DE-EQUIVARIANTIZATION OF HOPF ALGEBRAS

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Abstract. We study the de-equivariantization of a Hopf algebra by an affine group scheme and we apply Tannakian techniques in order to realize it as the tensor category of comodules over a coquasi-bialgebra. As an application we construct a family of coquasi-Hopf algebras $A(H, G, \Phi)$ attached to a coradically-graded pointed Hopf algebra H and some extra data.

Introduction

Actions of groups over abelian categories have been studied in recent years with the purpose of constructing, describing and studying categories with symmetries. For example, Gaitsgory [G] introduced the notion of the action of an affine group scheme G over a $\mathbb{C}$-linear abelian category $\mathcal{C}$ and the category of G-equivariant objects $\mathcal{C}^G$, called the equivariantization of $\mathcal{C}$ by G. The category $\mathcal{C}^G$ has an action of $\mathrm{Rep}(G)$ and the category of Hecke eigen-objects in $\mathcal{C}^G$ is again $\mathcal{C}$. In general, if $\mathrm{Rep}(G)$ acts on an abelian category $\mathcal{C}$, then the category of Hecke eigen-objects in $\mathcal{C}$ is called the de-equivariantization of $\mathcal{C}$ by G.

Equivariantization and de-equivariantization are standard techniques in theory of fusion categories [DGNO] and have been applied in geometric Langlands program [FG] and quantum groups [ArG].

Now, if $\mathcal{C}$ is a tensor category and the action of $\mathrm{Rep}(G)$ over $\mathcal{C}$ is tensorial, then the de-equivariantization has a natural tensor structure. A special but very important type of tensor categories are those equivalent to the category of corepresentations of a Hopf algebra, which include representations of algebraic groups, quantum groups, compact groups, etc. If $\mathcal{C}$ is the category of comodules (or finite dimensional modules) over a Hopf algebra, then $\mathcal{C}^G \to \mathcal{C} \to \mathrm{Vec}$ is a fiber functor on $\mathcal{C}^G$ (where $\mathcal{C}^G \to \mathcal{C}$ is the forgetful functor) and by Tannakian duality $\mathcal{C}^G$ is the category of comodules over Hopf algebra. Thus, the family of Hopf algebras is closed under equivariantization, in the sense that we obtain new categories which are equivalent to categories of corepresentations of Hopf algebras. This is not the case for the de-equivariantization process since the de-equivariantization of comodules over a Hopf algebra is not always equivalent to the category of corepresentations over a Hopf algebra (see Subsection 3.3, for concrete examples). However, under some mild conditions, it is always the category of corepresentations over a coquasi-bialgebra. As a consequence there exist coquasi-Hopf algebras not twist equivalent to Hopf algebras, which admit an equivariantization equivalent to a Hopf algebra. This phenomenon was used in [EG] to relate the Drinfeld doubles of some quasi-Hopf algebras with small quantum groups, and in [An1] in order to classify the family

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of basic quasi-Hopf algebras with cyclic group of one-dimensional representations, under some mild conditions.

In this paper we study the de-equivariantization of the category of comodules over a Hopf algebra by an affine group scheme and apply Tannakian techniques to realize the de-equivariantization as the tensor category of comodules over a coquasi-bialgebra.

We apply the construction to interpret the central extensions of Hopf algebras as a particular example, and an additional application to the context of pointed finite tensor categories, extending the family of examples obtained in [EG], [Ge], [An1].

The organization of the paper is the following. In Section 1 we recall the definitions related with the main construction of this paper. First, the relation between affine group schemes and commutative algebras, then co-quasi bialgebras, and finally the center of a tensor category. In Section 2 we build a co-quasi Hopf algebra which represents the tensor category obtained as the de-equivariantization of the category of co-representations of a Hopf algebra. To do this, we consider central braided Hopf bialgebras, which are in correspondence with inclusions of tensor categories of comodules over Hopf algebras with certain factorization through the center, making emphasis on the case of algebras of functions over an affine group (in particular, over finite groups). We then obtain the corresponding coquasi-Hopf algebra representing a de-equivariantization over the comodules of a Hopf algebra by a Tannakian reconstruction. Finally, Section 3 contains some applications of the previous results. The main one is the case of finite-dimensional pointed Hopf algebras, which gives place to a general construction of pointed coquasi-Hopf algebras, and consequently finite pointed tensor categories.

1. Preliminaries

In this section we recall some definitions and results on Hopf algebras, affine group schemes and coquasi-Hopf algebras. For further reading on these topics we direct the reader to [M], [W] and [S1] respectively. Throughout the paper we work over an arbitrary field $\Bbbk$. Algebras and coalgebras are always defined over $\Bbbk$. For a coalgebra (C, Δ, ε) we shall use Sweedler's notation omitting the sum symbol, that is $\Delta(c) = c_1 \otimes c_2$ for all $c \in C$. Similarly if (M, λ) is a left C-comodule, then $\lambda(m) = m_{-1} \otimes m_0 \in C \otimes M$ for all $m \in M$. The category of left C-comodules shall be denoted by ${}^C\mathcal{M}$.

1.1. **Affine group scheme and commutative Hopf algebras.**

Let $\Bbbk$-$\mathcal{A}lg$ denote the category of commutative $\Bbbk$-algebras and $\mathcal{G}rp$ the category of groups. An affine group scheme over $\Bbbk$ is a representable functor $G : \Bbbk\text{-}\mathcal{A}lg \to \mathcal{G}rp$. By Yoneda's lemma the commutative algebra that represents G is unique up to isomorphisms, and we shall denote it by $\mathcal{O}(G)$. The group structures on $G(A)$, $A \in \Bbbk\text{-}\mathcal{A}lg$, determine natural transformations

$$\begin{aligned} m &: G \times G \to G, \\ 1 &: Sp(\Bbbk) \to G, \\ i &: G \to G, \end{aligned}$$

and they define algebra maps

$$\begin{aligned}\Delta &: \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G),\\ \varepsilon &: \mathcal{O}(G) \to \Bbbk,\\ \mathcal{S} &: \mathcal{O}(G) \to \mathcal{O}(G),\end{aligned}$$

that give a Hopf algebra structure on $\mathcal{O}(G)$. Conversely, if K is a commutative Hopf algebra, then $Spec(K) : \Bbbk\text{-}\mathcal{A}lg \to \mathcal{S}et, A \mapsto Alg(K, A)$ is an affine group scheme with group structure given by the convolution product and this defines an anti-equivalence of categories between affine groups schemes over $\Bbbk$ and commutative Hopf algebras over $\Bbbk$.

Under this equivalence the category of representations of G is equivalent to the category of $\mathcal{O}(G)$-comodules, and quasi-coherent sheaves on G are $\mathcal{O}(G)$-modules.

1.2. **Coquasi-bialgebras.** A *coquasi-bialgebra* $(H, m, u, \omega, \Delta, \varepsilon)$ is a (coassociative) coalgebra (H, Δ, ε) together with coalgebra morphisms:

- the multiplication $m : H \otimes H \longrightarrow H$ (denoted $m(h \otimes g) = hg$),
- the unit $u : \Bbbk \longrightarrow H$ (where we call $u(1) = 1_H$),

and a convolution invertible element $\omega \in (H \otimes H \otimes H)^*$ such that for all $h, g, k, l \in H$:

$$\begin{aligned} h_1(g_1k_1)\omega(h_2, g_2, k_2) &= \omega(h_1, g_1, k_1)(h_2g_2)k_2\\ 1_H h &= h 1_H = h\\ \omega(h_1g_1, k_1, l_1)\omega(h_2, g_2, k_2l_2) &= \omega(h_1, g_1, k_1)\omega(h_2, g_2k_2, l_1)\omega(g_3, k_3, l_2)\\ \omega(h, 1_H, g) &= \varepsilon(h)\varepsilon(g).\end{aligned}$$

Note that $\omega(1_H, h, g) = \omega(h, g, 1_H) = \varepsilon(h)\varepsilon(g)$ for each $g, h \in H$.

A *coquasi-Hopf algebra* is a coquasi-bialgebra H endowed with a coalgebra anti-homomorphism $\mathcal{S} : H \longrightarrow H$ (the *antipode*) and elements α, $\beta \in H^*$ satisfying, for all $h \in H$:

$$\begin{aligned} \mathcal{S}(h_1)\alpha(h_2)h_3 &= \alpha(h)1_H\\ h_1\beta(h_2)\mathcal{S}(h_3) &= \beta(h)1_H\\ \varepsilon(h) &= \omega(h_1\beta(h_2), \mathcal{S}(h_3), \alpha(h_4)h_5)\\ &= \omega^{-1}(\mathcal{S}(h_1), \alpha(h_2)h_3\beta(h_4), \mathcal{S}(h_5)).\end{aligned}$$

The category of left H-comodules ${}^H\mathcal{M}$ is rigid and monoidal, where the tensor product is over the base field and the comodule structure of the tensor product is the codiagonal one. The associator is given by

$$\begin{aligned} \phi_{U,V,W} &: (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)\\ \phi_{U,V,W}((u \otimes v) \otimes w) &= \omega(u_{-1}, v_{-1}, w_{-1})u_0 \otimes (v_0 \otimes w_0)\end{aligned}$$

for $u \in U$, $v \in V$, $w \in W$ and $U, V, W \in {}^H\mathcal{M}$. The dual coactions are given by $\mathcal{S}$ and $\mathcal{S}^{-1}$, as in the case of Hopf algebras.

1.3. **The center construction and the category of Yetter-Drinfeld modules.** The center construction produces a braided monoidal category $\mathcal{Z}(\mathcal{C})$ from any monoidal category $\mathcal{C}$, see [K]. The objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(Y, c_{-,Y})$, where $Y \in \mathcal{C}$ and $c_{X,Y} : X \otimes Y \to Y \otimes X$ are isomorphisms natural in X satisfying $c_{X\otimes Y,Z} = (c_{XZ} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,Z})$ and $c_{I,Y} = \mathrm{id}_Y$, for all $X, Y, Z \in \mathcal{C}$. The braided monoidal structure is given in the following way:

- the tensor product is $(Y, c_{-,Y}) \otimes (Z, c_{-,Z}) = (Y \otimes Z, c_{-,Y\otimes Z})$, where
$$c_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{id}_Z) : X \otimes Y \otimes Z \to Y \otimes Z \otimes X,$$
for all $X \in \mathcal{C}$,
- the identity element is $(I, c_{-,I})$, $c_{Z,I} = \mathrm{id}_Z$
- the braiding is the morphism $c_{X,Y}$.

Let H be a Hopf algebra with bijective antipode. We shall denote by ${}^H\mathcal{M}$ the tensor category of left H-comodules. The category $\mathcal{Z}({}^H\mathcal{M})$ is braided equivalent to the category ${}^H_H\mathcal{YD}$ of left-left Yetter-Drinfeld modules, whose objects are left H-comodules and left H-modules M satisfying the condition

$$(h_1 \rightharpoonup m)_{-1} h_2 \otimes (h_1 \rightharpoonup m)_0 = h_1 m_{-1} \otimes h_2 \rightharpoonup m_0 \tag{1.1}$$

for all $m \in M$, $h \in H$. A Yetter-Drinfeld module N becomes an object in $\mathcal{Z}({}^H\mathcal{M})$ by

$$c_{M,N}(m \otimes n) = m_{-1} \rightharpoonup n \otimes m_0,$$

and inverse $c^{-1}_{M,N}(n \otimes m) = m_0 \otimes \mathcal{S}^{-1}(m_1) \rightharpoonup n$.

2. De-equivariantization of Hopf algebras

2.1. **Central inclusion and braided central Hopf subalgebras.** Let H be a Hopf algebra with bijective antipode. Let G be an affine group scheme over $\Bbbk$ and $\mathcal{O}(G)$ the Hopf algebra of regular functions over G.

A **central inclusion** of G in H is a braided monoidal inclusion $\iota : \mathrm{Rep}(G) \hookrightarrow \mathcal{Z}({}^H\mathcal{M}) \cong {}^H_H\mathcal{YD}$, such that the braiding of ${}^H_H\mathcal{YD}$ restricts to the usual symmetric braiding of $\mathrm{Rep}(G)$, and the composition $\mathrm{Rep}(G) \hookrightarrow {}^H_H\mathcal{YD} \to {}^H\mathcal{M}$ gives an inclusion.

In order to describe in Hopf-theoretical terms the central inclusions, we need the following concept.

Definition 2.1. Let H be a Hopf algebra. A *braided central Hopf subalgebra* of H is a pair (K, r), where $K \subset H$ is a Hopf subalgebra, and $r : H \otimes K \to \Bbbk$ is a bilinear form such that:

$$r(hh', k) = r(h', k_1) r(h, k_2), \tag{2.1}$$
$$r(h, kk') = r(h_1, k) r(h_2, k'), \tag{2.2}$$
$$r(h, 1) = \varepsilon(h), \quad r(1, k) = \varepsilon(k), \tag{2.3}$$
$$r(h_1, k_1) k_2 h_2 = h_1 k_1 r(h_2, k_2), \tag{2.4}$$
$$r(k, k') = \varepsilon(kk'), \tag{2.5}$$

for all $k, k' \in K$, $h, h' \in H$.

Remark 2.2. (1) The conditions (2.1), (2.2), (2.3), say that $r : H \otimes K \to \Bbbk$ is a Hopf skew pairing, so in particular r has a convolution-inverse
$$r^{-1}(h, k) = r(h, S(k)), \quad (h \in H, k \in K).$$

(2) The algebra K is commutative by (2.4) and (2.5).
(3) For all $V \in {}^H\mathcal{M}, W \in {}^K\mathcal{M}$, the map r defines a natural isomorphism $c_{V,W} : V \otimes W \to W \otimes V, v \otimes w \mapsto r(v_{-1}, w_{-1})w_0 \otimes v_0$ in ${}^H\mathcal{M}$, and these isomorphisms define a braided inclusion ${}^K\mathcal{M} \to \mathcal{Z}({}^H\mathcal{M}) = {}^H_H\mathcal{YD}$.
(4) The condition (2.5) implies that K is a commutative algebra in $\mathcal{Z}({}^H\mathcal{M})$.

For example, any central Hopf subalgebra $K \subset H$ is braided central with $r = \varepsilon_H \otimes \varepsilon_K$. Conversely, if $K \subset H$ is a braided central Hopf subalgebra with $r = \varepsilon_H \otimes \varepsilon_K$ then K is a central Hopf subalgebra.

Lemma 2.3. *Let $K \subset H$ be a braided central Hopf subalgebra. Then*

$$r(xh, k) = r(hx, k) = \varepsilon(x)r(h, k), \tag{2.6}$$

for all $x, k \in K, h \in H$,

Proof. It follows from conditions (2.1) and (2.5). □

The following result exhibits the relevance of braided central Hopf subalgebras.

Theorem 2.4. *Let H be a Hopf algebra and $K \subset H$ a commutative Hopf subalgebra. Then the following set of data are equivalent:*

(1) *A map $r : H \otimes K \to \Bbbk$ such that (K, r) is a braided central Hopf subalgebra of H.*
(2) *A braided monoidal functor $F : {}^K\mathcal{M} \to \mathcal{Z}({}^H\mathcal{M}) = {}^H_H\mathcal{YD}$ such that the composition with the forgetful functor $\mathcal{Z}({}^H\mathcal{M}) = {}^H_H\mathcal{YD} \to {}^H\mathcal{M}$ is an inclusion.*
(3) *A Hopf algebra map $\gamma : K \to (H^\circ)^{cop}$ with $\gamma(k)|_K = \varepsilon$ and*

$$\langle \gamma(k_1), h_1 \rangle k_2 h_2 = h_1 k_1 \langle \gamma(k_2), h_2 \rangle$$

for all $h \in H, k \in K$ (H° denotes the finite dual Hopf algebra).

Proof. (1) $\Rightarrow$ (2) This is item (3) in Remark 2.2.
(2) $\Rightarrow$ (3) Since every comodule is a colimit of finite dimensional comodules, the image of the monoidal functor ${}^K\mathcal{M} \to {}^H_H\mathcal{YD} \to {}_H\mathcal{M}$ lives in the tensor subcategory $\underline{{}_H\mathcal{M}}$ of H-modules that are colimits of finite dimensional H-modules, therefore it defines a monoidal functor $\gamma_* : {}^K\mathcal{M} \to {}^H_H\mathcal{YD} \to \underline{{}_H\mathcal{M}} \cong {}^{(H^\circ)^{cop}}\mathcal{M}$ that commutes with the forgetful functors, *i.e.*, the diagram of monoidal functors

$$\begin{tikzcd} {}^K\mathcal{M} \arrow[rr, "\gamma_*"] \arrow[dr] & & {}^{(H^\circ)^{cop}}\mathcal{M} \arrow[dl] \\ & \mathrm{Vec}_\Bbbk & \end{tikzcd}$$

commutes. By Tannakian duality (see [JS, Section 8, Proposition 4]), there is a unique Hopf algebra map $\gamma : K \to (H^\circ)^{cop}$ that induces γ_* and it is given by

$$h \rightharpoonup m = \langle \gamma(m_{-1}), h \rangle m_0,$$

for all $h \in H$, $m \in M$ and $M \in {}^K\mathcal{M}$.

It is enough to prove that

$$h_1 m_{-2} \langle \gamma(m_{-1}), h_2 \rangle \otimes m_0 = m_{-1} h_2 \langle \gamma(m_{-2}), h_1 \rangle \otimes m_0,$$

for all $m \in M$, $M \in {}^K\mathcal{M}$. Indeed, (1.1) implies that

$$\begin{aligned} h_1 m_{-2}\langle \gamma(m_{-1}), h_2\rangle \otimes m_0 &= h_1 m_{-2} \otimes \langle \gamma(m_{-1}), h_2\rangle m_0 \\ &= h_1 m_{-2} \otimes h_2 \rightharpoonup m_0 \\ &= (h_1 \rightharpoonup m)_{-1} h_2 \otimes (h_1 \rightharpoonup m)_0 \\ &= m_{-1} h_2 \otimes \langle \gamma(m_{-2}), h_1\rangle m_0 \\ &= m_{-1} h_2 \langle \gamma(m_{-2}), h_1\rangle \otimes m_0. \end{aligned}$$

(3) $\Rightarrow$ (1) The map $r(h,k) = \langle \gamma(k), h\rangle$ defines a braided central structure over K. $\square$

The following result in an immediate consequence of Theorem 2.4.

Corollary 2.5. *Let H be a Hopf algebra. There exists a bijective correspondence between central inclusions of G in H and braided central Hopf subalgebras K of H such that $K \cong \mathcal{O}(G)$ as Hopf algebras.* $\square$

2.2. **De-equivariantization of a Hopf algebra by an affine group scheme.** Let H be a Hopf algebra and G be an affine group scheme. Let $K \subset H$ a braided central Hopf subalgebra with $K = \mathcal{O}(G)$.

The algebra $\mathcal{O}(G)$ is a commutative algebra in the symmetric category $\mathrm{Rep}(G)$, and thus a commutative algebra in the braided tensor category ${}^H_H\mathcal{YD}$ (see Remark 2.2 item (4)). Therefore, the algebra $\mathcal{O}(G)$ is braided commutative.

We define the de-equivariantization ${}^H\mathcal{M}(G)$ of ${}^H\mathcal{M}$ by G, as the category of H-equivariant sheaves on G; that is, the category of left $\mathcal{O}(G)$-modules in ${}^H\mathcal{M}$.

Now, the category ${}^H_G\mathcal{M}_G$ of $\mathcal{O}(G)$-bimodules in ${}^H\mathcal{M}$ is a tensor category with the tensor product $M \otimes_{\mathcal{O}(G)} N$. We shall see in the next proposition that this tensor product induces a monoidal structure on ${}^H\mathcal{M}(G)$.

Proposition 2.6. *Let $V \in {}^H\mathcal{M}(G)$ with left $\mathcal{O}(G)$-module structure $\rightharpoonup: \mathcal{O}(G) \otimes V \to V$ and left H-comodule structure $\lambda : V \to \mathcal{O}(G) \otimes V, v \mapsto v_{-1} \otimes v_0$. The map $\leftharpoonup: V \otimes \mathcal{O}(G) \to V, v \leftharpoonup x = r(v_{-1}, x_1)x_2 \rightharpoonup v_0$, makes V an object in ${}^H_G\mathcal{M}_G$. This rule defines a fully faithful strict monoidal functor from ${}^H\mathcal{M}(G)$ to ${}^H_G\mathcal{M}_G$.*

Proof. Let $V \in {}^H\mathcal{M}(G)$ with left $\mathcal{O}(G)$-module structure $\rightharpoonup: \mathcal{O}(G) \otimes V \to V$ and left H-comodule structure $\lambda : V \to \mathcal{O}(G) \otimes V, v \mapsto v_{-1} \otimes v_0$.

We need to show that:

(1) The map $\leftharpoonup: V \otimes \mathcal{O}(G) \to V$ defines a right $\mathcal{O}(G)$-module structure: for any $v \in V$ and $x, y \in \mathcal{O}(G)$,

$$\begin{aligned} (v \leftharpoonup x) \leftharpoonup y &= (r(v_{-1}, x_1)x_2 \rightharpoonup v_0) \leftharpoonup y \\ &= r(v_{-1}, x_1) r((x_2 \rightharpoonup v_0)_{-1}, y_1) y_2 \rightharpoonup (x_2 \rightharpoonup v_0)_0 \\ &= r(v_{-2}, x_1) r(x_2 v_{-1}, y_1) y_2 \rightharpoonup (x_3 \rightharpoonup v_0) \\ &= r(v_{-2}, x_1) r(v_{-1}, y_1) y_2 x_2 \rightharpoonup v_0 \\ &= r(v_{-1}, y_1 x_1) y_2 x_2 \rightharpoonup v_0 \\ &= v \leftharpoonup (xy), \\ v \leftharpoonup 1 &= r(v_{-1}, 1) v_0 = \varepsilon(v_{-1}) v_0 = v. \end{aligned}$$

(2) The map $\leftharpoonup : V \otimes \mathcal{O}(G) \to V$ is a morphism in ${}^H\mathcal{M}$:

$$\begin{aligned}
(v \leftharpoonup x)_{-1} \otimes (v \leftharpoonup x)_0 &= r(v_{-2}, x)v_{-1} \otimes v_0,\\
v_{-1}x_1 \otimes v_0 \leftharpoonup x_2 &= v_{-2}x_1 \otimes r(v_{-1}, x_2)v_0\\
&= r(v_{-1}, x_2)v_{-2}x_1 \otimes v_0\\
&= r(v_{-2}, x_1)x_2v_{-1} \otimes v_0\\
&= r(v_{-2}, x_1)\varepsilon(x_2)v_{-1} \otimes v_0\\
&= r(v_{-2}, x)v_{-1} \otimes v_0.
\end{aligned}$$

(3) The maps $\leftharpoonup$ and $\rightharpoonup$ commute:

$$\begin{aligned}
(x \rightharpoonup v) \leftharpoonup y &= r((x \rightharpoonup v)_{-1}, y_1)y_2 \rightharpoonup (x \rightharpoonup v)_0\\
&= r(x_1v_{-1}, y_1)y_2 \rightharpoonup x_2 \rightharpoonup v_0\\
&= \varepsilon(x_1)r(v_{-1}, y_1)y_2 \rightharpoonup x_2 \rightharpoonup v_0\\
&= r(v_{-1}, y_1)y_2 \rightharpoonup x \rightharpoonup v_0\\
&= x \rightharpoonup (v \leftharpoonup y).
\end{aligned}$$

(4) For all $f : V \to W$ morphism in ${}^H\mathcal{M}(G)$, f is a morphism in ${}^H_G\mathcal{M}_G$: it is enough to prove that f is a right $\mathcal{O}(G)$-module morphism,

$$\begin{aligned}
f(v \leftharpoonup x) = f(r(v_1, x_1)x_2 \rightharpoonup v_0) &= r(v_1, x_1)x_2 \rightharpoonup f(v_0)\\
&= r(f(v)_{-1}, x_1)x_2 \rightharpoonup f(v)_0\\
&= f(v) \leftharpoonup x.
\end{aligned}$$

for all $x \in \mathcal{O}(G), v \in V$.

Therefore we have a well-defined fully faithful functor from ${}^H\mathcal{M}(G)$ to ${}^H_G\mathcal{M}_G$.

Let $V, W \in {}^H\mathcal{M}(G)$, $v \in V, w \in W, x \in \mathcal{O}(G)$, the calculation

$$\begin{aligned}
r((v \otimes_{\mathcal{O}(G)} w)_{-1}, x_1)x_2 \rightharpoonup (v \otimes_{\mathcal{O}(G)} w)_0 &= r(v_{-1}w_{-1}, x_1)(x_2 \rightharpoonup v_0) \otimes_{\mathcal{O}(G)} w_0\\
&= r(w_{-1}, x_1)r(v_{-1}, x_2)(x_3 \rightharpoonup v_0) \otimes_{\mathcal{O}(G)} w_0\\
&= r(w_{-1}, x_1)(v_0 \leftharpoonup x_2) \otimes_{\mathcal{O}(G)} w_0\\
&= r(w_{-1}, x_1)v_0 \otimes_{\mathcal{O}(G)} x_2 \rightharpoonup w_0\\
&= v \otimes_{\mathcal{O}(G)} (w \leftharpoonup x).
\end{aligned}$$

proves that the right $\mathcal{O}(G)$-action of $V \otimes_{\mathcal{O}(G)} W$ is induced by the left $\mathcal{O}(G)$-action; in other words, $\otimes_{\mathcal{O}(G)}$ defines a monoidal structure on ${}^H\mathcal{M}(G)$ such that ${}^H\mathcal{M}(G)$ is a tensor subcategory of ${}^H_G\mathcal{M}_G$. ☐

Definition 2.7. Let H be a Hopf algebra and G an affine group scheme, with a central inclusion of G in H. The category ${}^H\mathcal{M}(G)$ with the monoidal structure $\otimes_{\mathcal{O}(G)}$ is called the *de-equivariantization* of H by G.

2.3. **Tannakian reconstruction of ${}^H\mathcal{M}(G)$.** Let H be a Hopf algebra and $\mathcal{O}(G) \subset H$ be a braided central inclusion of G in H. We shall say that the central inclusion of G in H is *cleft* if there exists a convolution invertible $\mathcal{O}(G)$-linear map $\pi : H \to \mathcal{O}(G)$; such a map is called a *cointegral*.

Lemma 2.3 implies that $r : H \otimes \mathcal{O}(G) \to \Bbbk$ induces a well-defined map

$$H/\mathcal{O}(G)^+H \otimes \mathcal{O}(G) \to \Bbbk, \qquad \overline{h} \otimes k \mapsto r(h, k),$$

which we will denote again by r (here, $\mathcal{O}(G)^+ = \ker(\varepsilon)$ is the augmentation ideal). The goal of this section is to prove the following result.

Theorem 2.8. *Let H be a Hopf algebra and $(\mathcal{O}(G), r)$ a cleft braided central Hopf subalgebra with cointegral π such that $\varepsilon\pi = \varepsilon$ and $\pi(1) = 1$. Then the quotient coalgebra $Q := H/\mathcal{O}(G)^+H$ is a coquasi-bialgebra with multiplication and associator given by:*

$$m(a \otimes b) = \overline{j(a_1)j(b)_1}\, r(a_2, \pi(j(b)_2))$$
$$\omega(a \otimes b \otimes c) = r(a, \pi(j(b)_1 j(c)_1)) r(j(b)_2, \pi(j(c)_2)),$$

where $a, b, c \in Q$ and $j : Q \to H$ is given by $q \mapsto \pi^{-1}(q_1)q_2$. Moreover, there is a monoidal equivalence between ${}^H\mathcal{M}(G)$ and ${}^Q\mathcal{M}$.

Before giving a proof of the theorem, we will briefly explain the Tannakian reconstruction principle that we shall use. Let C be a coalgebra and ${}^C\mathcal{M}$ be the category of left C-comodules. Assume that ${}^C\mathcal{M}$ has a monoidal structure $\overline{\otimes} : {}^C\mathcal{M} \times {}^C\mathcal{M} \to {}^C\mathcal{M}$, $\alpha_{V,W,Z} : (V\overline{\otimes}W)\overline{\otimes}Z \to V\overline{\otimes}(W\overline{\otimes}Z)$, $1\overline{\otimes}V = V\overline{\otimes}1$ such that the underlying functor ${}^C\mathcal{M} \to \mathrm{Vect}_{\Bbbk}$ is a strict quasi-monoidal functor, *i.e.*, $V\overline{\otimes}W = V \otimes_{\Bbbk} W$ and $1 = \Bbbk$ as vector spaces, then C has a coquasi-bialgebra structure (m, ω) given by

$$m(a, b) = (a \otimes b)_{-1}\varepsilon((a \otimes b)_0), \tag{2.7}$$
$$\omega(a, b, c) = \varepsilon(\alpha(a \otimes b \otimes c)) \tag{2.8}$$

and the monoidal structure on ${}^C\mathcal{M}$ defined by the coquasi-bialgebra structure coincides with the monoidal structure $(\overline{\otimes}, \alpha, 1)$. See [K, Proposition XV. 1.2.] for a proof (in a dual version).

Proof of Theorem 2.8. From now on we fix a cointegral π such that $\varepsilon\pi = \varepsilon$ and $\pi(1) = 1$. Let $Q = H/\mathcal{O}(G)^+H$ be the quotient coalgebra of H, then by [DMR, Theorem 2.4] and [Sch, Theorem II] the functors

$$\widehat{\mathcal{V}} :\ {}^H\mathcal{M}(G) \to {}^Q\mathcal{M} \tag{2.9}$$
$$M \mapsto \overline{M} = M/K^+M \tag{2.10}$$
$$H\square_Q V \leftarrow\!\shortmid V, \tag{2.11}$$

define a category equivalence, where $\overline{M} = M/\mathcal{O}(G)^+M$ is a left Q-comodule with $\overline{m}_{-1} \otimes \overline{m}_0 = \overline{m_{-1}} \otimes \overline{m_0}$, and $H\square_Q V = \{\sum h \otimes v | \sum h_1 \otimes h_2 \otimes v = \sum h \otimes v_{-1} \otimes v_0\} \in {}^H\mathcal{M}(G)$ has as left H-comodule and a left $\mathcal{O}(G)$-module structures induced by the ones in the left factor of the tensor product.

By [S1, Lemma 3.3.5], for all $M, N \in {}^H\mathcal{M}(G)$ we have a linear isomorphism

$$\xi^\pi_{M,N} : \overline{M} \otimes \overline{N} \to \overline{M \otimes_{\mathcal{O}(G)} N}$$
$$\overline{m} \otimes \overline{n} \mapsto \overline{m \otimes_{\mathcal{O}(G)} \pi^{-1}(n_{-1})n_0}$$
$$r(m_{-1}, \pi(n_{-1})_1)\overline{\pi(n_{-1})_2 m_0} \otimes \overline{n_0} = \overline{m\pi(n_{-1})} \otimes \overline{n_0} \leftarrow\!\shortmid \overline{m \otimes_{\mathcal{O}(G)} n}$$

such that $(\mathcal{V}, \xi^\pi) :\ {}^H\mathcal{M}(G) \to \mathrm{Vec}_{\Bbbk}$, $M \mapsto \overline{M} := M/K^+M$ is a quasi-tensor functor. Then, using the equivalence (2.9), the category ${}^Q\mathcal{M}$ has a (unique) monoidal structure such the $\widehat{\mathcal{V}}$ is a monoidal equivalence and the following diagram of functors commutes

$$\begin{tikzcd}
{}^{H}\mathcal{M}(G) \arrow[rr, "\widehat{\mathcal{V}}"] \arrow[dr, "{(\mathcal{V},\xi)}"'] & & {}^{Q}\mathcal{M} \arrow[dl, "\mathcal{U}"] \\
& \mathrm{Vec}_{\Bbbk} &
\end{tikzcd}$$

Consequently the underlying functor $\mathcal{U}$ becomes a strict quasi-monoidal functor and we can apply Tannakian reconstruction.

A natural section $j : Q \to H$ for the canonical projection $\nu_H : H \to Q$ is given by

$$j(\overline{h}) = \pi^{-1}(h_{-1})h_0. \tag{2.12}$$

Fix a cointegral $\pi : H \to \mathcal{O}(G)$ such that $\pi(1) = 1$, and define j as in (2.12).

The Q-comodule structure on $Q \otimes Q$ is:

$$\begin{aligned}
(\overline{m} \otimes \overline{n})_{-1} \otimes (\overline{m} \otimes \overline{n})_0 &= \xi(\overline{m} \otimes \overline{n})_{-1} \otimes \xi^{-1}(\xi(\overline{m} \otimes \overline{n})_0) \\
&= (\overline{m \otimes j(\overline{n})})_{-1} \otimes \xi^{-1}\left((\overline{m \otimes j(\overline{n})})_0\right) \\
&= \overline{m_{-1}j(\overline{n})_{-1}} \otimes \xi^{-1}(\overline{m_0 \otimes j(\overline{n})_0}) \\
&= \overline{m_{-1}j(\overline{n})_{-1}} \otimes r(m_{0,-1}, \pi(j(\overline{n})_{0,-1})_1) \\
&\qquad \overline{\pi(j(\overline{n})_{0,-1})_2 m_{00}} \otimes \overline{j(\overline{n})_{00}} \\
&= \overline{m_{-2}j(\overline{n})_{-2}} \otimes r(\overline{m_{-1}}, \pi(j(\overline{n})_{-1})_1) \\
&\qquad \overline{\pi(j(\overline{n})_{-1})_2 m_0} \otimes \overline{j(\overline{n})_0} \\
&= \overline{j(\overline{m_{-2}})j(\overline{n})_{-2}} \otimes r(\overline{m}_{-1}, \pi(j(\overline{n})_{-1})_1) \\
&\qquad \overline{\pi(j(\overline{n})_{-1})_2 m_0} \otimes \overline{j(\overline{n})_0},
\end{aligned}$$

for all $m, n \in H$. Now, applying the formula (2.7), we have

$$\begin{aligned}
m(\overline{m} \otimes \overline{n}) &= \overline{j(\overline{m}_{-2})j(\overline{n})_{-2}}\, r(\overline{m}_{-1}, \pi(j(\overline{n})_{-1})_1)\varepsilon(\overline{\pi(j(\overline{n})_{-1})_2 m_0})\varepsilon(\overline{j(\overline{n})_0}) \\
&= \overline{j(\overline{m}_{-2})j(\overline{n})_{-1}}\, r(\overline{m}_{-1}, \pi(j(\overline{n})_0)_1)\varepsilon(\overline{\pi(j(\overline{n})_0)_2 m_0}) \\
&= \overline{j(\overline{m}_{-1})j(\overline{n})_{-1}}\, r(\overline{m}_0, \pi(j(\overline{n})_0)),
\end{aligned}$$

that is,

$$m(a \otimes b) = \overline{j(a_1)j(b)_1}\, r(a_2, \pi(j(b)_2)),\ \forall\, a, b \in Q.$$

The constraint of associativity of ${}^{Q}\mathcal{M}$, is defined by the commutativity of the diagram

$$\begin{tikzcd}
\overline{L} \otimes \overline{M} \otimes \overline{N} \arrow[rr, "\alpha_{\overline{L},\overline{M},\overline{N}}"] \arrow[d, "\xi_{L,M} \otimes \mathrm{id}"'] & & \overline{L} \otimes \overline{M} \otimes \overline{N} \arrow[d, "\mathrm{id} \otimes \xi_{M,N}"] \\
\overline{L \otimes_{\mathcal{O}(G)} M} \otimes \overline{N} \arrow[dr, "\xi_{L \otimes_{\mathcal{O}(G)} M, N}"'] & & \overline{L} \otimes \overline{M \otimes_{\mathcal{O}(G)} N} \arrow[dl, "\xi_{L, M \otimes_{\mathcal{O}(G)} N}"] \\
& \overline{L \otimes_{\mathcal{O}(G)} M \otimes_{\mathcal{O}(G)} N} &
\end{tikzcd}$$

Hence,

$$\begin{aligned}
\alpha(\bar{l}\otimes\overline{m}\otimes\overline{n}) &= (\mathrm{id}\otimes\xi^{-1}_{M,N})\circ\xi^{-1}_{L,M\otimes_{\mathcal{O}(G)}N}\circ\xi_{L\otimes_{\mathcal{O}(G)}M,N}\circ(\xi_{L,M}\otimes\mathrm{id})(\bar{l}\otimes\overline{m}\otimes\overline{n})\\
&= (\mathrm{id}\otimes\xi^{-1}_{M,N})\circ\xi^{-1}_{L,M\otimes_{\mathcal{O}(G)}N}(\overline{l\otimes j(\overline{m})\otimes j(\overline{n})})\\
&= (\mathrm{id}\otimes\xi^{-1}_{M,N})\Big(r(l_{-1},\pi((j(\overline{m})\otimes j(\overline{n}))_{-1})_1)\\
&\qquad \overline{\pi((j(\overline{m})\otimes j(\overline{n}))_{-1})_2 l_0}\otimes\overline{j(\overline{m})_0}\otimes\overline{j(\overline{n})_0}\Big)\\
&= r(l_{-1},\pi(j(\overline{m})_{-1}j(\overline{n})_{-1})_1)\overline{\pi(j(\overline{m})_{-1}j(\overline{n})_{-1})_2 l_0}\otimes\\
&\qquad \xi^{-1}_{M,N}\left(\overline{j(\overline{m})_0}\otimes\overline{j(\overline{n})_0}\right)\\
&= r(l_{-1},\pi(j(\overline{m})_{-1}j(\overline{n})_{-1})_1)r(j(\overline{m})_{0,-1},\pi(j(\overline{n})_{0,-1})_1)\\
&\qquad \overline{\pi(j(\overline{m})_{-1}j(\overline{n})_{-1})_2 l_0}\otimes\overline{\pi(j(\overline{n})_{0,-1})_2 j(\overline{m})_{00}}\otimes\overline{j(\overline{n})_{00}}\\
&= r(l_{-1},\pi(j(\overline{m})_{-2}j(\overline{n})_{-2})_1)r(j(\overline{m})_{-1},\pi(j(\overline{n})_{-1})_1)\\
&\qquad \overline{\pi(j(\overline{m})_{-2}j(\overline{n})_{-2})_2 l_0}\otimes\overline{\pi(j(\overline{n})_{-1})_2 j(\overline{m})_0}\otimes\overline{j(\overline{n})_0}
\end{aligned}$$

for all $l\in L, m\in M, n\in N$. Applying the formula (2.8),

$$\begin{aligned}
\omega(\bar{l}\otimes\overline{m}\otimes\overline{n}) &= r(l_{-1},\pi(j(\overline{m})_{-2}j(\overline{n})_{-2})_1)r(j(\overline{m})_{-1},\pi(j(\overline{n})_{-1})_1)\\
&\qquad \varepsilon(\overline{\pi(j(\overline{m})_{-1}j(\overline{n})_{-1})_2 l_0})\varepsilon(\overline{\pi(j(\overline{n})_{-1})_2 j(\overline{m})_0})\varepsilon(\overline{j(\overline{n})_0})\\
&= r(l,\pi(j(\overline{m})_{-1}j(\overline{n})_{-1}))r(j(\overline{m})_0,\pi(j(\overline{n})_0)),
\end{aligned}$$

that is

$$\omega(a\otimes b\otimes c) = r(a,\pi(j(b)_1 j(c)_1))r(j(b)_2,\pi(j(c)_2))$$

for all $a,b,c\in Q$.

□

Remark 2.9. (1) If $\pi: H\to K$ is any cointegral then $\pi'(h) := \pi(h_1)\varepsilon\pi^{-1}(h_2)$ is again a cointegral such that $\varepsilon\pi' = \varepsilon$.

(2) If $\pi: H\to K$ is a cointegral, then $\pi(1)\in H^{\times}$, and $\pi'(h) := \pi(h)\pi(1)^{-1}$ is again an cointegral such that $\pi'(1) = 1$.

(3) If H is finite dimensional Hopf algebra H, every Hopf subalgebra $K\subset H$ admits a cointegral $\pi: H\to K$.

Proposition 2.10. *If H is finite dimensional and G is a constant finite algebraic group, then the coquasi-bialgebra Q defined in Theorem 2.8 admits a coquasi-Hopf algebra structure.*

Proof. Since G is a constant finite group it follows by [DGNO, Theorem 4.18] that the de-equivariantization is a rigid monoidal category. Since Q is a quotient of H, Q is finite dimensional and by [S2, Theorem 3.1], Q is a coquasi-Hopf algebra.

□

3. Applications

In the last part of this work we apply the results of Section 2 to some particular cases. First we consider the category of G-graded vector spaces, for some group G. Second, we look at quotients of Hopf algebras by central Hopf subalgebras, and view them as de-equivariantizations. Finally we study a family of pointed finite-dimensional coquasi-Hopf algebras, whose dual algebras are a generalization of the quasi-Hopf algebras $A(H,s)$ in [An1].

3.1. **Baby example.** Let Γ be a discrete group, $G \subset \mathcal{Z}(\Gamma)$ a central subgroup of Γ, and $r : \Gamma \times G \to \Bbbk^*$ a bicharacter such that $r|_{G\times G} = 1$. Then the pair $(\Bbbk G, r)$ is a braided central Hopf subalgebra of $\Bbbk\Gamma$.

We shall fix a set of representatives of the right cosets of G in Γ, $Q \subset G$. Thus every element $\gamma \in \Gamma$ has a unique factorization $\gamma = gq$, $g \in G$, $q \in Q$. We assume $e \in Q$. The uniqueness of the factorization $\Gamma = GQ$ implies that there are well defined maps

$$\cdot : Q \times Q \to Q, \qquad \theta : Q \times Q \to G,$$

determined by the conditions

$$pq = \theta(p,q)p \cdot q, \qquad p,q \in Q.$$

The map θ is a 2-cocycle $\theta \in Z^2(\Gamma/G, G)$ where Γ/G acts trivially on G, since G is a central subgroup of Γ.

We define a map $\pi : \Gamma \to G$, $\gamma \mapsto x$, where $x \in G$ is the unique element such that $\gamma = xp$ with $p \in Q$, and $j : Q \to \Gamma$ is the inclusion. Now by Theorem 2.8, the de-equivariantization is defined as follows. Let K be the quotient group Γ/G, then the group algebra $\Bbbk K$ with the 3-cocycle

$$\omega(u,v,w) = r(u, \theta(v,w))$$

is a coquasi-Hopf algebra and ${}^{\Bbbk K}\mathcal{M}$ is tensor equivalent to ${}^{\Bbbk\Gamma}\mathcal{M}(\widehat{G})$, the de-equivariantization of ${}^{\Bbbk\Gamma}\mathcal{M}$ by the affine group scheme $\widehat{G}(-) = Alg(\Bbbk G, -)$.

Now, we will explain how this construction determines the same data of [An1, Example 2.2.6]. If Γ is abelian, the map $r : \Gamma \times G \to \Bbbk^*$ defines a group morphism $T : G \to \widehat{\Gamma}$, $x \mapsto r(-,x)$ such that $\langle T(x'), x\rangle = r(x,x') = 1$, for all $x, x' \in G$, thus it defines an inclusion of Vec_G as a Tannakian subcategory of $\mathcal{Z}(\mathrm{Vec}_\Gamma)$, and the 3-cocycle over K is:

$$\omega(u,v,w) = r(u,\theta(v,w)) = \langle T(\theta(v,w)), u\rangle.$$

3.2. **Second example: Central extension of Hopf algebras.** Let H be a Hopf algebra and (K,r) a braided central Hopf subalgebra. If $r(h,x) = \varepsilon(hx)$ for all $h \in H, x \in K$, then $K \subset H$ is a central Hopf subalgebra and this defines a central inclusion of the group scheme $G = Spec(K)$ in H. Also since K is central, K^+H is a Hopf ideal and $Q = H/K^+H$ is a quotient Hopf algebra of H.

Proposition 3.1. *Let H be a Hopf algebra and $K \subset H$ a cleft central Hopf subalgebra, then the de-equivariantization of ${}^H\mathcal{M}$ by $G = Spec(K)$ is tensor equivalent to the tensor category of comodules over the Hopf algebra $Q = H/K^+H$.*

Proof. The central Hopf subalgebra K is braided central with $r(h,k) = \varepsilon(hk)$ for all $h \in H, k \in K$. Then the product and coassociator in the coquasi-bialgebra defined in Theorem 2.8 are

$$\begin{aligned}
m(\overline{a}\otimes\overline{b}) &= \overline{j(\overline{a_1})j(\overline{b})_1}r(\overline{a_2},\pi(j(\overline{b})_2)) \\
&= \overline{j(\overline{a}_1)j(\overline{b})_1}\varepsilon(\overline{a}_2)\varepsilon(\pi(j(\overline{b})_2)) \\
&= \overline{j(\overline{a})j(\overline{b})} = \overline{j(\overline{a})j(\overline{b})} = \overline{ab}, \\
\omega(\overline{a}\otimes\overline{b}\otimes\overline{c}) &= r(\overline{a},\pi(j(\overline{b})_1 j(\overline{c})_1))r(j(\overline{b})_2,\pi(j(\overline{c})_2)) \\
&= \varepsilon(\overline{a})\varepsilon(\pi(j(\overline{b})_1 j(\overline{c})_1))\varepsilon(j(\overline{b})_2)\varepsilon(\pi(j(\overline{c})_2)) \\
&= \varepsilon(\overline{a}\overline{b}\overline{c}),
\end{aligned}$$

for all $a,b,c\in H, \overline{a},\overline{b},\overline{c}\in Q$. Then the coquasi-bialgebra structure is the Hopf algebra quotient structure, and then ${}^{Q}\mathcal{M}$ is tensor equivalent to the de-equivariantization by $Spec(K)$. □

The interesting point of the proposition above is that this provides a categorical interpretation of the tensor category ${}^{Q}\mathcal{M}$ in terms of de-equivariantization of an affine group scheme.

Example 3.2. Let G be a connected, simply connected complex simple Lie group, and let $\mathfrak{g}$ be its associated Lie algebra. In [ArG] the authors consider the following setting: an injective map of Hopf algebras $\iota : \mathcal{O}(G) \hookrightarrow A$, and a surjective map $\pi : A \to Q$, satisfying the conditions

i) $\pi\circ\iota(a) = \epsilon(a)1_Q$, for all $a\in\mathcal{O}(G)$;
ii) $A^{co\pi} = \mathcal{O}(G)$;
iii) $\ker\pi = \mathcal{O}(G)^{+}A$;
iv) either A is flat as $\mathcal{O}(G)$-module, or the functor $\mathrm{Ind} :\ {}^{Q}\mathcal{M} \to\ {}^{A}\mathcal{M}$ is exact and faithful.

They obtain an equivalence between the category ${}^{Q}\mathcal{M}$ and the de-equivariantization of ${}^{A}\mathcal{M}$ by G, see [ArG, Thm. 2.8]. Our results not only give an alternative proof of this equivalence but they also give that it is a tensor equivalence.

They apply the result to the following case. Let $l \geq 3$ be an odd integer, relative prime to 3 if $\mathfrak{g}$ contains a G_2-component, and let ζ be a complex primitive l-th root of 1. By $\mathcal{O}_\zeta(G)$ we denote the complex form of the quantized coordinate algebra of G at ζ and by $u_\zeta(\mathfrak{g})$ the Frobenius-Lusztig kernel of $\mathfrak{g}$ at ζ, see [DL] for definitions.

By [DL, Prop.6.4], $\mathcal{O}_\zeta(G)$ fits into the following cocleft central exact sequence

$$1\to\mathcal{O}(G)\to\mathcal{O}_\zeta(G)\to u_\zeta(\mathfrak{g})^{*}\to 1.$$

Then the tensor category of modules over the Frobenius-Lusztig kernel is a de-equivariantization of $\mathcal{O}_\zeta(G)$, which is the main result of [ArG]. Moreover, the main result in [AG] establishes that any quantum subgroup is obtained as a cocleft central exact sequence, similar to the previous one; that is, we can view these constructions as de-equivariantizations.

The same construction works for the restricted two parameter (pointed) quantum group $\widehat{u}_{\alpha,\beta}(\mathfrak{gl}_n)$ with the algebraic group GL_n, where $\alpha,\beta\in\Bbbk$ are such that $\alpha\beta^{-1}$ is a root of unity of order l, and $\alpha^l=\beta^l=1$. According to [Ga, Cor. 5.3, 5.15], we have a central extension of Hopf algebras

$$1\to\mathcal{O}(GL_n)\to\mathcal{O}_{\alpha,\beta}(GL_n)\to\widehat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^{*}\to 1.$$

Therefore Proposition 3.1 shows that the category of modules over $\widehat{u}_{\alpha,\beta}(\mathfrak{gl}_n)$ is the de-equivariantization of the category of comodules over $\mathcal{O}_{\alpha,\beta}(GL_n)$ by GL_n. A similar situation holds for any quantum subgroup of this quantum group.

3.3. **A generalization of the family of algebras $A(H,s)$.** In this subsection we shall assume that $\Bbbk$ is an algebraically closed field of characteristic zero.

Let Γ be a finite group and $\widehat{\Gamma} = Hom(\Gamma, \Bbbk^*)$. We consider a finite-dimensional coradically graded pointed Hopf algebra $H = \oplus_{n\geq 0} H_n$, with $G(H) = \Gamma$. We assume that H is generated as an algebra by Γ and H_1; this is always the case if Γ is abelian, see [An2, Theorem 4.15]. We fix a basis $x_1, \cdots, x_\theta$ of the space V of coinvariants of H_1, so $H \simeq \mathcal{B}(V)\#\Bbbk\Gamma$, where $\mathcal{B}(V)$ is the Nichols algebra associated to V, and $\Delta(x_i) = x_i \otimes g_i + 1 \otimes x_i$ for some $g_i \in \Gamma$, see [AS] for details about these facts.

Proposition 3.3. *Let $H = \bigoplus_{n=0}^{\infty} H_n$, Γ, $x_1, \cdots, x_\theta$ be as before. There exists a bijection between*

(a) *central braided Hopf subalgebras (K, r), and*

(b) *pairs (G, Φ), where G is a central subgroup of Γ, and $\Phi : G \to \widehat{\Gamma}$ is a morphism of groups such that*

$$\langle g', \Phi(g)\rangle = 1, \qquad g x_i g^{-1} = \langle g_i, \Phi(g)\rangle x_i,$$

for all $g, g' \in G$, $1 \leq i \leq \theta$.

The correspondence is given by defining $K = \Bbbk G$, and extending the evaluation map $< \cdot, \Phi(\cdot) >: \Gamma \times G \to \Bbbk$, linearly to $H_0 \otimes K$, and as zero over H_n, $n \geq 1$.

Proof. Given a central braided Hopf subalgebra $K \subset H$, we have that $K \subset H_0$ is commutative, so $K = \Bbbk G$ for some subgroup G of Γ. By (2.4), G is inside the center of Γ. By (2.1) and (2.2), we have a morphism of groups $\Phi : G \to \widehat{\Gamma}$ given by

$$< \gamma, \Phi(g) >:= r(\gamma, g), \qquad \gamma \in \Gamma, g \in G,$$

such that $\langle g', \Phi(g)\rangle = 1$ for all $g, g' \in G$. Now by (2.4) we also have that

$$r(x_i, g)\, g g_i + g x_i = r(g_i, g)\, x_i g + r(x_i, g)\, g,$$

so $r(x_i, g) = 0$, and $g x_i g^{-1} = r(g_i, g) x_i$ for all i and all $g \in G$, because H is graded and $g_i \neq 1$. As H is generated by skew primitive and group-like elements, we deduce that

$$r(x, k) = 0, \qquad \text{for all } k \in K, x \in H_n, n \geq 1.$$

The converse is easy to prove. □

Remark 3.4. Fix a set $Q \subset \Gamma$ of representatives of the right cosets of G in Γ. Note that the map $\pi : H \to K = \Bbbk G$ given as in Subsection 3.1 over H_0, and extended as 0 over the other components, is a cointegral for K.

Definition 3.5. Let $H = \bigoplus_{n\geq 0} H_n$ be a coradically graded finite-dimensional Hopf algebra such that $H_0 = k\Gamma$, where Γ is a finite group, and H is generated by group-like and skew-primitive elements. For each pair (G, Φ) as in the Proposition 3.3, we shall denote by $A(H, G, \Phi)$ the associated coquasi-Hopf algebra constructed by using Theorem 2.8.

Example 3.6. Let $H = \oplus_{n\geq 0} H_n$ be as above, where $G(H) = \Gamma$ is an abelian group. Therefore H is generated by the group-like elements and a finite set $x_1, \ldots, x_\theta$ of $(\gamma_i, 1)$-primitive elements, i.e.

$$\Delta(x_i) = x_i \otimes \gamma_i + 1 \otimes x_i, \qquad i \in \{1, \ldots, \theta\},$$

and also we can suppose that there are characters $\chi_i \in \widehat{\Gamma}$ such that

$$\gamma x_i \gamma^{-1} = \chi_i(\gamma) x_i, \qquad i \in \{1, \dots, \theta\}, \gamma \in \Gamma.$$

In this case, Φ must satisfy the condition $\chi_i(\gamma) = \langle \gamma_i, \Phi(\gamma) \rangle$ for all $i \in \{1, \dots, \theta\}$ and all $\gamma \in \Gamma$. Therefore $\Phi(\gamma)$ is uniquely determined (and possibly it does not exist) when the γ_i's generate Γ as a group.

The Nichols algebra $\mathcal{B}(V)$ admits a $\mathbb{N}^\theta$-gradation, and we can fix a basis B of $\mathcal{B}(V)$ whose elements are $\mathbb{N}^\theta$-homogeneous, and such that $1 \in B$. For each $\mathbf{x} \in B$ we denote $|\mathbf{x}| \in \mathbb{N}^\theta$ its degree, and

$$\gamma_{\mathbf{x}} := \gamma_1^{a_1} \cdots \gamma_\theta^{a_\theta}, \quad \chi_{\mathbf{x}} := \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \qquad \text{if } |\mathbf{x}| = (a_1, \dots, a_\theta) \in \mathbb{N}^\theta.$$

Therefore $\gamma \mathbf{x} \gamma^{-1} = \chi_{\mathbf{x}}(\gamma)\mathbf{x}$ for all $\gamma \in \Gamma$, and $\Delta(\mathbf{x})$ is written as the sum of $\mathbf{x} \otimes 1$ plus $\gamma_{\mathbf{x}} \otimes \mathbf{x}$ plus terms in intermediate degrees for the $\mathbb{N}$-gradation. Fix a set of representatives elements $q_1 = e, \dots, q_t \in \Gamma$ of Γ/G, so Q has a basis $(\overline{q_i \mathbf{x}})_{\mathbf{x} \in B, 1 \le i \le t}$. Therefore the multiplication and the associator of Q are given by:

$$\begin{aligned} m(q_i\mathbf{x}, q_j\mathbf{y}) &= r\big(q_i\gamma_{\mathbf{x}}, \pi(q_j\gamma_{\mathbf{y}})\big)\overline{q_i\mathbf{x} \cdot q_j\mathbf{y}} = \Phi\big(\pi(q_j\gamma_{\mathbf{y}})\big)(q_i\gamma_{\mathbf{x}})\overline{q_i\mathbf{x} \cdot q_j\mathbf{y}}, \\ \omega(q_i\mathbf{x}, q_j\mathbf{y}, q_k\mathbf{z}) &= r\big(q_i, \pi(q_jq_k)\big) r\big(q_j, \pi(q_k)\big)\delta_{\mathbf{x},1}\delta_{\mathbf{y},1}\delta_{\mathbf{z},1} \end{aligned}$$

for any $x, y, z \in B$ and $1 \le i, j, k \le t$.

More concretely, suppose that Γ is a cyclic group of order m^2, generated by γ, and that $x_1, \dots, x_\theta$ are the skew-primitive elements. Thus, if q is a primitive m^2-roof of unity, there are unique integers d_i, b_i, module m^2, such that

$$\chi_i(\gamma) = q^{d_i}, \qquad \Delta(x_i) = x_i \otimes \gamma^{b_i} + 1 \otimes x_i.$$

Set $G = \langle g \rangle$, where $g = \gamma^n$, so $G \simeq \mathbb{Z}_n$, and $\chi \in \widehat{\Gamma}$ such that $\chi(\gamma) = q$. A morphism $\Phi : G \to \widehat{\Gamma}$ is determined by an integer s (unique modulo n) such that $\Phi(g) = \chi^{ns}$. Therefore the conditions in Proposition 3.3 are satisfied for each element in

$$\Upsilon'(H) := \{s : \ 0 \le s \le n-1, \quad b_i s \equiv d_i(n), \forall i = 1, \dots, \theta\}.$$

A set of representatives of $\Gamma/G \simeq \mathbb{Z}_n$ is given by γ^i, $0 \le i \le n-1$.

Remark 3.7. If H is a finite dimensional Hopf algebra as in this example, then H^* is also of this type and $\Upsilon'(H^*) = \Upsilon(H)$, where $\Upsilon(H)$ was defined in [An1].

For each $s \in \Upsilon'(H)$ there exists a coquasi-Hopf algebra $A'(H, s)$. We identify the group of simple (one-dimensional) comodules with $\mathbb{Z}_n$, and the 3-cocycle determining the associator is

$$\omega(\gamma^i, \gamma^j, \gamma^k) = q^{nsi\left(j+k-(j+k)'\right)}, \qquad 0 \le i, j, k \le n-1,$$

where j' denotes the remainder of j in the division by n. Note that $A'(H, s)$ is dual to the quasi-Hopf algebra $A(H^*, s)$ of [An1], and these quasi-Hopf algebras include the examples in [Ge].

Example 3.8. We now consider de-equivariantizations of some pointed Hopf algebras related with small quantum groups by applying the previous construction.

We fix then a finite Cartan matrix $A = (a_{ij})_{1 \le i,j \le \theta}$ corresponding to a semisimple Lie algebra $\mathfrak{g}$, positive integers d_i, $1 \le i \le \theta$ such that they are the minimal ones satisfying $d_i a_{ij} = d_j a_{ji}$, and let Δ_+ be its set of positive roots and $M := |\Delta_+|$.

Fix also a root of unity q of order $N = mn$, $m, n > 1$, $q_{ij} := q^{d_i a_{ij}}$, $\{\alpha_i\}$ the canonical basis of $\mathbb{Z}^\theta$, $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times$ the bicharacter determined by $\chi(\alpha_i, \alpha_j) = q_{ij}$, $1 \le i, j \le \theta$. For each $\beta \in \Delta_+$, let $q_\beta := \chi(\beta, \beta)$ and $N_\beta := \operatorname{ord} q_\beta$.

We will describe the corresponding Nichols algebra $\mathcal{B}(V)$ of diagonal type attached to (q_{ij}) and the corresponding Hopf algebra obtained by bosonization by a particular abelian group. We refer to [An2, Theorems 1.25, 3.1] for the corresponding statements about the Nichols algebra. Fix a basis $x_1, \dots, x_\theta$ of V, the group $\Gamma = (\mathbb{Z}_N)^\theta$, with generators $\gamma_1, \dots, \gamma_\theta$ of each cyclic group of order N, and consider the realization of V as a Yetter-Drinfeld module with comodule structure determined by $\delta(x_i) = \gamma_i \otimes x_i$. Recall that the braided adjoint action of x_i has the following property:

$$(\operatorname{ad}_c x_i)y := x_i y - \chi(\alpha_i, \beta) y x_i, \quad \text{for each } y \in \mathcal{B}(V)\ \mathbb{Z}^\theta-\text{homogeneous of degree } \beta.$$

The associated finite-dimensional pointed Hopf algebra $H = \mathcal{B}(V)\#\Bbbk\Gamma$ is described as follows. As an algebra, it is generated by $\gamma_1, \dots, \gamma_\theta$, $x_1, \dots, x_\theta$, which satisfies the following relations:

$$\begin{aligned} &\gamma_i^N = 1, \qquad \gamma_i\gamma_j = \gamma_j\gamma_i, && \gamma_i x_j = q_{ij} x_j \gamma_i, \\ &(\operatorname{ad}_c x_i)^{1-a_{ij}} x_j = 0, \quad i \neq j, && x_\beta^{N_\beta} = 0, \quad \beta \in \Delta_+, \end{aligned}$$

if $N \geq 8$ (otherwise we need extra relations). Each x_β is an homogeneous element of $\mathcal{B}(V)$ of degree β, obtained for a fixed convex order on the roots $\beta_1 < \beta_2 < \cdots < \beta_M$, and $\mathcal{B}(V)$, resp. H, has a PBW basis B, resp. $\overline{B}$ as follows:

$$\begin{aligned} &\left\{ x_{\beta_M}^{b_M} \cdots x_{\beta_1}^{b_1} : 0 \leq a_i < N, 0 \leq b_j < N_{\beta_j} \right\}, \\ &\left\{ \gamma_1^{a_1} \cdots \gamma_\theta^{a_\theta} \mathbf{x} : 0 \leq a_i < N, \mathbf{x} \in B \right\}. \end{aligned}$$

The coproduct is determined by

$$\Delta(\gamma_i) = \gamma_i \otimes \gamma_i, \qquad \Delta(x_i) = x_i \otimes \gamma_i + 1 \otimes x_i.$$

We consider $n_i, m_i \in \mathbb{N}$ such that $N = n_i m_i$ for each $1 \leq i \leq \theta$. For each $a \in \mathbb{N}$, we denote by $\mu_i(a)$ the remainder of a on the division by n_i. Call $g_i = \gamma_i^{n_i}$, and let G be the subgroup of Γ generated by $g_1, \dots, g_\theta$. Therefore,

$$G \simeq \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_\theta}, \qquad G' := \Gamma/G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_\theta},$$

and a set of representatives of G' is given by $\gamma_1^{a_1}\gamma_2^{a_2} \cdots \gamma_\theta^{a_\theta}$, $0 \leq a_i < n_i$. With this information we can determine $j : Q \to H$ and $\pi : H \to \Bbbk G$ by

$$\begin{aligned} j(\overline{\gamma_1^{a_1} \cdots \gamma_\theta^{a_\theta} \mathbf{x}}) &= \gamma_1^{\mu_1(a_1)} \cdots \gamma_\theta^{\mu_\theta(a_\theta)} \mathbf{x}, \\ \pi(\gamma_1^{a_1} \cdots \gamma_\theta^{a_\theta} \mathbf{x}) &= \gamma_1^{a_1 - \mu_1(a_1)} \cdots \gamma_\theta^{a_\theta - \mu_\theta(a_\theta)} \varepsilon(\mathbf{x}), \end{aligned}$$

where $a_i \in \mathbb{N}$, $\mathbf{x} = x_{\beta_M}^{b_M} \cdots x_{\beta_1}^{b_1}$. By Proposition 3.3 we have that $\langle \gamma_j, \Phi(g_i) \rangle = \chi_j(g_i) = q_{ij}^{n_i}$ for each pair $1 \leq i, j \leq \theta$, so Φ is uniquely determined; since $\langle g_j, \Phi(g_i) \rangle = 1$ for all i, j, we need the extra conditions $N | n_i n_j$, which are equivalent to $m_i | n_j$. To determine explicitly the coquasi-Hopf algebra structure of Q, we consider the basis

$$\overline{B} := \left\{ \overline{\gamma_1^{a_1} \cdots \gamma_\theta^{a_\theta} \mathbf{x}} : 0 \leq a_i < n_i, \, \mathbf{x} = x_{\beta_M}^{b_M} \cdots x_{\beta_1}^{b_1} \right\}.$$

Given two elements $\mathbf{x},\mathbf{y}$ of $\mathcal{B}(V)$ of degree $(e_1,\dots,e_\theta),(f_1,\dots,f_\theta)\in\mathbb{N}_0^\theta$ respectively, and $0\le a_i,b_i<n_i$, we compute

$$m\left(\overline{\gamma_1^{a_1}\cdots\gamma_\theta^{a_\theta}\mathbf{x}},\overline{\gamma_1^{b_1}\cdots\gamma_\theta^{b_\theta}\mathbf{y}}\right)=\prod_{1\le i,j\le\theta} q_{ij}^{(b_j+f_j-\mu_j(b_j+f_j))(a_i+e_i)-b_je_i}\ \overline{\gamma_1^{\mu_1(a_1+b_1)}\cdots\gamma_\theta^{\mu_\theta(a_\theta+b_\theta)}\mathbf{x}\mathbf{y}},$$

For each $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{B}(V)$, $0\le a_i,b_i,c_i<n_i$, the associator is computed as

$$\omega\left(\overline{\gamma_1^{a_1}\cdots\gamma_\theta^{a_\theta}\mathbf{x}},\overline{\gamma_1^{b_1}\cdots\gamma_\theta^{b_\theta}\mathbf{y}},\overline{\gamma_1^{c_1}\cdots\gamma_\theta^{c_\theta}\mathbf{z}}\right)=\delta_{\mathbf{x},1}\delta_{\mathbf{y},1}\delta_{\mathbf{z},1}\prod_{1\le i,j\le\theta} q_{ij}^{(b_j+c_j-\mu_j(b_j+c_j))a_i}.$$

Note that we can obtain the quasi-Hopf algebras appearing in [EG] as the dual structures of the coquasi-Hopf algebras obtained for $\Gamma=(\mathbb{Z}_{n^2})^\theta$ and $G\cong(\mathbb{Z}_n)^\theta$ as a subgroup of Γ, i.e. $N=n^2$, $m_i=n_i=n$.

Example 3.9. Finally we consider some de-equivariantizations related with a Nichols algebra of diagonal type but not of Cartan type. Consider a braiding whose diagram is the last one of row 9 in [H, Table 1], and an associated $H=\mathcal{B}(V)\#\Bbbk\Gamma$. Here we fix $\Gamma=\mathbb{Z}_{9N}\times\mathbb{Z}_{18M}$, with generators γ_1, γ_2 of each cyclic subgroup, respectively, and q a root of unity of order 9. Using the presentation given for the corresponding Nichols algebra in [An2, Theorem 3.1], we can describe H as follows. As an algebra, it is generated by γ_1, γ_2, x_1, x_2, and relations

$$\gamma_1^{9N}=\gamma_2^{18M}=1,\qquad \gamma_1\gamma_2=\gamma_2\gamma_1,\qquad \gamma_ix_j\gamma_i^{-1}=q_{ij}x_j,$$
$$x_1^3=x_2^2=x_{12}^{18}=x_{112}^{18}=x_{112}y-qyx_{112}=0,$$

where $q_{11}=q^3$, $q_{12}=q^4=q_{21}$, $q_{22}=-1$, $(\mathrm{ad}_c x_i)y:=x_iy-(\gamma_iy\gamma_i^{-1})x$ for each $i=1,2$, and each $y\in\mathcal{B}(V)$, and we consider:

$$x_{12}=(\mathrm{ad}_c x_1)x_2,\qquad\qquad y=x_{112}x_{12}+x_{12}x_{112},$$
$$x_{112}=(\mathrm{ad}_c x_1)^2x_2,\qquad\qquad z=yx_{12}-q^2x_{12}y,$$

so by [An2, Theorem 1.25] $\mathcal{B}(V)$ has a PBW basis B as follows:

$$\left\{x_2^{b_6}x_{12}^{b_5}z^{b_4}y^{b_3}x_{112}^{b_2}x_1^{b_1}\,:\,0\le b_2,b_5<18, b_1,b_3\in\{0,1,2\}, b_4,b_6\in\{0,1\}\right\}.$$

The coproduct is determined by

$$\Delta(\gamma_i)=\gamma_i\otimes\gamma_i,\qquad \Delta(x_i)=x_i\otimes\gamma_i+1\otimes x_i.$$

Fix $n,n_1,m,m_1\in\mathbb{N}$ such that $9N=nn_1$ and $18M=mm_1$. For each $a\in\mathbb{N}$, we denote by a' (respectively, a'') the remainder on the division by n (respectively, m). Call $g_1=\gamma_1^n$, $g_2=\gamma_2^m$, and G the subgroup of Γ generated by g_1 and g_2. Therefore,

$$G\simeq\mathbb{Z}_{n_1}\times\mathbb{Z}_{m_1},\qquad G':=\Gamma/G\simeq\mathbb{Z}_n\times\mathbb{Z}_m,$$

and a set of representatives of G' are $\gamma_1^{a_1}\gamma_2^{a_2}$, $0\le a_1<n$, $0\le a_2<m$. Also, we can write explicitly:

$$j(\overline{\gamma_1^{a_1}\gamma_2^{a_2}\mathbf{x}})=\gamma_1^{a_1'}\gamma_2^{a_2''}\mathbf{x},\quad \pi(\gamma_1^{a_1}\gamma_2^{a_2}\mathbf{x})=\gamma_1^{a_1-a_1'}\gamma_2^{a_2-a_2''}\varepsilon(\mathbf{x})\in G,\qquad a_i\in\mathbb{N},\mathbf{x}\in B.$$

By Proposition 3.3 we have that

$$\langle\gamma_1,\Phi(g_1)\rangle=\chi_1(g_1)=q_{11}^n,\qquad \langle\gamma_2,\Phi(g_1)\rangle=\chi_2(g_1)=q_{12}^n,$$
$$\langle\gamma_1,\Phi(g_2)\rangle=\chi_1(g_2)=q_{21}^m,\qquad \langle\gamma_2,\Phi(g_2)\rangle=\chi_2(g_2)=q_{22}^m,$$

so Φ is uniquely determined, and moreover it tells us that m, n should satisfy $3|n$, $2|m$, $9|mn$, because $\langle g_j, \Phi(g_i)\rangle = 1$ for all $i, j \in \{1, 2\}$.

We compute the structure of the coquasi-Hopf algebra associated to this datum. Note that the following set is a basis of Q:

$$\overline{B} := \left\{ \overline{\gamma_1^{a_1}\gamma_2^{a_2}\mathbf{x}} : \, 0 \leq a_1 < n, \, 0 \leq a_2 < m, \, \mathbf{x} \in B \right\}.$$

Given $\mathbf{x}, \mathbf{y} \in B$ of degree $(e_1, e_2), (f_1, f_2) \in \mathbb{N}_0^2$ respectively, and $0 \leq a_1, b_1 < n$, $0 \leq a_2, b_2 < m$, we have that

$$\begin{aligned} m(\overline{\gamma_1^{a_1}\gamma_2^{a_2}\mathbf{x}}, \overline{\gamma_1^{b_1}\gamma_2^{b_2}\mathbf{y}}) = {} & q_{11}^{-b_1e_1} q_{12}^{(b_1+f_1-(b_1+f_1)')(a_2+e_2)-b_1e_2} \\ & q_{21}^{(b_2+f_2-(b_2+f_2)'')(a_1+e_1)-b_2e_1} q_{22}^{-b_2e_2} \overline{\gamma_1^{(a_1+b_1)'}\gamma_2^{(a_2+b_2)''}\mathbf{xy}}, \end{aligned}$$

where we use that $3|n$, $2|m$. And the associator is given by

$$\omega(\overline{\gamma_1^{a_1}\gamma_2^{a_2}\mathbf{x}}, \overline{\gamma_1^{b_1}\gamma_2^{b_2}\mathbf{y}}, \overline{\gamma_1^{c_1}\gamma_2^{c_2}\mathbf{z}}) = \delta_{\mathbf{x},1}\delta_{\mathbf{y},1}\delta_{\mathbf{z},1} q_{12}^{a_2(b_1+c_1-(b_1+c_1)')} q_{21}^{a_1(b_2+c_2-(b_2+c_2)'')},$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in B$, $0 \leq a_1, b_1, c_1 < n$, $0 \leq a_2, b_2, c_2 < m$.

References

[AG] N. Andruskiewitsch and G. García, *Finite subgroups of a simple quantum group*. Compositio Math. **145** (2009), 476–500.

[AS] ______, *Pointed Hopf algebras*, "New directions in Hopf algebras", MSRI series Cambridge Univ. Press; 1–68 (2002).

[An1] I. Angiono, *Basic quasi-Hopf algebras over cyclic groups*. Adv. Math. **225** (2010), 3545–3575.

[An2] I. Angiono, *On Nichols algebras of diagonal type*. J. Reine Angew. Math., to appear.

[ArG] S. Arkhipov and D. Gaitsgory, *Another realization of the category of modules over the small quantum group*. Adv. Math. **173** (2003), 114–143.

[DMR] S. Dăscălescu, G. Militaru and Ş. Raianu, *Crossed coproducts and cleft coextensions*. Comm. Algebra, **24** (1996), 1229–1243.

[DL] C. De Concini and V. Lyubashenko, *Quantum function algebra at roots of 1*. Adv. Math. **108** (1994), 205–262.

[DGNO] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik, *On braided fusion categories I*. Selecta Math. **16** (2010), no. 1, 1119.

[EG] P. Etingof, S. Gelaki, *The small quantum group as a quantum double*. J. Algebra **322** (2009), 2580–2585.

[FG] E. Frenkel and D. Gaitsgory, *Localization of $\mathfrak{g}$-modules on the affine Grassmannian*. Ann. of Math. **170** (2009), 1339–1381.

[G] D. Gaitsgory, *The notion of category over an algebraic stack*, preprint, math.AG/0507192.

[Ga] G. García, *Quantum subgroups of $GL_{\alpha,\beta}(n)$*. J. Algebra **324** (2010), 1392-1428.

[Ge] S. Gelaki, *Basic quasi-Hopf algebras of dimension n^3*. J. Pure Appl. Algebra **198** (2005), 165–174.

[H] I. Heckenberger, *Classification of arithmetic root systems*. Adv. Math. **220** (2009), 59–124.

[JS] A. Joyal and R. Street, *An introduction to Tannaka duality and quantum groups*. in Part II of Category Theory, Proceedings, Como 1990. Lec. Notes in Mathematics **1488**, (1991), 411–492.

[K] C. Kassel, *Quantum Groups*. Graduate Texts in Mathematics, **155**, Springer-Verlag, New York (1995).

[M] S. Montgomery, *Hopf Algebras and Their Actions on Rings*. CBMS Conf. Math. Publ., **82**, Amer. Math. Soc., Providence (1993).

[S1] P. Schauenburg, *Hopf bimodules, coquasibialgebras, and an exact sequence of Kac*. Adv. Math. **165** (2002) 194–263.

[S2] P. Schauenburg, *Two characterizations of finite quasi-Hopf algebras*. J. Algebra **273** (2004), 538–550.

[Sch] H.-J. Schneider, *Principal homegeneous spaces for arbitrary Hopf algebras*. Israel J. Math., **72** (1990), 167–231.

[W] W.C. Waterhouse, *Introduction to Affine Group Schemes*. Graduate Texts in Mathematics **66**, Springer-Verlag, New York (1979).

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