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The logic Ł$^\bullet$

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The algebraic category MV$^\bullet$ is the image of MV, the category whose objects are the MV-algebras, by the equivalence K$^\bullet$ (cf. [7, 8]). In this paper we define the logic Ł$^\bullet$ whose Lindenbaum algebra is an MV$^\bullet$-algebra (object of MV$^\bullet$), and establish a link between Ł$^\bullet$ and the infinite valued Łukasiewicz logic Ł. We define $\boldsymbol{c}$U-operators, that have properties of universal quantifiers, and establish a bijection that maps an MV-algebra endowed with a U-operator (cf. [20–22]) into an MV$^\bullet$-algebra endowed with a $\boldsymbol{c}$U-operator. This map extends to a functor that is a categorical equivalence.

1 Introduction

Since we are working on ideas and results of the paper [8], we recommend that the reader have the mentioned paper at hand while reading this work. All the residuated lattices considered in this paper are distributive and commutative, so we shall omit mentioning these two conditions in the sequel, assuming them as given. Recall that a residuated lattice is said to be integral if it is bounded above by the unit of the product. All the categories considered in this paper have an underlying variety (cf. [1]), so we shall use the same notation in both cases.

Many years ago, the *constructive logic with strong negation* (cf. [24]) was created for logical reasons: the intuitionistic negation does not have "good" constructive properties. From the beginning, constructive logic with strong negation was closely related to intuitionistic logic. The algebraic counterpart of that logic was studied in first place for Rasiowa, and further defined for equations by Brignole and Monteiro (cf. [4]). They call the algebraic models *Nelson algebras*. Also, the algebraic approach of the relationship between intuitionism and constructive logic with strong negation appears: a characterization of Nelson algebras as pairs of disjoint elements of Heyting algebras (cf. [13, 28]). In [9], Cignoli proves that this characterization can be formalized from a categorical point of view as an adjunction between the categories of Heyting algebras and Nelson algebras, which restricted to centered Nelson algebras becomes an equivalence. More recently, Spinks and Veroff have shown that Nelson algebras are equivalent to a subvariety of residuated lattices that they call *Nelson residuated lattices*. This result proves that constructive logic with strong negation can be considered as a substructural logic.

In [5], the authors obtain many results about constructive logic with strong negation and some of its axiomatic extensions. For example, they give an algebraic proof of the equivalence between Nelson algebras and Nelson residuated lattices, and they prove a Glivenko-like theorem relating constructive logic with strong negation and three valued Łukasiewicz logic.

In [7], the authors proved that the equivalence between Heyting and Nelson algebras (in fact between Heyting algebras and Nelson residuated lattices) can be lifted to an equivalence between the category IRL_0 of integral residuated lattices with bottom, and a category of involutive residuated lattices called DRL', whose objects are $\boldsymbol{c}$-differential residuated lattices satisfying the condition (CK$^\bullet$) (cf. [7, 8]). Later it was proved in [8] that a $\boldsymbol{c}$-differential residuated lattice satisfies the characterizing condition of DRL' if and only if we can define a unary operation κ that satisfies some "quantifier-like" conditions. The category MV of MV-algebras can be considered as a full subcategory of IRL_0, and MV$^\bullet$ is the image of MV under the mentioned equivalence (cf. [8]).

The categorical equivalence between IRL_0 and DRL' given in [7] relies on a generalization of a standard construction of an involutive residuated lattice from an integral residuated lattice (cf. [27]).

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The general purpose of our present work is to "lift" the relationship between intuitionistic logic and constructive logic with strong negation to some kind of link between logics associated to IRL_0 and DRL'. This paper is a first step in that research. We study here a logic $\text{Ł}^\bullet$, whose algebraic models are the objects of $\mathrm{MV}^\bullet$, and its relationship with the infinite valued Łukasiewicz logic Ł associated to MV. We also study some operators in $\mathrm{MV}^\bullet$-algebras, called $\boldsymbol{c}$U-operators, that have properties of universal quantifiers. They correspond to U-operators in MV-algebras.

In § 2 we present some basic results about the categories DRL' and $\mathrm{MV}^\bullet$ which we shall need later. In § 3 we define the extension $\mathrm{RW}\kappa$ of the system RW (cf. [14]) and we prove that it is complete in DRL'. The Lindenbaum-Tarski algebra of $\text{Ł}^\bullet$, extension of $\mathrm{RW}\kappa$, is an $\mathrm{MV}^\bullet$-algebra. We obtain some results about the relationship between the logical system Ł of Łukasiewicz and our system $\text{Ł}^\bullet$. Finally, a new and unexplored field is that of § 4, where an extra operator, called $\boldsymbol{c}$U-operator, is added to the involutive residuated lattices previously investigated, expressing a possible notion of universal quantifier. Indeed, we establish a bijection that maps an MV-algebra endowed with a U-operator (cf. [20–22]) into a $\mathrm{MV}^\bullet$-algebra endowed with a $\boldsymbol{c}$U-operator. This map extends to a functor (that we also call $\mathsf{K}^\bullet$) that is a categorical equivalence.

2 Preliminary definitions and results

Let $\langle A, \wedge, \vee, ., \rightarrow, 0, 1\rangle$ be an object in IRL_0. We define $\mathsf{K}^\bullet(A)$ in the following way:

$$\mathsf{K}^\bullet(A) = \{(a, b) \in A \times A^0 : a.b = 0\},$$

where A^0 is the dual order of A (cf. [7], § 7]). For $A \in \mathrm{IRL}_0$, define the following operations in $\mathsf{K}^\bullet(A)$:

$$\begin{aligned}
(a, b) \vee (d, e) &:= (a \vee d, b \wedge e),\\
(a, b) \wedge (d, e) &:= (a \wedge d, b \vee e),\\
\sim (a, b) &:= (b, a),\\
(a, b) * (d, e) &:= (a.d, (a \rightarrow e) \wedge (d \rightarrow b)),\\
(a, b) \rightarrow (d, e) &:= ((a \rightarrow d) \wedge (e \rightarrow b), a.e).
\end{aligned}$$

An involutive residuated lattice is an algebra $\mathbf{T} = \langle T, \wedge, \vee, *, \rightarrow, \sim, 1\rangle$ such that

1. $\langle T, \wedge, \vee, *, \rightarrow, 1\rangle$ is a residuated lattice,
2. $\sim$ is an involution of the lattice that is a dual automorphism; i.e., $\sim\sim x = x$ for all $x \in T$, and
3. $x * y \leq z$ iff $x \leq\, \sim (y * (\sim z))$.

Note that in this case we have that $\sim (y * (\sim z)) = y \rightarrow z$.

An involutive residuated lattice is said to be centered if it has a distinguished element, called a center, that is, a fixed point for the involution. A *$\boldsymbol{c}$-differential residuated lattice* is an integral involutive residuated lattice with bottom 0 and center $\boldsymbol{c}$, satisfying the following condition [7, Definition 7.2]:

For any $x, y \in T$, $(x * y) \wedge \boldsymbol{c} = ((x \wedge \boldsymbol{c}) * y) \vee (x * (y \wedge \boldsymbol{c}))$.

We denote the category of $\boldsymbol{c}$-differential residuated lattices by DRL, where 0 and $\boldsymbol{c}$ are in the signature of the algebras of DRL.

The algebra $\langle \mathsf{K}^\bullet(A), \wedge, \vee, *, \rightarrow, \sim, (0, 1), (1, 0), (0, 0)\rangle$ is an object of DRL, where $(0, 1)$ is the bottom, $(1, 0)$ is the top and $\boldsymbol{c} = (0, 0)$. The assignment $A \mapsto \mathsf{K}^\bullet(A)$ extends to a functor $\mathsf{K}^\bullet : \mathrm{IRL}_0 \rightarrow \mathrm{DRL}$.

For any $T \in \mathrm{DRL}$, consider $\mathsf{C}(T) := \{x \in T \ : \ x \geq \boldsymbol{c}\}$. This defines a functor $\mathsf{C} : \mathrm{DRL} \rightarrow \mathrm{IRL}_0$ which is left adjoint to $\mathsf{K}^\bullet$ [7, Theorem 7.6].

The adjunction $\mathsf{C} \dashv \mathsf{K}^\bullet : \mathrm{IRL}_0 \to \mathrm{DRL}$ restricts to an equivalence $\mathsf{C} \dashv \mathsf{K}^\bullet : \mathrm{IRL}_0 \to \mathrm{DRL}'$ [7, Corollary 7.8], where DRL' is the subcategory whose objects T satisfy the following condition:

For every pair of elements $z, w \in T$ such that $z, w \geq \boldsymbol{c}$ and $z * w \leq \boldsymbol{c}$,

(CK$^\bullet$) there exists $x \in T$ such that $x \vee \boldsymbol{c} = z$ and $\sim x \vee \boldsymbol{c} = w$.

Let T be an algebra in DRL'. Then there is a map $\kappa : T \to T$ such that satisfies the following two conditions:

(1) $\kappa x \vee \boldsymbol{c} = \boldsymbol{c} \to x$,

(2) $\kappa x \wedge \boldsymbol{c} = x \wedge \boldsymbol{c}$.

Conversely, if T is an object of DRL in which there exists an operator κ that satisfies the previous equations, then (CK$^\bullet$) holds on T [8, Theorem 1]. In the following, we use DRL' to denote the category whose objects have a unary operator κ in its signature satisfying the corresponding equations.

Let $\mathrm{MV}^\bullet$ be the full subcategory of DRL' whose objects $\langle T, \wedge, \vee, *, \to, \sim, \kappa, 0, 1, \boldsymbol{c}\rangle$ satisfy the following equations:

(Inv$^\bullet$) $\sim \kappa x = \kappa \sim \kappa x$,

(QHey$^\bullet$) $\boldsymbol{c} * x * (x \to (y \vee \boldsymbol{c})) = \boldsymbol{c} * (x \wedge y)$,

(Lin$^\bullet$) $(x \to y) \vee (y \to x) \geq \boldsymbol{c}$

Recall that an MV-algebra is term equivalent to an integral residuated lattice with bottom $\langle A, \vee, \wedge, ., \to, \neg, 0, 1\rangle$ that satisfies the following equations:

(Inv) $\neg\neg x = x$ (here, $\neg x = x \to 0$),

(Lin) $(x \to y) \vee (y \to x) = 1$,

(QHey) $x.(x \to y) = x \wedge y$.

For $T \in \mathrm{MV}^\bullet$ we have that $\kappa(T) = \{x \in T : \kappa x = x\}$. The assignment $T \mapsto \kappa(T)$ extends to a functor $\mathscr{K} : \mathrm{MV}^\bullet \to \mathrm{MV}$. If $g : T \to S$ is a morphism in $\mathrm{MV}^\bullet$, then $\mathscr{K}(g) : \kappa(T) \to \kappa(S)$ is the morphism in MV given by the restriction of g to $\kappa(T)$. For $A \in \mathrm{MV}$, the assignment $A \mapsto \mathsf{K}^\bullet(A)$ extends to a functor $\mathsf{K}^\bullet : \mathrm{MV} \to \mathrm{MV}^\bullet$. If $f : A \to B$ is a morphism in MV, then $\mathsf{K}^\bullet(f) : \mathsf{K}^\bullet(A) \to \mathsf{K}^\bullet(B)$ is the morphism in $\mathrm{MV}^\bullet$ given by $(\mathsf{K}^\bullet(f))(a, b) = (f(a), f(b))$. The adjunction $\mathscr{K} \dashv \mathsf{K}^\bullet : \mathrm{MV} \to \mathrm{MV}^\bullet$ is an equivalence [8, Corollary 15]. For every $A \in \mathrm{MV}$ we have the isomorphism $\alpha : A \to \kappa(\mathsf{K}^\bullet(A))$ given by $\alpha(a) = (a, \neg a)$, and for every $T \in \mathrm{MV}^\bullet$ we have the isomorphism $\beta : T \to \mathsf{K}^\bullet(\kappa(T))$ given by $\beta(x) = (\lambda x, \lambda \sim x)$, where $\lambda x = \sim \kappa \sim x$.

If $A \in \mathrm{MV}$, then $\kappa : \mathsf{K}^\bullet(A) \to \mathsf{K}^\bullet(A)$ is given by $\kappa(a, b) = (\neg b, b)$.

Consider the real interval [0, 1] endowed with its structure of MV-algebra. The image of [0, 1] under $\mathsf{K}^\bullet$ is the triangle in Figure 1.

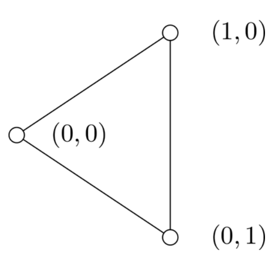


Fig. 1 Image of [0,1] under $\mathsf{K}^\bullet$

3 A system based on the system RW

In this section we study the varieties DRL$'$ and MV$^\bullet$ from a logical approach.

Throughout this section, we assume that a fixed countably infinite set of variables is given. The formal system RW has signature $\{\cdot, \rightarrow, \wedge, \vee, \sim, 1\}$. The ranks of these symbols are just as for involutive residuated lattices. For convenience, we define also a derived binary connective $\leftrightarrow$ by

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

The axioms (A1) to (A14) and rules (R1) and (R2) of RW, taken from [3] (cf. also [14, 15]), are as follows, where x, y and z are three distinct variables:

(A1) $x \rightarrow x$,

(A2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))$,

(A3) $x \rightarrow ((x \rightarrow y) \rightarrow y)$ (or $(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z))$),

(A4) $(x \wedge y) \rightarrow x$,

(A5) $(x \wedge y) \rightarrow y$,

(A6) $((x \rightarrow y) \wedge (x \rightarrow z)) \rightarrow (x \rightarrow (y \wedge z))$,

(A7) $x \rightarrow (x \vee y)$,

(A8) $y \rightarrow (x \vee y)$,

(A9) $((x \rightarrow z) \wedge (y \rightarrow z)) \rightarrow ((x \vee y) \rightarrow z)$,

(A10) $(x \wedge (y \vee z)) \rightarrow ((x \wedge y) \vee (x \wedge z))$,

(A11) $(\sim\sim x) \rightarrow x$,

(A12) $(x \rightarrow\sim y) \rightarrow (y \rightarrow\sim x)$,

(A13) $(x \rightarrow (y \rightarrow z)) \leftrightarrow ((x \cdot y) \rightarrow z)$,

(A14) $x \leftrightarrow (1 \rightarrow x)$,

(R1) $x, x \rightarrow y \rhd y$ (MP, modus ponens),

(R1) $x, y \rhd x \wedge y$ (A, adjunction).

Let $\mathcal{F}_{\mathrm{RW}}$ be the set of formulas of the system RW. We define the notions of proof and theorem on **RW** as usual. A *proof* of a formula φ in $\mathcal{F}_{\mathrm{RW}}$ is a finite nonempty sequence of formulas $\varphi_0, \varphi_1, ..., \varphi_{n-1} = \varphi$ in $\mathcal{F}_{\mathrm{RW}}$ such that for each $i < n$ one of the following is true:

(i) φ_i is a substitution instance of one of the axioms (A1) to (A14) above;
(ii) for some $j, k < i$, φ_k is $\varphi_j \rightarrow \varphi_i$;
(iii) φ_i is $\varphi_j \wedge \varphi_k$ for some $j, k < i$.

The formula φ in $\mathcal{F}_{\mathrm{RW}}$ is called a *theorem* of RW if it has a proof in RW. Let $\Gamma \subseteq \mathcal{F}_{\mathrm{RW}}$. Then we consider the *deducibility relation* of RW as follows: $\Gamma \vdash_{\mathrm{RW}} \varphi$ iff there is a finite sequence of formulas $\varphi_0, \varphi_1, ..., \varphi_{n-1} = \varphi$ in $\mathcal{F}_{\mathrm{RW}}$ such that for each $i < n$, either $\varphi \in \Gamma$ or one of the conditions (i), (ii), (iii) in the earlier definition of proofs and theorems is true. In this case we say that φ is a consequence of Γ.

The variety of involutive residuated lattices, called iCDRL, models the deducibility relation of RW in the sense of the following definition of [2]: a deductive system S is called algebrizable if it has an equivalent algebraic semantics K. We can get an axiomatization of K from [2, Theorem 2. 17]. The notation $\mathrm{K} \models \alpha \approx \beta$ means that the equation $\alpha \approx \beta$ holds in every algebra of K.

3.1 The system RWκ

We define the formal system RWκ, which signature is $\{\cdot, \to, \wedge, \vee, \sim, \kappa, 0, 1, \boldsymbol{c}\}$. The axioms of this system are (A1) to (A14) from above, (A15) to (A19), and the rules are (R1), (R2), where the axioms (A15) to (A19) are as follows:

(A15) $(x \cdot y) \to (x \wedge y)$,
(A16) $\boldsymbol{c} \leftrightarrow \sim \boldsymbol{c}$,
(A17) $((x \cdot y) \wedge \boldsymbol{c}) \leftrightarrow ((x \cdot (y \wedge \boldsymbol{c})) \vee (y \cdot (x \wedge \boldsymbol{c})))$,
(A18) $(\kappa x \wedge \boldsymbol{c}) \leftrightarrow (x \wedge \boldsymbol{c})$,
(A19) $(\kappa x \vee \boldsymbol{c}) \leftrightarrow (\boldsymbol{c} \to x)$.

In every integral residuated lattice $\langle T, \wedge, \vee, ., \to, 1\rangle$ we have that $x.y \leq x \wedge y$ for any $x, y \in T$. This is the motivation of axiom (A15). Moreover, the axiom (A15) allows us to prove the theorem $x \to (y \to (x \wedge y))$, from where it is possible to eliminate rule (R2).

Let $\mathcal{F}$ be the set of formulas of RWκ. We write $\equiv$ for the following equivalence relation defined on the set $\mathcal{F}$, where $\vdash$ is the deducibility relation of RWκ: $\alpha \equiv \beta$ iff $\vdash \alpha \leftrightarrow \beta$.

Lemma 3.1

(i) *Let $\alpha, \beta, \gamma, \eta \in \mathcal{F}$ such that $\alpha \equiv \beta$ and $\gamma \equiv \eta$. Then $\alpha \vee \gamma \equiv \beta \vee \eta$ and $\alpha \wedge \gamma \equiv \beta \wedge \eta$.*
(ii) *Let $\alpha, \beta \in \mathcal{F}$, and suppose that there is $\gamma \in \mathcal{F}$ such that $\alpha \wedge \gamma \equiv \beta \wedge \gamma$ and $\alpha \vee \gamma \equiv \beta \vee \gamma$. Then $\alpha \equiv \beta$.*
(iii) $\kappa(\boldsymbol{c}) \equiv 1$.

P r o o f . The items (i) and (ii) follow immediately. In order to prove (iii), by (A18) we have that $\kappa(\boldsymbol{c}) \wedge \boldsymbol{c} \equiv \boldsymbol{c} \wedge 1$, and by (A19) we obtain $\kappa(\boldsymbol{c}) \vee \boldsymbol{c} \equiv \boldsymbol{c} \vee 1$. Hence, by (ii), we conclude that $\kappa(\boldsymbol{c}) \equiv 1$. □

Proposition 3.2

(a) Let $\hat{\mathbf{c}}$ satisfy the axioms (A16) to (A19). Then $\vdash \boldsymbol{c} \leftrightarrow \hat{\mathbf{c}}$.
(b) Let $\alpha \in \mathcal{F}$. Suppose that $\hat{\kappa}$ satisfies the axioms (A18) and (A19). Then $\vdash \kappa\alpha \leftrightarrow \hat{\kappa}\alpha$.

P r o o f . First we prove (a). By Lemma 3.1 and axiom (A18) we obtain that $\boldsymbol{c} \equiv \boldsymbol{c} \wedge \hat{\mathbf{c}}$. In a similar way we obtain that $\hat{\mathbf{c}} \equiv \boldsymbol{c} \wedge \hat{\mathbf{c}}$. Thus, $\boldsymbol{c} \equiv \hat{\mathbf{c}}$, i.e., $\vdash \boldsymbol{c} \leftrightarrow \hat{\mathbf{c}}$. Now we prove (b). By (A18) and (A19) we have that $\kappa\alpha \wedge \boldsymbol{c} \equiv \hat{\kappa}\alpha \wedge \boldsymbol{c}$ and $\kappa\alpha \vee \boldsymbol{c} \equiv \hat{\kappa}\alpha \vee \boldsymbol{c}$. Therefore by Lemma 3.1 we conclude that $\kappa\alpha \equiv \hat{\kappa}\alpha$, i.e., $\vdash \kappa\alpha \leftrightarrow \hat{\kappa}\alpha$. □

By Proposition 3.2 and [6, Theorem 1] (cf. also [14, Lemma 8.4]), we deduce that the variety DRL$'$ is the equivalent algebraic semantics for the consequence relation $\vdash$. In particular, we have the following two facts:

(i) Let $\alpha, \beta \in \mathcal{F}$. Then $\vdash \alpha \leftrightarrow \beta$ iff DRL$' \models \alpha \approx \beta$. Equivalently, for any $\alpha \in \mathcal{F}$ we have that $\vdash \alpha$ iff DRL$' \models \alpha \approx 1$.
(ii) If $A_{\mathrm{DRL}'} = \mathcal{F}/\!\equiv$, then $A_{\mathrm{DRL}'} \in$ DRL$'$ and in consequence it is the free algebra in countably infinite generators on the variety DRL$'$.

Let $\mathbf{T} = \langle T, \wedge, \vee, *, \to, \sim, \kappa, 0, 1, \boldsymbol{c}\rangle$ be an algebra of DRL$'$. We consider the reduct algebra $\mathbf{R}(\mathbf{T}) = \langle T, \wedge, \vee, *, \to, \sim, 0, 1, \boldsymbol{c}\rangle$.

Corollary 3.3 *The function κ is compatible in* $\mathbf{R}(\mathbf{T})$.

P r o o f . Consider the system S, where the axioms are (A1) to (A14), (A15) to (A17), and the only deduction rule is modus ponens. Let K be the variety characterized by the equations of iCDRL plus the equations associated to the axioms (A15) to (A17). By [14, Lemma 8.4], K is the equivalent algebraic semantics for the deducibility relation of S. The conclusion of the corollary follows from Proposition 3.2 and [6, Theorem 4]. □

Finally we give a weak deduction theorem (or local deduction theorem) for RWκ, which can be deduced from [15, Corollary 2.15].

Theorem 3.4 *Let $\alpha, \beta \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$. Then $\Gamma \cup \{\alpha\} \vdash \beta$ iff there is $n \in \mathbb{N}$ such that $\Gamma \vdash \alpha^n \to \beta$.*

3.2 The system Ł$^\bullet$

We define the formal system Ł$^\bullet$ whose signature is the same as that of RWκ. The axioms of this system are the axioms of RWκ, (A20), (A21), (A22), and the rule is (MP), where the axioms (A20), (A21) and (A22) are as follows:

(A20) $\sim \kappa x \leftrightarrow \kappa(\sim \kappa x)$,

(A21) $\boldsymbol{c} \to ((x \to y) \vee (y \to x))$,

(A22) $((x \cdot \boldsymbol{c}) \cdot (x \to (y \vee \boldsymbol{c})) \leftrightarrow ((x \wedge y) \cdot \boldsymbol{c})$.

The variety MV$^\bullet$ is the equivalent algebraic semantics for the deducibility relation $\vdash_{Ł^\bullet}$ of the system Ł$^\bullet$.

We abuse notation and we also write $\equiv$ for the corresponding equivalence relation defined on $\mathcal{F}$: $\alpha \equiv \beta$ iff $\vdash_{Ł^\bullet} \alpha \leftrightarrow \beta$. In particular, $A_{\mathrm{MV}^\bullet} = \mathcal{F}/\equiv$ is the free algebra in countably infinite generators on the variety MV$^\bullet$. For $\alpha \in \mathcal{F}$ we write $\overline{\alpha}$ for the class of α relative to the equivalence relation $\equiv$. Let A_{MV} be the free algebra in countably infinite generators on the variety MV.

Now we study relation between A_{MV} and $A_{\mathrm{MV}^\bullet}$. We start with some preliminary lemmas.

Lemma 3.5 *Let $\langle T, \wedge, \vee, *, \to, \sim, \kappa, 0, 1, \boldsymbol{c}\rangle \in$ MV$^\bullet$ and let $x, y \in T$. Then the following conditions hold:*

(a) $\kappa 0 = 0$ *and* $\kappa(\boldsymbol{c}) = \kappa 1 = 1$,
(b) $\kappa(x \wedge y) = \kappa(x) \wedge \kappa(y)$,
(c) $\kappa(x \vee y) = \kappa(x) \vee \kappa(y)$,
(d) $\kappa(x * y) = (\lambda x * \kappa y) \vee (\lambda y * \kappa x)$,
(e) $\kappa(x + y) = \kappa(x) + \kappa(y)$, *where* $x + y = \sim((\sim x) * (\sim y))$,
(f) $\lambda(x * y) = \lambda x * \lambda y$,
(g) $\kappa(x \to y) = \lambda x \to \kappa y$,
(h) $\sim \kappa x \leq \kappa \sim x$.

P r o o f. It follows from results of [8]. Other possible proof follows from the categorical equivalence between MV and MV$^\bullet$, and the fact that for $A \in$ MV the map $\kappa : \mathsf{K}^\bullet(A) \to \mathsf{K}^\bullet(A)$ is given by $\kappa(a, b) = (\neg b, b)$. □

Lemma 3.6 *Let T be in MV$^\bullet$ generated by $\{y_1, y_2, ...\}$ and let $X = \{\kappa y_1, \kappa y_2, ...\} \cup \{\kappa \sim y_1, \kappa \sim y_2, ...\}$. If $v \in T$, then κv and λv can be expressed as a combination of elements of X. Moreover, $\kappa(T)$ is generated by X.*

P r o o f. We give a proof by induction on the complexity $\mathrm{c}(v)$ of v.

First suppose that $\mathrm{c}(v) = 0$. Hence, v must be 1, 0 or $\boldsymbol{c}$. If $v = 1$, then $\kappa v = 1 = \kappa y_1 \to \kappa y_1$. If $v = 0$, then $\kappa v = 0 = \sim(\kappa y_1 \to \kappa y_1)$. If $v = \boldsymbol{c}$, then $\kappa v = 1 = \kappa y_1 \to \kappa y_1$. Besides $\lambda 0 = \lambda \boldsymbol{c} = \kappa 0$ and $\lambda 1 = \kappa 1$.

Second, suppose that, for $\mathrm{c}(v) < n$, κv and λv can be expressed as a combination of elements of X. Let v be an element of T such that $\mathrm{c}(v) = n$. We consider the following cases:

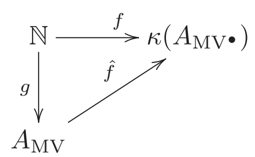
(1) $v = \sim w$. Then we can decompose κw in terms of X because $\mathrm{c}(w) = n - 1$. So, $\kappa v = \kappa \sim w = \sim \lambda w$ and $\lambda v = \sim \kappa w$ can be decomposed too.
(2) $v = w * z$. We have that $\kappa v = \lambda w * \kappa z \vee \kappa w * \lambda z$, and by inductive hypotheses κv can be decomposed. Besides we have that $\lambda v = \lambda w * \lambda z$, so it can be decomposed too.
(3) $v = w \to z$. This case follows from the two precedents.
(4) $v = w \wedge z$. We have that $\kappa v = \kappa w \wedge \kappa z$ and $\lambda v = \lambda w \wedge \lambda z$, so the result follows.
(5) $v = w \vee z$. This case is similar to the previous one. □

Remark 3.7 Let $\mathbb{N}$ be the set of natural numbers, and for $n \in \mathbb{N}$ let y_n be the nth propositional variable of Ł$^\bullet$. We consider the function $f : \mathbb{N} \to \kappa(A_{\mathrm{MV}^\bullet})$ given by $f(2n) = \kappa(\overline{y_n})$ and $f(2n - 1) = \kappa(\sim \overline{y_n})$. Let y_i, y_j be two distinct propositional variables and suppose that $\kappa(\overline{y_i}) = \kappa(\overline{y_j})$. Then the equation $\kappa(x) = \kappa(y)$ would be true in every algebra of $\mathrm{MV}^\bullet$, which is false. Thus, $\kappa(\overline{y_i}) \neq \kappa(\overline{y_j})$. In a similar way we can prove that $\kappa(\overline{y_i}) \neq \kappa(\sim \overline{y_j})$, and that $\kappa(\sim \overline{y_i}) \neq \kappa(\sim \overline{y_j})$. Hence, f is an injective function and the set X as above theorem is countably infinite.

The isomorphism $\beta : T \longrightarrow \mathsf{K}^\bullet(\kappa(T))$, for T in $\mathrm{MV}^\bullet$, maps $u \mapsto (\lambda u, \lambda \sim u)$. So, in particular, $\lambda u * \lambda \sim u = 0$, equality that can be rewritten in terms of κ and $+$ as: $\kappa u + \kappa \sim u = 1$. This equality holds in $\kappa(T)$. For this reason the set X of the above lemma is not a free system of generators. For example, let $h : X \to \{0, 1\}$ be the function given by $h(x) = 0$ for every $x \in X$. It follows from the equation $\sim \kappa x \leq \kappa \sim x$ that the function can not be extended to an homomorphism $\hat{h} : \kappa(A_{\mathrm{MV}^\bullet}) \to \{0, 1\}$ in MV.

Theorem 3.8 *There is an ideal I in A_{MV} such that $\kappa(A_{\mathrm{MV}^\bullet}) \cong A_{\mathrm{MV}}/I$.*

P r o o f. Let $g : \mathbb{N} \to A_{\mathrm{MV}}$ be the function given by $g(n) = |x_n|$, where $|x_n|$ is the class of the propositional variable x_n relative to the canonical equivalence relation of Ł. Let f be the function given in Remark 3.7. By Lemma 3.6 the set X generates $\kappa(A_{\mathrm{MV}^\bullet})$, so we have that there is an epimorphism $\hat{f} : A_{\mathrm{MV}} \to \kappa(A_{\mathrm{MV}^\bullet})$ in MV which extends f and such that the following diagram commutes:



Therefore $\kappa(A_{\mathrm{MV}^\bullet}) \cong A_{\mathrm{MV}}/\mathrm{Ker}(\hat{f})$, where $\mathrm{Ker}(\hat{f}) = \{|\alpha| \in A_{\mathrm{MV}} : \hat{f}(|\alpha|) = \overline{0}\}$. □

It is well known that $\mathrm{MV} = \mathbb{V}([0, 1])$ (cf. [11]). Then it follows from the categorical equivalence between MV and $\mathrm{MV}^\bullet$ the following

Proposition 3.9 *The variety* $\mathrm{MV}^\bullet$ *is generated by* $\mathsf{K}^\bullet([0, 1])$. *Equivalently, an equation holds in every algebra of* $\mathrm{MV}^\bullet$ *iff it holds in* $\mathsf{K}^\bullet([0, 1])$.

Let X be an countably infinite set of propositional variables $\{x_1, x_2, ...\}$. Any function $v : X \to T$ with $T \in \mathrm{MV}^\bullet$ (called a *T-valuation*) can be extended to a unique homomorphism $\overline{v} : \mathcal{F} \to T$. Let $\alpha \in \mathcal{F}$ and $T \in \mathrm{MV}^\bullet$. We say that α is a *T-tautology* if for any T-valuation v: $\overline{v}(\alpha) = 1$.

We have the following result about completeness of Ł$^\bullet$:

Corollary 3.10 *Let $\alpha \in \mathcal{F}$. Then, $\vdash_{Ł^\bullet} \alpha$ iff α is a $\mathsf{K}^\bullet([0, 1])$-tautology.*

4 On U-operators in MV and MV$^\bullet$

The algebraic treatment of classical reasoning involves the notions of propositional functions and quantifiers. Consider the following example:

$$\frac{\begin{array}{c}\text{“All men are mortal”}\\ \text{“Some Greeks are men”}\end{array}}{\text{“Some Greeks are mortal”}}$$

Concerning this inference, Halmos says “Within the framework of Boolean algebras alone, it is not possible to formulate the inference that allows, . . ., the conclusion ‘Some Greeks are mortal’ ” ; instead, “a study of what ‘some’ and ‘all’ mean” is necessary [18, p. 20].

Roughly speaking, he defined propositional functions (like “x is mortal”, “x is even”, etc.) from some set X (for example, men, natural numbers, etc.) to a Boolean algebra B (of propositions) and existential quantifiers as certain algebraic operators. The application of an existential operator to a propositional function produces a constant, that is, a proposition of B. He also defines the dual notion of universal quantifier.

In the original approach of Halmos, a quantifier ∇ is a map satisfying the following conditions:

(H1) $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$,

(H2) $\nabla 0 = 0$,

(H3) $x \leq \nabla x$.

These conditions imply that the image of ∇ is a subalgebra and coincides with the set of fixed points of ∇. This two results seem to be essential.

The notion of quantifier has been generalized from there in many ways. For example, the definition of a Q-distributive lattice is given in [10] and in [23] a quantifier for De Morgan algebras is defined.

In [12], the authors define a quantifier in MV-algebras as a map such that (E1) to (E6) hold, where (E1) is (H3) above, and

(E2) $\nabla(x \vee y) = \nabla x \vee \nabla y$,
(E3) $\nabla\neg\nabla x = \neg\nabla x$,
(E4) $\nabla(\nabla x \oplus \nabla y) = \nabla x \oplus \nabla y$,
(E5) $\nabla(x \odot x) = \nabla x \odot \nabla x$, and
(E6) $\nabla(x \oplus x) = \nabla x \oplus \nabla x$.

More references about quantifiers on MV-algebras can be found in [16,17,25,26]. It follows from results of [8] that for $\langle T, \wedge, \vee, *, \to, \sim, 0, 1, \boldsymbol{c}\rangle \in \mathrm{MV}^{\bullet}$ the map κ satisfies the conditions (E1) to (E4), (E6) and the inequality $\kappa(x * x) \le \kappa x * \kappa x$. By this reason we say that κ satisfies some "quantifier-like" conditions.

In [22], the authors give a construction to obtain Wajsberg algebras from themselves, and this construction is analyzed when the original algebra is endowed with a U-operator, which expresses a possible notion of universal quantifier. The U-operators on Wajsberg algebras have been studied in [19–21].

In this section we build up a bijection between the set of U-operators of a MV-algebra A and some subset of U-operators of $\mathsf{K}^{\bullet}(A)$, the $\boldsymbol{c}$U–operators, where we define U-operators in algebras of $\mathrm{MV}^{\bullet}$ as in the case of MV-algebras. Also, we study $\boldsymbol{c}$-relatively complete subalgebras of algebras of $\mathrm{MV}^{\bullet}$, that are in a bijective relationship with the $\boldsymbol{c}$U-operators. To end the section we extend the categorical equivalence $\mathscr{K} \dashv \mathsf{K}^{\bullet}$(cf. [8]) by defining two extended functors, that we shall call also $\mathscr{K}$ and $\mathsf{K}^{\bullet}$, between the categories $\mathcal{MV}_{\mathrm{U}}$, whose objects are pairs formed by an MV-algebra and a U-operator, and $\mathcal{MV}^{\bullet}{}_{c\mathrm{U}}$, whose objects are pairs formed by an object of $\mathrm{MV}^{\bullet}$ and a $\boldsymbol{c}$U-operator.

4.1 U-operators

Definition 4.1 Let $A \in \mathrm{MV}$. A function $\forall : A \to A$ is a U-operator if it has the following properties:

(U1) $\forall a \to a = 1$,
(U2) $\forall(\forall a \to b) = \forall a \to \forall b$.

Every U-operator on a MV-algebra has the following properties:

(U4) $\forall\forall a = \forall a$,
(U5) $a \le b$ implies $\forall a \le \forall b$,
(U6) $\forall(a \wedge b) = \forall a \wedge \forall b$,
(U7) $\forall(a \vee \forall b) = \forall a \vee \forall b$,
(U8) $\forall\neg\forall a = \neg\forall a$,
(U9) $\forall(a.\forall b) = \forall a.\forall b$,
(U10) $\forall a.\exists b = \exists(b.\forall a) = \forall(a.\exists b)$, where $\exists a = \neg\forall\neg a$,
(U11) $\forall 0 = 0, \forall 1 = 1$,
(U12) $\forall\exists a = \exists a, \exists\forall a = \forall a$.

Let $T \in \mathrm{MV}^{\bullet}$. We say that a function $\forall : T \to T$ is a *U-operator* if it satisfies the conditions (U1) and (U2) above.

Proposition 4.2 *Let $A \in \mathrm{MV}$, and let $\forall : A \to A$ be a U-operator.*

(a) The function $\Delta : \mathsf{K}^{\bullet}(A) \to \mathsf{K}^{\bullet}(A)$ given by $\Delta(a, b) = (\forall a, \exists b)$ is a U-operator.
(b) The function $\lambda : \mathsf{K}^{\bullet}(A) \to \mathsf{K}^{\bullet}(A)$, which is given by $\lambda(a, b) = (a, \neg a)$, is a U-operator.
(c) $\Delta = \lambda$ iff A is trivial.

Proof. First we prove (a). Let $(a, b) \in \mathsf{K}^\bullet(A)$, so $a \leq \neg b$. Then, $\forall a \leq \forall\neg b$ and in consequence we obtain $\forall a.\exists b = 0$. Therefore Δ is a well defined map.

In order to prove that Δ satisfies (U1), let $(a, b) \in \mathsf{K}^\bullet(A)$. As $\forall a.b \leq \forall a.\exists b = 0$, we have that $\forall a.\exists b = 0$. Then

$$\begin{aligned}\Delta(a, b) \to (a, b) &= (\forall a, \exists b) \to (a, b)\\ &= ((\forall a \to a) \wedge (b \to \exists b), (\forall a).b)\\ &= (1, 0).\end{aligned}$$

Now let $(a, b), (d, e) \in \mathsf{K}^\bullet(A)$. Then

$$\begin{aligned}\Delta(\Delta(a, b) \to (d, e)) &= \Delta((\forall a, \exists b) \to (d, e))\\ &= \Delta((\forall a \to d) \wedge (e \to \exists b), (\forall a).e)\\ &= (\forall((\forall a \to d) \wedge (e.\exists b)), \exists((\forall a).e))\\ &= ((\forall a \to \forall d) \wedge (\forall e \to \exists b), \forall a.\exists e).\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\Delta(a, b) \to \Delta(d, e) &= (\forall a, \exists b) \to (\forall d, \exists e))\\ &= (\forall a \to \forall d) \wedge (\exists e \to \exists b), \forall a.\exists e).\end{aligned}$$

Therefore Δ satisfies the condition (U2).

Now we prove (b). It is immediate that λ satisfies (U1). In order to prove that λ satisfies the condition (U2), let $(a, b), (d, e) \in \mathsf{K}^\bullet(A)$. Then $a.(a \to d).e \leq d.e = 0$, so $a \to d \leq e \to \neg a$. Then we obtain that

$$\begin{aligned}\lambda(\lambda(a, b) \to (d, e)) &= \lambda((a, \neg a) \to (d, e))\\ &= \lambda((a \to d) \wedge (e \to \neg a), a.e)\\ &= \lambda(a \to d, a.e)\\ &= (a \to d, \neg(a \to d))\\ &= (a \to d, a.\neg d).\end{aligned}$$

Besides we have that

$$\begin{aligned}\lambda(a, b) \to \lambda(d, e) &= (a, \neg a) \to (d, \neg d)\\ &= (a \to d) \wedge (\neg d \to \neg a), a.\neg d)\\ &= (a \to d, a.\neg d).\end{aligned}$$

Thus, λ satisfies (U2).

Finally we prove (c). Suppose that $\Delta = \lambda$ and let $a \in A$. Then $\forall a = a$ because $(a, \neg a) \in \mathsf{K}^\bullet(A)$. Thus, $\forall$ is the identity function. Besides $b = \neg a$ for every $(a, b) \in \mathsf{K}^\bullet(A)$. In particular, as $(0, 0) \in \mathsf{K}^\bullet(A)$ we obtain $0 = 1$. Hence, $A = \{0\}$. The converse is immediate. □

Let $A \in \mathrm{MV}$. If $\forall : A \to A$ is a U-operator, then $\Delta(\lambda(a, b)) = \lambda(\Delta(a, b)) = (\forall a, \neg\forall a)$ for every $(a, b) \in \mathsf{K}^\bullet(A)$.

Definition 4.3 Let $T \in \mathrm{MV}^\bullet$ and let $\Omega : T \to T$ be a function. We consider the following conditions for every $u \in T$:

(i) $\Omega(u \vee \boldsymbol{c}) = \Omega(u) \vee \boldsymbol{c}$,
(ii) $\Omega(u \wedge \boldsymbol{c}) = \Omega(u) \wedge \boldsymbol{c}$,
(iii) $\Omega(u * \boldsymbol{c}) = \Omega(u) * \boldsymbol{c}$.

A U-operator satisfying conditions (i), (ii) and (iii) is called a $\boldsymbol{c}$U-operator.

Remark 4.4 Let $T \in \mathrm{MV}^{\bullet}$, let $\Omega : T \to T$ be a $\boldsymbol{c}$U-operator and let $u \in T$. Then $u \geq \boldsymbol{c}$ (respectively $u \leq \boldsymbol{c}$) implies $\Omega(u) \geq \boldsymbol{c}$ (respectively $\Omega(u) \leq \boldsymbol{c}$). Hence, $\Omega(\boldsymbol{c}) = \boldsymbol{c}$.

As usual, if X is a set we consider the map $\pi_1 : X \times X \to X$ given by $\pi_1(a, b) = a$.

Theorem 4.5 *Let $A \in \mathrm{MV}$. Then $\forall \longmapsto \Delta$ is a bijection between the set of U-operators of A and the set of $\boldsymbol{c}$U-operators of $\mathsf{K}^{\bullet}(A)$.*

Proof. Let $\forall : A \to A$ be a U-operator. By Proposition 4.2 we have that the map $\Delta : \mathsf{K}^{\bullet}(A) \to \mathsf{K}^{\bullet}(A)$ given by $\Delta(a, b) = (\forall a, \neg\forall\neg b)$ is a U-operator. Moreover, we have that

$$\Delta((a, b) \vee \boldsymbol{c}) = \Delta(a, 0) = (\forall a, 0) = \Delta(a, b) \vee \boldsymbol{c},$$

$$\Delta((a, b) \wedge \boldsymbol{c}) = \Delta(0, b) = (0, \exists b) = \Delta(a, b) \wedge \boldsymbol{c},$$

$$\Delta((a, b) * \boldsymbol{c}) = \Delta(0, \neg a) = (0, \neg\forall a) = (\forall a, \exists b) * \boldsymbol{c} = \Delta(a, b) * \boldsymbol{c}.$$

Suppose that $\forall_1 : A \to A$ and $\forall_2 : A \to A$ are U-operators such that $(\forall_1 a, \exists_1 b) = (\forall_2 a, \exists_2 b)$ for every $(a, b) \in \mathsf{K}^{\bullet}(A)$. Let $a \in A$. As $(a, \neg a) \in \mathsf{K}^{\bullet}(A)$, we obtain $\forall_1 a = \forall_2 a$. Hence, $\forall_1 = \forall_2$. Thus, $\forall \longmapsto \Delta$ is injective.

On the other hand, let $\Omega : \mathsf{K}^{\bullet}(A) \to \mathsf{K}^{\bullet}(A)$ be a $\boldsymbol{c}$U-operator. We define the function $\forall : A \to A$ as $\forall a = \pi_1\Omega(a, 0)$. This function is a U-operator. In order to prove it, let $a \in A$. By Remark 4.4 we have that $\Omega(a, 0) = \Omega((a, 0) \vee \boldsymbol{c}) = (\forall a, 0)$. Thus, $\forall a \to a = 1$ because $(\forall a, 0) \to (a, 0) = (1, 0)$. Let us prove that $\forall$ satisfies (U2). Let $a, b \in A$. Then $(\forall a \to b, 0) = (\forall a, 0) \to (b, 0) = \Omega(a, 0) \to (b, 0)$. Hence

$$\begin{aligned}
(\forall(\forall a \to b), 0) &= \Omega(\Omega(a, 0) \to (b, 0)) \\
&= \Omega(a, 0) \to \Omega(b, 0) \\
&= (\forall a, 0) \to (\forall b, 0) \\
&= (\forall a \to \forall b, 0).
\end{aligned}$$

Thus, $\forall(\forall a \to b) = \forall a \to \forall b$.

Now we prove that $\Omega = \Delta$. First, we shall see that $\Omega(u) = \Delta(u)$ for $u \leq \boldsymbol{c}$ and for $u \geq \boldsymbol{c}$. Indeed, $\Omega(a, 0) = (\forall a, 0) = \Delta(a, 0)$ and

$$\begin{aligned}
\Omega(0, b) &= \Omega((\neg b, 0) * \boldsymbol{c}) \\
&= \Omega(\neg b, 0) * \boldsymbol{c} \\
&= (\forall\neg b, 0) * \boldsymbol{c} \\
&= (0, \exists b) \\
&= \Delta(0, b).
\end{aligned}$$

Finally we show that $\Omega(a, b) \vee \boldsymbol{c} = \Delta(a, b) \vee \boldsymbol{c}$ and $\Omega(a, b) \wedge \boldsymbol{c} = \Delta(a, b) \wedge \boldsymbol{c}$. In fact,

$$\Omega(a, b) \vee \boldsymbol{c} = \Omega(a, 0) = (\forall a, 0) = \Delta(a, b) \vee \boldsymbol{c},$$

$$\Omega(a, b) \wedge \boldsymbol{c} = \Omega(0, b) = (0, \exists b) = \Delta(a, b) \wedge \boldsymbol{c}.$$

Therefore the result follows by distributivity. □

Let $A \in \mathrm{MV}$. Then $\lambda : \mathsf{K}^{\bullet}(A) \to \mathsf{K}^{\bullet}(A)$ is a $\boldsymbol{c}$U-operator iff A is trivial.

4.2 Relatively complete subalgebras of algebras of $\mathrm{MV}^{\bullet}$

Using a standard argument (cf. [18] for the case of monadic Boolean algebras), it is possible to prove that, for an MV-algebra A, there is a bijection between the set of U-operators on A and the set of subalgebras of A that are *relatively complete*. A subalgebra A_0 is called relatively complete if for every $x \in A$ the set $A_0 \cap (x]$ has a maximum. We build up a bijection between the set of $\boldsymbol{c}$U-operators of an algebra T of $\mathrm{MV}^{\bullet}$ and some subset of subalgebras relatively complete of T, where we define these subalgebras as in the case of MV-algebras.

We state in what follows some properties of U-operators and $\boldsymbol{c}$U-operators on algebras of $\mathrm{MV}^{\bullet}$.

Lemma 4.6 *Let Ω be a U-operator on T. Then, the following properties hold:*

(U_{01}) $\Omega 0 = 0$ *and* $\Omega 1 = 1$,

(U4) $\Omega\Omega u = \Omega u$,

(U5) $u \leq v$ *implies* $\Omega u \leq \Omega v$,

(U8) $\Omega \sim \Omega u = \sim \Omega u$.

P r o o f. From (U1) we have that $\Omega 0 \leq 0$. From (U1) and (U2): $\Omega 1 = \Omega(\Omega u \to u) = \Omega u \to \Omega u = 1$.

Let us now prove (U4) by using (U_{01}) and (U2). We have that $\Omega\Omega u = \Omega(1 \to \Omega u) = \Omega(\Omega 1 \to \Omega u) = \Omega 1 \to \Omega u = 1 \to \Omega u = \Omega u$. Suppose $u \leq v$. Then $\Omega u \leq u \leq v$, so $\Omega u \to v = 1$, from where $\Omega(\Omega u \to v) = 1 = \Omega u \to \Omega v$. So, $\Omega u \leq \Omega v$, That is, (U5) holds.

To see (U8), we have by (U2) and (U_{01}) that $\Omega \sim \Omega u = \Omega(\Omega u \to 0) = \Omega u \to \Omega 0 = \Omega u \to 0 = \sim \Omega u$. □

Lemma 4.7 *Let Ω be a $\boldsymbol{c}$U-operator on T. Then, the following properties hold:*

(U6) $\Omega(u \wedge v) = \Omega u \wedge \Omega v$,

(U7) $\Omega(u \vee \Omega v) = \Omega u \vee \Omega v$,

(U9) $\Omega u * \Omega v = \Omega w$, *for some* $w \in T$.

P r o o f. The first two equalities follow from (U6) and (U7) for MV-algebras, because we can assume that $T \cong \mathsf{K}^{\bullet}(A)$ with A an MV-algebra, and that every $\boldsymbol{c}$U-operator has the form $\Omega(a, b) = (\forall a, \exists b)$.

We have $\Omega u * \Omega v = \Omega(a, b) * \Omega(d, e) = (\forall a . \forall b, (\forall a \to \exists e) \wedge (\forall d \to \exists b))$. For MV-algebras we can see that $\forall x \to \exists y = \exists(x \to \exists y)$ and $\exists r \wedge \exists s = \exists(r \wedge \exists s)$. From these equalities we deduce that $(\forall a \to \exists e) \wedge (\forall d \to \exists b) = \exists((a \to \exists e) \wedge \exists(d \to \exists b))$. Also, by (U9), $\forall a . \forall b = \forall(a . \forall b)$. Thus, for $w = (a . \forall b, (a \to \exists e) \wedge \exists(d \to \exists b))$ we have $\Omega u * \Omega v = \Omega w$. □

Now we give a bijection between $\boldsymbol{c}$U-operators and certain relatively complete subalgebras that we shall call $\boldsymbol{c}$*-relatively complete*.

Definition 4.8 Let T be an algebra in $\mathrm{MV}^{\bullet}$, T_0 a subalgebra of T. We say that T_0 is a $\boldsymbol{c}$-relatively complete subalgebra of T if the following conditions hold:

(RC) For every $u \in T$ the set $T_0 \cap (u]$ has a maximum.

($\vee$) For every $u \in T$, if $t \in T_0 \cap (u \vee \boldsymbol{c}]$ then there is $s \in T_0 \cap (u]$ such that $t \leq s \vee \boldsymbol{c}$.

($*$) For every $u \in T$, if $t \in T_0 \cap (u * \boldsymbol{c}]$ then there is $s \in T_0 \cap (u]$ such that $t \leq s * \boldsymbol{c}$.

Lemma 4.9 *Let Ω be a $\boldsymbol{c}$U-operator on T, T an algebra in* $\mathrm{MV}^{\bullet}$. *Then, the image of T by Ω, noted $\Omega(T)$, is a $\boldsymbol{c}$-relatively complete subalgebra of T.*

P r o o f. First we prove that $\Omega(T)$ is a subalgebra. In fact, from Lemma 4.6 and condition (U8), we have that Ω is closed by $\sim$. By Lemma 4.7 we have that Ω is closed by $\wedge$, $\vee$ and $*$. The property $(\mathbf{U_{01}})$ proves that 0 and 1 are in $\Omega(T)$, and by the definition of items (i) and (ii) of $\boldsymbol{c}$U-operators we have also that $\boldsymbol{c} \in \Omega(T)$. We have that $\Omega(T)$ is closed by κ because $\kappa\Omega(a, b) = \Omega\kappa(a, b)$.

To see (RC), we shall show that, for every $u \in T$, $\Omega u = \max(\Omega T \cap (u])$. In first place, $\Omega u \in \Omega(T) \cap (u]$. Second, suppose $t \in \Omega T \cap (u]$. Then $t = \Omega t$ (by (U4)) and $t \leq u$, from where $t \leq \Omega u$, thus proving $\Omega u = \max(\Omega(T) \cap (u])$.

Finally we prove ($\vee$) and ($*$). Let $t \in \Omega(T) \cap (u \vee \boldsymbol{c}]$, so $t = \Omega t$ and $t \leq u \vee \boldsymbol{c}$. Then, by (U5), Lemma 4.6 and by the fact that Ω is a $\boldsymbol{c}$U-operator, we have that $t \leq \Omega(u \vee \boldsymbol{c}) = \Omega u \vee \boldsymbol{c}$. If we take $s = \Omega u$, then ($\vee$) holds. To see ($*$), replace $\vee$ by $*$. □

Lemma 4.10 *Let T_0 be a relatively complete subalgebra of T and suppose that there exists the function $\Omega : T \to T$ given by $\Omega u = \max(T_0 \cap (u])$ for every $u \in T$. Then, Ω is a U-operator on T. Moreover, if T_0 satisfies the conditions $(\vee)$ and $(*)$ above then Ω is a $\boldsymbol{c}$U-operator on T.*

Proof. In first place, we can see that $t = \max(T_0 \cap (t]) = \Omega t$ for every $t \in T_0$. In particular, $\Omega \boldsymbol{c} = \boldsymbol{c}$, because being T_0 a subalgebra, $\boldsymbol{c} \in T_0$. It is obvious that (U1) holds for Ω, because $\Omega u \leq u$. Let us prove that $\Omega u \to \Omega v = \max(T_0 \cap (\Omega u \to v]) = \Omega(\Omega u \to v)$, that is, (U2). The expression $\Omega u \to \Omega v$ is in T_0, because T_0 is a subalgebra, and in $(\Omega u \to v]$, by (U1). Let $t \in T_0 \cap (\Omega u \to v]$. We shall prove that $t \leq \Omega u \to \Omega v$. We have that $t * \Omega u \leq v$ and $t * \Omega u \in T_0$. Then, $\Omega(t * \Omega u) = t * \Omega u$ and, by (U5), $t * \Omega u \leq \Omega v$, from where $t \leq \Omega u \to \Omega v$. Thus, $\Omega u \to \Omega v$ is the maximum.

Suppose that T_0 satisfies the conditions $(\vee)$ and $(*)$. First we can prove that Ω satisfies condition (ii) of $\boldsymbol{c}$U-operators. Indeed, by (U5), $\Omega(u \wedge \boldsymbol{c}) \leq \Omega u$ and $\Omega(u \wedge \boldsymbol{c}) \leq \Omega \boldsymbol{c} = \boldsymbol{c}$, so, $\Omega(u \wedge \boldsymbol{c}) \leq \Omega u \wedge \boldsymbol{c}$. But $\Omega u \wedge \boldsymbol{c} \in (T_0 \cap (u \wedge \boldsymbol{c}])$, from where $\Omega u \wedge \boldsymbol{c} \leq \Omega(u \wedge \boldsymbol{c})$. Then, $\Omega u \wedge \boldsymbol{c} = \Omega(u \wedge \boldsymbol{c})$. Now we are going to prove that $\Omega(u \vee \boldsymbol{c}) = \Omega u \vee \boldsymbol{c}$. We have that $\Omega u \vee \boldsymbol{c} \in T_0$ and $\Omega u \vee \boldsymbol{c} \leq u \vee \boldsymbol{c}$. We shall see that it is the maximum of $T_0 \cap (u \vee \boldsymbol{c}]$. If $t \in T_0 \cap (u \vee \boldsymbol{c}]$, for condition $(\vee)$ there exists $s \in T_0 \cap (u]$ such that $t \leq s \vee \boldsymbol{c}$. But $s \leq \Omega u$, then $t \leq \Omega u \vee \boldsymbol{c}$. In a similar way we can prove, by using $(*)$, that $\Omega(u * \boldsymbol{c}) = \Omega u * \boldsymbol{c}$. □

Straightforward computations prove the following:

Corollary 4.11 *Let $T \in$ MV$^\bullet$. There is a bijection between the set of $\boldsymbol{c}$-relatively complete subalgebras of T and the set of $\boldsymbol{c}$U-operators of T.*

4.3 Extending the categorical equivalence between MV and MV$^\bullet$

In what follows we shall extend the categorical equivalence $\mathscr{K} \dashv \mathsf{K}^\bullet$ (cf. [8]). In fact, we define two extended functors, that we shall call also $\mathscr{K}$ and $\mathsf{K}^\bullet$, between the category $\mathcal{MV}_{\mathrm{U}}$ whose objects are pairs formed by an MV-algebra and a U-operator, and the category $\mathcal{MV}^\bullet{}_{c\mathrm{U}}$ whose objects are pairs formed by an object of MV$^\bullet$ and a $\boldsymbol{c}$U-operator. A morphism $f : (A, \forall) \longrightarrow (A', \forall')$ in $\mathcal{MV}_{\mathrm{U}}$ is a morphism $f : A \longrightarrow A'$ in MV such that $f(\forall x) = \forall'(fx)$ for any $x \in A$. Analogous definition for morphisms in $\mathcal{MV}^\bullet{}_{c\mathrm{U}}$.

Lemma 4.12 *Let T be in* MV$^\bullet$*, Ω a $\boldsymbol{c}$U-operator. Then, $\Omega(\kappa u) = \kappa(\Omega u)$ and $\Omega(\lambda u) = \lambda(\Omega u)$.*

Proof. We only need to prove that $\Omega(\kappa u)$ satisfies the equations of $\kappa(\Omega u)$ because κ is uniquely determined. We have that $\Omega(\kappa u) \wedge \boldsymbol{c} = \Omega(\kappa u \wedge \boldsymbol{c}) = \Omega(u \wedge \boldsymbol{c}) = \Omega u \wedge \boldsymbol{c}$ and $\Omega(\kappa u) \vee \boldsymbol{c} = \Omega(\kappa u \vee \boldsymbol{c}) = \Omega(\boldsymbol{c} \to u) = \Omega(\Omega \boldsymbol{c} \to u) = \Omega \boldsymbol{c} \to \Omega u = \boldsymbol{c} \to \Omega u$. Analogous for λ. □

Lemma 4.12 allow us to give the following

Definition 4.13 The functor $\mathsf{K}^\bullet : \mathcal{MV}_{\mathrm{U}} \longrightarrow \mathcal{MV}^\bullet{}_{c\mathrm{U}}$ is defined by the assignment $(A, \forall) \mapsto (\mathsf{K}^\bullet(A), \Delta)$, where $\Delta = \forall \times \exists$ restricted to $\mathsf{K}^\bullet(A)$ (cf. Theorem 4.5). The functor $\mathscr{K} : \mathcal{MV}^\bullet{}_{c\mathrm{U}} \longrightarrow \mathcal{MV}_{\mathrm{U}}$ is defined by the assignment $(T, \Omega) \mapsto (\kappa(T), \Omega_{\kappa(T)})$, where $\Omega_{\kappa(T)}$ is the restriction of Ω to $\kappa(T)$.

Consider $(A, \forall)$ in $\mathcal{MV}_{\mathrm{U}}$ and apply to this object the composition of the two functors $\mathsf{K}^\bullet$ and $\mathscr{K}$. The result is $(\kappa(\mathsf{K}^\bullet(A)), \Delta_{\kappa(\mathsf{K}^\bullet(A))})$, where $\Delta_{\kappa(\mathsf{K}^\bullet(A))}$ is the restriction of $\forall \times \exists$ to $\kappa(\mathsf{K}^\bullet(A))$. On the other way, for an object (T, Ω) in $\mathcal{MV}^\bullet{}_{c\mathrm{U}}$, we obtain by the composition of the functors $\mathscr{K}$ and $\mathsf{K}^\bullet$ the object $(\mathsf{K}^\bullet(\kappa(T)), \Delta)$, where $\Delta = \Omega_{\kappa(T)} \times \sim \Omega_{\kappa(T)} \sim$.

Theorem 4.14 *Let $(A, \forall) \in \mathcal{MV}_{\mathrm{U}}$ and Let $(T, \Omega) \in \mathcal{MV}^\bullet{}_{c\mathrm{U}}$.*

1. *The isomorphism $\varphi : A \to \kappa(\mathsf{K}^\bullet(A))$ in* MV *extends to an isomorphism $\varphi : (A, \forall) \to (\kappa(\mathsf{K}^\bullet(A)), \Delta_{\kappa(\mathsf{K}^\bullet(A))})$.*
2. *The isomorphism $\beta : T \to \mathsf{K}^\bullet(\kappa(T))$ in* MV$^\bullet$ *extends to an isomorphism $\beta : (T, \Omega) \to (\mathsf{K}^\bullet(\kappa(T)), \Delta)$.*

Proof. The first isomorphism is given by the assignment $\varphi(a) = (a, \neg a)$ (cf. [8]). It is easy to see that φ is also a morphism in $\mathcal{MV}_{\mathrm{U}}$. In fact, $\varphi(\forall a) = (\forall a, \neg\forall a)$ and $\Delta_{\kappa(\mathsf{K}^\bullet(A))}(\varphi a) = (\forall a, \exists\neg a) = (\forall a, \neg\forall a)$.

The map β is defined by $\beta u = (\lambda u, \lambda \sim u)$ (cf. [8]). Then, $\beta(\Omega u) = (\lambda(\Omega u), \lambda(\sim \Omega u)) = (\lambda(\Omega u), \sim \kappa(\Omega u))$. By Lemma 4.12 we obtain that $\Delta(\beta u) = \Delta(\lambda u, \lambda \sim u) = (\Omega(\lambda u), \sim \Omega \sim (\lambda \sim u)) = (\Omega(\lambda u), \sim \Omega(\kappa u)) = \beta(\Omega u)$. □

A moment's reflection shows the following

Corollary 4.15 *There exists a categorical equivalence between* $\mathcal{MV}_{\mathrm{U}}$ *and* $\mathcal{MV}^{\bullet}{}_{c\mathrm{U}}$.

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