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Split Partial Isometries

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Abstract A partial isometry V is said to be a *split* partial isometry if $\mathcal{H} = R(V) + N(V)$, with $R(V) \cap N(V) = \{0\}$ ($R(V)$ = range of V, $N(V)$ = null-space of V). We study the topological properties of the set $\mathcal{I}_0$ of such partial isometries. Denote by $\mathcal{I}$ the set of all partial isometries of $\mathcal{B}(\mathcal{H})$, and by $\mathcal{I}_N$ the set of normal partial isometries. Then

$$\mathcal{I}_N \subset \mathcal{I}_0 \subset \mathcal{I},$$

and the inclusions are proper. It is known that $\mathcal{I}$ is a C^∞-submanifold of $\mathcal{B}(\mathcal{H})$. It is shown here that $\mathcal{I}_0$ is open in $\mathcal{I}$, therefore is has also C^∞-local structure.
We characterize the set $\mathcal{I}_0$, in terms of metric properties, existence of special pseudo-inverses, and a property of the spectrum and the resolvent of V. The connected components of $\mathcal{I}_0$ are characterized: $V_0, V_1 \in \mathcal{I}_0$ lie in the same connected component if and only if

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$$\dim R(V_0) = \dim R(V_1) \text{ and } \dim R(V_0)^\perp = \dim R(V_1)^\perp.$$

This result is known for normal partial isometries.

Keywords Partial isometries · Projections · Idempotents

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1 Introduction

Partial isometries were first defined by John von Neumann, as the "argument"' part of the polar decomposition of closed linear operators om Hilbert spaces. Halmos and collaborators [8] studied same topological features of the set $\mathcal{I}$ of all partial isometries of a fixed Hilbert space $\mathcal{H}$.

In this paper a class of partial isometries is studied. We say that v is a *split partial isometry* if $\mathcal{H}$ is the direct sum of its range $R(V)$ and its null-space $N(V)$. The set $\mathcal{I}_0$ of all such partial isometries is a proper subset of $\mathcal{I}$, which contains properly the set $\mathcal{I}_N$ of normal partial isometries (i.e. $R(V) = N(V)^\perp$), and, a fortiori, contains the set $\mathcal{P}$ of all orthogonal projections in $\mathcal{H}$.

Let us fix some notation. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the space of bounded operators acting in $\mathcal{H}$, $Gl(\mathcal{H})$ the group of invertible operators, and $\mathcal{U}(\mathcal{H})$ the unitary group of $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$ is an operator, we denote by $R(A)$ its range, by $N(A)$ its null-space, and by $\sigma(A)$ its spectrum. Two closed subspaces $\mathcal{S}, \mathcal{T}$ of $\mathcal{H}$ are said to be in direct sum if $\mathcal{S}+\mathcal{T} = \mathcal{H}$ and $\mathcal{S}\cap\mathcal{T} = \{0\}$, in symbols, $\mathcal{S}\dot{+}\mathcal{T} = \mathcal{H}$ (we shall reserve the notation $\mathcal{S} \oplus \mathcal{T} = \mathcal{H}$ for the case when the subspaces are orthogonal). A direct sum splitting as above gives rise to an *idempotent* operator in $\mathcal{B}(\mathcal{H})$: $E(s+t) = s$ and $(1-E)(s+t) = t$. E shall be called a projection when $\mathcal{S}$ and $\mathcal{T}$ are orthogonal, and denoted $E = P_{\mathcal{S}}$.

As said above, $\mathcal{I}$ is the set of partial isometries of $\mathcal{H}$, i.e.

$$\mathcal{I} = \{V \in \mathcal{B}(\mathcal{H}) : V \text{ is isometric between } N(V)^\perp \text{ and } R(V)\}.$$

Equivalently, V^*V and / or VV^* are projections. In that case V^*V is the projection onto $N(V)^\perp$ (also called the initial space of V), and VV^* is the projection onto $R(V)$ (the final space of V). There are several papers dealing with the structure of $\mathcal{I}$, topological or geometrical, among them [1,2,8,10,11].

We shall study here a class of partial isometries, which we shall call *split isometries* and denote by $\mathcal{I}_0$, namely

$$\mathcal{I}_0 = \{V \in \mathcal{I} : N(V)\dot{+}R(V) = \mathcal{H}\}.$$

Examples of split isometries are selfadjoint projections, partial isometries whose range and null-spaces are mutually orthogonal (=normal partial isometries), and partial isometries which appear in the polar decomposition of an oblique projection [5]. It is apparent that this class $\mathcal{I}_0$ is invariant under inner conjugation by unitary operators.

The contents of the paper are the following. Section 2 contains further notations, preliminaries, results on partial isometries and several characteristic properties of the set $\mathcal{I}_0$ of split partial isometries. For instance, $V \in \mathcal{I}_0$ if and only if it admits a commuting pseudo-inverse, or if 0 is a pole of order one of the resolvent (Theorem 2.2). Some of these properties are based in a theorem by Buckholtz [3] on pairs of orthogonal projections P,Q such that $R(P)\dot{+}R(Q) = \mathcal{H}$. In Sect. 3 we examine the local structure of $\mathcal{I}_0$. It is shown that $\mathcal{I}_0$ is a submanifold of $\mathcal{B}(\mathcal{H})$. Also it is shown that a partial isometry lying close enough to a projection, belongs to $\mathcal{I}_0$: (Theorem 3.5) if $V \in \mathcal{I}$ and P a projection with $\|V - P\| < 1/3$, then $V \in \mathcal{I}_0$. In Sect. 4 we study the relationship between $\mathcal{I}_0$ and the set $\mathcal{I}_N$ of normal partial isometries. We prove that each $V \in \mathcal{I}_0$ gives rise to a unique selfadjoint operator X_V, with $\|X_V\| < \pi/2$, which is co-diagonal with respect to the initial projection of V, such that $e^{-iX_V}V$ is normal. Therefore $\mathcal{I}_0$ decomposes as pairs $(X_V, e^{-iX_V}V)$. This implies that the space of split partial isometries has the same homotopy type as the space of normal partial isometries. For instance, it is shown that if $V_0, V_1 \in \mathcal{I}_0$ verify

$$\dim R(V_0) = \dim R(V_1) \;\text{ and }\; \dim R(V_0)^{\perp} = \dim R(V_1)^{\perp},$$

then they can be joined by a smooth curve in $\mathcal{I}_0$.

2 Split Partial Isometries

The following result is known, and will be useful below. We transcribe as it was stated by Buckholtz in [3]

Lemma 2.1 *Let $\mathcal{R}$, $\mathcal{K}$ be closed subspaces in $\mathcal{H}$. Then*

$$\mathcal{R} \dot{+} \mathcal{K} = \mathcal{H}$$

if and only if

$$P_{\mathcal{R}} - P_{\mathcal{K}} \in Gl(\mathcal{H}).$$

if and only if

$$\|P_{\mathcal{R}} + P_{\mathcal{K}} - 1\| < 1.$$

In that case, the idempotent onto $\mathcal{R}$ induced by the decomposition is $E = P_{\mathcal{R}}(P_{\mathcal{R}} - P_{\mathcal{K}})^{-1}Q$.

See, for instance, [3] and [4].

The next result gives several characterizations of the class of split isometries.

Theorem 2.2 *Let $V \in \mathcal{I}$, a non invertible partial isometry. Then the following are equivalent:*

1. $V \in \mathcal{I}_0$.
2. $\|V^*V - VV^*\| < 1$.

3. $V^*V + VV^* - 1 \in Gl(\mathcal{H})$.
4. *There exists* $W \in \mathcal{B}(\mathcal{H})$ *such that*

$$WVW = W, \;\; VWV = V, \;\; \textit{and} \;\; WV = VW.$$

   Such W *is unique with these properties.*
5. *There exist* $S, R \in \mathcal{B}(\mathcal{H})$ *with* S *invertible and* R *an idempotent, such that* $V = SR = RS$.
6. *There exists an invertible operator* T *which commutes with* V*, such that* $V = VTV$.
7. $0 \in \sigma(V)$ *is isolated, and it is a pole of order* 1 *of the resolvent of* V.

Proof Let us first prove the equivalences, then the additional property. (1) is equivalent to (2) or (3) by the Lemma above: put $\mathcal{R} = R(V)$ and $\mathcal{K} = N(V)$.

Suppose (3), i.e. $V^*V + VV^* - 1 \in Gl(\mathcal{H})$, and let $C = (V^*V + VV^* - 1)^{-1}$. Using that $VV^*V = V$ and $V^*VV^* = V^*$, one has that

$$V = V(V^*V + VV^* - 1)C = V^2V^*C,$$

and that

$$V = C(V^*V + VV^* - 1)V = CV^*V^2.$$

Note that this implies that

$$VV^*C = CV^*V. \tag{1}$$

Indeed, $VV^*C = CV^*V^2V^*C = CV^*(V^2V^*C) = CV^*V$. This intertwining property of C and the two formulas above imply the identities

$$V = V^2V^*C = VCV^*V \;\text{ and }\; V = CV^*V^2 = VV^*CV. \tag{2}$$

Multiplying the first identity in (2) on the right by V^* gives

$$VV^* = VCV^*VV^* = VCV^*. \tag{3}$$

Multiplying the second identity in (2) on the left by V^* gives

$$V^*V = V^*VV^*CV = V^*CV. \tag{4}$$

Put $W = CV^*C = CV^*VV^*C$. Then, by (3) and (4),

$$VWV = VCV^*CV = VV^*CV = VV^*V = V$$

and

$$WVW = CV^*CVCV^*C = CV^*VCV^*C = CV^*VV^*C = CV^*C.$$

Finally, using again also (1)

$$VW = VCV^*C = VV^*C = CV^*V = CV^*CV = WV.$$

Let us prove now that this last property (that V has a commuting pseudoinverse W) implies that $V \in \mathcal{I}_0$. Note that $Q = WV = VW$ is an idempotent operator, with $N(Q) = N(WV) = N(V)$ and $R(Q) = R(VW) = R(V)$, and the proof follows.

Suppose (4) holds. Let $R = VW$ and $S = V + 1 - VW$. Then $R^2 = R$ and S is invertible with $S^{-1} = W + 1 - WV$. Clearly,

$$V = SR = RS.$$

This proves (5).

Suppose (5) holds. Then (6) follows with $T = S^{-1}$. In fact, $VT = TV$ and

$$VTV = VS^{-1}V = VS^{-1}SR = VR = SR^2 = SR = V.$$

Suppose (6) holds. Then (4) follows with $W = T^2V$. Indeed, $VW = WV$ and

$$VWV = VT^2V^2 = VTVTV = VTV = V; \quad WVW = T^2V^2T^2V = T^2V = W.$$

(4) $\Longrightarrow$ (1). Since, $VW = WV$, the identity $1 = VW + (1 - VW) = VW + (1 - VW)$, show (1) holds.

(1) is equivalent to (7) (see [14], Theorems 10.1 and 10.2)

That the commuting pseudoinverse, when it exists, is unique, is known (see, for instance, [9]). □

As it was noted in the introduction, a partial isometry V is normal if and only if $N(V) \oplus R(V) = \mathcal{H}$.

Remark 2.3 1. Let $T \in \mathcal{B}(\mathcal{H})$ with Moore–Penrose inverse $T^\dagger$. Then

$$TT^\dagger = T^\dagger T \iff \mathcal{H} = N(T) \oplus R(T).$$

Indeed, suppose $TT^\dagger = T^\dagger T$. Then $R(T) = R(TT^\dagger)$ and $N(T) = N(T^\dagger T) = N(TT^\dagger)$. Since $TT^\dagger$ is a orthogonal projection, we have $\mathcal{H} = N(T) \oplus R(T)$. Conversely, suppose $\mathcal{H} = N(T) \oplus R(T)$; then the orthogonal complement of $N(T)$ is $R(T)$ and therefore $TT^\dagger = P_{R(T)} = P_{N(T)^\perp} = T^\dagger T$.

2. Using the Lemma above, note that if $V \in \mathcal{I}_0$, then the idempotent onto $R(V)$ given by the decomposition $R(V) \dot{+} N(V) = \mathcal{H}$ is

$$VV^*CV^*V = VV^*C = CV^*V.$$

3. By the theorem above (for instance, condition 2), it is clear that $V \in \mathcal{I}_0$ if and only if $V^* \in \mathcal{I}_0$.

4. Denote by $\mathcal{Q}$ the set of idempotents in $\mathcal{B}(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H})$, let V_T be the (unique) partial isometry in the polar decomposition of T, $T = V_T|T|$ (with initial space $R(T)^\perp$ and final space $\overline{R(T)}$). It is easy to prove that the map $\alpha : \mathcal{I}_0 \to \mathcal{Q}$, defined by $\alpha(V) = VV^*C = CV^*V$ is surjective, and that the map $\beta : \mathcal{Q} \to \mathcal{I}_0$ defined by $\beta(E) = V_E$ is a right inverse of α, i.e., $\alpha(\beta(E)) = E$ for every $E \in \mathcal{Q}$. It is apparent that α is continuous. Continuity of the map β was proved in [5].

Theorem 2.4 *Let $V \in \mathcal{I}_0$. Then $V^2 \in \mathcal{I}$ if and only if V is normal.*

Proof Clearly, V normal implies $V^2 \in \mathcal{I}$. Suppose $V^2 \in \mathcal{I}$. Let us first prove that $VV^{*2}V$ is orthogonal projection. Indeed, $(VV^{*2}V)^2 = VV^{*2}VVV^{*2}V = VV^{*2}V^2V^{*2}V = VV^{*2}V$. And, since $\|VV^{*2}V\| \leq 1$, $VV^{*2}V$ is an orthogonal projection. Thus, in particular $VV^{*2}V = (VV^{*2}V)^* = V^*V^2V^*$. We claim that $N(V) = N(V^*)$. Let $x \in N(V)$. Then, $V^*V^2V^*x = VV^{*2}Vx = 0$. Thus, $V^2V^*x \in N(V^*) \cap R(V^2) = N(V^*) \cap R(V) = \{0\}$ (since $V \in \mathcal{I}_0$). Therefore, $V^2V^*x = 0$ and thus, $V^*x \in N(V^2) \cap R(V^*) = N(V) \cap R(V^*) = \{0\}$. That is $x \in N(V^*)$ and $N(V) \subseteq N(V^*)$. The other inclusion follows by symmetry. Finally, we have, $N(V) = N(V^*) = R(V)^\perp$ and thus $\mathcal{H} = N(V) \oplus R(V)$, i.e. V is normal. □

The next result characterizes the operators $T \in \mathcal{B}(\mathcal{H})$ such that the partial isometry in the polar decomposition belongs to $\mathcal{I}_0$. Recall that the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ is the factorization $T = V|T|$, where V is a partial isometry such that $N(V) = N(T)$ and $|T| = (T^*T)^{1/2}$. It can be shown that V is uniquely determined by these properties, and it will be denoted V_T. Moreover, it holds that $R(V_T) = \overline{R(T)}$ and $T = |T^*|V_T$.

Proposition 2.5 *Given $T \in \mathcal{B}(\mathcal{H})$, V_T belongs to $\mathcal{I}_0$ if and only if $\mathcal{H} = \overline{R(T)} \dot{+} N(T)$.*

Proof By the definition of $\mathcal{I}_0$, if $V_T \in \mathcal{I}_0$ then $\mathcal{H} = R(V_T) \dot{+} N(V_T) = \overline{R(T)} \dot{+} N(T)$. The converse is evident. □

As remarked in Sect. 1, one has the strict inclusions

$$\mathcal{P} \subset \mathcal{I}_N \subset \mathcal{I}_0 \subset \mathcal{I}.$$

It is apparent that the first inclusion is strict. Let us write a simple example of a non normal partial isometry in $\mathcal{I}_0$. Let $\mathcal{S}, \mathcal{T}$ be two non orthogonal subspaces such that $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{H}$. Then $dim\ \mathcal{S} = dim\ \mathcal{T}^\perp$. Pick $\{\xi_i : i \in I\}$ and $\{\eta_i : i \in I\}$ orthonormal bases of $\mathcal{S}$ and $\mathcal{T}^\perp$, respectively. Define $V\eta_i = \xi_i$ and $V|_\mathcal{T} = 0$. Then $V \in \mathcal{I}_0 \setminus \mathcal{I}_N$. Finally, let $\mathcal{S} \subset \mathcal{H}$ be an infinite dimensional closed subspace such that $\mathcal{S}^\perp$ is also infinite dimensional, and let W be isometric between $\mathcal{S}$ and $\mathcal{S}^\perp$. Then $W \in \mathcal{I} \setminus \mathcal{I}_0$.

3 Local Structure of $\mathcal{I}_0$

In this section we examine the local structure of $\mathcal{I}_0$. First we note that $\mathcal{I}_0$ is a differentiable manifold. In [2] it was shown the set $\mathcal{I}$ is a C^∞-submanifold of $\mathcal{B}(\mathcal{H})$. Then the following is apparent:

Corollary 3.1 *The set $\mathcal{I}_0$ is a C^∞-submanifold of $\mathcal{B}(\mathcal{H})$*

Proof By the characterization of $\mathcal{I}_0$ in the Theorem of the previous section, it is clear that $\mathcal{I}_0$ is open in $\mathcal{I}$, which is a complemented C^∞-submanifold of $\mathcal{B}(\mathcal{H})$ (see [1]). □

The following Lemma will be useful. First recall the basic fact that unitary operators close enough to the identity have unique logarithms, in the following sense: if $U \in \mathcal{U}(\mathcal{H})$ and $\|U - 1\| < 2$, then there exists a unique $X \in \mathcal{B}(\mathcal{H})$ with $X^* = X$ and $\|X\| < \pi$ such that $U = e^{iX}$.

Lemma 3.2 *Let $A, X \in \mathcal{B}(\mathcal{H})$ with $X^* = X$ and $\|X\| \leq \pi$. If $\|e^{iX}A - A\| < R$, then*

$$\|e^{itX}A - A\| < R,$$

for all t with $|t| \leq 1$

Proof First note that $\|e^{iX}A - A\| < R$ implies that

$$\|e^{-iX}A - A\| = \|e^{-iX}(A - e^{iX}A)\| = \|e^{iX}A - A\| < R.$$

Let $\xi \in \mathcal{H}, \xi \neq 0$, and consider $f_\xi(t) = \|e^{itX}\xi - \xi\|^2$. Apparently,

$$\dot{f}_\xi(t) = -2\ Re\left(i\langle Xe^{itX}\xi, \xi\rangle\right).$$

We claim that $\dot{f}_\xi(t) \geq 0$ for $0 \leq t \leq 1$ and $\dot{f}_\xi(t) \leq 0$ for $-1 \leq t \leq 0$. Suppose first that X has finite spectrum, i.e.

$$X = \sum_{j=1}^{n} \alpha_j P_j,$$

with P_j mutually orthogonal selfadjoint projections, and $\alpha_j \in \mathbb{R}$ with $|\alpha_j| \leq \pi$. Put $\xi_j = P_j\xi$. Then $X\xi_j = \alpha_j\xi_j$ and $e^{itX}\xi_j = e^{it\alpha_j}\xi_j$. Then

$$\dot{f}_\xi(t) = -2Re\left(i\left\langle\sum_{j=1}^{n}\alpha_j e^{it\alpha_j}\xi_j, \sum_{k=1}^{n}\xi_k\right\rangle\right) = -2\ Re\left(i\sum_{j=1}^{n}\alpha_j e^{it\alpha_j}\|\xi_j\|^2\right).$$

Note that

$$-2\ Re(i\alpha_j e^{it\alpha_j}) = \alpha_j \sin(t\alpha_j) = |\alpha_j|\sin(t|\alpha_j|).$$

Since $|\alpha_j| \leq \pi$ for all $j = 1, \ldots, n$, $\dot{f}_\xi(t) \geq 0$, if $0 \leq t \leq 1$, and $\dot{f}_\xi(t) \geq 0$, if $-1 \leq t \leq 0$. Thus the assertion is true in this case.

For an arbitrary $X = X^*$, there exists a sequence $X_k = X_k^*$ with X_k of finite spectrum and $\|X_k\| \leq \pi$, such that $\|X_k - X\| \to 0$. Since for each X_k it holds that $-Re(i\langle X_k e^{itX_k}\xi, \xi\rangle) \geq 0$ for $0 \leq t \leq 1$, then also

$$-Re\left(i\langle Xe^{itX}\xi, \xi\rangle\right) \geq 0, \quad \text{for } 0 \leq t \leq 1.$$

It follows that $f_\xi(t) = \|e^{itX}\xi - \xi\|$ is non decreasing for $t \in [0, 1]$. Analogously, $f_\xi(t)$ is non increasing in $[-1, 0]$. If $\eta \in \mathcal{H}$, put $\xi = A\eta$. Then $\|e^{itX}A\eta - A\eta\|$ is non decreasing in $[0, 1]$, and non increasing in $[-1, 0]$. By hypothesis, $\|e^{iX}A - A\| < R$, thus there exists $\delta > 0$ such that $\|e^{iX}A - A\| < R - \delta$. Then

$$\|e^{iX}A\eta - A\eta\| < (R - \delta)\|\eta\|$$

Therefore

$$\|e^{itX}A\eta - A\eta\| < (R - \delta)\|\eta\|, \quad \text{for } t \in [-1, 1],$$

and thus $\|e^{itX}A - A\| \leq R - \delta < R$, for $t \in [-1, 1]$. □

Lemma 3.3 *Let P be a selfadjoint projection and U a unitary operator. Then if*

$$\|UP - P\| < 1,$$

it holds that

$$UR(P)\dot{+}N(P) = \mathcal{H}.$$

Proof Suppose that U verifies the condition above. Let us check first that $U(R(P)) \cap N(P) = \{0\}$. Suppose otherwise, that there exists $\xi \in \mathcal{H}$ such that $\|P\xi\| = 1$ and $UP\xi \in N(P)$, i.e. $PUP\xi = 0$. Then

$$\begin{aligned}&1 > \|UP - P\|^2 \geq \|UP\xi - P\xi\|^2 = \|UP\xi\|^2 + \|P\xi\|^2 - 2Re\langle UP\xi, P\xi\rangle \\ &= 2 - 2Re\langle PUP\xi, \xi\rangle = 2,\end{aligned}$$

a contradiction.

Let us check now that $U(R(P)) + N(P) = \mathcal{H}$. Suppose that there exists a unitary vector η orthogonal to both subspaces. Then $\eta \perp UP\xi$ for all $\xi \in \mathcal{H}$ and $\eta \perp N(P)$. The latter condition means that $P\eta = \eta$, and putting $\xi = \eta$ in the former means that $0 = \langle UP\eta, \eta\rangle = \langle UP\eta, P\eta\rangle$. This leads to a contradiction with the same computation as above. This implies that the sum is dense in $\mathcal{H}$. Let us check that it is closed. Let $\xi_n \in \mathcal{H}$ be a sequence in $U(R(P)) + N(P)$ which converges to ξ. Then there exist $\eta_n, \psi_n \in \mathcal{H}$ such that $\xi_n = UP\eta_n + (1 - P)\psi_n$. Then $PUP\eta_n \to P\xi$. Note that

$$\|PUP - P\| = \|P(UP - P)\| \leq \|UP - P\| < 1.$$

This implies that PUP is an invertible operator in $\mathcal{B}(R(P))$. In particular, this implies that the sequence $P\eta_n$ is convergent, and therefore also the sequence $UP\eta_n$. Thus also the sequence $(1-P)\psi_n$ is convergent, and this implies that the sum is closed. □

The next result estimates how close a partial isometry V must be to $P_{N(V)^\perp}$, in order to belong to $\mathcal{I}_0$. Note that $\|V - P_{N(V)^\perp}\| = \|V - P_{R(V)}\|$. Indeed,

$$\begin{aligned}&\|V - P_{N(V)^\perp}\|^2 = \|V - V^*V\|^2 \\ &= \|(V - V^*V)(V^* - V^*V)\| = \|VV^* - V^* - V + V^*V\|,\end{aligned}$$

and

$$\begin{aligned}&\|V - P_{R(V)}\|^2 = \|V - VV^*\|^2 = \|(V^* - VV^*)(V - VV^*)\| \\ &= \|V^*V - V - V^* + VV^*\|.\end{aligned}$$

Corollary 3.4 *Let V be a partial isometry. If*

$$\|V - P_{N(V)^\perp}\| < 1$$

(*or equivalently* $\|V - P_{R(V)}\| < 1$) *then $V \in \mathcal{I}_0$. Moreover, in this case there exists a smooth curve $V(t)$ in $\mathcal{I}_0$, $t \in [0,1]$ of the form $V(t) = e^{itX}P_{N(V)^\perp}$, such that $V(0) = P_{N(V)^\perp}$ and $V(1) = V$. Analogously, one can find a curve of the form $V'(t) = P_{R(V)}e^{itY}$ joining V and $P_{R(V)}$.*

Proof The hypothesis that $\|V - P_{N(V)^\perp}\| < 1$ implies the existence of a unitary operator U such that $V = UP_{N(V)^\perp}$. Indeed, in Prop. 3.1 of [2], it was proved that if two partial isometries V_1, V_2 verify $\|V_1 - V_2\| < 1$, then there exist unitaries U_1, U_2 such that $V_2 = U_1V_1U_2^*$. We may apply this result to $V_1 = P_{N(V)^\perp}$ and $V_2 = V$:

$$V = U_1P_{N(V)^\perp}U_2^*.$$

Note that

$$V^*V = U_2P_{N(V)^\perp}U_1^*U_1P_{N(V)^\perp}U_2^* = U_2P_{N(V)^\perp}U_2^*,$$

i.e. U_2 commutes with $P_{N(V)^\perp}$. Therefore $V = U_1U_2^*P_{N(V)^\perp}$.

There exists $X^* = X$ with $\|X\| \le \pi$ such that $U = e^{iX}$. Put $V(t) = e^{itX}P_{N(V)^\perp}$. Clearly $V(t)$ is smooth, $V(0) = P_{N(V)^\perp}$ and $V(1) = V$. Moreover, by the above lemmas, $e^{itX}(R(P)) \dot{+} N(P) = \mathcal{H}$. Since $e^{itX}(R(P)) = R(e^{itX}P) = R(V(t))$ and $N(P) = N(e^{itX}P) = N(V(t))$, this shows that $V(t) \in \mathcal{I}_0$.

Since also V lies in the same connected component of $P = P_{R(V)}$, then there exists a unitary operator W such that $V = P_{R(V)}W^*$. Thus $\|PW^* - P\| = \|WP - P\| < 1$. Then, by the lemma,

$$WR(P) \dot{+} N(P) = R(V^*) \dot{+} N(V^*) = \mathcal{H},$$

i.e. $V^* \in \mathcal{I}_0$, and thus $V \in \mathcal{I}_0$. The construction of $V(t)$ is similar as in the previous case. $\square$

The next result shows that if V is close enough to an arbitrary projection, then V lies in $\mathcal{I}_0$.

We shall use results from [2], concerning the structure of $\mathcal{I}$ as a homogeneous space of $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$, by means of the action

$$(U, W) \cdot V = UVW^*, \quad U, W \in \mathcal{U}(\mathcal{H}), \quad V \in \mathcal{I}_0.$$

For instance, it holds that if $V_0, V_1 \in \mathcal{I}$ verify that $\|V_1 - V_2\| < 1$, then there exist unitary operators γ, ν, which are polynomials in V_i, V_j^*, such that $V_2 = \gamma V_1 \nu^*$.

Theorem 3.5 *If V is a partial isometry and P is a projection such that $\|V - P\| < 1/3$, then $V \in \mathcal{I}_0$. Moreover, there is smooth curve $V(t) \in \mathcal{I}_0$ such that $V(0) = P$ and $V(1) = V$.*

Proof We recall the construction of the alluded γ and ν, for the case $V_1 = P$ and $V_2 = V$, simpler than in [2], because V_1 is a projection, and the distance between the partial isometries is less than $1/2$ (rather than less than 1). Put

$$P' = V^*V \quad \text{and} \quad Q' = VV^*.$$

Note that

$$\|P - P'\| \le \|V^*V - V^*P\| + \|V^*P - P\| \le \|V - P\| + \|V^* - P\| < \frac{2}{3} < 1.$$

Analogously $\|Q' - P\| < 1$. Projections at norm distance less that 1 are unitarily equivalent, and the unitaries can be chosen as smooth functions in terms of the projections (see for instance [13]). In this case, there are unitaries ν and σ such that

$$\nu P \nu^* = P' \quad \text{and} \quad \sigma P \sigma^* = Q'.$$

The cross section $\mu_P(V)$ in [2] performing $\mu_P(V) \cdot P = V$ is given by $\mu_P(V) = (\gamma, \nu)$, where γ is

$$\gamma = V\nu P + \sigma(1 - P).$$

Then

$$\|\nu^*\gamma P - P\| = \|\gamma P - \nu P\| = \|V\nu P - \nu P\|.$$

Note that $\nu P = P'\nu$, so that $V\nu P = VP'\nu = VV^*V\nu = V\nu$. Thus the term above equals

$$\begin{aligned}\|V\nu - \nu P\| &\le \|V\nu - p\nu\| + \|P\nu - \nu P\| = \|V - P\| + \|P - \nu P \nu^*\| \\ &\le 3\|V - P\| < 1.\end{aligned}$$

Then $\|\nu^*\gamma P - P\| < 1$, which by the above Lemma implies that

$$\mathcal{H} = \nu^*\gamma R(P) \dot{+} N(P) = \gamma R(P) \dot{+} \nu N(P).$$

Note that $\gamma R(P) = R(\gamma P \nu^*) = R(V)$ and $\nu N(P) = N(\gamma P \nu^*) = N(V)$, and then $V \in \mathcal{I}_0$.

Moreover

$$\|V - P_{N(V)}\| = \|\gamma P \nu^* - \nu P \nu^*\| = \|\nu^*\gamma P - P\| < 1,$$

which by the above result implies that V and $P_{N(V)}$ can be joined by a smooth curve inside $\mathcal{I}_0$. On the other hand, P and $P_{N(V)} = \nu P \nu^*$ can also be joined by a smooth curve inside the manifold of selfadjoint projections [13], which is a submanifold of $\mathcal{I}_0$. □

Corollary 3.6 *Let V_1, V_2 be partial isometries with $dim N(V_1) = dim N(V_2) = codim R(V_1) = codim R(V_2)$, and let P_1 and P_2 be projections such that $\|V_i - P_i\| \leq 1/3$ for $i = 1, 2$. Then V_i lie in the same connected component of $\mathcal{I}_0$*

Proof Both V_1 and V_2 lie in $\mathcal{I}_0$ by the above Proposition. Clearly the projections P_1 and P_2 are unitarily equivalent, therefore they can be joined by a continuous curve. On the other hand, the above Proposition also states that V_1 can be joined to P_1 by means of a continuous curve inside $\mathcal{I}_0$, and the same holds for V_2 and P_2. Thus V_1 and V_2 can be joined by a continuous curve inside $\mathcal{I}_0$. □

4 The Relationship with Normal Partial Isometries

In this section we study topologic properties of $\mathcal{I}_0$, for instance, we characterize the connected components. It will be useful to recall how the connected components of $\mathcal{I}$ [10] and $\mathcal{I}_N$ [2] are parametrized. The connected components of $\mathcal{I}$ are identified by three numbers $\iota, \kappa, \nu \in \mathbb{N}_0 \cup \{\infty\}$:

$$\mathcal{I}^{\nu}_{\iota,\kappa} = \{V \in \mathcal{I} : dim R(V) = \iota,\ dim N(V) = \kappa,\ dim R(V)^{\perp} = \nu\},$$

with the obvious restrictions (for instance, if $\iota < \infty$, then $\nu = \infty$, etc.). If V lies in $\mathcal{I}_N$ or in $\mathcal{I}_0$, apparently $\kappa = \nu$, therefore

$$\mathcal{I}_N \subset \mathcal{I}_0 \subset \cup_{\iota,\kappa} \mathcal{I}^{\kappa}_{\iota,\kappa}.$$

These balanced connected components $\mathcal{I}^{\kappa}_{\iota,\kappa}$, are characterized by the fact that they contain projections [2]: for each pair ι, κ, there is an orthogonal projection $P_{\iota,\kappa}$ (in fact, a whole connected component of projections) such that

$$\mathcal{I}^{\kappa}_{\iota,\kappa} = \{U P_{\iota,\kappa} W^* : U, W \in \mathcal{U}(\mathcal{H})\}.$$

An example of a non balanced isometry is the unilateral shift, or any isometry. In [2] it was shown that these numbers ι, κ parametrize the connected components of $\mathcal{I}_N$, more precisely, the connected components $(\mathcal{I}_N)_{\iota,\kappa}$ are:

$$(\mathcal{I}_N)_{\iota,\kappa} = \mathcal{I}_N \cap \mathcal{I}^{\kappa}_{\iota,\kappa}.$$

We shall see below that the same happens for $\mathcal{I}_0$.

In a previous work [1], the first two authors studied the geometry of the set $\mathcal{I}_N$ of normal partial isometries, i.e., partial isometries such that the initial space V^*V and the final space VV^* coincide. As remarked above, $\mathcal{I}_N \subset \mathcal{I}_0$ is a smooth submanifold. In this section we shall study the topological properties of $\mathcal{I}_0$ relating it to $\mathcal{I}_N$

Let us recall the following fact from the differential geometry of the space of projections, or Grassmannian of $\mathcal{H}$, denoted by $\mathcal{P}$, as developed by Corach, Porta and Recht [6,13]:

Remark 4.1 1. The tangent space $(T\mathcal{P})_{\mathcal{P}}$ of $\mathcal{P}$ at P consists of selfadjoint operators X which are co-diagonal with respect to P: $PXP = (1-P)X(1-P) = 0$.

2. The manifold $\mathcal{P}$ is a homogeneous space of $\mathcal{U}(\mathcal{H})$, by means of the action $U \cdot P = UPU^*$. If $P(t)$ is a curve of projections, the parallel transport $X(t)$ of a tangent vector X along $P(t)$, with $X(t_0) = X$, is given by

$$X(t) = \Gamma(t)X\Gamma(t)^*,$$

where $\Gamma(t)$ is the curve of unitaries obtained as the unique solution of the linear equation

$$\begin{cases} \dot{\Gamma}(t) = (\dot{P}(t)P(t) - P(t)\dot{P}(t))\Gamma(t) \\ \Gamma(t_0) = 1. \end{cases} \tag{5}$$

Additionally, the curve $\Gamma(t)$ lifts $P(t)$:

$$\Gamma(t)P(0)\Gamma(t)^* = P(t).$$

3. If P_0, P_1 are selfadjoint projections such that $\|P_0 - P_1\| < 1$ then there exists a unique $X \in \mathcal{B}(\mathcal{H})$ with $X^* = X$, $\|X\| < \pi/2$, which is P_0-codiagonal

$$P_0XP_0 = (1-P_0)X(1-P_0) = 0,$$

such that
   (a) $e^{iX}P_0e^{-iX} = P_1$.
   (b) The curve $\rho(t) = e^{itX}P_0e^{-itX}$, $t \in [0,1]$ is the shortest curve of projections joining P_0 and P_1 (among rectifiable curves).
   (c) If we fix P_0, the map which sends $P_1 \mapsto X$ is smooth. It is in fact the inverse of the exponential map of the Grassmann manifold.

Note that the second equivalent condition established in Theorem 2.2 states that if $V \in \mathcal{I}_0$ then

$$\|V^*V - VV^*\| < 1.$$

Therefore, by the above cited result, there exists a unique selfadjoint operator $X_V \in \mathcal{B}(\mathcal{H})$ such that

1. $\|X_V\| < \pi/2$.
2. X_V is V^*V-codiagonal.
3. $e^{iX_V}V^*Ve^{-iX_V} = VV^*$.
4. The map $V \mapsto X_V$ is smooth.

In particular, these conditions imply that the unitary e^{iX_V} maps $N(V)^\perp$ onto $R(V)$. It follows that $e^{-iX_V}V$ is a partial isometry with initial and final space $N(V)^\perp$, i.e. $e^{-iX_V}V \in \mathcal{I}_N$. If $A \in \mathcal{I}_N$, put $P_A = A^*A = AA^*$. Let us denote by

$$\mathcal{E} = \{(X, A) : A \in \mathcal{I}_N, X^* = X, \|X\| < \pi/2, X \text{ is co-diagonal with respect to } P_A\}.$$

Consider $\mathcal{E}$ with the topology induced by the norm in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. Therefore the following map is defined

$$\Delta : \mathcal{I}_0 \to \mathcal{E}, \quad \Delta(V) = (X_V, e^{-iX_V}V). \tag{6}$$

Theorem 4.2 *The map Δ is a homeomorphism.*

Proof Note that Δ is clearly continuous. We claim that its inverse is the map Π

$$\Pi : \mathcal{E} \to \mathcal{I}_0, \quad \Pi(X, A) = e^{iX}A.$$

Apparently Π is the restriction to $\mathcal{E}$ of a continuous map defined in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ with an identical formula. We must check first that Π takes values in $\mathcal{I}_0$. Put $V = \Pi(X, A) = e^{iX}A$. Then, using that $AA^* = A^*A = P_A$,

$$V^*V - VV^* = A^*A - e^{iX}AA^*e^{-iX} = \frac{1}{2}\{(2p_A - 1) - e^{iX}(2P_A - 1)e^{-iX}\}.$$

Since X is P_A co-diagonal, it is elementary to verify that X anti-commutes with $2P_A - 1$:

$$X(2P_A - 1) = -(2P_A - 1)X.$$

Thus $e^{iX}(2P_A - 1)e^{-iX} = e^{2iX}(2P_A - 1)$. It follows that

$$\|V^*V - VV^*\| = \frac{1}{2}\|(2P_A - 1)(1 - e^{2iX})\| = \frac{1}{2}\|1 - e^{2iX}\|,$$

where the last equality follows because $2P_A - 1$ is a unitary operator. As remarked at the beginning of Sect. 2, since $\|2X\| < \pi$, $\|1 - e^{2iX}\| < 2$ and thus $\|V^*V - VV^*\| < 1$, i.e. $V \in \mathcal{I}_0$. If $V \in \mathcal{I}_0$, it is apparent that $\Pi(\Delta(V)) = V$. Let $(X, A) \in \mathcal{E}$ and put $V = e^{iX}A$. Then $V^*V = A^*A$ and

$$VV^* = e^{iX}AA^*e^{-iX} = e^{iX}A^*Ae^{-iX} = e^{iX}V^*Ve^{-iX}.$$

Since X is $P_A = V^*V$-co-diagonal, and $\|X\| < \pi/2$, by the uniqueness property of the logarithm remarked above, it follows that $X_V = X$, and therefore $\Delta\Pi(X, A) = \Delta(V) = (X, e^{-iX}V) = (X, A)$. □

As recalled above, the connected components of $\mathcal{I}_N$ are parametrized by the projections: two normal partial isometries lie in the same connected component of $\mathcal{I}_N$ if and only if their final (=initial) projections are unitarily equivalent. Moreover, one has the following fact:

Proposition 4.3 *Let $P(t), t \in [0, 1]$ be a smooth curve of projections. Let $A_0, A_1 \in \mathcal{I}_N$ such that $A_i^*A_i = P(i)$ for $i = 0, 1$. Then there exists a continuous curve $A(t) \in \mathcal{I}_N$ such that $A^*(t)A(t) = A(t)A^*(t) = P(t)$, $A(0) = A_0$ and $A(1) = A_1$.*

Proof Let us construct a continuous (in fact it will be smooth) curve $A(t) \in \mathcal{I}_N$, $t \in [0, 1/2]$ with

$$A(0)=P(0),\ \ A(1/2)=P(1/2)\ \ \text{and}\ A^*(t)A(t)=A(t)A^*(t)=P(t),\ \ t \in [0, 1/2].$$

Let $\Gamma(t)$ be the solution of Eq. (5) with $\Gamma(0) = 1$. Then $\Gamma(t)$ lifts $P(t)$: $\Gamma(t)P(0)\Gamma^*(t) = P(t)$. The operator A_0 is a unitary operator in $R(P(0))$, thus there exists a selfadjoint operator X_0 which acts in $R(P(0))$, i.e. $P(0)X_0P(0) = X_0$, such that $A_0 = e^{iX_0}$. Since $\Gamma(t)$ lifts $P(t)$, it follows that $X_t = \Gamma(t)X_0\Gamma(t)^*$ acts in $R(P(t))$:

$$P(t)X_tP(t) = \Gamma(t)P(0)X_0P(0)\Gamma^*(t) = \Gamma(t)X_0\Gamma^*(t) = X_t.$$

It follows that $A(t) = P(t)e^{i(1-2t)X_t}$ is a smooth curve, such that for each $t \in [0, 1/2]$, $A(t)$ is a unitary in $R(P(t))$, or in other words, $A(t) \in \mathcal{I}_N$, with $A^*(t)A(t) = A(t)A^*(t) = P(t)$, such that $A(0) = A_0$ and $A(1/2) = P(1/2)$. Analogously, one constructs a smooth curve $A(t)$ for $y \in [1/2, 1]$ such that $A(t) \in \mathcal{I}_N$, $A^*(t)A(t) = A(t)A^*(t) = P(t)$, $A(1/2) = P(1/2)$ and $A(1) = A_1$. Adjoining both paths, one obtains a continuous path as required (in fact smooth, except eventually at $t = 1/2$). □

The next result shows that each connected component of $\mathcal{I}_0$ is the intersection of $\mathcal{I}_0$ with a component of $\mathcal{I}$.

Theorem 4.4 *If $V_0, V_1 \in \mathcal{I}_0$ lie in the same connected component of $\mathcal{I}$, then there is a smooth curve in $\mathcal{I}_0$ joining them.*

Proof Since $\mathcal{I}$ is a smooth submanifold of $\mathcal{B}(\mathcal{H})$ (see for instance [1]), if V_0, V_1 lie in the same connected component of $\mathcal{I}$, then there exists a smooth curve $V(t)$ in $\mathcal{I}$ with $V(0) = V_0$ and $V(1) = V_1$. Let $P(t) = V^*(t)V(t)$, which is a smooth curve in the Grassmannian $\mathcal{P}$. By the above theorem, to V_0 and V_1 correspond selfadjoint operators X_{V_0} and X_{V_1} and normal partial isometries A_0 and A_1. In particular, as seen above, X_{V_0}, being $V_0^* V_0 = P(0)$ co-diagonal, is a tangent vector in $(T\mathcal{P})_{P(0)}$. Let $X(t)$ be the parallel transport of X_{V_0} along the smooth curve $P(t)$, which consists of $P(t)$ co-diagonal selfadjoint operators. Note that since $X(t) = \Gamma(t) X_{V_0} \Gamma(t)^*$ and $\Gamma(t)$ are unitary operators,

$$\|X(t)\| = \|X_{V_0}\| < \pi/2.$$

Also note that $A_i^* A_i = A_i A_i^* = P(i)$, for $i = 0, 1$. By the above result on $\mathcal{I}_N$, there exists a smooth curve $A(t) \in \mathcal{I}_N$, with $A(0) = A_0$, $A(1) = A_1$ and

$$A(t)^* A(t) = A(t) A(t)^* = P(t).$$

Then the pairs $\alpha(t) = (X(t), A(t))$ form a continuous curve in $\mathcal{E}$, with initial point (X_{V_0}, A_0). At $t = 1$, $X(1)$ may be different than X_{V_1}, however both selfadjoint operators are $P(1)$ co-diagonal, and have norms less than $\pi/2$. Consider then the curve $\beta(t) = (tX_{V_1} + (1-t)X(1), A_1)$, $t \in [0, 1]$. The selfadjoint operators $tX_{V_1} + (1-t)X(1)$ are $P(1)$ co-diagonal, because X_{V_1} and X_1 are. Moreover,

$$\|tX_{V_1} + (1-t)X(1)\| \leq t\|X_{V_1}\| + (1-t)\|X(1)\| < \pi/2.$$

Therefore this curve $\beta(t)$ also lies in $\mathcal{E}$. Adjoining α and β one obtains a continuous curve in $\mathcal{E}$ which joins (X_{V_0}, A_0) and (X_{V_1}, A_1). By the above theorem, this induces a continuous curve in $\mathcal{I}_0$, joining V_0 and V_1. Since $\mathcal{I}_0$ is a submanifold of $\mathcal{B}(\mathcal{H})$, this implies the existence of a smooth curve joining them. □

In Corollary 3.4 it was shown that if $V \in \mathcal{I}$ and $\|V - P_{R(V)}\| < 1$, then $V \in \mathcal{I}_0$. The following corollary is related to his property.

Corollary 4.5 *If $V \in \mathcal{I}_0$, then V lies in the same component of $\mathcal{I}_0$ as $P_{R(V)}$. The same holds for $P_{N(V)^\perp}$.*

Proof Clearly V and $P_{R(V)}$ have the same range. They also have the same nullity. Indeed, the null-space of $P_{R(V)}$ is $R(V)^\perp = N(V^*)$. Since also $V^* \in \mathcal{I}_0$, $R(V^*) = N(V)^\perp$ is a supplement for this space. It follows that $N(V)$ and $N(V^*)$ have a common supplement, namely $N(V)^\perp$. Therefore $dim\ N(V) = dim\ N(V^*) = dim\ R(V)^\perp$. It follows that V and $P_{R(V)}$ lie in the same connected component of $\mathcal{I}$. Therefore, by the above theorem, they lie in the same connected component of $\mathcal{I}_0$. The proof for $P_{N(V)^\perp}$ is analogous. □

Consider the map

$$\rho : \mathcal{I}_0 \to \mathcal{I}_N, \quad \rho(V) = e^{-iX_V} V.$$

Note that, via the homeomorphism Δ, it corresponds to the projection onto the second coordinate:

$$\mathcal{E} \to \mathcal{I}_N, \quad (X, A) \mapsto A.$$

It is clearly a retraction. Therefore one may use it to compare the homotopy groups of $\mathcal{I}_0$ and $\mathcal{I}_N$. Note that the fibre over each element $A \in \mathcal{I}_N$ is contractible, it identifies with the open ball of radius $\pi/2$ of the Banach space

$$\{X \in \mathcal{B}(\mathcal{H}) : X^* = X,\, P_A X P_A = (1 - P_A)X(1 - P_A) = 0\}.$$

In [2] it was shown that if $A \in \mathcal{I}_N$, then

$$\pi_1(\mathcal{I}_N, A) \simeq \pi_1(\mathcal{U}(R(A))),$$

which is trivial if $dim R(A) > 1$, and equal to $\mathbb{Z}$ if $dim R(A) = 1$. Therefore:

Corollary 4.6 *If $V \in \mathcal{I}_0$, then*

$$\pi_1(\mathcal{I}_0, V) = 0 \quad \textit{if } \ dim R(V) > 1,$$

and

$$\pi_1(\mathcal{I}_0, V) = \mathbb{Z} \quad \textit{if}\, dim R(V) = 1.$$

In the case when both the range and the kernel are infinite dimensional, one can prove that $\mathcal{I}_0$ is contractible. In order to do so, let us recall from [2] the fibre bundle μ_P. If P is a projection, denote by

$$H_P = \mathcal{U}(R(P)) \times \mathcal{U}(N(P))$$

regarded as a subgroup of $\mathcal{U}(\mathcal{H})$ (i.e., the group of unitaries which commute with P). Note that each connected component of $\mathcal{I}_N$ (and therefore also of $\mathcal{I}_0$) contains selfadjoint projections. Let $\mathcal{I}_N^P$ be the connected component of $\mathcal{I}_N$ which contains P. Then

$$\mu_P : \mathcal{U}(\mathcal{H}) \times H_P \to \mathcal{I}_N^P, \quad \mu_P(U, \Omega) = U\Omega P U^*$$

is a locally trivial fiber bundle (Proposition 4.3 of [2]). The fibre is

$$\mathcal{F} = H_P \times \mathcal{U}(N(P)).$$

If both $N(P)$ and $R(P)$ are infinite dimensional, by Kuiper's theorem [7], $\mathcal{U}(\mathcal{H}), \mathcal{U}(N(P)), \mathcal{U}(R(P))$ (and therefore also H_P) are contractible. Therefore $\mathcal{I}_N^P$ has trivial homotopy group of all orders. By the remarks, the same happens for the connected component of P in $\mathcal{I}_0$. Then:

Corollary 4.7 *Let $V \in \mathcal{I}_0$ such that $R(V)$ and $R(V)^\perp$ (equivalently, $N(V)$ and $N(V)^\perp$) are infinite dimensional. Then the connected component of $\mathcal{I}_0$ containing V is contractible.*

Proof By Corollary 4.5, the connected component of $\mathcal{I}_0$ containing V, contains also $P_{R(V)}$. By the above remark, this connected component of $\mathcal{I}_0$ is homotopically equivalent to $\mathcal{I}_N^P$ for $P = P_{R(V)}$, which has infinite dimensional rank and nullity. Therefore this component $\mathcal{I}_0$ has trivial homotopy of all orders. Since it is a differentiable manifold, it is contractible by Palais's theorem [12]. □

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