The Images of the Clique Operator and Its Square are Different

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Abstract: Let $\mathcal{G}$ be the class of all graphs and K be the clique operator. The validity of the equality $K(\mathcal{G}) = K^2(\mathcal{G})$ has been an open question for several years. A graph in $K(\mathcal{G})$ but not in $K^2(\mathcal{G})$ is exhibited here. © 2013 Wiley Periodicals, Inc. J. Graph Theory 77: 39–57, 2014

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1 INTRODUCTION

Dealing with clique graphs is not an easy task and most of the problems about them prove complicated. One example is the problem of clique graph recognition: given a graph, determine whether it is a clique graph or not. This problem has been proved to be NP-complete [1] and thus, given the state of affairs, no efficient clique graph recognition algorithm is known.

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Working on iterated clique graphs (graphs obtained by applying the clique operator to a graph more than once) might be even more difficult and not much is known about techniques to determine whether a graph is an iterated clique graph or not.

Let $\mathcal{G}$ be the class of all graphs, K be the clique operator and K^2 be the composition of K with itself. In view of the previous paragraph, it is not surprising to learn that it was unknown whether $K(\mathcal{G}) = K^2(\mathcal{G})$. The main goal of this paper is to prove that O_4, the clique graph of the octahedral graph O_3, is in $K(\mathcal{G})$ but not in $K^2(\mathcal{G})$, thus establishing the falseness of the equality.

For that purpose, some definitions and basic properties are given in Section 2, and the graphs in $K^{-1}(O_4)$ are described in Section 3 thanks to the fact that the octahedron is an induced subgraph of every graph in $K^{-1}(O_4)$. A demonstration of the equality $K^{-1}(O_4) \cap K(\mathcal{G}) = \emptyset$ that uses the terminology developed in Section 4 follows in Section 5, thus establishing that $O_4 \notin K^2(\mathcal{G})$.

2 DEFINITIONS, BASICS, AND GOALS

For a simple graph G, the set of vertices of G is denoted by $V(G)$, and $E(G)$ denotes the set of its edges. A graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. The subgraph *induced* by a subset A of $V(G)$, denoted by $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G. We say that G' is an *induced subgraph* of G if there exists a subset A of $V(G)$ such that $G[A] = G'$.

For a vertex $v \in V(G)$, the *(closed) neighborhood* of v, denoted by $N[v]$ or $N_G[v]$, is the set composed of v and the vertices adjacent to it. If w is another vertex and $N[v] = N[w]$, then we say that v and w are *twins*, symbolized by $v \sim w$.

Let $\mathcal{F}$ be a family of nonempty sets of vertices of a graph. If $F \in \mathcal{F}$, then F is called a *member* of $\mathcal{F}$. If $v \in \bigcup_{F \in \mathcal{F}} F$, then we say that v is a *vertex* of $\mathcal{F}$. The family $\mathcal{F}$ is *Helly* if the intersection of all the members of every subfamily of pairwise intersecting sets is not empty. The *intersection graph* of $\mathcal{F}$, denoted $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint.

Let A be a set of vertices of $\mathcal{F}$. The notation $A \ll \mathcal{F}$ is used to indicate that there exists $F \in \mathcal{F}$ such that $A \subseteq F$. Similarly, if $v_1 ... v_n$ is a sequence of vertices of $\mathcal{F}$, then $v_1 ... v_n \ll \mathcal{F}$ denotes that $\{v_1, ..., v_n\} \ll \mathcal{F}$. Since the Helly property will be used very frequently, it is suitable to state the following simple proposition:

Proposition 2.1. *Let $\mathcal{F}$ be a Helly family and $E_1, ..., E_n$ be pairwise intersecting sets such that $E_i \ll \mathcal{F}$ for $1 \leq i \leq n$. Then, there exists a vertex v such that $E_i \cup \{v\} \ll \mathcal{F}$ for $1 \leq i \leq n$.*

Proof. Let $F_1, ..., F_n$ be members of $\mathcal{F}$ such that $E_i \subseteq F_i$ for $1 \leq i \leq n$. Then, $F_1, ..., F_n$ are pairwise intersecting sets. Since $\mathcal{F}$ is Helly, there exists a vertex v such that $v \in F_i$ for all i between 1 and n. Therefore, $E_i \cup \{v\} \subseteq F_i$ and hence $E_i \cup \{v\} \ll \mathcal{F}$ for every value of i such that $1 \leq i \leq n$. ■

A subset C of $V(G)$ is *complete* if its elements are pairwise adjacent vertices. A *clique* is defined to be a maximal complete set. The family of all the cliques of G is denoted by $\mathcal{C}(G)$. If $\mathcal{C}(G)$ is Helly, then we say that G is a *clique-Helly* graph.

The *clique graph* of G is defined as the intersection graph of $\mathcal{C}(G)$. Let $\mathcal{G}$ be the class of all graphs. The function $K : \mathcal{G} \to \mathcal{G}$ assigning to each graph its clique graph is called the *clique operator*. Every graph in $K(\mathcal{G})$ is called a *clique graph*.

The most classical characterization of clique graphs is due to Roberts and Spencer:

Theorem 2.2 ([6]). *Let G be a graph. Then, G is a clique graph if and only if there exists a family $\mathcal{F}$ of complete sets of G satisfying the following properties:*

1. *$\mathcal{F}$ covers all the edges of G, that is, for all $vw \in E(G)$, we have $vw \ll \mathcal{F}$.*
2. *$\mathcal{F}$ is Helly.*

Define the *two-section graph* $S(\mathcal{F})$ of a family $\mathcal{F}$ as the graph whose vertex set is $\bigcup_{F \in \mathcal{F}} F$, where v and w are adjacent in $S(\mathcal{F})$ if and only if there exists $F \in \mathcal{F}$ such that $\{v, w\} \subseteq F$. This concept allows us to rewrite Theorem 2.2 as follows:

Theorem 2.3. *Let G be a graph. Then, G is a clique graph if and only if there exists a Helly family $\mathcal{F}$ such that $S(\mathcal{F}) = G$.*

For a given graph G, we use $K^{-1}(G)$ instead of $K^{-1}(\{G\})$ to denote the set of all graphs that have G as their clique graph (up to isomorphism).

Call $\mathcal{F}$ a *separating family* if, for every ordered pair (v, w) of vertices of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $v \in F$ and $w \notin F$. The following characterization of $K^{-1}(G)$ can be given:

Theorem 2.4 ([3]). *Let G be a clique graph. Then, $K^{-1}(G)$ is composed of all the graphs of the form $L(\mathcal{F})$, where $\mathcal{F}$ is a Helly and separating family such that $S(\mathcal{F}) = G$.*

The exponential notation K^n will indicate the composition of the clique operator with itself n times, where n is a natural number larger than 0. The operator K^0 is set equal to the identity function on $\mathcal{G}$. For $i \geq 0$, the *i-th iterated clique graph* of G is defined to be the graph $K^i(G)$. Thus, $K^i(\mathcal{G})$ is the class of *i-th iterated clique graphs*.

The following result can be easily derived from the definitions and basic set theory:

Proposition 2.5. *Let G be a graph. Then, $G \in K^2(\mathcal{G})$ if and only if $K^{-1}(G) \cap K(\mathcal{G}) \neq \emptyset$.*

For $n \geq 3$, define the *n-dimensional octahedron* O_n as the graph such that $V(O_n) = \{1, 2, ..., 2n\}$ and $E(O_n) = \{ij : i \neq j \wedge |i - j| \neq n\}$. If $v \in V(O_n)$, then the definition implies that $N[v]$ fails to contain only one vertex of the graph, called here the *opposite* of v, which will be denoted by v' in the sequel. For example, upon considering the definition of O_3 or its picture in Fig. 1, we infer that $1' = 4$, $2' = 5$, $3' = 6$, $4' = 1$, $5' = 2$, and $6' = 3$. Back to the general case, it can also be inferred that O_n has a total of 2^n cliques, each containing n vertices.

As an example of application of Theorem 2.3, let us prove by contradiction that O_3 is not a clique graph. Suppose to the contrary that there exists a Helly family $\mathcal{F}$ of sets of vertices of O_3 such that $S(\mathcal{F}) = O_3$. Since 12, 13 and 23 are edges of O_3, these edges are in the two-section graph of $\mathcal{F}$. If $\{1, 2, 3\}$ were not in $\mathcal{F}$, then $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ should belong to $\mathcal{F}$ to ensure that 12, 13, and 23 are edges of $S(\mathcal{F})$. But $\mathcal{F}$ would not be Helly in this case because $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are pairwise intersecting and the intersection of the three is empty. Therefore, $\{1, 2, 3\}$ is necessarily in $\mathcal{F}$.

We conclude from the same reasoning that $\{2, 4, 6\}$ and $\{3, 4, 5\}$ are in $\mathcal{F}$. Thus, we obtain a new contradiction, because $\{1, 2, 3\}$, $\{2, 4, 6\}$, and $\{3, 4, 5\}$ are pairwise intersecting but the intersection of the three is empty.

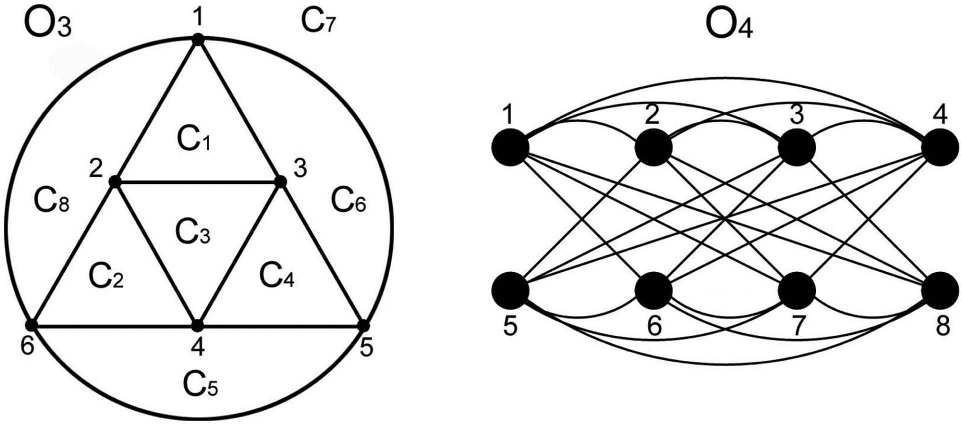


FIGURE 1. O_3 and its clique graph O_4. The cliques of O_3 are labeled.

Therefore, a family like $\mathcal{F}$ cannot exist and hence O_3 is not a clique graph.

It is not difficult to verify that $K(O_3) = O_4$. Therefore, $O_4 \in K(\mathcal{G})$. It is interesting to note that O_3 is an induced subgraph of O_4. In Fig. 1, the set $\{1, 2, 3, 5, 6, 7\}$ of vertices of O_4 induces O_3. This fact reveals that it is possible for a graph with a nonclique induced subgraph to be a clique graph. In other words, the class of clique graphs is not hereditary.

The problem this paper considers is the validity of the equality $K(\mathcal{G}) = K^2(\mathcal{G})$. This problem has been open for at least 15 years, with one early paper and one early mention about it appearing in [3, 7]. It is clear that $K^2(\mathcal{G}) \subseteq K(\mathcal{G})$, but determining whether the other inclusion is true or not is the difficult part.

If the equality were true, then the main conclusion would be that the concepts of clique graph and iterated clique graph are equivalent. To show this, note that for $n \geq 1$ we would have

$$K^n(\mathcal{G}) = K^{n-1}(K(\mathcal{G})) = K^{n-1}(K^2(\mathcal{G})) = K^{n+1}(\mathcal{G})$$

so, by induction, $K^n(\mathcal{G}) = K(\mathcal{G})$ for all n larger than or equal to 1.

However, the objective of this paper is to prove that the equality is false, result that is stated below as a theorem.

Theorem 2.6 (Difference Theorem)**.** $K(\mathcal{G}) \neq K^2(\mathcal{G})$.

Most researchers who considered the problem were inclined to think that $K(\mathcal{G}) \neq K^2(\mathcal{G})$ and there were some partial results in that respect. For example, it was known that, assuming the Difference Theorem to be true, every graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ should have at least eight vertices [4]. In fact, one of the candidates to be a graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ was O_4, which has just eight vertices. However, all the attempts to prove that $O_4 \in K(\mathcal{G}) \setminus K^2(\mathcal{G})$ were unsuccessful, this being one of the main reasons why there are not any further papers on the problem. Nor can a reference to a paper where it is conjectured that $O_4 \in K(\mathcal{G}) \setminus K^2(\mathcal{G})$ be given.

One of the main difficulties that existed to approach the problem was the fact that (even now) there exists no known algorithm to decide whether a graph is in $K^2(\mathcal{G})$. There are two alternatives for a graph G not to be in $K^2(\mathcal{G})$. One is that G is not a clique graph. This possibility is generally difficult to evaluate due to the NP-completeness of the clique graph recognition problem. Otherwise, when it is known that G is in $K(\mathcal{G})$, there exists an infinite number of graphs having their clique graphs equal to G. In this case, it is necessary to prove that none of these graphs is a clique graph to ensure that $G \notin K^2(\mathcal{G})$. So, if proving that a single graph is not a clique graph is usually complicated, then dealing with an infinite family of graphs makes things much worse.

The Difference Theorem will be proved by establishing that O_4 is indeed in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$. In view of Proposition 2.5, a proof of the following theorem will suffice:

Theorem 2.7. $K^{-1}(O_4) \cap K(\mathcal{G}) = \emptyset$.

That is why the next section is about describing the graphs in $K^{-1}(O_4)$.

3 THE GRAPHS IN $K^{-1}(O_4)$

In order to study the structure of the graphs in $K^{-1}(O_4)$, it will be highly useful to prove that every graph in $K^{-1}(O_4)$ has O_3 as an induced subgraph. In view of Theorem 2.4, it is sufficient to verify that O_3 is an induced subgraph of the intersection graph of every Helly family $\mathcal{F}$ such that $S(\mathcal{F}) = O_4$. Recall that, by the definition of two-section graph, the condition $S(\mathcal{F}) = O_4$ implies that the elements of every $F \in \mathcal{F}$ are pairwise adjacent vertices of O_4.

We will first consider those $\mathcal{F}$ that are minimal in the sense that no proper subfamily of $\mathcal{F}$ has O_4 as its two-section graph. From Lemma 3.1 to Lemma 3.5, the families considered will be assumed to satisfy this minimality.

Lemma 3.1. *No set with exactly two vertices is a member of $\mathcal{F}$.*

Proof. Suppose to the contrary that a and b are adjacent vertices of O_4 such that $\{a, b\} \in \mathcal{F}$. As every clique of O_4 has size four, we can take another vertex c that is adjacent to both a and b. Since $S(\mathcal{F}) = O_4$, it follows that $ab \ll \mathcal{F}$, $ac \ll \mathcal{F}$ and $bc \ll \mathcal{F}$. Thus, by Proposition 2.1, there exists a vertex d such that $abd \ll \mathcal{F}$, $acd \ll \mathcal{F}$ and $bcd \ll \mathcal{F}$. If $d \neq a$ and $d \neq b$, then the fact that $abd \ll \mathcal{F}$ implies that $S(\mathcal{F}) = S(\mathcal{F} \setminus \{\{a, b\}\})$. This contradicts the minimality of $\mathcal{F}$. Similarly, when $d = a$ or $d = b$, the contradiction arises from the fact that $bcd \ll \mathcal{F}$ or $acd \ll \mathcal{F}$, respectively.

Therefore, $\{a, b\} \notin \mathcal{F}$. ■

Lemma 3.2. *No set with exactly three vertices is a member of $\mathcal{F}$.*

Proof. Suppose to the contrary that a, b, and c are distinct vertices of O_4 such that $\{a, b, c\} \in \mathcal{F}$. Hence, a and b are both adjacent to c. By the structure of O_4, both a and b are adjacent to c' (the opposite of c) as well. Since $S(\mathcal{F}) = O_4$, it follows that $ab \ll \mathcal{F}$, $ac' \ll \mathcal{F}$, and $bc' \ll \mathcal{F}$. Thus, by Proposition 2.1, there exists a vertex d such that $abd \ll \mathcal{F}$, $ac'd \ll \mathcal{F}$, and $bc'd \ll \mathcal{F}$. It is clear that $d \neq c$ because $ac'd \ll \mathcal{F}$ and c is not adjacent to c'. If $d \neq a$ and $d \neq b$, then it follows from $abd \ll \mathcal{F}$ that ab is also an edge of $S(\mathcal{F} \setminus \{\{a, b, c\}\})$. The same conclusion can be obtained from $bc'd \ll \mathcal{F}$ or $ac'd \ll \mathcal{F}$ in case that $d = a$ or $d = b$, respectively.

We can prove in a similar fashion that ac and bc are edges of $S(\mathcal{F} \setminus \{\{a, b, c\}\})$. Thus, $S(\mathcal{F}) = S(\mathcal{F} \setminus \{\{a, b, c\}\})$, which contradicts the minimality of $\mathcal{F}$.

Therefore, $\{a, b, c\}$ cannot be a member of $\mathcal{F}$. ■

Corollary 3.3. $|F| = 4$ *for all* $F \in \mathcal{F}$.

Lemma 3.4. *Let* F *and* F' *be two members of* $\mathcal{F}$. *Then,* $|F \cap F'|$ *is even.*

Proof. We first prove that $|F \cap F'| \neq 1$.

Suppose to the contrary that $F = \{a, b, c, d\}$ and that $F \cap F' = \{a\}$, that is, $F' = \{a, b', c', d'\}$. Take $F'' \in \mathcal{F}$ such that $\{a', b\} \subseteq F''$. Then, neither c' nor d' can be elements of F''; otherwise F, F', and F'' would be pairwise intersecting sets such that $F \cap F' \cap F'' = \emptyset$. Therefore, by Corollary 3.3, $F'' = \{a', b, c, d\}$.

As $abcd \ll \mathcal{F}$, $ab'c'd' \ll \mathcal{F}$ and $b'c \ll \mathcal{F}$, we deduce from Proposition 2.1 that there exists a vertex v such that $abcdv \ll \mathcal{F}$, $ab'c'd'v \ll \mathcal{F}$ and $b'cv \ll \mathcal{F}$. Since $abcdv \ll \mathcal{F}$, it follows that v cannot be equal to a', b', c', or d'. Similarly, v cannot be equal to b, c, or d because $ab'c'd'v \ll \mathcal{F}$. Therefore $v = a$, which yields that $ab'c \ll \mathcal{F}$.

If we now consider that $abcd \ll \mathcal{F}$, $ab'c'd' \ll \mathcal{F}$, and $bc' \ll \mathcal{F}$, then Proposition 2.1 yields that $abc' \ll \mathcal{F}$. It is a consequence of similar reasonings that $abd' \ll \mathcal{F}$.

Let $\mathcal{F}' = \{H \in \mathcal{F} : H = F'' \vee \{a, b', c\} \subseteq H \vee \{a, b, c'\} \subseteq H \vee \{a, b, d'\} \subseteq H\}$. The members of this subfamily are pairwise intersecting. As $\mathcal{F}$ is Helly, there exists a common vertex v for all the members of $\mathcal{F}'$. Since $\bigcup_{F \in \mathcal{F}'} F = V(O_4)$ and all the members of $\mathcal{F}'$ are complete sets of O_4, we have that $N[v] = V(O_4)$. By the structure of O_4, there is no vertex with that neighborhood, so we have a contradiction.

Therefore, $|F \cap F'| \neq 1$.

Now suppose that $|F \cap F'| = 3$, with $F = \{a, b, c, d\}$ and $F' = \{a, b, c, d'\}$. Let F'' be a member of $\mathcal{F}$ such that $\{a', b'\} \subseteq F''$. If $F'' = \{a', b', c', d\}$ or $F'' = \{a', b', c, d'\}$, then $|F \cap F''| = 1$. If $F'' = \{a', b', c, d\}$ or $F'' = \{a', b', c', d'\}$, then $|F' \cap F''| = 1$. Both cases contradict the previous part of the proof.

Therefore, $|F \cap F'|$ equals 0, 2, or 4. ■

Lemma 3.5. *Let* $\{a, b, c, d\}$ *be a member of* $\mathcal{F}$. *Then, the set of opposites* $\{a', b', c', d'\}$ *is also a member of* $\mathcal{F}$.

Proof. Suppose to the contrary that $\{a', b', c', d'\} \notin \mathcal{F}$. Since $a'b' \ll \mathcal{F}$, $a'c' \ll \mathcal{F}$, and $a'd' \ll \mathcal{F}$ and, by the previous lemma, the intersection between $\{a, b, c, d\}$ and every member of $\mathcal{F}$ containing $\{a', b'\}$, $\{a', c'\}$, or $\{a', d'\}$ must have even cardinality, $\{a', b', c, d\}, \{a', c', b, d\}, \{a', d', b, c\} \in \mathcal{F}$. These three sets and $\{a, b, c, d\}$ form a subfamily of $\mathcal{F}$ of pairwise intersecting sets with no common vertex, thus contradicting that $\mathcal{F}$ is Helly.

Therefore, $\{a', b', c', d'\} \in \mathcal{F}$. ■

It is now possible to prove the first major result.

Theorem 3.6. *Let* G *be a graph such that* $K(G) = O_4$. *Then,* O_3 *is an induced subgraph of* G.

Proof. By Theorem 2.4, there exists a Helly family $\mathcal{F}$ such that $L(\mathcal{F}) = G$ and $S(\mathcal{F}) = O_4$. Among all the subfamilies of $\mathcal{F}$ with two-section graph equal to O_4, take $\mathcal{F}'$ minimal with respect to inclusion. The case that $\mathcal{F} = \mathcal{F}'$ is not discarded.

Let $\{a, b, c, d\}$ be a member of $\mathcal{F}'$. Then, by Lemma 3.5, $\{a', b', c', d'\} \in \mathcal{F}'$, and Lemma 3.4 implies that every other member of $\mathcal{F}'$ has exactly two elements in common with $\{a, b, c, d\}$. We can suppose without loss of generality that $\{a, b, c', d'\} \in \mathcal{F}'$, so $\{a', b', c, d\} \in \mathcal{F}'$ as well.

Let F be a member of $\mathcal{F}'$ such that $\{a, b'\} \subseteq F$. Since $|\{a, b, c, d\} \cap F| = 2$, it follows that $F = \{a, b', c, d'\}$ or $F = \{a, b', c', d\}$.

If $F = \{a, b', c, d'\}$, then $\{a', b, c', d\} \in \mathcal{F}'$ and $\{a, b, c, d\}$,$\{a', b', c', d'\}$,$\{a, b, c', d'\}$, $\{a', b', c, d\}$,$\{a, b', c, d'\}$, $\{a', b, c', d\}$ induce O_3 as a subgraph of $L(\mathcal{F})$. The reasoning is similar if $F = \{a, b', c', d\}$.

Therefore, O_3 is an induced subgraph of G. ■

It is interesting to note that, in case that $G \neq O_3$, Theorem 3.6 tells us that G has an induced subgraph with fewer vertices, namely O_3, whose clique graph also equals O_4.

Now that we know that every graph in $K^{-1}(O_4)$ has O_3 as an induced subgraph, let us see what effects this fact has on the structure of the cliques of the graphs in $K^{-1}(O_4)$.

Let G be any graph such that $K(G) = O_4$ and let V' be a subset of $V(G)$ such that $G[V'] = O_3$. Define the *clique-to-clique* function f as follows: for each clique C of $G[V']$, let $f(C)$ be a clique of G containing C. The next property will be very useful for us:

Proposition 3.7. *The clique-to-clique function f is a bijection between $\mathcal{C}(G[V'])$ and $\mathcal{C}(G)$.*

Proof. Let C and C' be two different cliques of $G[V']$. Then, there exists a vertex u such that $u \in C \setminus C'$ because otherwise we would have that $C \subseteq C'$. As u is not in C', there must be a vertex $v \in C'$ that is not adjacent to u, because otherwise $C' \cup \{u\}$ would be a complete set larger than C'. It follows that $u \in f(C)$ and $v \in f(C')$, thus forcing $f(C)$ and $f(C')$ to be different. Hence, f is a one to one function from $\mathcal{C}(G[V])$ to $\mathcal{C}(G)$.

Since the clique graphs of $G[V']$ and G are both equal to O_4, we have that $|\mathcal{C}(G[V'])| = |\mathcal{C}(G)| = 8$. Therefore, f is a bijection. ■

In words, Proposition 3.7 means that the cliques of G are obtained as an extension of the cliques of $G[V']$ (that extension is not necessary for cliques of $G[V']$ that are also cliques of G). Some important structural consequences of this fact are found below:

Proposition 3.8. *Let G be a graph in $K^{-1}(O_4)$ and V' be a subset of $V(G)$ such that $G[V'] = O_3$ and v, w be two vertices of G. Then:*

(a) $N[v] \cap V' \neq \emptyset$.
(b) *If* $N[v] \cap V' \subseteq N[w] \cap V'$*, then* $N[v] \subseteq N[w]$.
(c) *If* $N[v] \cap V' = N[w] \cap V'$*, then* $v \sim w$.
(d) $N[v] \cap V' \neq V'$.

Proof. It is a consequence of Proposition 3.7 that every clique of G contains vertices of V'. Let C be a clique of G containing v. Thus, $C \cap V' \neq \emptyset$ implies $N[v] \cap V' \neq \emptyset$. This proves (a).

Now suppose that $N[v] \cap V' \subseteq N[w] \cap V'$ and let u be any element of $N[v]$. Take $C \in \mathcal{C}(G)$ such that $\{u, v\} \subseteq C$. Let f be the clique-to-clique function of the previous proposition. Since $f^{-1}(C) \subseteq N[v] \cap V' \subseteq N[w] \cap V'$, we infer that $f^{-1}(C) \cup \{w\}$ is a complete set of G. Let C' be a clique of G such that $f^{-1}(C) \cup \{w\} \subseteq C'$. This inclusion and the definition of f imply that $f^{-1}(C) \cup f^{-1}(C') \subseteq C'$. Hence, $f^{-1}(C) \cup f^{-1}(C')$ is a complete set of G, and of $G[V']$ as well. Furthermore, $f^{-1}(C)$ and $f^{-1}(C')$ are cliques of $G[V']$. The maximality involved in the definition of clique and the inclusions $f^{-1}(C) \subseteq f^{-1}(C) \cup f^{-1}(C')$ and $f^{-1}(C') \subseteq f^{-1}(C) \cup f^{-1}(C')$ imply that $f^{-1}(C) = f^{-1}(C') = f^{-1}(C) \cup f^{-1}(C')$. Hence $C = f(f^{-1}(C)) = f(f^{-1}(C')) = C'$. Since we knew that $u \in C$ and $w \in C'$, we have that $\{u, w\} \subseteq C$. Thus $u \in N[w]$.

It follows that $N[v] \subseteq N[w]$, which proves part (b). Part (c) is a direct consequence of (b).

In order to prove (d), suppose to the contrary that $N[v] \cap V' = V'$. We can use part (b) to deduce that $N[u] \subseteq N[v]$ for all $u \in V(G)$, that is, $N[v] = V(G)$. Thus, v is in every clique of G and $K(G)$ is a complete graph, which contradicts that $K(G) = O_4$. Therefore, $N[v] \cap V' \neq V'$. ■

Proposition 3.8, part (c) especially, will play a very important role in the proof of Theorem 2.7.

4 CLASSIFYING FAMILIES OF CLIQUES OF O_3

The information obtained from Section 3 leaves us in a good position to prove that $O_4 \notin K^2(\mathcal{G})$. Since the proof will involve many structures of the graph O_3, it is appropriate to have this section to define them and show some of their characteristics.

A *castle* is defined to be a subfamily $\mathcal{A}$ of $\mathcal{C}(O_3)$ such that $|\mathcal{A}| = 3$ and $|C \cap C'| = 1$ for every pair C, C' of distinct members of $\mathcal{A}$. Castles can always be characterized as in the next proposition:

Proposition 4.1. *Let $\mathcal{A}$ be a castle of O_3. Then, there exists $C' \in \mathcal{C}(O_3)$ such that $\mathcal{A} = \{C \in \mathcal{C}(O_3) : |C \cap C'| = 2\}$.*

Proof. Let C_1, C_2, C_3 be the members of $\mathcal{A}$ and v_1, v_2, v_3 be vertices such that $v_1 \in C_1 \cap C_2$, $v_2 \in C_1 \cap C_3$, and $v_3 \in C_2 \cap C_3$.

Suppose that $v_1 = v_2$. Then, $v_1 \in C_1 \cap C_2 \cap C_3$ and, by the inclusion-exclusion principle, we can get that $|C_1 \cup C_2 \cup C_3| = 7$, thus contradicting that O_3 has only six vertices. Therefore, $v_1 \neq v_2$. Similarly, $v_i \neq v_j$ for $i \neq j$.

It also holds that $\{v_1, v_2, v_3\}$ is complete. Since every clique of O_3 has exactly three vertices, $\{v_1, v_2, v_3\}$ is a clique, call it C'. It is easy to verify that the family $\{C \in \mathcal{C}(O_3) : |C \cap C'| = 2\}$ consists of three cliques. As C_1, C_2, C_3 are members of this family, the equality in the statement of this proposition follows. ■

Proposition 4.1 is useful for counting purposes. It is a consequence of it that O_3 has eight different castles, that is, there are as many castles as cliques O_3 has.

A family $\mathcal{A}$ of cliques is said to be *castled* if it contains a castle as a subfamily. We have the following result regarding castled families:

Proposition 4.2. *Let $\mathcal{A}$ be a subfamily of $\mathcal{C}(O_3)$ such that $|\mathcal{A}| \geq 5$. Then, $\mathcal{A}$ is castled.*

Proof. We prove the contrapositive.

Suppose that $\mathcal{A}$ is not castled. For each $C \in \mathcal{C}(O_3)$, let $\mathcal{A}_C = \{C' \in \mathcal{C}(O_3) : |C \cap C'| = 2\} \cap \mathcal{A}$. In words, the members of $\mathcal{A}_C$ are the cliques that are in $\mathcal{A}$ and share two vertices with C.

Since each clique shares two vertices with three other cliques in O_3, we have that $|\mathcal{A}| = \frac{1}{3} \sum_{C \in \mathcal{C}(O_3)} |\mathcal{A}_C|$.

Note that $|\mathcal{A}_C| \neq 3$ for every $C \in \mathcal{C}(O_3)$; otherwise $\mathcal{A}_C$ would be a castle, thus contradicting that $\mathcal{A}$ is not castled. We now consider two cases:

- $|\mathcal{A}_C| \leq 1$ for all $C \in \mathcal{C}(O_3)$:
  Since O_3 has eight cliques, the formula we found for $|\mathcal{A}|$ yields that $|\mathcal{A}| \leq \frac{8}{3}$.

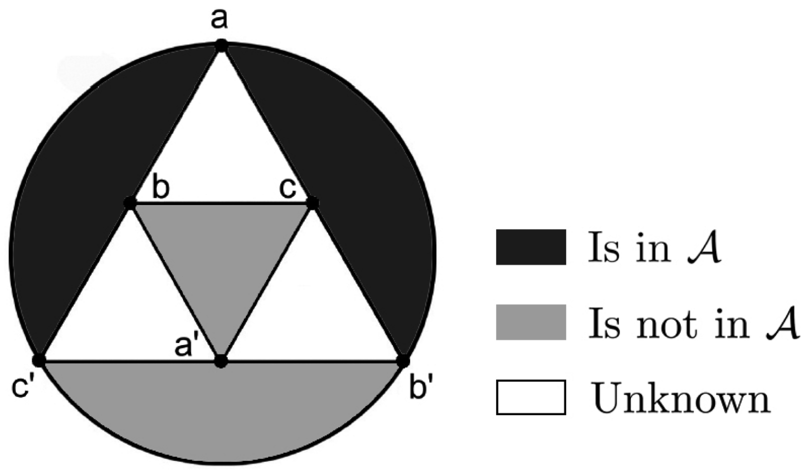


FIGURE 2. A graphical representation of the second case of the proof of Proposition 4.2.

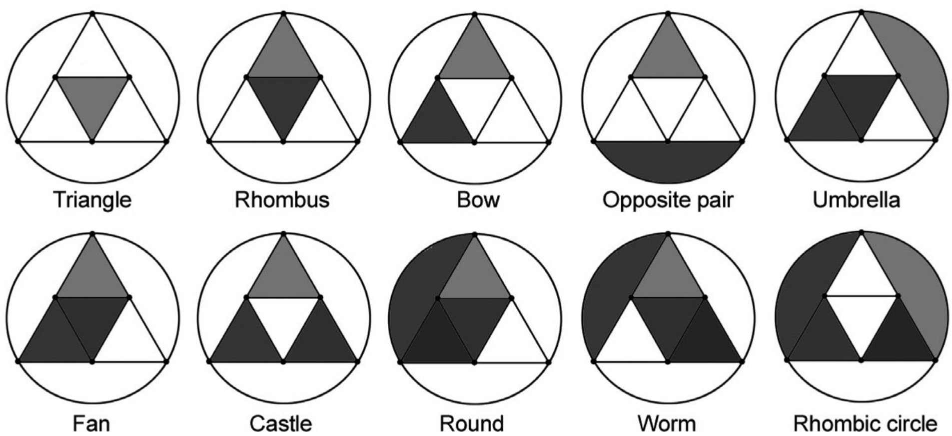


FIGURE 3. Graphical representation of the families defined in this section.

- There exists $C \in \mathcal{C}(O_3)$ such that $|\mathcal{A}_C| = 2$:
  Let $C = \{a, b, c\}$ and suppose without loss of generality that $\mathcal{A}_C = \{\{a, b', c\}, \{a, b, c'\}\}$. This equality implies that $\{a', b, c\}$ is not in $\mathcal{A}$. Furthermore, $\{a', b', c'\}$ is not in $\mathcal{A}$, otherwise this clique together with $\{a, b', c\}$ and $\{a, b, c'\}$ would form a castle contained in $\mathcal{A}$. Thus, $\mathcal{A}_{\{a',b',c\}} = \{a, b', c\}$ and $\mathcal{A}_{\{a',b,c'\}} = \{a, b, c'\}$ (see Fig. 2). Hence $|\mathcal{A}_{\{a',b',c\}}| = |\mathcal{A}_{\{a',b,c'\}}| = 1$.
  We can now use the formula for $|\mathcal{A}|$ to obtain that $|\mathcal{A}| \leq \frac{1}{3}(2.6 + 1.2) = \frac{14}{3}$.

In either case, $|\mathcal{A}| < 5$, as desired. ■

Proposition 4.2 is equivalent to stating that every noncastled subfamily of $\mathcal{C}(O_3)$ has at most four members. The following classification for non-castled subfamilies is proposed (see also Figure 3):

We just say that $\mathcal{A}$ is *empty* if $|\mathcal{A}| = 0$. It is a *triangle* if $|\mathcal{A}| = 1$.

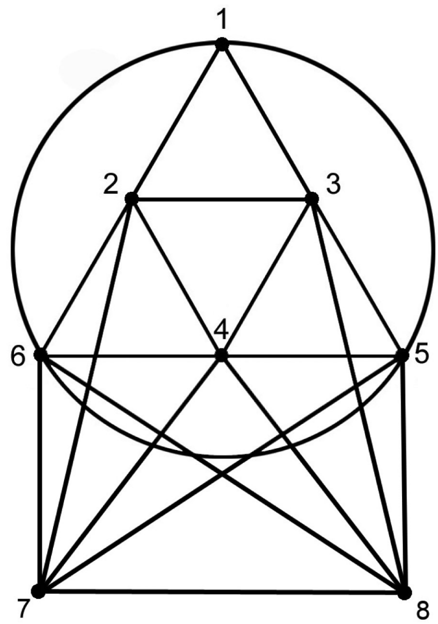


FIGURE 4. A graph in $K^{-1}(O_4)$. The set {1,2,3,4,5,6} induces O_3. The families $\mathcal{N}_7$ and $\mathcal{N}_8$ are both rhombi. Vertices 7 and 8 are adjacent because {4, 5, 6} is in $\mathcal{N}_7 \cap \mathcal{N}_8$.

In case that $|\mathcal{A}| = 2$, with $\mathcal{A} = \{C, C'\}$, the subfamily $\mathcal{A}$ is a *rhombus* if $|C \cap C'| = 2$, is a *bow* if $|C \cap C'| = 1$ or is an *opposite pair* if $C \cap C' = \emptyset$.

In case that $|\mathcal{A}| = 3$, the subfamily $\mathcal{A}$ is an *umbrella* if it contains an opposite pair or is a *fan* if the intersection of all its members is nonempty.

In case that $|\mathcal{A}| = 4$, we say that $\mathcal{A}$ is a *round* if the intersection of all its members is nonempty; $\mathcal{A}$ is a *worm* if its members can be listed in such a way that two of them are consecutive if and only if they share two vertices; and $\mathcal{A}$ is a *rhombic circle* if it contains two distinct opposite pairs.

The terminology that has just been introduced allows a complete and quite short characterization of $K^{-1}(O_4)$:

Lemma 4.3. *The collection $\mathfrak{C}$ of all triangles, rhombi, and rounds of O_3 is Helly.*

Proof. For each clique $\{x, y, z\}$ of O_3, let $\mathcal{T}_{xyz}$ be the triangle whose only member is that clique.

There exists a correspondence between the edges of O_3 and its rhombi. For each $xy \in E(O_3)$, let $\mathcal{R}_{xy}$ be the rhombus $\{\{x, y, z\}, \{x, y, z'\}\}$, where z is one of the vertices of O_3 adjacent to both x and y.

Finally, for each $x \in V(O_3)$, let $\mathcal{R}_x$ be the round such that the intersection of all its members is $\{x\}$.

Thus $\mathfrak{C} = \{\mathcal{T}_{xyz}\}_{\{x,y,z\}\in\mathcal{C}(O_3)} \cup \{\mathcal{R}_{xy}\}_{xy\in E(O_3)} \cup \{\mathcal{R}_x\}_{x\in V(O_3)}$.

It is not difficult to prove that the intersection of two members of $\mathfrak{C}$ is not empty if and only if we cannot find opposite vertices in their indexes. Furthermore, given a clique C

of O_3 and a member of $\mathfrak{C}$, we have that C is in that member if and only if C contains all the vertices in the index of the member.

Let $\mathfrak{C}'$ be a subcollection of $\mathfrak{C}$ whose members are pairwise intersecting. By the previous paragraph, the set of vertices appearing in at least one index of a member of $\mathfrak{C}'$ is a complete set of O_3. Let $\{a, b, c\}$ be a clique of O_3 containing that complete set. Then, $\{a, b, c\}$ is in all the members of $\mathfrak{C}'$.

Therefore, $\mathfrak{C}$ is Helly. ■

Proposition 4.4. *Let G be a graph and V' be a subset of $V(G)$ such that $G[V'] = O_3$. Then, $K(G) = O_4$ if and only if the following two conditions are satisfied:*

1. *For all $v \in V(G)$, the family $\mathcal{N}_v$ of cliques of $G[V']$ contained in $N[v]$ is a triangle, a rhombus or a round.*
2. *For all $v, w \in V(G)$ such that $v \neq w$, we have that v and w are adjacent in G if and only if $\mathcal{N}_v \cap \mathcal{N}_w \neq \emptyset$.*

Proof. Assume that $K(G) = O_4$. Let v be a vertex of G and f be the clique-to-clique function. Take a clique $C \in \mathcal{C}(G)$ such that $v \in C$. Then $f^{-1}(C) \subseteq N[v]$ and hence $\mathcal{N}_v$ is not empty.

If $|\mathcal{N}_v| = 1$, then $\mathcal{N}_v$ is a triangle.

Now suppose that $|\mathcal{N}_v| = 2$. Then, $\mathcal{N}_v$ cannot be a bow, since the set of vertices in at least one member of a fixed bow contains three cliques of $G[V']$. Nor can it be an opposite pair, because that would make v contradict part (d) of Proposition 3.8. Therefore, $\mathcal{N}_v$ is a rhombus.

Otherwise, by the same reasons as in the previous paragraph, $\mathcal{N}_v$ cannot be an umbrella, a fan, a worm, a rhombic circle or be castled. Hence, $\mathcal{N}_v$ is a round.

Consider two different vertices v and w of G. Suppose that $\mathcal{N}_v \cap \mathcal{N}_w \neq \emptyset$. Let $C \in \mathcal{N}_v \cap \mathcal{N}_w$. Thus, $f(C)$ is a clique of G containing v and w, so the two vertices are adjacent. Conversely, suppose that v and w are adjacent and let C' be a clique of G containing both vertices. Then, $f^{-1}(C') \in \mathcal{N}_v \cap \mathcal{N}_w$. Therefore, $\mathcal{N}_v \cap \mathcal{N}_w \neq \emptyset$.

Conversely, suppose that conditions 1 and 2 of the proposition are satisfied. For every $C \in \mathcal{C}(G[V'])$, let $C' = \{v \in V(G) : C \in \mathcal{N}_v\}$. The next step is to prove that $\mathcal{C}(G) = \{C' : C \in \mathcal{C}(G[V'])\}$.

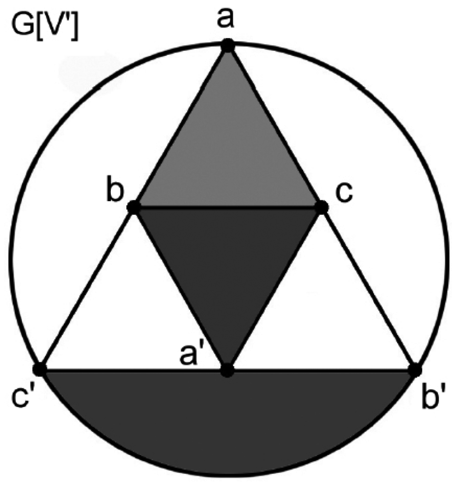
Let C_1 be a clique of $G[V']$. It is clear from condition 2 that C_1' is a complete set of vertices of G. Let C_2 be another clique of $G[V']$. Since for $i \in \{1, 2\}$ no vertex of $V' \setminus C_i$ is adjacent to all the vertices of C_i, it holds that $C_i' \cap V' = C_i$. Thus, $C_1' \neq C_2'$ and neither contains the other.

Let D be a clique of G. By conditions 1 and 2, $\{\mathcal{N}_v\}_{v \in D}$ is a collection whose members are pairwise intersecting and such that every member is a triangle, a rhombus or a round of $G[V']$. By Lemma 4.3, there exists $C \in \mathcal{C}(G[V'])$ that is in every member of $\{\mathcal{N}_v\}_{v \in D}$. Therefore, $D \subseteq C'$. As D is a clique of G, it follows that $C' = D$.

We conclude from the previous paragraph that $\mathcal{C}(G)$ is a subfamily of $\{C' : C \in \mathcal{C}(G[V'])\}$. Moreover, we already know that no member of $\{C' : C \in \mathcal{C}(G[V'])\}$ contains another member. Therefore, $\mathcal{C}(G) = \{C' : C \in \mathcal{C}(G[V'])\}$.

Now, given cliques C_1 and C_2 of $G[V']$, we prove that $C_1 \cap C_2 \neq \emptyset$ if and only if $C_1' \cap C_2' \neq \emptyset$. Since $C_i \subseteq C_i'$ for $i \in \{1, 2\}$, it is true that $C_1 \cap C_2 \neq \emptyset$ implies $C_1' \cap C_2' \neq \emptyset$.

Now suppose that $C_1' \cap C_2' \neq \emptyset$. Let $v \in C_1' \cap C_2'$. Then, $C_1' \cup C_2' \subseteq N[v]$ and hence $C_1 \cup C_2 \subseteq N[v]$.



$$F_1 = \{a, a', b, c\},\ F_2 = \{b, c'\},\ F_3 = \{a', b', c'\}$$

FIGURE 5. If $\mathcal{F} = \{\{a, a', b, c\}, \{b, c'\}, \{a', b'.c'\}\}$, then $\mathcal{C}(\mathcal{F}, V')$ is an umbrella composed of the cliques $\{a, b, c\}$, $\{a', b, c\}$, and $\{a', b', c'\}$. The first two cliques are contained in F_1 and the last clique is contained in F_3.

If $C_1 \cap C_2 = \emptyset$, then $C_1 \cup C_2 = V'$. This implies that $V' \subseteq N[v]$, which contradicts part (d) of Proposition 3.8. Therefore, $C_1 \cap C_2 \neq \emptyset$.

We can now conclude that $K(G) = K(G[V'])$. Therefore, $K(G) = K(O_3) = O_4$. ■

An example of a graph in $K^{-1}(O_4)$, and how it satisfies the conditions of Proposition 4.4, appears in Figure 4. Although Proposition 4.4 tells us in full detail what the graphs of $K^{-1}(O_4)$ are like, we will not use it for the proof of Theorem 2.7. Proposition 3.8 will be enough for our purposes.

5 PROOF OF THEOREM 2.7

We will prove that $K^{-1}(O_4) \cap K(\mathcal{G}) = \emptyset$ by contradiction.

Let us suppose to the contrary that there exists a graph G in $K(\mathcal{G}) \cap K^{-1}(O_4)$. As $G \in K(\mathcal{G})$, there exists by Theorem 2.3 a Helly family $\mathcal{F}$ of sets of vertices of G such that $S(\mathcal{F}) = G$. On the other side, as $G \in K^{-1}(O_4)$, we have that O_3 is an induced subgraph of G.

For each set $V' = \{a, a', b, b', c, c'\}$ such that $G[V'] = O_3$, let $\mathcal{C}(\mathcal{F}, V') = \{C \in \mathcal{C}(G[V']) : C \ll \mathcal{F}\}$ (see example in Fig. 5). We also choose, for each clique $\{x, y, z\}$ of $G[V']$, a vertex v_{xyz} that is in every member of $\mathcal{F}$ that contains $\{x, y\}$, $\{x, z\}$, or $\{y, z\}$. Such a vertex can always be found because $\mathcal{F}$ is Helly.

Remark 1. It is useful to note that two of these vertices are adjacent (or equal) if their indexes share two vertices. For example, $v_{xyz} \in N[v_{xyz'}]$ because both v_{xyz} and $v_{xyz'}$ are in every member of $\mathcal{F}$ containing $\{x, y\}$.

In order to obtain a contradiction, we will prove that $\mathcal{C}(\mathcal{F}, V')$ cannot equal any of the families defined in Section 4, which will be done by considering several cases that appear below. The main strategy to succeed will be using Proposition 3.8 to find twin vertices in V' for the vertices of the form v_{xyz}. This will allow to find contradictions quite directly to discard some of the cases. Otherwise, it will allow to find other induced O_3 in G and see how their cliques are covered by $\mathcal{F}$. As a result, there will be a reduction to the previous discarded cases.

Some of the reasonings to be used will be formulated as lemmas. They will appear between cases as they become necessary.

$\mathcal{C}(\mathcal{F}, V')$ is not castled:

Suppose to the contrary that C_1, C_2, C_3 are members of $\mathcal{C}(\mathcal{F}, V')$ forming a castle. Let F_1, F_2, F_3 be members of $\mathcal{F}$ such that $C_i \subseteq F_i$ for $1 \leq i \leq 3$. Then, F_1, F_2, F_3 are pairwise intersecting and, as $\mathcal{F}$ is Helly, we can take $v \in V(G)$ such that $v \in F_1 \cap F_2 \cap F_3$. We can deduce from Proposition 4.1 that $C_1 \cup C_2 \cup C_3 = V'$, so $V' \subseteq F_1 \cup F_2 \cup F_3$. Since $F_1 \cup F_2 \cup F_3$ is contained in the neighborhood of v in $S(\mathcal{F})$, we infer that V' is contained in the neighborhood of v in G, thus contradicting part (d) of Proposition 3.8.

Therefore $\mathcal{C}(\mathcal{F}, V')$ is not castled.

Lemma 5.1. *Let w, x, y, z be four different vertices of V' such that $\{x, y, z\}$ is a clique of $G[V']$. Then,*

1. $\{x, y, z\} \subseteq N[v_{xyz}]$
2. *If $wxy \ll \mathcal{F}$, then $w \in N[v_{xyz}]$.*

Proof.

1. Since x and y are adjacent in G and $S(\mathcal{F}) = G$, there exists $F \in \mathcal{F}$ such that $\{x, y\} \subseteq F$. By the definition of v_{xyz}, it follows that $\{x, y, v_{xyz}\} \subseteq F$. Hence, x and y are in the neighborhood of v_{xyz} in $S(\mathcal{F})$, that is, $\{x, y\} \subseteq N_G[v_{xyz}]$. We can similarly prove that $z \in N_G[v_{xyz}]$.
2. Assume that $wxy \ll \mathcal{F}$. Let $F \in \mathcal{F}$ be such that $\{w, x, y\} \subseteq F$. By the definition of v_{xyz}, it follows that $\{w, x, y, v_{xyz}\} \subseteq F$. Hence, w is in the neighborhood of v_{xyz} in $S(\mathcal{F})$, that is, $w \in N_G[v_{xyz}]$.

■

$\mathcal{C}(\mathcal{F}, V')$ does not contain an umbrella:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ contains an umbrella with members $\{a, b, c\}$, $\{a', b', c\}$ and $\{a', b', c'\}$. Then, by Lemma 5.1 and the facts that $abc \ll \mathcal{F}$ and $a'b'c \ll \mathcal{F}$, we have that $\{a, a', b, b', c\} \subseteq N[v_{ab'c}]$. We deduce from part (c) of Proposition 3.8 that $v_{ab'c} \sim c$.

Furthermore, $\{a, a', b', c'\} \subseteq N[v_{ab'c'}]$ due to Lemma 5.1 and the fact that $a'b'c' \ll \mathcal{F}$, and $v_{ab'c'} \in N[v_{ab'c}]$ due to Remark 1. Since $N[v_{ab'c}] = N[c]$, we have that $v_{ab'c'} \in N[c]$. It follows that $\{a, a', b', c, c'\} \subseteq N[v_{ab'c'}]$ and hence, by Proposition 3.8, $v_{ab'c'} \sim b'$.

We can prove with similar reasonings that $v_{a'bc} \sim c$ and that $\{a', b, b', c'\} \subseteq N[v_{a'bc'}]$. By Remark 1, $v_{a'bc}$ and $v_{a'bc'}$ are adjacent, so c is also in $N[v_{a'bc'}]$. Consequently, by Proposition 3.8, $v_{a'bc'} \sim a'$.

We can infer from Lemma 5.1 that $\{a, b, c, c'\} \subseteq N[v_{abc'}]$. Moreover, by Remark 1, $v_{abc'}$ is adjacent to $v_{ab'c'}$. This fact implies that $v_{abc'}$ is adjacent to b' as well. Thus, Proposition 3.8 yields that $v_{abc'} \sim a$. This contradicts that $v_{a'bc'}$ and $v_{abc'}$ are neighbors because we had obtained that these vertices are twins of a' and a, respectively.

Therefore, $\mathcal{C}(\mathcal{F}, V')$ does not contain an umbrella.

Given the classification of Section 4, the families that contain umbrellas are worms, rhombic circles and umbrellas themselves. Therefore, $\mathcal{C}(\mathcal{F}, V')$ belongs to none of those classes.

Lemma 5.2. *Let x, y, z be vertices of G such that $x \in N[y]$ and $z \in V'$. Then, $z \in N[x] \cap N[y]$ or $z' \in N[x] \cap N[y]$.*

Proof. Since x and y are adjacent, there exists a clique C in G that contains x and y. Let f be the clique-to-clique function. Then, $f^{-1}(C)$ is a clique of $G[V']$ contained in $N[x] \cap N[y]$. By the structure of O_3, we have that $z \in f^{-1}(C)$ or $z' \in f^{-1}(C)$. Hence $z \in C$ or $z' \in C$, which is sufficient to complete the proof. ■

Lemma 5.3. *Let x, y, z be three vertices in V' such that $\{x, y, z\}$ is a clique of $G[V']$ and $xyz \ll \mathcal{F}$. Then, $xyv_{xyz'} \ll \mathcal{F}$, $xv_{xyz'}z \ll \mathcal{F}$ and $v_{xyz'}yz \ll \mathcal{F}$.*

Proof. Let $F \in \mathcal{F}$ be such that $\{x, y, z\} \subseteq F$. By the definition of $v_{xyz'}$, we have that $\{x, y, z, v_{xyz'}\} \subseteq F$, from which the lemma follows. ■

<u>$\mathcal{C}(\mathcal{F}, V')$ is not a round</u>:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ is a round with members $\{a, b, c\}$, $\{a, b, c'\}$, $\{a, b', c\}$, and $\{a, b', c'\}$. Then, by Lemma 5.1, $\{a, a', b', c\} \subseteq N[v_{a'b'c}]$ and $\{a, a', b, c\} \subseteq N[v_{a'bc}]$. As $v_{a'b'c} \in N[v_{a'bc}]$, we can apply Lemma 5.2 to infer that $b \in N[v_{a'b'c}] \cap N[v_{a'bc}]$ or $b' \in N[v_{a'b'c}] \cap N[v_{a'bc}]$. If the former holds, then $\{a, a', b, b', c\} \subseteq N[v_{a'b'c}]$. If the latter holds, then $\{a, a', b, b', c\} \subseteq N[v_{a'bc}]$. Thus, by Proposition 3.8, $v_{a'b'c} \sim c$ or $v_{a'bc} \sim c$.

If $v_{a'b'c}$ is a twin of c, let $V'' = \{a, a', b, b', v_{a'b'c}, c'\}$. Then, $G[V''] = O_3$ and $\{a, b, c'\}$ and $\{a, b', c'\}$ are members of $\mathcal{C}(\mathcal{F}, V'')$. We infer from the definition of $v_{a'b'c}$ that $\{a', b', v_{a'b'c}\} \in \mathcal{C}(\mathcal{F}, V'')$. Furthermore, Lemma 5.3 and $ab'c \ll \mathcal{F}$ imply that $\{a, b', v_{a'b'c}\} \in \mathcal{C}(\mathcal{F}, V'')$. Thus, $\mathcal{C}(\mathcal{F}, V'')$ contains a worm of $G[V'']$ and hence contains an umbrella, which contradicts the previous case of the proof.

If $v_{a'bc} \sim c$, let now $V'' = \{a, a', b, b', v_{a'bc}, c'\}$. Thus, $\{a, b, c'\}$ and $\{a, b', c'\}$ are again members of $\mathcal{C}(\mathcal{F}, V'')$. By the definition of $v_{a'bc}$, we have that $\{a', b, v_{a'bc}\}$ is another member of $\mathcal{C}(\mathcal{F}, V'')$. Furthermore, Lemma 5.3 and $abc \ll \mathcal{F}$ imply that $\{a, b, v_{a'bc}\} \in \mathcal{C}(\mathcal{F}, V'')$. Hence we have the same contradiction as in the previous paragraph.

Therefore, $\mathcal{C}(\mathcal{F}, V')$ is not a round.

It is possible to conclude from the cases considered so far that $|\mathcal{C}(\mathcal{F}, V')|$ cannot be larger than 3. Now we study the remaining possibilities for the cardinality of $|\mathcal{C}(\mathcal{F}, V')|$.

<u>$|\mathcal{C}(\mathcal{F}, V')| \neq 3$</u>:

It is already known that $\mathcal{C}(\mathcal{F}, V')$ is not an umbrella. Therefore, it remains to investigate the possibility that $\mathcal{C}(\mathcal{F}, V')$ is a fan.

<u>$\mathcal{C}(\mathcal{F}, V')$ is not a fan</u>:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ is a fan with members $\{a, b, c\}$, $\{a, b', c\}$, and $\{a, b', c'\}$. Then, by Lemma 5.1, we have that $\{a, a', b', c\} \subseteq N[v_{a'b'c}]$ and $\{a, a', b, c\} \subseteq N[v_{a'bc}]$. Since $v_{a'b'c} \in N[v_{a'bc}]$, we can apply Lemma 5.2 to infer that $b \in N[v_{a'b'c}] \cap N[v_{a'bc}]$ or $b' \in N[v_{a'b'c}] \cap N[v_{a'bc}]$. We deduce from Proposition 3.8 that $v_{a'b'c} \sim c$ or $v_{a'bc} \sim c$. We can infer that $v_{a'b'c} \sim b'$ or $v_{a'b'c'} \sim b'$ through a similar chain of reasonings.

If $v_{a'bc} \sim c$, set $V'' = \{a, a', b, b', v_{a'bc}, c'\}$. Then, $\{a, b', c'\} \in \mathcal{C}(\mathcal{F}, V'')$. By the definition of $v_{a'bc}$, Lemma 5.3 and $abc \ll \mathcal{F}$, we have that $\{a', b, v_{a'bc}\}$ and $\{a, b, v_{a'bc}\}$ are other members of $\mathcal{C}(\mathcal{F}, V'')$. Thus, $\mathcal{C}(\mathcal{F}, V'')$ contains an umbrella of $G[V'']$, which is a contradiction.

If $v_{a'b'c'} \sim b'$, set $V'' = \{a, a', b, v_{a'b'c'}, c, c'\}$. Then, $\{a, b, c\} \in \mathcal{C}(\mathcal{F}, V'')$. By the definition of $v_{a'b'c'}$, Lemma 5.3 and $ab'c' \ll \mathcal{F}$, we have that $\{a', v_{a'b'c'}, c'\}$ and $\{a, v_{a'b'c'}, c'\}$ are other members of $\mathcal{C}(\mathcal{F}, V'')$. We thus obtain the same contradiction as in the previous paragraph.

It is not possible that neither $v_{a'bc} \sim c$ nor $v_{a'b'c'} \sim b'$; otherwise $v_{a'b'c} \sim c$ and $v_{a'b'c} \sim b'$, which is a contradiction.

Therefore, $\mathcal{C}(\mathcal{F}, V')$ is not a fan.

$|\mathcal{C}(\mathcal{F}, V')| \neq 2$:

$\mathcal{C}(\mathcal{F}, V')$ is not a bow:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ is a bow and that $\{a, b, c\}$ and $\{a, b', c'\}$ are its members. The proof will be organized as a series of statements, each having its justification between brackets.

(i) $v_{ab'c} \sim a$	[$abc \ll \mathcal{F}$, $ab'c' \ll \mathcal{F}$, Lemma 5.1 and Proposition 3.8]
(ii) $\{a', b', c\} \subseteq N[v_{a'b'c}]$	[Lemma 5.1]
(iii) $v_{abc'} \sim a$	[$abc \ll \mathcal{F}$, $ab'c' \ll \mathcal{F}$, Lemma 5.1 and Proposition 3.8]
(iv) $\{a, a', b, c\} \subseteq N[v_{a'bc}]$	[$abc \ll \mathcal{F}$ and Lemma 5.1]
(v) $\{a, a', b', c'\} \subseteq N[v_{a'b'c'}]$	[$ab'c' \ll \mathcal{F}$ and Lemma 5.1]
(vi) $\{a', b, c'\} \subseteq N[v_{a'bc'}]$	[Lemma 5.1]
(vii) $v_{ab'c} \in N[v_{a'b'c}]$	[Remark 1]
(viii) $\{a, a', b', c\} \subseteq N[v_{a'b'c}]$	[(i), (ii) and (vii)]
(ix) $v_{abc'} \in N[v_{a'bc'}]$	[Remark 1]
(x) $\{a, a', b, c'\} \subseteq N[v_{a'bc'}]$	[(iii), (vi) and (ix)]
(xi) $v_{a'b'c'} \in N[v_{a'b'c}]$	[Remark 1]
(xii) $v_{a'b'c'} \sim b'$ or $v_{a'b'c} \sim b'$	[(v), (viii), (xi), Lemma 5.2 and Proposition 3.8]
(xiii) $v_{a'bc} \in N[v_{a'b'c}]$	[Remark 1]
(xiv) $v_{a'bc} \sim c$ or $v_{a'b'c} \sim c$	[(iv), (viii), (xiii), Lemma 5.2 and Proposition 3.8]
(xv) $v_{a'bc} \in N[v_{a'bc'}]$	[Remark 1]
(xvi) $v_{a'bc} \sim b$ or $v_{a'bc'} \sim b$	[(iv), (x), (xv), Lemma 5.2 and Proposition 3.8]
(xvii) $v_{a'b'c'} \in N[v_{a'bc'}]$	[Remark 1]
(xviii) $v_{a'b'c'} \sim c'$ or $v_{a'bc'} \sim c'$	[(v), (x), (xvii), Lemma 5.2 and Proposition 3.8]

If $v_{a'b'c'} \sim b'$, combine (xiv), (xvi), and (xviii) to obtain that $v_{a'bc'} \sim c'$, $v_{a'bc} \sim b$ and $v_{a'b'c} \sim c$. Set $V'' = \{a, a', v_{a'bc}, b', c, v_{a'bc'}\}$. Then, by Lemma 5.3 and the definitions of $v_{a'bc}$ and $v_{a'bc'}$, we have that $\{a, v_{a'bc}, c\}$, $\{a', v_{a'bc}, c\}$, and $\{a', v_{a'bc}, v_{a'bc'}\}$ are members of $\mathcal{C}(\mathcal{F}, V'')$. Thus, $|\mathcal{C}(\mathcal{F}, V'')| \geq 3$, which is a contradiction.

If $v_{a'b'c} \sim b'$, then the combination of (xiv), (xvi), and (xviii) now yields that $v_{a'bc} \sim c$, $v_{a'bc'} \sim b$, and $v_{a'b'c'} \sim c'$. Set $V'' = \{a, a', b, v_{a'b'c}, c, v_{a'b'c'}\}$. Then, $\{a, b, c\} \in \mathcal{C}(\mathcal{F}, V'')$. The sets $\{a', v_{a'b'c}, c\}$ and $\{a', v_{a'b'c}, v_{a'b'c'}\}$ are also members of $\mathcal{C}(\mathcal{F}, V'')$ because of the definitions of $v_{a'b'c}$ and $v_{a'b'c'}$. Hence we have the same contradiction as in the previous paragraph.

Therefore, $\mathcal{C}(\mathcal{F}, V')$ is not a bow.

$\mathcal{C}(\mathcal{F}, V')$ is not an opposite pair:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ is an opposite pair with members $\{a, b, c\}$ and $\{a', b', c'\}$. Then, by Lemma 5.1, we have that $\{a, b, b', c\} \subseteq N[v_{ab'c}]$, $\{a, a', b', c'\} \subseteq N[v_{ab'c'}]$, and $\{a', b', c, c'\} \subseteq N[v_{a'b'c}]$.

It is clear from Remark 1 that $v_{ab'c}$ is adjacent to both $v_{ab'c'}$ and $v_{a'b'c}$. Furthermore, any member of $\mathcal{F}$ containing $\{a', b', c'\}$ must have $v_{ab'c'}$ and $v_{a'b'c}$ as elements. We conclude that $\{v_{ab'c}, v_{ab'c'}, v_{a'b'c}\}$ is a complete set. Let C be a clique of G containing $\{v_{ab'c}, v_{ab'c'}, v_{a'b'c}\}$ and f be the clique-to-clique function.

We now prove that $C \neq f(\{a, b, c\})$. Suppose to the contrary that the equality holds. Then, the vertices a, b and c are in $N[v_{ab'c'}]$ because $v_{ab'c'} \in C$. Combine this fact with what we knew from the beginning to infer that $V' \subseteq N[v_{ab'c'}]$, thus contradicting part (d) of Proposition 3.8. Therefore, $C \neq f(\{a, b, c\})$.

We can conclude after similar reasonings that $C \neq f(\{a, b, c'\})$, $C \neq f(\{a', b, c\})$, $C \neq f(\{a', b, c'\})$, and $C \neq f(\{a', b', c'\})$.

As a consequence of Proposition 3.7, the remaining possibilities are $C = f(\{a, b', c'\})$, $C = f(\{a, b', c\})$, and $C = f(\{a', b', c\})$.

If $C = f(\{a, b', c'\})$, then $\{a, b, b', c, c'\} \subseteq N[v_{ab'c}]$ and hence $v_{ab'c} \sim a$. Set $V'' = \{v_{ab'c}, a', b, b', c, c'\}$. Thus, $\{a', b', c'\} \in \mathcal{C}(\mathcal{F}, V'')$ and we get from Lemma 5.3 and the definition of $v_{ab'c}$ that $\{v_{ab'c}, b, c\}$ and $\{v_{ab'c}, b', c\}$ are other members of $\mathcal{C}(\mathcal{F}, V'')$.

If $C = f(\{a, b', c\})$, then $\{a, a', b', c, c'\} \subseteq N[v_{ab'c'}]$ and hence $v_{ab'c'} \sim b'$. Set $V'' = \{a, a', b, v_{ab'c'}, c, c'\}$. Thus, $\{a, b, c\} \in \mathcal{C}(\mathcal{F}, V'')$ and we get from the definition of $v_{ab'c'}$ and Lemma 5.3 that $\{a, v_{ab'c'}, c'\}$ and $\{a', v_{ab'c'}, c'\}$ are other members of $\mathcal{C}(\mathcal{F}, V'')$.

If $C = f(\{a', b', c\})$, then $\{a, a', b, b', c\} \subseteq N[v_{ab'c}]$ and hence $v_{ab'c} \sim c$. Set $V'' = \{a, a', b, b', v_{ab'c}, c'\}$. Thus, $\{a', b', c'\} \in \mathcal{C}(\mathcal{F}, V'')$ and we get from Lemma 5.3 and the definition of $v_{ab'c}$ that $\{a, b, v_{ab'c}\}$ and $\{a, b', v_{ab'c}\}$ are other members of $\mathcal{C}(\mathcal{F}, V'')$.

We have that $|\mathcal{C}(\mathcal{F}, V'')| \geq 3$ in every case, which is a contradiction. Therefore, $\mathcal{C}(\mathcal{F}, V')$ is not an opposite pair.

<u>$\mathcal{C}(\mathcal{F}, V')$ is not a rhombus</u>:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ is a rhombus and that $\{a, b, c\}$ and $\{a, b', c\}$ are its members. Then,

(i) $\{a, b', c, c'\} \subseteq N[v_{ab'c'}]$ [$ab'c \ll \mathcal{F}$ and Lemma 5.1]
(ii) $\{a, a', b', c\} \subseteq N[v_{a'b'c}]$ [$ab'c \ll \mathcal{F}$ and Lemma 5.1]
(iii) $\{a, b, c, c'\} \subseteq N[v_{abc'}]$ [$abc \ll \mathcal{F}$ and Lemma 5.1]
(iv) $\{a, a', b, c\} \subseteq N[v_{a'bc}]$ [$abc \ll \mathcal{F}$ and Lemma 5.1]
(v) $v_{ab'c'} \in N[v_{abc'}]$ [Remark 1]
(vi) $v_{a'b'c} \in N[v_{a'bc}]$ [Remark 1]
(vii) $v_{ab'c'} \sim a$ or $v_{abc'} \sim a$ [(i), (iii), (v), Lemma 5.2 and Proposition 3.8]
(viii) $v_{a'b'c} \sim c$ or $v_{a'bc} \sim c$ [(ii), (iv), (vi), Lemma 5.2 and Proposition 3.8]

Combining (vii) and (viii) yields four possibilities.

If $v_{ab'c'} \sim a$ and $v_{a'b'c} \sim c$, set $V'' = \{v_{ab'c'}, a', b, b', v_{a'b'c}, c'\}$. By the definitions of $v_{ab'c'}$ and $v_{a'b'c}$, we have that $\{v_{ab'c'}, b', c'\}$ and $\{a', b', v_{a'b'c}\}$ are members of $\mathcal{C}(\mathcal{F}, V'')$. Since $ab'c \ll \mathcal{F}$, we deduce that $\{v_{ab'c'}, b', v_{a'b'c}\}$ is another member of $\mathcal{C}(\mathcal{F}, V'')$. Consequently, $|\mathcal{C}(\mathcal{F}, V'')| \geq 3$.

If $v_{abc'} \sim a$ and $v_{a'bc} \sim c$, set $V'' = \{v_{abc'}, a', b, b', v_{a'bc}, c'\}$. By the definitions of $v_{abc'}$ and $v_{a'bc}$, we have that $\{v_{abc'}, b, c'\}$ and $\{a', b, v_{a'bc}\}$ are members of $\mathcal{C}(\mathcal{F}, V'')$. Since $abc \ll \mathcal{F}$, we deduce that $\{v_{abc'}, b, v_{a'bc}\}$ is another member of $\mathcal{C}(\mathcal{F}, V'')$, so $|\mathcal{C}(\mathcal{F}, V'')| \geq 3$ for this case as well.

If $v_{ab'c'} \sim a$ and $v_{a'bc} \sim c$, set $V'' = \{v_{ab'c'}, a', b, b', v_{a'bc}, c'\}$. By the definitions of $v_{ab'c'}$ and $v_{a'bc}$, we have that $\{v_{ab'c'}, b', c'\}$ and $\{a', b, v_{a'bc}\}$ are members of $\mathcal{C}(\mathcal{F}, V'')$. We conclude that $\mathcal{C}(\mathcal{F}, V'')$ is an opposite pair or that $|\mathcal{C}(\mathcal{F}, V'')| \geq 3$.

If $v_{abc'} \sim a$ and $v_{a'b'c} \sim c$, set $V'' = \{v_{abc'}, a', b, b', v_{a'b'c}, c'\}$. By the definitions of $v_{abc'}$ and $v_{a'b'c}$, we have that $\{v_{abc'}, b, c'\}$ and $\{a', b', v_{a'b'c}\}$ are members of $\mathcal{C}(\mathcal{F}, V'')$. The conclusion is identical to that of the previous case.

Each of the four cases resulted in a contradiction. Therefore, $\mathcal{C}(\mathcal{F}, V')$ is not a rhombus.

$|\mathcal{C}(\mathcal{F}, V')| \neq 1$:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ is a triangle consisting of $\{a, b, c\}$.

Apply Lemma 5.1 to obtain that $\{a, b, b', c\} \subseteq N[v_{ab'c}]$, $\{a, b', c'\} \subseteq N[v_{ab'c'}]$, $\{a', b', c\} \subseteq N[v_{a'b'c}]$, $\{a, b, c, c'\} \subseteq N[v_{abc'}]$ and $\{a, a', b, c\} \subseteq N[v_{a'bc}]$. Furthermore, $abc \ll \mathcal{F}$ implies that $abcv_{ab'c}v_{abc'}v_{a'bc} \ll \mathcal{F}$. Thus, $\{v_{ab'c}, v_{ab'c'}, v_{abc'}\}$ and $\{v_{ab'c}, v_{a'b'c}, v_{a'bc}\}$ are complete. Let C and C' be cliques of G such that $\{v_{ab'c}, v_{ab'c'}, v_{abc'}\} \subseteq C$ and $\{v_{ab'c}, v_{a'b'c}, v_{a'bc}\} \subseteq C'$.

Reason like in the case that $\mathcal{C}(\mathcal{F}, V')$ is not an opposite pair to deduce that C is equal to $f(\{a, b, c\})$, $f(\{a, b', c'\})$, $f(\{a, b', c\})$, or $f(\{a, b, c'\})$ and that C' is equal to $f(\{a, b, c\})$, $f(\{a, b', c\})$, $f(\{a', b', c\})$, or $f(\{a', b, c\})$.

If $C = C' = f(\{a, b, c\})$, then $\{a, b, b', c, c'\} \subseteq N[v_{ab'c'}]$ and $\{a, a', b, b', c\} \subseteq N[v_{a'b'c}]$. It follows from Proposition 3.8 that $v_{ab'c'} \sim a$ and $v_{a'b'c} \sim c$. Set $V'' = \{v_{ab'c'}, a', b, b', v_{a'b'c}, c'\}$. Thus, we deduce from the definitions of $v_{ab'c'}$ and $v_{a'b'c}$ that $\{v_{ab'c'}, b', c'\}$ and $\{a', b', v_{a'b'c}\}$ are members of $\mathcal{C}(\mathcal{F}, V'')$.

If $C = f(\{a, b', c'\})$, then $\{a, b, b', c, c'\} \subseteq N[v_{ab'c}]$ and hence $v_{ab'c} \sim a$. Set $V'' = \{v_{ab'c}, a', b, b', c, c'\}$. By the definition of $v_{ab'c}$, we have that $\{v_{ab'c}, b', c\} \in \mathcal{C}(\mathcal{F}, V'')$. Moreover, $abc \ll \mathcal{F}$ and Lemma 5.3 imply that $\{v_{ab'c}, b, c\}$ is another member of $\mathcal{C}(\mathcal{F}, V'')$.

If $C = f(\{a, b', c\})$, then $\{a, b, b', c, c'\} \subseteq N[v_{abc'}]$ and hence $v_{abc'} \sim a$. Set $V'' = \{v_{abc'}, a', b, b', c, c'\}$. By the definition of $v_{abc'}$, we have that $\{v_{abc'}, b, c'\} \in \mathcal{C}(\mathcal{F}, V'')$. Moreover, $abc \ll \mathcal{F}$ and Lemma 5.3 imply that $\{v_{abc'}, b, c\}$ is another member of $\mathcal{C}(\mathcal{F}, V'')$.

If $C = f(\{a, b, c'\})$, then $\{a, b, b', c, c'\} \subseteq N[v_{ab'c}]$. Hence $v_{ab'c} \sim a$, case which has already been considered.

If $C' \neq f(\{a, b, c\})$, then C' is equal to $f(\{a, b', c\})$, $f(\{a', b', c\})$, or $f(\{a', b, c\})$. Reasoning like in the previous cases, it can be verified that $v_{ab'c} \sim c$ or $v_{a'bc} \sim c$.

If $v_{ab'c} \sim c$, set $V'' = \{a, a', b, b', v_{ab'c}, c'\}$. By the definition of $v_{ab'c}$, we have that $\{a, b', v_{ab'c}\} \in \mathcal{C}(\mathcal{F}, V'')$. Moreover, $abc \ll \mathcal{F}$ and Lemma 5.3 imply that $\{a, b, v_{ab'c}\}$ is another member of $\mathcal{C}(\mathcal{F}, V'')$.

If $v_{a'bc} \sim c$, set $V'' = \{a, a', b, b', v_{a'bc}, c'\}$. By the definition of $v_{a'bc}$, we have that $\{a', b, v_{a'bc}\} \in \mathcal{C}(\mathcal{F}, V'')$. Moreover, $abc \ll \mathcal{F}$ and Lemma 5.3 imply that $\{a, b, v_{a'bc}\}$ is another member of $\mathcal{C}(\mathcal{F}, V'')$.

We got that $|\mathcal{C}(\mathcal{F}, V'')| \geq 2$ in every case, which is a contradiction. Therefore, $\mathcal{C}(\mathcal{F}, V')$ is not a triangle.

$\mathcal{C}(\mathcal{F}, V')$ is not empty:

Suppose to the contrary that $\mathcal{C}(\mathcal{F}, V')$ is empty. We know that $v_{abc} \in N[v_{ab'c}]$, so we can take $C \in \mathcal{C}(G)$ such that $\{v_{abc}, v_{ab'c}\} \subseteq C$. It is clear that $C \neq f(\{a', b', c'\})$ and $C \neq f(\{a', b, c'\})$; otherwise V' would be contained in the neighborhood of v_{abc} or $v_{ab'c}$, thus contradicting Proposition 3.8.

If $C = f(\{a, b', c'\})$, then $\{a, b, b', c, c'\} \subseteq N[v_{abc}]$ and hence $v_{abc} \sim a$. Set $V'' = \{v_{abc}, a', b, b', c, c'\}$. Thus, we get from the definition of v_{abc} that $\{v_{abc}, b, c\}$ is a member of $\mathcal{C}(\mathcal{F}, V'')$, which is a contradiction. Therefore, $C \neq f(\{a, b', c'\})$.

We can conclude from similar reasonings that $C \neq f(\{a', b', c\})$, $C \neq f(\{a, b, c'\})$, and $C \neq f(\{a', b, c\})$. Therefore, $C = f(\{a, b, c\})$ or $C = f(\{a, b', c\})$.

Suppose without loss of generality that $C = f(\{a, b, c\})$. Then, $\{a, b, b', c\} \subseteq N[v_{ab'c}]$. Let $C' \in \mathcal{C}(G)$ be such that $\{v_{ab'c'}, v_{ab'c}\} \subseteq C'$. Reasoning as before, we can infer that $C' = f(\{a, b', c'\})$ or $C' = f(\{a, b', c\})$.

If $C' = f(\{a, b', c'\})$, then $\{a, b, b', c, c'\} \subseteq N[v_{ab'c}]$ and hence $v_{ab'c} \sim a$. We set $V'' = \{v_{ab'c}, a', b, b', c, c'\}$ to conclude that $\{v_{ab'c}, b', c\}$ is a member of $\mathcal{C}(\mathcal{F}, V'')$, which is a contradiction. Therefore, $C' = f(\{a, b', c\})$ and hence $\{a, b', c, c'\} \subseteq N[v_{ab'c'}]$.

We now write a couple of statements based on arguments like those of the previous paragraph.

Vertices $v_{a'b'c}$ and $v_{ab'c}$ are both contained in $f(\{a, b', c\})$ and hence $\{a, a', b', c\} \subseteq N[v_{a'b'c}]$.

Vertices $v_{a'b'c'}$ and $v_{ab'c'}$ are both contained in $f(\{a, b', c'\})$ and hence $\{a, a', b', c'\} \subseteq N[v_{a'b'c'}]$.

Now consider the vertices $v_{a'b'c'}$ and $v_{a'b'c}$. If both vertices are contained in $f(\{a', b', c\})$, then $\{a, a', b', c, c'\} \subseteq N[v_{a'b'c'}]$ and hence $v_{a'b'c'} \sim b'$. Set $V'' = \{a, a', b, v_{a'b'c'}, c, c'\}$. We get from the definition of $v_{a'b'c'}$ that $\{a', v_{a'b'c'}, c'\} \in \mathcal{C}(\mathcal{F}, V'')$, which is a contradiction.

If $v_{a'b'c'}$ and $v_{a'b'c}$ are contained in $f(\{a', b', c'\})$, then $\{a, a', b', c, c'\} \subseteq N[v_{a'b'c}]$ and hence $v_{a'b'c} \sim b'$. This yields a contradiction similar to the one of the previous paragraph.

All the contradictions we found leave no other alternative. Therefore, $\mathcal{C}(\mathcal{F}, V')$ is not empty.

Overall, there is no set that $\mathcal{C}(\mathcal{F}, V')$ can equal. This contradiction comes from assuming the existence of a graph in $K(\mathcal{G}) \cap K^{-1}(O_4)$. Therefore, $K(\mathcal{G}) \cap K^{-1}(O_4) = \emptyset$. Q.E.D.

6 CONCLUDING REMARKS

After proving that $K(\mathcal{G}) \neq K^2(\mathcal{G})$, many other questions can naturally arise. We can be more general and wonder, given any natural number n, if the equality $K^n(\mathcal{G}) = K^{n+1}(\mathcal{G})$ is true.

It is not difficult to prove that $K(O_n) = O_{2^{n-1}}$ [2, 5], so all the iterated clique graphs of O_3 are octahedral. It is tempting to conjecture that $K^n(O_3) \in K^n(\mathcal{G}) \setminus K^{n+1}(\mathcal{G})$ for $n \geq 1$, which would imply that the equality $K^n(\mathcal{G}) = K^{n+1}(\mathcal{G})$ is always false.

It is also interesting to consider the intersection $\bigcap_{n=1}^{\infty} K^n(\mathcal{G})$. This class contains, for example, all the clique Helly graphs, but we are far from knowing what all the graphs it contains are.

A deeper knowledge on iterated clique graphs should be developed to gain insight into these and other questions.

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