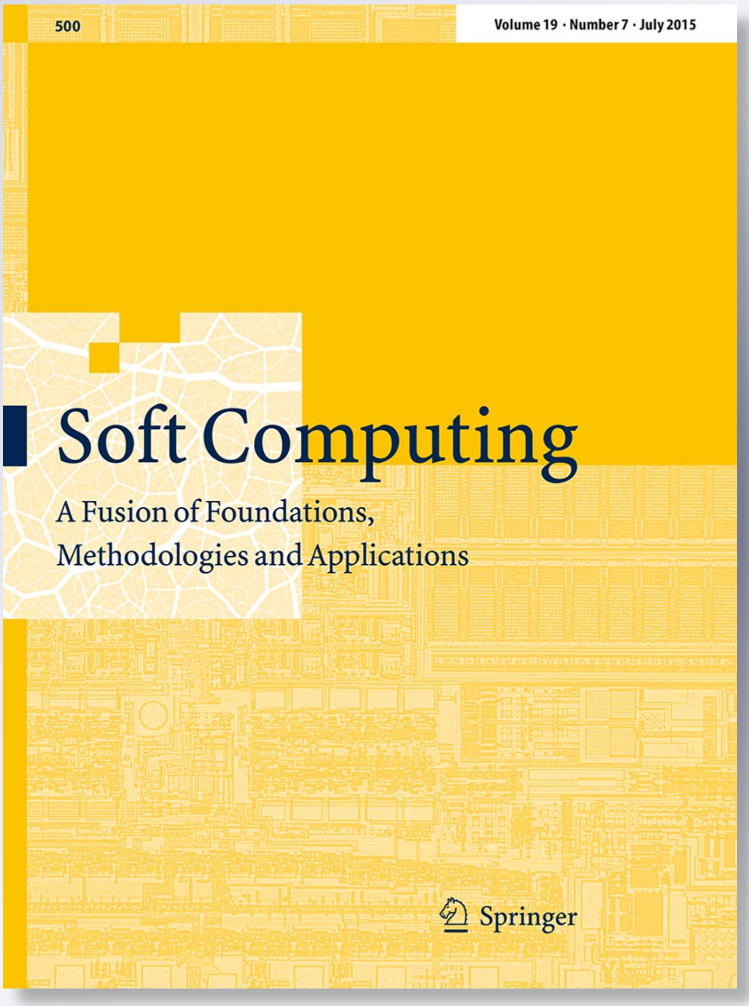
$$\sigma $$ σ -Ideals in distributive pseudocomplemented residuated lattices

Sergio A. Celani

Soft Computing
A Fusion of Foundations,
Methodologies and Applications

ISSN 1432-7643
Volume 19
Number 7

Soft Comput (2015) 19:1773-1777
DOI 10.1007/s00500-015-1592-x



Soft Comput (2015) 19:1773–1777
DOI 10.1007/s00500-015-1592-x

FOUNDATIONS

σ-Ideals in distributive pseudocomplemented residuated lattices

Sergio A. Celani

Published online: 29 January 2015

Abstract In this paper we shall introduce the notion of σ-ideals in the variety of pseudocomplemented residuated lattices. We shall also give some characterizations of the stonean pseudocomplemented residuated lattices.

Keywords Distributive residuated lattices · Pseudocomplemented lattices · Stonean residuated lattices

1 Introduction

It is well known that a subset I of a Boolean algebra $\boldsymbol{A}$ is an ideal iff the subset $\neg(I) = \{\neg a \mid a \in I\}$ is a filter, where $\neg a$ is the negation of a. For distributive pseudocomplemented lattices we do not have an equivalent result, but we can identify an interesting class of ideals where there exists a similar result. A *σ-ideal* in a distributive pseudocomplemented lattice $\boldsymbol{A}$ is an ideal I such that $I = \{a \mid \exists x \in I\ (a^* \vee x = 1)\}$, where a^* is the pseudocomplement of a. If I is a σ-ideal, then there exists a filter F such that $I = ((F)^*] = \{a \mid \exists f \in F : (a \leq f^*)\}$. This class of ideals was introduced by Cornish 1977 (see also Cornish 1973) to study congruences and sheaf representations of distributive pseudocomplemented lattices, and to give some characterizations of Stone lattices.

A distributive pseudocomplemented *residuated* lattice is a distributive residuated lattice $\boldsymbol{A} = \langle A, \vee, \wedge, \circ, \rightarrow, 0, 1\rangle$ such that it satisfies the equation $x \wedge \neg x = 0$, where $\neg x = x \rightarrow 0$ (see Torrens 2011; Chajda et al. 2007; Cignoli 2008; Cignoli and Esteva 2009). The main objective of this paper is to introduce the notion σ-ideal in the variety of distributive pseudocomplemented residuated lattices and give a characterization of stonean distributive residuated lattice in terms of σ-ideals.

The paper is organized as follows. In Sect. 2 we will recall some notions that will be needed in the sequel. In Sect. 3 we shall define the notion of σ-ideal in the variety of distributive pseudocomplemented residuated lattices. We shall give some properties of this class of ideals, and we will give characterizations of the class of distributive pseudocomplemented residuated lattices that satisfy the Stone identity.

2 Preliminaries

For basic notions in residuated lattices we refer to Chajda et al. (2007), Galatos et al. (2007), Höhle (1995) and Turunen (1999), and for basic concepts in distributive lattices we refer to Balbes and Dwinger (1974). First, we recall the definition of a distributive residuated lattice.

Definition 1 An *integral bounded residuated lattice-ordered commutative monoid, or distributive residuated lattice*, for short, is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \circ, \rightarrow, 0, 1\rangle$ of type (2, 2, 2, 2, 0, 0) satisfying the following conditions:

R1 $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
R2 $\langle A, \circ, 1\rangle$ is a commutative monoid,
R3 $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$,
R4 $(x \circ (x \rightarrow y)) \vee y = y$,

Communicated by L. Spada.

This research was supported by the CONICET under Grant PIP 112-201101-00636.

S. A. Celani (✉)
Departamento de Matemáticas, Facultad de Ciencias Exactas, Univ. Nac. del Centro, CONICET, Pinto 399, 7000 Tandil, Argentina
e-mail: scelani@exa.unicen.edu.ar

R5 $x \to (y \to z) = (x \circ y) \to z$,
R6 $x \to (x \vee y) = 1$.

We denote by $\mathcal{DRL}$ the variety of distributive residuated lattices.

In the next lemma we list, for further reference, some well-known properties which we will use throughout this paper. Proofs of these properties can be found, for example in Höhle (1995).

Proposition 1 *Let $A \in \mathcal{DRL}$. Then the following conditions hold for all $x, y, z \in A$:*

1. $x \leq y$ *iff* $x \to y = 1$.
2. $z \circ x \leq y$ *iff* $z \leq x \to y$.
3. *If* $x \leq y$, *then* $x \circ z \leq y \circ z$.
4. $x \circ y \leq x, y$.

Proof 1. If $x \leq y$, then by condition R6 of Definition 1, $x \to (x \vee y) = x \to y = 1$. If $x \to y = 1$, then by the condition R4 of Definition 1 $(x \circ (x \to y)) \vee y = (x \circ 1) \vee y = y$. The property 2 follows by the condition R5 of Definition 1. The others properties are well known. □

Let $A \in \mathcal{DRL}$. We define a unary operation $\neg$ by

$\neg x = x \to 0$,

for each $x \in A$. As usual this operation is called the *negation* operation. The following properties are well known.

Proposition 2 *Let $A \in \mathcal{DRL}$. The following identities and quasi-identities hold true in A :*

1. $x \leq y$, *then* $\neg y \leq \neg x$,
2. $\neg x = \neg\neg\neg x$,
3. $x \leq \neg\neg x$,
4. $x \to \neg y = y \to \neg x$,
5. $x \to y \leq \neg y \to \neg x$,
6. $x \circ \neg x = 0$,
7. $\neg(x \vee y) = \neg x \wedge \neg y$.
8. $\neg\neg x \circ \neg\neg y \leq \neg\neg(x \circ y)$.

Definition 2 (Torrens 2011; Cignoli 2008; Cignoli and Esteva 2009) A *distributive pseudocomplemented residuated lattice* is a distributive residuated lattice A such that $x \wedge \neg x = 0$, for all $x \in A$.

Let $A \in \mathcal{DRL}$. If $x \wedge \neg x = 0$, then $\neg x$ is the pseudocomplement of x in A as a lattice, i.e., $y \wedge x = 0$ if and only if $y \leq \neg x$. Indeed, suppose $y \wedge x = 0$. Then $y \circ x \leq y \wedge x = 0$, and since $\neg x = x \to 0$, we have $y \leq \neg x$.

The variety of *distributive* pseudocomplemented residuated lattices will be denoted by $\mathcal{DPRL}$.

An *implicative filter* or *i-filter* of a bounded residuated lattice A is a subset $F \subseteq A$ satisfying the following conditions:

F1 if $a \in F$ and $a \leq b$, then $b \in F$,
F2 if $a, b \in F$, then $a \circ b \in F$.

Alternatively, an implicative filter can be defined by properties F1 and F3 : if $x \in F$ and $x \to y \in F$, then $y \in F$. An implicative filter is *proper* if $F \neq A$, i.e., if $0 \notin F$. We note that every i-filter F is a filter, i.e., F is closed under $\wedge$.

The set of all i-filters of A is denoted by $\mathrm{Fi}^{\circ}(A)$. For a subset $X \subseteq A$, we denote by $\langle X \rangle$ the i-filter generated by X, i.e., $\langle X \rangle = \bigcap \{F \in \mathrm{Fi}^{\circ}(A) : X \subseteq F\}$. If $X = \{a\}$, we write $\langle a \rangle$ by $\langle \{a\} \rangle$. We note that

$$\langle X \rangle = \{a \in A : \exists x_0, \ldots, x_k \in X \, (x_0 \circ \cdots \circ x_k \leq a)\}.$$

In particular, $\langle x \rangle = \{a \in A : x^n \leq a$, for some $n \geq 0\}$, for $x \in A$. If $F \in \mathrm{Fi}^{\circ}(A)$ and $a \in A$, then $\langle F \cup \{a\} \rangle = \{x \in A : \exists f \in F \exists n \in \mathbb{N} \, (f \circ a^n \leq x)\}$.

A filter P of A is *prime* if $a \vee b \in P$, implies that $a \in P$ or $b \in P$, for all $a, b \in A$. We denote by $\mathrm{Pr}(A)$ the set of all prime filters of A. A *prime i-filter* is an i-filter P such that P is prime. We denote by $\mathrm{Pr}^{\circ}(A)$ the set of all prime i-filters of A.

The set of all filters of the bounded distributive lattice $\langle A, \vee, \wedge, 0, 1 \rangle$ is denoted by $\mathrm{Fi}(A)$. The set of all lattice ideals of $\langle A, \vee, \wedge, 0, 1 \rangle$ is denoted by $\mathrm{Id}(A)$. The filter (ideal) generated by a subset X is denoted by $F(X)$ $(I(X))$.

Let $A \in \mathcal{DPRL}$. A filter (i-filter) U is *maximal*, if it is proper and there are not other filters (i-filters) contained between U and A. In the theory of distributive pseudocomplemented lattices is well known that a proper filter U is maximal if $\forall a \in A$ ($a \notin U$ iff $\neg a \in U$). We note that if U is a maximal filter, then U is prime.

The following result, called prime i-filter theorem is fundamental in all that follows.

Theorem 1 *Let $A \in \mathcal{DRL}$. Let $F \in \mathrm{Fi}^{\circ}(A)$ and $I \in \mathrm{Id}(A)$ such that $F \cap I = \emptyset$. Then there exists $P \in \mathrm{Pr}^{\circ}(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof See Galatos et al. (2007). □

Corollary 1 *Let $A \in \mathcal{DRL}$. Let $F \in \mathrm{Fi}^{\circ}(A)$.*

1. *For each $a \notin F$ there exists $P \in \mathrm{Pr}^{\circ}(A)$ such that $a \notin P$ and $F \subseteq P$.*
2. $F = \bigcap \{P \in \mathrm{Pr}^{\circ}(A) : F \subseteq P\}$.
3. *If $a \neq 1$ there exists a maximal i-filter U such that $a \notin U$.*

Lemma 1 *Let $A \in \mathcal{DPRL}$. Let $P \in \mathrm{Pr}(A)$.*

1. *If $\neg a \notin P$ then there exists $Q \in \mathrm{Pr}(A)$ such that $P \subseteq Q$ and $a \in Q$.*
2. *If $\neg a \notin P$ then there exists a maximal i-filter U such that $P \subseteq U$ and $a \in U$.*

Proof (1) Let $\neg a \notin P$. Consider the filter $F(P \cup \{a\})$. Then $0 \notin F(P \cup \{a\})$, otherwise there exists $p \in P$ such that $p \wedge a = 0$. Since $p \circ a \leq p \wedge a = 0$, we get that $p \leq \neg a$, but this implies that $\neg a \in P$, which is a contradiction. Thus the filter $F(P \cup \{a\})$ is proper. Then by the prime filter theorem for distributive lattices (see Balbes and Dwinger 1974) there exists $Q \in \Pr(A)$ such that $P \subseteq Q$ and $a \in Q$.

(2) It follows by (1). □

Lemma 2 *Let $A \in \mathcal{DPRL}$. If $a \leq \neg b_1 \vee \cdots \vee \neg b_n$, then $\neg a \in \langle\{b_1, \ldots, b_n\}\rangle$.*

Proof Assume that $a \leq \neg b_1 \vee \cdots \vee \neg b_n$. If $\neg a \notin \langle\{b_1, \ldots, b_n\}\rangle$, by Theorem 1 there exists $P \in \Pr^{\circ}(A)$ such that $b_1, \ldots, b_n \in P$, and $\neg a \notin P$. By Lemma 1, there exists $Q \in \Pr(A)$ such that $P \subseteq Q$, and $a \in Q$. So, $\neg b_1 \vee \cdots \vee \neg b_n \in Q$, and as Q is prime, $\neg b_i \in Q$ for some $1 \leq i \leq n$. Then $b_i, \neg b_i \in Q$, and this implies that $b_i \wedge \neg b_i = 0 \in Q$, which is impossible. Thus, $\neg a \in \langle\{b_1, \ldots, b_n\}\rangle$. □

3 σ-Ideals in $\mathcal{DPRL}$

Let $A \in \mathcal{DPRL}$. For each ideal I of A we consider the following set

$$\sigma(I) = \{a \in A \mid (\neg a] \vee I = A\},$$

where we recall that $(x]$ is the principal ideal generated by x. We note that $a \in \sigma(I)$ iff there exists $x \in I$ such that $\neg a \vee x = 1$.

Lemma 3 *Let $A \in \mathcal{DPRL}$. Let I be an ideal of A. Then $\sigma(I)$ is an ideal such that $\sigma(I) \subseteq I$.*

Proof It is clear that $\sigma(I)$ is a decreasing subset of A such that $0 \in \sigma(I)$. Let $a, b \in \sigma(I)$, i.e., $(\neg a] \vee I = A$ and $(\neg b] \vee I = A$. As $1 \in A$, there exists $x, y \in I$ such that $\neg a \vee x = 1$ and $\neg b \vee y = 1$. We prove that $a \vee b \in \sigma(I)$. Suppose that $a \vee b \notin \sigma(I)$, i.e., $(\neg(a \vee b)] \vee I \neq A$. So, there exists $c \in A$ such that $c \notin (\neg(a \vee b)] \vee I$. From Theorem 1 there exists $P \in \Pr(A)$ such that $((\neg(a \vee b)] \vee I) \cap P = \emptyset$, and $c \in P$. So, $\neg(a \vee b) \notin P$ and $I \cap P = \emptyset$. By Lemma 1 there exists $Q \in \Pr(A)$ such that $P \subseteq Q$ and $a \vee b \in Q$. As $I \cap P = \emptyset$, we have that $x, y \notin P$. Then, $\neg a, \neg b \in P$. As $a \vee b \in Q$ and Q is prime, $a \in Q$ or $b \in Q$. In the first case we obtain that $\neg a, a \in Q$, and as Q is a filter, $\neg a \wedge a = 0 \in Q$, which is impossible. If $b \in Q$, then we obtain also a contradiction. Thus, $(\neg(a \vee b)] \vee I = A$, i.e., $a \vee b \in \sigma(I)$.

We prove that $\sigma(I) \subseteq I$. If $a \in \sigma(I)$ but $a \notin I$, there exists $P \in \Pr(A)$ such that $P \cap I = \emptyset$ and $a \in P$. As $\neg a \vee x = 1$ for some $x \in I$, we get that $\neg a \in P$, which is a contradiction. Thus, $\sigma(I) \subseteq I$.

Definition 3 Let $A \in \mathcal{DPRL}$. Let I be an ideal of A. We shall say that I is a σ*-ideal* if $I = \sigma(I)$.

Now we prove that each σ-ideal is generated by a set $\neg(F) = \{\neg f : f \in F\}$, where F is an i-filter.

Proposition 3 *Let $A \in \mathcal{DPRL}$. For each ideal I there exists $F \in \mathrm{Fi}^{\circ}(A)$ such that $\sigma(I) = I(\neg(F))$.*

Proof Let I be an ideal. Consider the set

$$F = \{a \in A \mid (\neg\neg a] \vee I = A\}.$$

We prove that F is an i-filter. It is clear that F is increasing. Let $a, b \in F$. Then there exists $x, y \in I$ such that $\neg\neg a \vee x = \neg\neg b \vee y = 1$. Let $z = x \vee y \in I$. Then $\neg\neg a \vee z = \neg\neg b \vee z = 1$. So,

$$(\neg\neg a \vee z) \circ (\neg\neg b \vee z) = z \vee (\neg\neg a \circ \neg\neg b) = 1,$$

and as $\neg\neg a \circ \neg\neg b \leq \neg\neg(a \circ b)$, we have that $z \vee (\neg\neg(a \circ b)) = 1$, i.e., $a \circ b \in F$. Therefore, $F \in \mathrm{Fi}^{\circ}(A)$.

We prove that $I(\neg(F)) \subseteq \sigma(I)$. Let $x \in I(\neg(F))$. Then there are elements $f_1, f_2, \ldots, f_n \in A$ such that $x \leq \neg f_1 \vee \cdots \vee \neg f_n$ and $(\neg\neg f_i] \vee I = A$, for each $1 \leq i \leq n$. Then there exist $y_1, \ldots, y_n \in I$ such that

$$1 = \neg\neg f_1 \vee y_1 = \cdots = \neg\neg f_n \vee y_n.$$

We prove that $\neg x \vee y_1 \vee \cdots \vee y_n = 1$. Suppose the contrary. Then there exists $P \in \Pr(A)$ such that $\neg x \notin P$ and $y_1 \vee \cdots \vee y_n \notin P$. By Lemma 1 there exists $Q \in \Pr(A)$ such that $P \subseteq Q$ and $x \in Q$. So, $\neg\neg f_i \in P$, for all $1 \leq i \leq n$. So, $\neg f_1 \vee \cdots \vee \neg f_n \in Q$, and Q is prime, $\neg f_i \in Q$ for some $1 \leq i \leq n$. From $\neg f_i, \neg\neg f_i \in Q$ we obtain that $\neg f_i \wedge \neg\neg f_i = 0 \in Q$, which is impossible. Therefore, $\neg x \vee y_1 \vee \cdots \vee y_n = 1$. As $y_1 \vee \cdots \vee y_n \in I$, we have that $x \in \sigma(I)$.

We prove that $\sigma(I) \subseteq I(\neg(F))$. Let $x \in \sigma(I)$. Then $\neg x \vee y = 1$ for some $y \in I$. So, $\neg\neg(\neg x) \vee y = \neg x \vee y = 1$. Then $\neg x \in F$, and thus $\neg\neg x \in \neg(F)$. Since $x \leq \neg\neg x$, we get that $x \in I(\neg(F))$. □

A *stonean* residuated lattice is a distributive bounded residuated lattice A satisfying the Stone equation

$$\neg x \vee \neg\neg x = 1. \tag{1}$$

The variety of distributive stonean residuated lattices is denoted by $\mathcal{DSRL}$. Now we give different characterizations of distributive stonean residuated lattices.

We recall that in a stonean residuated lattice A is valid the following equation

$$\neg\neg(x \circ y) = \neg\neg x \wedge \neg\neg y.$$

(see Cignoli 2008). By this equation we deduce that $\neg\neg(x^n) = \neg\neg(x \circ \cdots \circ x) = \neg\neg x$, and consequently we get that

$$\neg(x^n) = \neg x.$$

This equation will be used in the following results.

Lemma 4 *Let $\mathbf{A} \in \mathcal{DSRL}$. Let $P \in \mathrm{Fi}^{\circ}(A)$. Then P is maximal iff $\neg a \in P$, when $a \notin P$.*

Proof Assume that $P \in \mathrm{Fi}^{\circ}(A)$ is maximal. Let $a \notin P$. Then $0 \in \langle P \cup \{a\}\rangle$. So there exists $p \in P$ and $n \in \mathbb{N}$ such that $p \circ a^n = 0$. So, $p \leq \neg a^n = \neg a \in P$.

Let $G \in \mathrm{Fi}^{\circ}(A)$ such that $P \subseteq G$. If there exists $a \in G - P$, then $\neg a \in P$. So, $a, \neg a \in G$, and as G is an i-filter, $0 = a \circ \neg a \in G$, and thus $G = A$. □

Proposition 4 *Let $\mathbf{A} \in \mathcal{DSRL}$. For each proper σ-ideal I there exists a maximal i-filter U such that $I \cap U = \emptyset$.*

Proof Let I be a σ-ideal. By Proposition 3 there exists a i-filter F such that $I(\neg(F)) = I$. We prove that $I(\neg(F)) \cap F = \emptyset$. Suppose that there exists $a \in I(\neg(F)) \cap F$. Then there exist $f_1, \ldots, f_n \in F$ such that $a \leq \neg f_1 \vee \cdots \vee \neg f_n$. By Lemma 2, $\neg a \in \langle\{f_1, \ldots, f_n\}\rangle \subseteq F$. As $a \in F$, we have that $0 = a \wedge \neg a \in F$, which is a contradiction. Then, $I(\neg(F)) \cap F = \emptyset$. By Theorem 1 there exists $P \in \mathrm{Pr}^{\circ}(A)$ such that $F \subseteq P$ and $I(\neg(F)) \cap P = \emptyset$. We prove that P is maximal filter. Let $a \notin P$. By Lemma 4 we need to prove that $\neg a \in P$. As $\langle P \cup \{a\}\rangle \cap I(\neg(F)) \neq \emptyset$, there exists $b \in A$ and there exist $f_1, \ldots, f_n \in F$ such that

$$b \leq \neg f_1 \vee \cdots \vee \neg f_n \text{ and } p \circ a^n \leq b.$$

Then $\neg b \in \langle\{f_1, \ldots, f_n\}\rangle \subseteq F \subseteq P$ and

$$p \leq a^n \to b \leq \neg b \to \neg a^n = \neg b \to \neg a \in P.$$

As P is an i-filter, we have that $\neg a \in P$. Thus, P is maximal. □

Theorem 2 *Let $\mathbf{A} \in \mathcal{DPRL}$. Then $\mathbf{A}$ is stonean iff $I(\neg(F))$ is a σ-ideal, for each $F \in \mathrm{Fi}(A)$.*

Proof Let $a \in A$. Consider the ideal $I(\neg\neg a)$. It is easy to see that $I(\neg\neg a) = I(\neg([\neg a))$. By hypothesis, $I(\neg([\neg a))$ is a σ-ideal. As $a \in I(\neg\neg a)$ then there exists $x \in I(\neg\neg a)$ such that $\neg a \vee x = 1$. But this implies that $\neg a \vee \neg\neg a = 1$, i.e., $\mathbf{A}$ is stonean.

Let $F \in \mathrm{Fi}(A)$. Let $I = I(\neg(F))$. By Lemma 3, $\sigma(I) \subseteq I$. We need to prove the inclusion $I \subseteq \sigma(I)$. Let $a \in I$. Then there are elements $f_1, f_2, \ldots, f_n \in F$ such that $a \leq \neg f_1 \vee \cdots \vee \neg f_n$. From Lemma 2 we have $\neg a \in \langle\{f_1, f_2, \ldots, f_n\}\rangle \subseteq F$. Then, $\neg\neg a \in \neg(F) \subseteq I(\neg(F))$. So, $1 = \neg a \vee \neg\neg a \in (\neg a] \vee I$. Thus, $(\neg a] \vee I = A$, i.e., $a \in \sigma(I)$. □

Corollary 2 *Let $\mathbf{A} \in \mathcal{DSRL}$. An ideal I is a σ-ideal if and only if there exists $F \in \mathrm{Fi}(A)$ such that $I = I(\neg(F))$.*

Proof If I is a σ-ideal, then by Proposition 3 there exists a filter F such that $I = \sigma(I) = I(\neg(F))$. Conversely, if there exists a filter F such that $I = I(\neg(F))$, then by Theorem 2, I is a σ-ideal.

Recall that a proper ideal I of a bounded distributive lattice $\mathbf{A}$ is minimal iff $I^c = A - I$ is a maximal filter.

Theorem 3 *Let $\mathbf{A} \in \mathcal{DPRL}$. Then $\mathbf{A}$ is stonean iff each minimal prime is a σ-ideal.*

Proof Assume that $\mathbf{A}$ is stonean. Let I be a minimal prime ideal. We prove that $I \subseteq \sigma(I)$. Let $a \in I$. Then $a \notin I^c$, and by Lemma 4, $\neg a \notin I$. As $\neg a \wedge \neg\neg a = 0 \in I$, and I is prime, $\neg\neg a \in I$. As $\neg a \vee \neg\neg a = 1$, we get that $a \in \sigma(I)$.

Assume that there exists $a \in A$ such that $\neg a \vee \neg\neg a \neq 1$. Then by the prime filter theorem, there exists $P \in \mathrm{Pr}(A)$ such that $\neg a \vee \neg\neg a \notin P$. Since every prime filter is contained in a maximal filter, we have that there exists a proper maximal filter U such that $P \subseteq U$. If $\neg a \in U$, then $a \notin U$. As $I = U^c$ is a proper minimal ideal, and by hypothesis I is a σ-ideal, we have that there exists $x \in I$ such that $\neg a \vee x = 1$. Since P is prime and $\neg a \notin P$, we get that $x \in P \subseteq U$, which is a contradiction. Thus, $\neg a \notin U$, i.e., $\neg a \in U^c = \sigma(U^c)$. Then there exists $x \notin U$ such that $\neg a \vee x = 1$. But $\neg a \vee x = 1 \in U$ and as it is prime, $\neg a \in U$ or $x \in U$, which is impossible. Thus, $\neg a \vee \neg\neg a = 1$, and $\mathbf{A}$ is stonean. □

Definition 4 Let $\mathbf{A} \in \mathcal{DSRL}$. Let I be an ideal of A. We shall say that I is an *α-ideal* if $\neg\neg a \in I$, whenever $a \in I$.

We note that as $a \leq \neg\neg a$, for all $a \in A$, an ideal I of A is an α-ideal iff $\forall a \in A$ ($a \in I$ iff $\neg\neg a \in I$).

It is not hard to prove this adaptation of the prime ideal theorem.

Proposition 5 *Let $\mathbf{A} \in \mathcal{DPRL}$. Let $F \in \mathrm{Fi}(A)$ and I be an α-ideal such that $F \cap I = \emptyset$. Then there exists a prime α-ideal J such that $F \cap J = \emptyset$ and $I \subseteq J$.*

Lemma 5 *Let $\mathbf{A} \in \mathcal{DSRL}$. If I is a prime α-ideal I, then $(\neg(I^c)]$ is a prime ideal.*

Proof Let I be a prime α-ideal. We first prove that $(\neg(I^c)] \subseteq I$. Let $a \in A$ and $x \notin I$ such that $a \leq \neg x$. Then $x \leq \neg\neg x \leq \neg a$. So, $\neg a \notin I$, and since $\neg a \wedge \neg\neg a = 0 \in I$ and I is prime, $\neg\neg a \in I$. So, $a \in I$, because I is decreasing.

Since $1 \in I^c$, we have that $0 = \neg 1 \in (\neg(I^c)]$. Let $a, b \in (\neg(I^c)]$. Then there are elements $x, y \notin I$ such that $a \leq \neg x$ and $b \leq \neg y$. Let $z = x \vee y$. As I is an ideal, $z \notin I$, and since $a \leq \neg z$ and $b \leq \neg z$, we get that $a \vee b \leq \neg z$. Thus, $(\neg(I^c)]$ is an ideal.

We prove that $(\neg(I^c)]$ is prime. Let $a \wedge b \in (\neg(I^c)]$. Then $a \wedge b \in I$. As I is prime, we can assume that $a \in I$, and taking into account that I is an α-ideal, we get that $\neg\neg a \in I$. Since $\neg a \vee \neg\neg a = 1 \notin I$, $\neg a \notin I$. So, $\neg\neg a \in \neg(I^c) \subseteq (\neg(I^c)]$. Thus, $(\neg(I^c)]$ is a prime ideal.

Theorem 4 *Let $\mathbf{A} \in \mathcal{DPRL}$. Then the following conditions are equivalent:*

1. *$\mathbf{A}$ is stonean,*
2. *$(\neg(I^c)]$ is a prime ideal for each prime α-ideal I.*

Proof The direction (1) $\Rightarrow$ (2) follows by the previous lemma. We prove (2) $\Rightarrow$ (1). Suppose that there exists $a \in A$ such that $\neg a \vee \neg\neg a \neq 1$. Let us consider the ideal $(\neg a \vee \neg\neg a]$. As $\neg\neg(\neg a \vee \neg\neg a) = \neg a \vee \neg\neg a$, we have that I is a proper α-ideal. Then by Proposition 5 there exists a prime α-ideal J such that $I \subseteq J$. We note that $\neg a$, $\neg\neg a \in J$. By hypothesis the ideal $(\neg(J^c)]$ is a prime ideal. It is clear that $(\neg(J^c)] \subseteq J$. We prove that $a, \neg a \notin (\neg(J^c)]$. If $a \in (\neg(J^c)]$, then there exists $d \notin J$ such that $a \leq \neg d$. So, $d \leq \neg\neg d \leq \neg a$, and as $\neg a \in J$ and J is decreasing, we obtain that $d \in J$, which is a contradiction. If $\neg a \in (\neg(J^c)]$, then there exists $g \notin J$ such that $\neg a \leq \neg g$. So, $g \leq \neg\neg g \leq \neg\neg a$, and as $\neg\neg a \in J$, we obtain that $g \in J$, which is an absurd. Thus, $a, \neg a \notin (\neg(J^c)]$. Since $(\neg(J^c)]$ is prime, $a \wedge \neg a = 0 \notin (\neg(J^c)]$, which is impossible, because $(\neg(J^c)]$ is an ideal. Therefore, $\neg a \vee \neg\neg a = 1$, for all $a \in A$. □

Theorem 5 *Let $\mathbf{A} \in \mathcal{DPRL}$. Then $\mathbf{A}$ is stonean iff $I = \sigma(I)$ for every α-ideal I.*

Proof $\Rightarrow$) Let I be an α-ideal. Let $a \in I$. Then $\neg\neg a \in I$. As A is stonean, $\neg a \vee \neg\neg a = 1$. So, $a \in \sigma(I)$. Thus, $I = \sigma(I)$.

$\Leftarrow$) Let $a \in A$. Consider the principal ideal $(\neg\neg a] = I$. It is clear that I is an α-ideal. As $a \leq \neg\neg a$, we get that $a \in I = \sigma(I)$. Then there exists $x \leq \neg\neg a$ such that $\neg a \vee x = 1$. Thus, $\neg a \vee \neg\neg a = 1$, and consequently A is stonean. □

References

Balbes R, Dwinger PH (1974) Distributive lattices. University of Missouri Press, Missouri

Castaño D, Díaz Varela JP, Torrens A (2011) Free-decomposability in varieties of pseudocomplemented residuated lattices. Studia Logica 98:223–235

Chajda I, Halaš R, Kühr J (2007) Semilattice structures. In: Research and exposition in mathematics, vol 30. Heldermann

Cignoli R (2008) Free algebras in varieties of Stonean residuated lattices. Soft Comput 12:315–320

Cignoli R, Esteva F (2009) Commutative integral bounded residuated lattices with an added involution. Ann Pure Appl Log 161:150–160

Cornish W (1973) Congruences on distributive pseudocomplemented lattices. Bull Aust Math Soc 8:161–179

Cornish W (1977) O-ideales, congruences and sheaf representation of distributive lattices. Rev Roum Math Pures Appl 22:1059–1067

Galatos N, Jipsen P, Kowalski T, Ono H (2007) Residuated lattices: an algebraic glimpse at substructural logics. In: Studies in logics and the foundations of mathematics, vol 151. Elsevier

Höhle U (1995) Commutative, residuated l-monoids, in non-classical logics and their applications to fuzzy subsets. Kluwer Academic Publishers, Dordrecht

Turunen E (1999) Mathematics behind fuzzy logic. Advances in soft computing. Physica-Verlag, Heidelberg