$ for all $P \in X(A/\theta)$.

We are ready to prove the following results.

Proposition 5.11. *Let $A \in \mathcal{DN}$ and let $\mathcal{F}(A)$ be the dual space of A. Let $\theta \in \mathrm{Con}(A)$. Then $\langle Y_\theta, \mathcal{K}_{Y_\theta}\rangle$ is an N-space.*

Proof. We prove that $\varphi(a)^c \cap Y_\theta$ is compact in the topology $\mathcal{T}_{Y_\theta}$, for each $\varphi(a)^c \in \mathcal{K}_A$. Since $\mathcal{K}_A$ is a basis of $\mathcal{T}_A$, it suffices to take $\{\varphi(b)^c : b \in B\} \subseteq \mathcal{K}_A$ such that $\varphi(a)^c \cap Y_\theta \subseteq \bigcup\{\varphi(b)^c \cap Y_\theta : b \in B\}$ for some $B \subseteq A$. We prove that there exist $b_1, \ldots, b_n \in B$ with $Y_\theta \cap \varphi(a)^c \subseteq (\varphi(b_1)^c \cap Y_\theta) \cup \cdots \cup (\varphi(b_n)^c \cap Y_\theta)$. Consider $B/\theta = \{b/\theta : b \in B\}$, so $(a/\theta] \cap F(B/\theta) \neq \emptyset$. Suppose the contrary; then there exists $Q \in X(A/\theta)$ such that $a/\theta \in Q$ and $Q \cap F(B/\theta) = \emptyset$. Then $q_\theta^{-1}(Q) \in X(A)$ and $q_\theta^{-1}(Q) \in \varphi(a)^c \cap Y_\theta \subseteq \bigcup\{\varphi(b)^c \cap Y_\theta : b \in B\}$. Therefore, there exists $b_i \in B$ such that $q_\theta^{-1}(Q) \in \varphi(b_i)^c$, i.e., $b_j \in q_\theta^{-1}(Q)$. Thus, $q_\theta(b_j) = b_j/\theta \in Q$, which is a contradiction because $Q \cap F(B/\theta) = \emptyset$. So, we have proved there are $b_1, \ldots, b_n \in B$ such that $b_1 \wedge \cdots \wedge b_n$ exists and $b_1/\theta \wedge \cdots \wedge b_n/\theta \leq a/\theta$. We see that $\varphi(a)^c \cap Y_\theta \subseteq (\varphi(b_1)^c \cap Y_\theta) \cup \cdots \cup (\varphi(b_n)^c \cap Y_\theta)$. Let $P \in Y_\theta \cap \varphi(a)^c$. Then $a \in P$ and $P = q_\theta^{-1}(Q)$ for some $Q \in X(A/\theta)$. Thus, $q_\theta(a) = a/\theta \in Q$ and $(b_1 \wedge \cdots \wedge b_n)/\theta \in Q$. Since Q is prime, there is b_j for some j, such that $b_j/\theta \in Q$, i.e., $b_i \in q_\theta^{-1}(Q) = P$. So, we have $P \in \varphi(b_i)^c$ for some $b_i \in \{b_1, \ldots, b_n\}$. It follows that $P \in (\varphi(b_1)^c \cap Y_\theta) \cup \cdots \cup (\varphi(b_n)^c \cap Y_\theta)$ and that $\varphi(a)^c \cap Y_\theta$ is compact in the topology $\mathcal{T}_{Y_\theta}$. Therefore, Y_θ is a $\mathcal{K}$-subset.

To complete the proof, let $\{\varphi(b_i) \cap Y_\theta : b_i \in B\}$ and $\{\varphi(c_j) \cap Y_\theta : c_j \in C\}$ be non-empty families of $D_{\mathcal{K}_{Y_\theta}}(Y_\theta)$ such that

$$\bigcap\{\varphi(c_j) \cap Y_\theta : c_j \in C\} \subseteq \bigcup\{\varphi(b_i) \cap Y_\theta : b_i \in B\}.$$

Let $B/\theta = \{b/\theta : b \in B\}$ and $C/\theta = \{c/\theta : c \in C\}$. If $I(B/\theta) \cap F(C/\theta) = \emptyset$, then there exists $Q \in X(A/\theta)$ such that $I(B/\theta) \subseteq Q$ and $Q \cap F(C/\theta) = \emptyset$. Then $q_\theta^{-1}(Q) = P \in Y_\theta$. As $I(B/\theta) \subseteq Q$, so $P \notin \bigcup\{\varphi(b_i) \cap Y_\theta : b_i \in B\}$. On the other hand, since $Q \cap F(C/\theta) = \emptyset$, we have $P \in \bigcap\{\varphi(c_j) \cap Y_\theta : c_j \in C\}$, which is a contradiction. Thus, $I(B/\theta) \cap F(C/\theta) \neq \emptyset$, so there are $b_1, \ldots, b_n \in B$ and $c_1, \ldots, c_k \in C$ such that $c_1 \wedge \cdots \wedge c_k$ exists and $c_1/\theta \wedge \cdots \wedge c_k/\theta \leq b_1/\theta \vee \cdots \vee b_n/\theta$. Finally, it is easy to see that

$$\bigcap_{j=1}^{k} (\varphi(c_j) \cap Y_\theta) \subseteq \bigcup_{i=1}^{n} (\varphi(b_i) \cap Y_\theta).$$

So, by Theorem 5.10, $\langle Y_\theta, \mathcal{K}_{Y_\theta}\rangle$ is an N-space. □

The above results motivate the following definition.

Definition 5.12. Let $\langle X, \mathcal{K}\rangle$ be an N-space and let $Y \subseteq X$. We shall say that Y is an *N-subspace* if the pair $\langle Y, \mathcal{K}_Y\rangle$ is an N-space. The set of all N-subspaces of X will be denoted by $\mathcal{S}(X)$.

Let $A \in \mathcal{DN}$ and let Y be a subset of A. Define the binary relation $\theta(Y) \subseteq A \times A$ by $(a, b) \in \theta(Y)$ iff $\varphi(a)^c \cap Y = \varphi(b)^c \cap Y$.

Lemma 5.13. *Let $A \in \mathcal{DN}$. Then the binary relation $\theta(Y)$ is a congruence of A.*

Theorem 5.14. *Let $A \in \mathcal{DN}$ and let $\mathcal{F}(A)$ be the dual space of A. Then the mapping $F \colon \mathcal{S}(X(A)) \to \mathrm{Con}(A)$ defined by $F(Y) = \theta(Y)$ is an dual isomorphism.*

Proof. By Lemma 5.13, F is well defined. Let $Y_1, Y_2 \in \mathcal{S}(X(A))$ such that $\theta(Y_1) = \theta(Y_2)$. Suppose that $Y_1 \nsubseteq Y_2$, i.e., that there exists $P \in Y_1$ with $P \notin Y_2$. Consider the set

$$\mathcal{F} = \bigcap\{\varphi(b) \cap Y_2 : \varphi(b) \notin H(P)\} \cap \bigcap\{\varphi(a)^c \cap Y_2 : \varphi(a) \in H(P)\}.$$

If $\mathcal{F} \neq \emptyset$, then exists $Q \in \mathcal{F}$ and $H(P) = H(Q)$. Thus, since H is 1-1, we have $P = Q \in Y_2$, which is a contradiction. Therefore, $\mathcal{F} = \emptyset$ and

$$\bigcap\{\varphi(b) \cap Y_2 : \varphi(b) \notin H(P)\} \subseteq \bigcup\{\varphi(a) \cap Y_2 : \varphi(a) \in H(P)\}.$$

Since Y_2 is an N-subspace, Proposition 3.9 implies there exist $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ such that $b_1 \wedge \cdots \wedge b_k$ exists and

$$(\varphi(b_1) \cap Y_2) \cap \cdots \cap (\varphi(b_k) \cap Y_2) \subseteq (\varphi(a_1) \cap Y_2) \cup \cdots \cup (\varphi(a_n) \cap Y_2).$$

Let $b = b_1 \wedge \cdots \wedge b_k$ and $a = a_1 \vee \cdots \vee a_n$. So, $\varphi(b) \cap Y_2 \subseteq \varphi(a) \cap Y_2$. Thus, $\varphi(a)^c \cap Y_2 \subseteq \varphi(b)^c \cap Y_2$ and the pair $(a \vee b, a) \in \theta(Y_2) = \theta(Y_1)$. Then $\varphi(a \vee b)^c \cap Y_1 = \varphi(a)^c \cap Y_1$. Since $P \in \varphi(a)^c \cap Y_1$, we have $P \in \varphi(a \vee b)^c$, i.e., $a \vee b \in P$. As P is an ideal, $b \in P$, which is a contradiction because $\varphi(b) \notin H(P)$. This shows that F is 1-1.

Now we prove F is onto. For $\theta \in \mathrm{Con}(A)$, let $Y_\theta = \{q_\theta^{-1}(P) : P \in X(A/\theta)\}$. By Proposition 5.11, Y_θ is an N-subspace of $X(A)$. We see that $\theta(Y_\theta) = \theta$. Let $(a,b) \in \theta$. If $Q \in \varphi(a) \cap Y_\theta$, then $a \notin Q$ and there exists $P \in X(A/\theta)$ such that $Q = q_\theta^{-1}(P)$. Thus, $q_\theta(a) = a/\theta \notin P$. Since $a/\theta = b/\theta$, we have $q_\theta(b) \notin P$. So, $b \notin q_\theta^{-1}(P) = Q$ and $Q \in \varphi(b) \cap Y_\theta$. Analogously, we obtain $\varphi(b) \cap Y_\theta \subseteq \varphi(a) \cap Y_\theta$, and therefore $\varphi(a) \cap Y_\theta = \varphi(b) \cap Y_\theta$. So, $\varphi(a)^c \cap Y_\theta = \varphi(b)^c \cap Y_\theta$ and $(a,b) \in \theta(Y_\theta)$. Conversely, let $(a,b) \in \theta(Y_\theta)$. Then $\varphi(a)^c \cap Y_\theta = \varphi(b)^c \cap Y_\theta$. Let $P \in X(A/\theta)$. We have

$$\begin{aligned} q_\theta(a) \notin P \quad &\text{iff} \quad a \notin q_\theta^{-1}(P) && \text{iff} \quad q_\theta^{-1}(P) \notin \varphi(a)^c \\ &\text{iff} \quad q_\theta^{-1}(P) \notin \varphi(a)^c \cap Y_\theta = \varphi(b)^c \cap Y_\theta && \text{iff} \quad q_\theta^{-1}(P) \notin \varphi(b)^c \\ &\text{iff} \quad b \notin q_\theta^{-1}(P) && \text{iff} \quad q_\theta(b) \notin P, \end{aligned}$$

i.e., $q_\theta(a) \in P$ iff $q_\theta(b) \in P$ for all $P \in X(A/\theta)$. We prove $q_\theta(a) = q_\theta(b)$. Suppose that $q_\theta(a) \nleq q_\theta(b)$. Then $(q_\theta(b)] \cap [q_\theta(a)) = \emptyset$ and by Theorem 2.8, there exists $Q \in X(A/\theta)$ with $(q_\theta(b)] \subseteq Q$ and $Q \cap [q_\theta(a)) = \emptyset$. So, $q_\theta(b) \in Q$, but $q_\theta(a) \in Q$, which is a contradiction. Thus, $q_\theta(a) \leq q_\theta(b)$. Analogously, $q_\theta(b) \leq q_\theta(a)$ and $q_\theta(a) = q_\theta(b)$. Then $a/\theta = b/\theta$ and $(a,b) \in \theta$. □

5.3. Subalgebras. As usual, by a subalgebra of a nearlattice A is meant a subset of A closed under the ternary operation m. The lattice of subalgebras of A will be denoted by $\mathrm{Sub}(A)$.

Definition 5.15. Let $\langle X, \mathcal{K}\rangle$ be an N-space. A subset $\emptyset \neq \mathcal{L} \subseteq \mathcal{K}$ will be called an N-*basic* set if for any $U, V, W \in \mathcal{L}$, $(U \cap W) \cup (V \cap W) \in \mathcal{L}$.

Given an N-space $\langle X, \mathcal{K}\rangle$, let $NB(X)$ denote $\{\mathcal{L} \subseteq \mathcal{K} : \mathcal{L}$ is an N-basic set$\}$.

Lemma 5.16. *Let $\langle X, \mathcal{K}\rangle$ be an N-space. Then $\langle NB(X), \subseteq\rangle$ is a lattice.*

For $A \in \mathcal{DN}$, let $T(B)$ denote $\{\varphi(b)^c : b \in B\}$, for each $B \in \mathrm{Sub}(A)$.

Proposition 5.17. *Let $A \in \mathcal{DN}$. The mapping $T\colon \mathrm{Sub}(A) \to NB(X(A))$ is an order preserving function.*

Proof. Let $B \in \mathrm{Sub}(A)$. It is clear that $T(B) \subseteq \mathcal{K}_A$. If $U, V, W \in T(B)$, then there are $a, b, c \in B$ such that $U = \varphi(a)^c$,$V = \varphi(b)^c$, and $W = \varphi(c)^c$. Thus,

$$(U \cap W) \cup (V \cap W) = [\varphi(a)^c \cap \varphi(c)^c] \cup [\varphi(b)^c \cap \varphi(c)^c] = \varphi(m(a,b,c))^c.$$

Since B is a subalgebra of A, $m(a,b,c) \in B$ and $(U \cap W) \cup (V \cap W) \in T(B)$. So, $T(B)$ is an N-basic set of $X(A)$. It is easy to show that the function T preserves the order. □

Let $A \in \mathcal{DN}$ and $\mathcal{L} \in NB(X(A))$; consider $S(\mathcal{L}) = \{a \in A : \varphi(a)^c \in \mathcal{L}\}$. We have the following lemma.

Lemma 5.18. *Let $A \in \mathcal{DN}$ and $\mathcal{L} \in NB(X(A))$. Then $S(\mathcal{L}) \in \mathrm{Sub}(A)$.*

Proof. We will prove that $S(\mathcal{L})$ is closed under the ternary operation m. Let $a, b, c \in S(\mathcal{L})$. Since $\mathcal{L}$ is an N-basic set and $\varphi(a)^c, \varphi(b)^c, \varphi(c)^c \in \mathcal{L}$, we have $[\varphi(a)^c \cap \varphi(c)^c] \cup [\varphi(b)^c \cap \varphi(c)^c] \in \mathcal{L}$. But

$$\begin{aligned}[\varphi(a)^c \cap \varphi(c)^c] \cup [\varphi(b)^c \cap \varphi(c)^c] &= \varphi(a \vee c)^c \cup \varphi(b \vee c)^c \\ &= \varphi((a \vee c) \wedge_c (b \vee c))^c = \varphi(m(a, b, c))^c.\end{aligned}$$

So, $m(a, b, c) \in S(\mathcal{L})$. □

Theorem 5.19. *Let $A \in \mathcal{DN}$. Then the lattice of subalgebras of A is isomorphic to the lattice of N-basic subsets of $\mathcal{K}_A$.*

Proof. Let $B \in \mathrm{Sub}(A)$. Then $a \in S(T(B))$ iff $\varphi(a)^c \in T(B)$ iff there exists $b \in B$ such that $\varphi(a)^c = \varphi(b)^c$ iff $a = b$. So, $a \in B$ and $S(T(B)) = B$.

Conversely, let $\mathcal{L} \in TB(X(A))$. Then $U \in T(S(\mathcal{L}))$ iff there exists $a \in S(\mathcal{L})$ such that $U = \varphi(a)^c$ iff $U \in \mathcal{L}$. Thus, $T(S(\mathcal{L})) = \mathcal{L}$. □

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Sergio Celani and Ismael Calomino

CONICET and Departamento de Matemáticas, Facultad de Ciencias Exactas, Univ. Nac. del Centro, Pinto 399, 7000 Tandil, Argentina

e-mail: `scelani@exa.unicen.edu.ar`

e-mail: `calomino@exa.unicen.edu.ar`