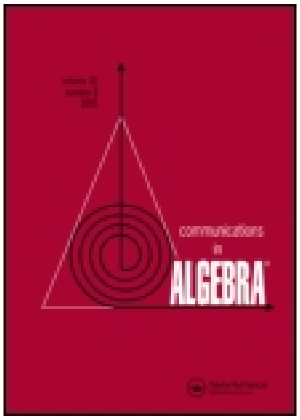

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EXTENSIONS OF JACOBSON'S LEMMA

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Jacobson's Lemma, relating different kinds of non singularity of $ca - 1$ and $ac - 1$, extends to $ba - 1$ whenever $aca = aba$.

Key Words: Common spectral properties; Jacobson's lemma; Operator equation.

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0. INTRODUCTION

Suppose A is a ring, with identity 1, or more generally an additive category: we shall write,

$$A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}, \tag{0.1}$$

for the invertible group, with

$$A^{-1}_{left} = \{a \in A : 1 \in Aa\}, \qquad A^{-1}_{right} = \{a \in A : 1 \in aA\}, \tag{0.2}$$

the left- and right-invertibles, and

$$A^{o}_{left} = \{a \in A : a^{-1}(0) = \{0\}\}, \qquad A^{o}_{right} = \{a \in A : a_{-1}(0) = \{0\}\}, \tag{0.3}$$

the monomorphisms and epimorphisms, with

$$a^{-1}(0) = \{x \in A : ax = 0\}, \qquad a_{-1}(0) = \{x \in A : ax = 0\},$$

respectively, the left and the right annihilator of $a \in A$.

In a Banach algebra these are the elements that are either not left zero divisors or not right zero divisors; in the category of bounded operators between Banach

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spaces these are the operators that are either not one one or not dense. Now *Jacobson's Lemma* [1, 5, 12] says that if $a, c \in A$ then

$$ac - 1 \in A^{-1} \Longleftrightarrow ca - 1 \in A^{-1}. \tag{0.4}$$

Indeed (0.4) holds separately for the left and the right invertibles of (0.2), as well as for the non zero-divisors of (0.3): for example, there is implication

$$c'(ac - 1) = 1 \Longrightarrow (cc'a - 1)(ca - 1) = 1. \tag{0.5}$$

The formula of (0.5) will also convert a right inverse, or a generalized inverse, for $ac - 1$ into one for $ca - 1$. In this note, we generalize (0.4) and many of its relatives from $ca - 1$ to certain $ba - 1$: specifically we will suppose

$$aba = aca. \tag{0.6}$$

Three special cases are of interest: the case

$$b = c, \tag{0.7}$$

which will give Jacobson's lemma; the case in which

$$aba = aca = a, \tag{0.8}$$

in which both b and c are generalized inverses of $a \in A$; and the case

$$aba = a^2, \tag{0.9}$$

in which $c = 1$. This last case goes back to Vidav [16], cf [2, 14, 15]; in particular, Schmoeger [14] shows that (0.9) holds if there are idempotents $p = p^2, q = q^2$ for which $a = qp, b = pq$.

The central results in this note are of course pure algebra: but in the neighboring realm of topological algebra they have very close relatives, and we take the opportunity to extend our purely algebraic observations to their topological analogues.

1. INVERTIBILITY

Jacobson's Lemma is primarily about invertibility, covering both left, right, and indeed generalized invertibility. The proof of our extension involves one specfic act of proof, and then a curious logical syllogism:

Theorem 1. *If* $a, b, c \in A$ *satisfy* (0.6) *then*

$$ac - 1 \in A^{-1} \Longleftrightarrow ba - 1 \in A^{-1} \tag{1.1}$$

and

$$ca \in A^{-1}_{left} \Longleftrightarrow ba \in A^{-1}_{left}, \qquad ac \in A^{-1}_{right} \Longleftrightarrow ab \in A^{-1}_{right}. \tag{1.2}$$

Proof. Towards (1.1) we claim

$$ac - 1 \in A^{-1}_{left} \Longrightarrow ba - 1 \in A^{-1}_{left}, \tag{1.3}$$

and conversely

$$ba - 1 \in A^{-1}_{left} \Longrightarrow ac - 1 \in A^{-1}_{left}. \tag{1.4}$$

The basic act of proof is (1.3): if $c' \in A$ then in the presence of (0.6) there is implication

$$\begin{aligned} c'(ac-1) &= 1 \Longrightarrow a = c'(ac-1)a = c'a(ba-1) \\ &\Longrightarrow ba - 1 = bc'a(ba-1) - 1 \Longrightarrow 1 = (bc'a - 1)(ba-1). \end{aligned} \tag{1.5}$$

This applies when $c = b$ and then continues to hold after interchanging a and b: this in particular gives Jacobson's Lemma (0.5). Interchanging b and c in (1.3), and also in (0.5), now completes (1.4). The analogue of (1.3) for right invertibility follows by reversal of multiplication, applied however to the converse (1.4), after interchange of c and b.

For the first part of (1.2), we observe, in the presence of (0.6),

$$(ca)^2 \in Aba. \tag{1.6}$$

The converse is a simple interchange of b and c, and then the second part is reversal of products.

Alternatively, for (1.2), notice that if (0.6) holds and either ca or ba is left invertible, then $ca = ba$.

Theorem 1 is familiar [1, 6, 14] when $c = b$ (0.7), and is obtained by Schmoeger [14, 15] when $c = 1$ (0.9). We have not been able to extend Theorem 1 from semi invertibility to "regularity," in the sense of having a generalized inverse. We cannot interchange ca and ac in (1.2): if for example $a = u$ and $b = c = v$ with

$$vu = 1 \neq uv, \tag{1.7}$$

then ca is invertible while ac is neither left nor right invertible. We also cannot interchange A^{-1}_{left} and A^{-1}_{right}, and, hence, replace them both by the invertible group A^{-1}, in (1.2): for example if (1.7) holds then

$$(a = v, b = 1, c = uv) \Longrightarrow (aba = v^2 = aca, (ba)u = 1, (1 - uv)(ca) = 0). \tag{1.8}$$

The same example shows that we cannot replace (1.3) by inclusion

$$A(ac - 1) \subseteq A(ba - 1). \tag{1.9}$$

2. MONOMORPHISM

The analogue of Theorem 1 holds for mono- and epimorphisms:

Theorem 2. *If* (0.6) *holds then*

$$ac - 1 \in A^o_{left} \Longleftrightarrow ba - 1 \in A^o_{left}, \tag{2.1}$$

and

$$ac - 1 \in A^o_{right} \Longleftrightarrow ba - 1 \in A^o_{right}. \tag{2.2}$$

Also

$$ca \in A^o_{left} \Longleftrightarrow ba \in A^o_{left} \tag{2.3}$$

and

$$ac \in A^o_{right} \Longleftrightarrow ab \in A^o_{right}. \tag{2.4}$$

Proof. The basic act of proof here is forward implication in (2.1). If $x \in A$ then

$$(ba - 1)x = 0 \Longrightarrow \big(x = bax \text{ and } (ac - 1)ax = a(ba - 1)x = 0\big). \tag{2.5}$$

Thus,

$$a(ba - 1)^{-1}(0) \subseteq (ac - 1)^{-1}(0), \tag{2.6}$$

and, hence,

$$(ac - 1)^{-1}(0) \Longrightarrow (ba - 1)^{-1}(0).$$

It follows ([1, Proposition 2])

$$(ba - 1)^{-1}(0) \subseteq b(ac - 1)^{-1}(0). \tag{2.7}$$

This, in particular, establishes forward implication in (2.1). Now the same logic as for Theorem 1 now supplies the backward implication. Also forward implication in (2.2) follows from (2.1) by "reversal of products." Finally, for forward implication in (2.3), it follows from (1.6) that

$$(ba)^{-1}(0) \subseteq (ca)^{-2}(0), \tag{2.8}$$

while

$$(ca)^{-1}(0) = \{0\} \Longrightarrow (ca)^{-2}(0) = \{0\}.$$

□

If (0.6) holds and either ca or ba is monomorphic then again it follows $ca = ba$.

3. TOPOLOGICAL ZERO DIVISORS

If A is a normed algebra, or category, then the non zero divisors of (0.3) can be replaced by non topological zero divisors. We recall that a linear mapping $T : X \to Y$ between normed spaces is said to be *bounded below* if there is $k > 0$ for which

$$\| \cdot \| \leq k \| T(\cdot) \|, \tag{3.1}$$

and that in a normed algebra $a \in A$ is a *left topological zero divisor*, or a *right topological zero divisor* precisely when the left multiplication, or right multiplication,

$$L_a : x \mapsto ax, \qquad R_a : x \mapsto xa$$

is not bounded below. Evidently for $T : X \to Y$ there is implication

$$left\ invertible \implies bounded\ below \implies one\ one.$$

We shall write

$$A^{\bullet}_{left} = \{a \in A : L_a : \quad \text{bounded below}\}, \qquad A^{\bullet}_{right} = \{a \in A : R_a : \quad \text{bounded below}\}; \tag{3.2}$$

evidently

$$A^{-1}_{left} \subseteq A^{\bullet}_{left} \subseteq A^{o}_{left}, \tag{3.3}$$

and similarly with "right" in place of "left." Thus, there is also a "quantitative" version of Theorem 2:

Theorem 3. *If* (0.6) *holds then*

$$ac - 1 \in A^{\bullet}_{left} \iff ba - 1 \in A^{\bullet}_{left} \tag{3.4}$$

and

$$ac - 1 \in A^{\bullet}_{right} \iff ba - 1 \in A^{\bullet}_{right}. \tag{3.5}$$

Also,

$$ca \in A^{\bullet}_{left} \iff ba \in A^{\bullet}_{left} \tag{3.6}$$

and

$$ac \in A^{\bullet}_{right} \iff ab \in A^{\bullet}_{right}. \tag{3.7}$$

Proof. The act of proof is forward implication in (3.4): if $k > 0$ there is implication

$$\| \cdot \| \leq k \| (ac - 1)(\cdot) \| \implies \| \cdot \| \leq (\|b\| k \|a\| + 1) \| (ba - 1)(\cdot) \| \tag{3.8}$$

and

$$\|\cdot\| \leq k\|(ac)(\cdot)\| \Longrightarrow \|\cdot\| \leq k^2\|c\| \, \|a\|\|(ba)(\cdot)\|; \tag{3.9}$$

for example, if for arbitrary $x \in A$ we have $\|x\| \leq k\|(ac-1)x\|$ then for arbitrary x

$$\|ax\| \leq k\|(ac-1)ax\| \leq k\|a\| \, \|(ba-1)x,$$

and, hence,

$$\|x\| \leq \|b\| \, \|ax\| + \|(ba-1)x\| \leq (\|a\|k\|b\|+1)\|(ba-1)x\|.$$

$\square$

Alternatively passage from A to the "enlargement" ([5, Definition 1.9.2])

$$\mathbf{Q}(A) = \ell_\infty(A)/c_0(A) \tag{3.10}$$

has the effect of recognizing topological zero divisors $a \in A$ as giving zero divisors $\mathbf{Q}(a) \in \mathbf{Q}(A)$. The details of the construction are unimportant: all that matters is ([5] Theorem 3.3.5) that if $T : X \to Y$ then

$$T \textit{ bounded below } \Longrightarrow \mathbf{Q}(T) \textit{ one one } \Longrightarrow \mathbf{Q}(T) \textit{ bounded below.} \tag{3.11}$$

It follows that if $a \in A$ then

$$a \in A^{\bullet}_{left} \Longleftrightarrow \mathbf{Q}(a) \in \mathbf{Q}(A)^{o}_{left}. \tag{3.12}$$

Thus, Theorem 3 is a consequence of Theorem 2 applied to the enlargement.

4. SURJECTIVITY

The analogue of (1.4) and (2.2) hold, for linear operators, with right invertibility, or epimorphisms, replaced by the property of being "surjective," or onto:

Theorem 4. *If* (0.6) *holds with $a : X \to Y$ and $b, c : Y \to X$ then*

$$(ca-1)X = X \Longleftrightarrow (ab-1)Y = Y. \tag{4.1}$$

Proof. If (0.6) holds then

$$b^{-1}(ca-1)X \subseteq (ab-1)Y, \qquad (ca-1)X \subseteq a^{-1}(ab-1)Y. \tag{4.2}$$

Indeed,

$$\begin{aligned} by = (ca-1)x &\Longrightarrow aby = a(ca-1)x = (ab-1)ax \\ &\Longrightarrow y = aby - (ab-1)y = (ab-1)(ax-y) \end{aligned}$$

and

$$x = (ca - 1)w \Longrightarrow ax = a(ca - 1)w = (ab - 1)aw;$$

now the first part of (4.2) gives forward implication in (4.1). $\square$

Alternatively, this follows by applying (2.2) to the category of all linear operators on linear spaces. However, we can "quantify" Theorem 4 to give the analogue for "openness" between normed spaces, and then "almost openness" ([5, Definition 3.4.1]):

Theorem 5. *If* (0.6) *holds there is implication*

$$ca - 1 \text{ relatively open } \Longrightarrow ab - 1 \text{ relatively open}. \tag{4.3}$$

Proof. Here $a : X \to Y$ is said to be "relatively open" if there is $k > 0$ for which

$$\forall\, x \in X\ \exists\, x' \in X\ :\ ax = ax' \quad \text{with} \|x'\| \leq k\ \|ax\|; \tag{4.4}$$

thus, "open" means relatively open and onto. Following the argument of (4.2) suppose $ca - 1$ is relatively open: then there is $k > 0$ for which

$$by = (ca - 1)x \Longrightarrow by = (ca - 1)x' \text{ with } \|x'\| \leq k\|by\|.$$

It now follows

$$y \in (ab - 1)Y \Longrightarrow y = aby - (ab - 1)y = (ab - 1)(ax' - y)$$

with

$$\|ax' - y\| \leq \|a\|\ \|x'\| + \|y\| \leq (\|a\|k\|b\| + 1)\|y\|.$$

$\square$

Of course on Banach spaces openness and almost openness revert to the property of being onto. Between normed spaces an operator $a : X \to Y$ is dense iff its dual $a^* : Y^* \to X^*$ is one one, and is almost open iff its dual is bounded below: thus, we can also derive "right" nonsingularity results from "left."

It is clear that (4.2) continues to hold if we replace the ranges of $ca - 1$ and $ab - 1$ by their closures. It is not, however, clear that the closed range property transfers:

$$(ac - 1)Y = \mathrm{cl}\ (ac - 1)Y \Longleftrightarrow (ba - 1)X = \mathrm{cl}\ (ba - 1)X? \tag{4.5}$$

Certainly if $(ac - 1)Y$ is closed in Y then

$$x = \lim_n (ba - 1)x_n \Longrightarrow ax = \lim_n (ac - 1)ax_n = (ac - 1)y,$$

giving $x = b(ac - 1)y - (ba - 1)x$. In the special case (0.8), with $b = c$, it now follows ([1] Theorem 5)

$$x = (ca - 1)(cy - x).$$

5. FREDHOLM THEORY

When $A = B(X)$ is the bounded operators on a Banach space, or indeed the category of all bounded operators, then Theorem 1 becomes a theorem about invertibility of operators; if instead A is either a Calkin algebra or the "Calkin category" then it is a theorem about being Fredholm. Thus, the analogue of Theorem 1 holds for left and right Fredholmness. The analogue of Theorem 1 also holds for upper semi-Fredholmness: this follows from Theorem 2 together with ([5, Definition 5.7.4]) an "essential" version of the enlargement,

$$\mathbf{P}(A) = \ell_\infty(A)/m(A),$$

for which

$$T \textit{ upper semi Fredholm } \Longleftrightarrow \mathbf{P}(T) \textit{ one one}. \tag{5.1}$$

If $ac - 1$ is Fredholm, in the category of bounded linear operators on Banach spaces, so that with (0.6) also $ba - 1$ is Fredholm, then (2.7) shows that in addition $ac - 1$ and $ba - 1$ have the same *nullity*, and then dually the same *defect*, and of course the same *index*: thus, we learn that if (0.6) holds then,

$$ac - 1 \textit{ Weyl} \Longleftrightarrow ba - 1 \textit{ Weyl}. \tag{5.2}$$

This does not appear to survive in a more abstract context:

Theorem 6. *If $T : A \to D$ is a (unital) homomorphism, or more generally an additive functor, and if $a, b, c \in A$ satisfy (0.6), then*

$$ac - 1 \textit{ left or right } T \textit{ Fredholm} \Longleftrightarrow ba - 1 \textit{ left or right } T \textit{ Fredholm}. \tag{5.3}$$

Proof. We say that $a \in A$ is "T Fredholm" when $Ta \in D$ is invertible; now apply Theorem 1 to Ta, Tb, Tc in D. □

Theorem 6 hardly needed stating, but enables us to observe that the corresponding result for "T Weyl" is not clear. Suppose that (0.6) holds and that

$$ac - 1 = e + u \in A^{-1}_{left} + T^{-1}(0) \textit{ with } e'e - 1 = 0 = T(u) : \tag{5.4}$$

then, as from the argument for Theorem 1,

$$1 = (be'a - 1)(ba - 1) - be'ua, \quad \Longrightarrow (be'a - 1)(ba - 1) \in 1 + T^{-1}(0) \subseteq T^{-1}D^{-1}. \tag{5.5}$$

In other words, all we learn is that $ba - 1$ is left T Fredholm.

We have also been unable to decide whether, in the presence of (0.6), if $a \in \mathrm{cl}\, A^{-1}$ then there is implication

$$ac - 1 \in \mathrm{Exp}(A) \Longleftrightarrow ba - 1 \in \mathrm{Exp}(A). \tag{5.6}$$

Here

$$\mathrm{Exp}(A) = \{e^{c_1} e^{c_2} \dots e^{c_n} : n \in \mathbf{N}, c \in A^n\} \tag{5.7}$$

is the subgroup of A^{-1} generated by the exponentials, which coincides with the connected component of the identity. When $c = b$ this is a result of Murphy ([12, Proposition 4.3]).

6. DRAZIN INVERTIBILITY

On the other hand, "Drazin invertibility" transfers: we recall [8, 10] that if

$$ba^2 = a = a^2 c, \tag{6.1}$$

then (0.8) holds: there is equality $ba = ac$ and in fact bac is a "group inverse" for $a \in A$. This motivates:

Theorem 7. *If $a, b, c \in A$ satisfy* (0.6) *then there is implication*

$$ac - 1 \in A(ac - 1)^2 \Longleftrightarrow ba - 1 \in A(ba - 1)^2 \tag{6.2}$$

and

$$ac - 1 \in (ac - 1)^2 A \Longleftrightarrow ba - 1 \in (ba - 1)^2 A. \tag{6.3}$$

Proof. We argue, for forward implication in (6.2),

$$\begin{aligned} ac - 1 \;&=\; c'(ac-1)^2 \Longrightarrow a(ba-1) = c'(ac-1)^2 a = c'a(ba-1)^2 \\ &\Longrightarrow ba(ba-1) = bc'a(ba-1)^2 \;\Longrightarrow\; ba - 1 = (bc'a - 1)(ba-1)^2. \end{aligned}$$

□

More generally $a \in A$ is *Drazin invertible* if some power a^k has a group inverse. To extend Theorem 7, we argue (cf [4, Lemma 2.1]) that if (0.6) holds and $f \in \mathrm{Poly}$ is a polynomial

$$f(ac)a = af(ba) \;\; : \tag{6.4}$$

note that (6.4) is clear for constants and the coordinate, and transfers to sums and products of polynomials. Now if $k \in \mathbf{N}$ we can argue, extending Theorem 2.2 of [4],

$$(ac-1)^k = c'(ac-1)^{k+1} \Longrightarrow a(ba-1)^k = (ac-1)^k a = c'a(ba-1)^{k+1},$$

giving

$$(ba-1)^k = ba(ba-1)^k - (ba-1)^{k+1} = (bc'a-1)(ba-1)^{k+1}.$$

7. SPECTRAL THEORY

If A is a real or a complex linear algebra (or category) then (1.3) immediately implies, for arbitrary non zero scalar λ,

$$ac - \lambda \in A^{-1}_{left} \Longrightarrow ba - \lambda \in A^{-1}_{left}, \tag{7.1}$$

and, similarly, (2.1) implies

$$ac - \lambda \in A^{o}_{left} \Longrightarrow ba - \lambda \in A^{o}_{left}. \tag{7.2}$$

If we now define the *spectrum* of $a \in A$ by setting

$$\sigma(a) = \sigma^{left}(a) \cup \sigma^{right}(a), \tag{7.3}$$

where

$$\sigma^{left}(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{left}\}, \qquad \sigma^{right}(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{right}\} \tag{7.4}$$

then Theorem 1 can be restated in terms of the spectrum:

Theorem 8. *If* (0.6) *holds then*

$$\sigma(ac)\backslash\{0\} = \sigma(ba)\backslash\{0\}. \tag{7.5}$$

Proof. (7.5) holds separately for the left and the right spectrum; inclusion one way follows from the implication (1.3) together with (7.1) and (7.2), and the logical syllogism of that argument converts this inclusion to equality. □

Theorem 8 has obvious analogues in which the spectrum is replaced by "point" and "approximate point" spectrum. From Theorem 8 it follows that, in the presence of (0.6), there is equality

$$\mathrm{acc}\ \sigma(ac) = \mathrm{acc}\ \sigma(ba), \tag{7.6}$$

where we write acc(K) for the *accumulation points* of $K \subseteq \mathbf{C}$. Considering the status of the point $0 \in \mathbf{C}$, it is now clear that if (0.6) holds then either neither or both ac and ba have a *Koliha-Drazin inverse*.

8. LOCAL SPECTRA

Theorems 1 and 2 have analogues in which invertibility or injectivity is replaced by *local one-one-ness*, also known as the "single valued extension property" [10, p. 14; 13, p. 139]. We shall say that $a \in A$ is *locally one-one* [7, 8] if there is implication

$$(a-z)f(z) \equiv 0 \Longrightarrow f(z) \equiv 0, \tag{8.1}$$

whenever $f : U \to A$ is holomorphic on an open neighborhood U of $0 \in \mathbf{C}$. Thus, $a \in A$ has the single valued extension property at $\lambda \in \mathbf{C}$ if and only if ([7] Theorem 9) $a - \lambda \in A$ is locally one-one. The local analogue of Theorem 2 makes no distinction between zero and non zero points:

Theorem 9. *If A is a Banach linear category and if* (0.6) *holds then, for arbitrary* $\lambda \in \mathbf{C}$,

$$ac - \lambda \text{ locally one} - \text{one} \iff ba - \lambda \text{ locally one} - \text{one}. \tag{8.2}$$

Proof. We can virtually copy out the proof of (2.1): writing $z : \mathbf{C} \to \mathbf{C}$ for the complex coordinate we have near $\lambda \in \mathbf{C}$,

$$\begin{aligned}(ba - z)f(z) \equiv 0 \Longrightarrow (ac - z)af(z) \equiv a(ba - z)f(z) \equiv 0 \Longrightarrow af(z) \\ \equiv 0 \Longrightarrow zf(z) \equiv baf(z) \equiv 0.\end{aligned}$$

When $\lambda = 0$ this argument works on a deleted neighbourhood. □

Theorem 9 applied to the enlargement $\mathbf{Q}(A)$ gives [8] something very close to the analogue of Theorem 2 for "Bishop's property (β)."

In the category of bounded operators we shall call $y \in Y$ a *holomorphic range point* of $a : X \to X$ if there exists $f : U \to X$ holomorphic on an open neighbourood of $0 \in \mathbf{C}$ for which

$$(a - z)f(z) \equiv y \; ; \tag{8.3}$$

the set $a^{\omega}(X)$ of its holomorphic range points is called the *transfinite range* or "coeur analytique" of $a : X \to X$. With this notation the intersection $a^{\omega}(X)_{\cap}a^{-1}(0)$, known [7, 8] as the *holomorphic kernel*, vanishes if and only if a is locally one one. Now we can replace the ranges of $ca - 1$ and $ab - 1$ in (4.1) by their holomorphic ranges; similarly, we can replace the null spaces in (2.7) by holomorphic kernels.

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