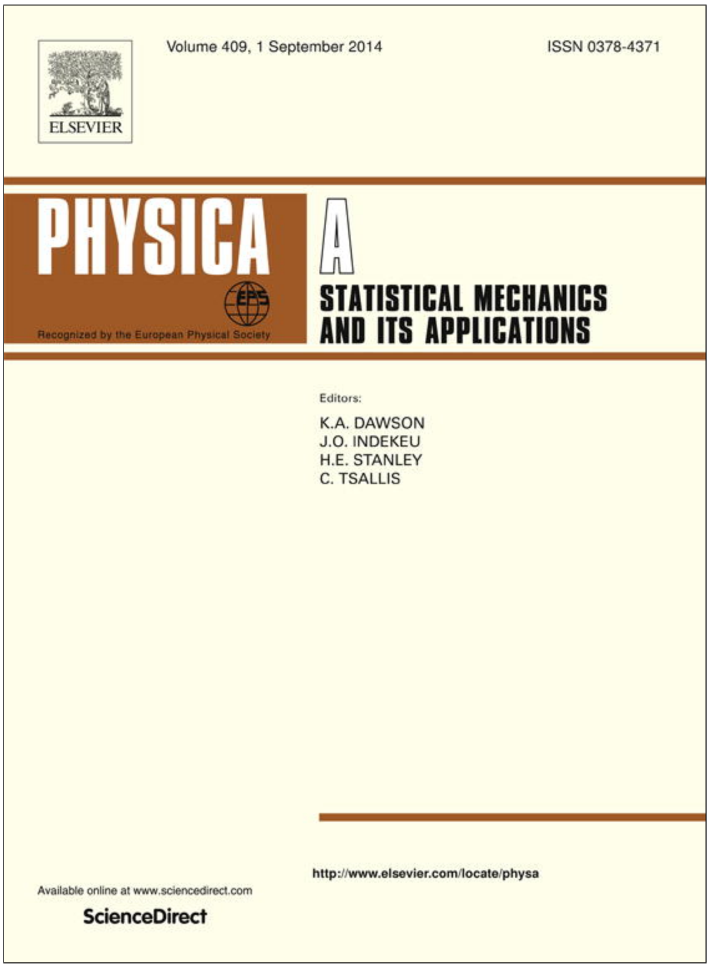
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Exponential distributed time-delay nonlinear models: Monte Carlo simulations☆

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HIGHLIGHTS

- An exponential distributed time-delay nonlinear model is numerically simulated.
- First passage time statistics for a distributed delay system have been worked out.
- We report the mean first passage time as a function of the mean time-delay.
- The crossover and a power-law behavior for the mean first passage time are shown.

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ABSTRACT

The stochastic dynamics toward the final attractor in an exponential distributed time-delay nonlinear model is studied, in the small noise approximation. The passage time statistic for this non-Markovian type of system has been worked out using Monte Carlo simulations. We report the mean first passage time $\langle t_e \rangle_{MC}$ from the unstable state as a function of the mean time-delay $\epsilon \equiv \lambda^{-1}$. We have compared our Monte Carlo simulations for $\lambda \gg 1$ against previous results (Cáceres, 2008) and we have found excellent agreement in the adiabatic regime. The crossover for $\lambda \sim 1$ and a power-law behavior $\langle t_e \rangle_{MC} \sim \lambda^{-\nu}$ for $\lambda \ll 1$ have also been found in agreement with recent theoretical predictions (Cáceres, 2014).

1. Introduction

It has been shown that the stochastic dynamics in nonlinear models can be studied in a mean-field approximation by characterizing the first passage time statistics of the random escape problem [1–4], if the model has a distributed time-delay structure the stochastic problem turns to be a non-Markovian problem but it can also be studied in a similar way [5]. In particular it has been proved that the Mean First Passage Time (MFPT) from the linear unstable state characterizes the onset of the pattern formation independently of the structure of the saturation. However, this MFPT strongly depends on the mean time-delay parameter occurring in the model. In the present paper we wish to present a Monte Carlo simulation for an exponential distributed time-delay nonlinear model, these results are going to be compared with the aforementioned theoretical predictions [5].

The relaxation analysis associated to time-delay nonlinear problems are under continuous investigation and will be the subject of our program. The first obvious complication in a delay model is the lack of a mathematical theory to tackle the passage time statistics (escape times) for a general non-Markovian problem [6–8]; the second difficulty in the dynamical analysis is the non-linear nature of the problem [3,9]. As far as we know, this is the first time that a systematic approach

☆ This work is dedicated to the memory of Prof. Dr. C.E. Budde.

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has been presented to solve the characterization of the onset-time from an initial unstable state toward the attractor in distributed time-delay nonlinear stochastic models.

In the present paper we are going to focus on the numerical simulations to study the stochastic dynamics and the escape time from the unstable point in an exponential distributed time-delay Logistic model, allowing in this way to characterize the general mechanism of relaxation of the system. We have compared the present numerical results against recent theoretical predictions [5] showing a very good agreement. In particular the adiabatic regime has shown to be in excellent agreement with the non-delay case [1,4]. The important point in studying the present delay Monte Carlo simulations will be, therefore, to compare a general theoretical prediction against a concrete nonlinear non-Markovian model. We emphasize that using symmetry arguments, the present conclusions (from the distributed time-delay Logistic model) apply also to the analysis of an exponential distributed time-delay bistable stochastic flux, as we will show.

2. The distributed time-delay nonlinear model

Consider the following exponential distributed time-delay problem:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = F(N, N_G), \tag{1}$$
$$N_G(t) = \int_0^\infty G(s)N(t-s)\mathrm{d}s, \qquad \int_0^\infty G(s)\mathrm{d}s = 1, \quad G(s) = \lambda \mathrm{e}^{-\lambda s}.$$

Note that the definition of $N_G(t)$ involves a weighted average of $N(t)$ over different past times s, this definition is different from a Mori-like time-convolution between the $N(t)$ and a given kernel (not necessarily positive).

Because the distribution $G(s)$ is an exponential the variable $N_G(t)$ fulfills a closed differential equation in terms of N and N_G. To prove this, from Eq. (1) we first write $N_G(t)$ in the alternative way

$$N_G(t) = \int_{-\infty}^t G(t-s)N(s)\mathrm{d}s, \tag{2}$$

then taking the time derivative in Eq. (2) we get

$$\frac{\mathrm{d}N_G}{\mathrm{d}t} = G(0)N(t) + \int_{-\infty}^t \left(\frac{\mathrm{d}}{\mathrm{d}t}G(t-s)\right) N(s)\mathrm{d}s.$$

The initial condition for $N_G(0)$ is related to a pre-function $\varphi(t)$ for $t \in (-\infty, 0)$. This is simple to see from the fact that $N_G(0) = \int_{-\infty}^0 G(-s)N(s)\mathrm{d}s = \int_0^\infty G(s)\varphi(-s)\mathrm{d}s$. Now using the fact that the $G(t)$ fulfills $\dot{G}(t) = -\lambda G(t)$ we finally arrive to $\dot{N}_G = \lambda\,(N - N_G)$. Therefore the distributed delay system of equations (1) can be replaced by the set of differential equations

$$\frac{\mathrm{d}N}{\mathrm{d}t} = F(N, N_G), \tag{3}$$
$$\frac{\mathrm{d}N_G}{\mathrm{d}t} = \lambda\,[N(t) - N_G(t)]\,, \quad \lambda > 0. \tag{4}$$

The situation when $G(s)$ is a biparametric exponential model (with a sharp peak around the delay time t_D) can also be tackled in a similar way [5]. Note that from Eqs. (3) and (4) $N_G(t)$ is a slave variable when $\lambda \gg 1$. Here by definition $G(t)$ is positive and can be thought as a probability distribution, then the mean time-delay characterizing the time-lag in an exponential distributed model is:

$$\epsilon = \int_0^\infty tG(t)\mathrm{d}t = 1/\lambda.$$

In particular, when $\lambda \to \infty$ we recover the usual non-delay case. In general, we may expect that if the time-scale ϵ is longer than the “natural period” of the system, large amplitude oscillations will result. On the other hand, the time-lag could have an increasingly destabilizing effect on $N(t)$ giving rise first to damped oscillations and then possibly to divergent oscillations.

In general putting $N(t) = N^*\,(1 + n(t))$ and $N_G(t) = N^*\,(1 + n_G(t))$ where $F(N^*, N^*) = 0$ and calling $a = -\partial_N F|_{N=N_G=N^*}$; $b = -\partial_{N_G} F|_{N=N_G=N^*}$ the linear stability analysis, around the attractor N^*, for a perturbation like $n(t) \propto \exp(-ct + i\omega t)$ leads to the conclusion that the rate of change dn/dt is affected by $n(t)$ and the weighted average $n_G(t)$ in the form: $\dot{n} = -an - bn_G$, where the damped rate c and the frequency of oscillations ω are related by

$$c + i\omega = \frac{\lambda}{2}\left[\left(1 + \frac{a}{\lambda}\right) \pm \sqrt{\left(1 + \frac{a}{\lambda}\right)^2 - \frac{4}{\lambda}\,(a+b)}\,\right], \tag{5}$$

therefore oscillations may happen ($\omega \neq 0$) if and only if: $\left(1 + \frac{a}{\lambda}\right)^2 < \frac{4}{\lambda}\,(a+b)\,.$

In the Logistic model $F(N, N_G) = r\left[N_G(t) - \frac{N(t)^2}{K}\right]$ with exponential distributed time-delay we get: $a = -\partial_N F|_{N=N_G=N^*} = 2r$, $b = -\partial_{N_G} F|_{N=N_G=N^*} = -r$. Therefore $(a+b) = r > 0$ so the system is stable and does not present oscillations for any value of λ. This result is in direct contrast to the fixed birth time-delay model (leading to $\dot{n} = -2rn(t) + rn(t-\tau_D)$) for which increasing τ_D affects both stability and the type of damping [10]. In the present work we will be interested in the analysis of the Logistic system. However other distributed time-delay dynamical models can also be studied in a similar fashion, for example the archetypal bistable flux: $\dot{N} = N_G - N^3$.

3. Stochastic approach in the exponential distributed time-delay model

Fluctuations are ubiquitous in real systems and we are often forced to consider stochastic perturbation in the dynamical systems because some parameters may fluctuate in time or because there are degrees of freedom which are not considered in the model [6–8,11,12]. The simplest way to introduce these fluctuations into the dynamical system (3) and (4) may, therefore, be written in the form

$$\frac{dN}{dt} = F(N, N_G) + \sqrt{\theta}\xi(t), \tag{6}$$

$$\frac{dN_G}{dt} = \lambda\left[N(t) - N_G(t)\right], \quad \lambda > 0, \tag{7}$$

where $\xi(t)$ is a zero mean Gaussian white noise characterized by the correlation $\langle\xi(t)\xi(t')\rangle = \delta\left(t - t'\right)$, and $\sqrt{\theta}$ is the noise parameter.

If we consider $F(N, N_G) = \left[N_G(t) - N(t)^3\right]$ in Eq. (6) we can study the onset-time of the pattern-formation in a distributed time-delay bistable flux. Due to the symmetry of this archetypal bistable flux the onset-time, from the non-equilibrium initial state $N(0) = 0$, can also be studied if we consider the Logistic model $F(N, N_G) = r\left[N_G(t) - \frac{N(t)^2}{K}\right]$ with the reflecting Boundary Conditions (BC) at the origin. Therefore in what follows of the present work we will focus in this last situation (so we can compare the present results against previous non-delay Monte Carlo simulations of the Logistic equation [4]). This is so because, for small noise, if the unstability is linear the dominant value of the onset-time does not depend on the saturation term (see Appendix A); therefore in the present work we will consider the set of stochastic differential equations (SDEs) (6) and (7) with

$$F(N, N_G) = r\left[N_G(t) - \frac{N(t)^2}{K}\right]. \tag{8}$$

This system is going to be solved under the restriction $N(t) \geq 0$. In order to accomplish this fact we will impose reflecting BC on the realizations when $N(t) = 0$.

In a population model the positive or negative character of the Gaussian noise represents gain or lose in the time-variation of $dN(t)/dt$ due to the extrinsic factors. The present additive noise is not tempted to describe "extraneous predation", Eqs. (6)–(8) represent a possible noise-additive character in a delay Logistic deterministic dynamics. This additive noise model is a plausible ansatz when the unspecified random contributions are more important at low density [5].

When $\lambda \gg 1$, from Eqs. (6)–(8), introducing an adiabatic perturbation theory [13] into the corresponding $2D$ Fokker–Planck (FP) representation it is possible to write down an associated $1D$ FP [11] for the marginal process $N(t)$

$$\partial_t P\left(N, t|N(0)\right) = \left[\partial_N U'(N) + \frac{\theta}{2}\partial_N^2\right] P\left(N, t|N(0)\right), \tag{9}$$

where

$$U(N) = -\frac{r\left(1 - \epsilon r\right)}{2}N^2 + \frac{r\left(1 - \epsilon r\right)}{3K}N^3 + \mathcal{O}\left(\epsilon^2\right). \tag{10}$$

As before, $\epsilon = \lambda^{-1}$ is the mean time-delay from the exponential distribution $G(s)$, thus $U(N)$ is a perturbed delay-dependent adiabatic potential [5]. Thus Eq. (9) represents a genuine Markovian adiabatic approximation.

Note that the FP equation (9) is equivalent of adding noise (white and Gaussian) to the deterministic dynamics in the form

$$dN = -U'(N)dt + \sqrt{\theta}dW(t), \quad N(t) \geq 0, \tag{11}$$

where $dW(t)$ is a Wiener differential. In order to study the realizations $N(t)$ of this SDE, the Wiener paths have to be worked out with care to assure that $N(t) \geq 0$, this issue was considered using the Stochastic Paths Perturbation Approach (SPPA) [4]. In the context of the FP equation (9) it means that we will be interested in a reflecting BC at $N = 0$.

3.1. The mean first passage time in the Markovian approximation

To study the first passage time statistics for a non-Markov process is a complex task, and to calculate the passage time statistics for a general 2D Markov problem is also a non-trivial problem [6]. In this section we are going to use the previous adiabatic approximation, Eq. (9), to calculate the Mean First Passage Time (MFPT) in the Markovian approximation. We will show that this value is the characteristic time-scale to reach the domain of attraction in the small noisy version of Eq. (1), i.e., the scaling-time to characterize the stochastic relaxation toward the attractor.

It has been proved that from a linear instability there is a universal scaling-time which characterizes the transition from the microscopic disorder to macroscopic order. The pioneer calculations were done by using a stochastic self-consistent scaling theory [3,14], and from that result this onset time has been called Suzuki's scaling-time τ_S. The dominant contribution in terms of the noise intensity $\sqrt{\theta}$ is given by: $\tau_S \propto \ln(1/\theta)$. However, if the instability were nonlinear this scaling-time would follow a different law in terms of the noise intensity. This fact can also be seen from the self-consistent scaling theory [3], and from the SPPA [15–18].

Alternatively, by using the first passage time theory for a Markovian process it is also possible to prove that asymptotically Suzuki's scaling-time is just the MFPT calculated from $N = 0$ to N_f. To get an analytical expression for this MFPT we now introduce an asymptotic calculation in the small noise parameter $\sqrt{\theta}$. For small noise the *dominant* scaling-time for the onset of the macroscopic order $\mathcal{O}(1)$ is universal and independent of the nonlinear saturation term, therefore the dominant contribution in the small parameter θ has been shown to be

$$\tau(N_f) = \frac{1}{2\mathcal{A}}\left(\ln\frac{N_f^2\mathcal{A}}{\theta} - \psi\left(\frac{1}{2}\right) + \cdots\right), \tag{12}$$

where $\psi\left(\frac{1}{2}\right) \simeq -1.9351$ is the Digamma function $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ and $\mathcal{A}$ is the rate in the linear instability term. Corrections to this dominant contribution will depend on the saturation term. A Markovian perturbative calculation is presented in Appendix A to evaluate this corrections explicitly in the Logistic model. On the other hand, in Appendix B we calculate this Suzuki's scaling time using the SPPA for a bistable stochastic flux. There, in addition we present the first passage time distribution $\mathcal{P}(t_e)$. Then, the stochastic dynamics toward the final attractor in an exponential distributed time-delay bistable model can be studied in the instanton-like approximation.

Hence the scaling-time for the relaxation in a structured nonlinear model (with an exponential distributed time-delay) is characterized in the Markov approximation by the curvature of the perturbed delay-dependent adiabatic potential around the unstable point, i.e., the scaling-time is given by (12) with $\mathcal{A} \equiv \mathcal{A}(\epsilon) = U''(N = 0)$, see (10). However, what about if we want to consider a situation far away from the adiabatic approximation, in this case the system is dominated by its non-Markovian character and this problem has been tackled using a different technique.

4. Concerning the non-adiabatic calculation of the MFPT

A non-adiabatic analysis of the stochastic escape times from the unstable state can be carried out by introducing the SPPA into the dynamics of exponential distributed time-delay models [5]. In the small noise approximation the SPPA consists of obtaining information about the first passage time statistics without solving the Fokker–Planck equation. This is done by analyzing the stochastic realizations of the process under study when they are written in terms of Wiener paths [1,4,15,17,18]. From the SPPA in the *small noise approximation* the MFPT from $N = 0$ to $N = N_f$ can be obtained analytically from Eqs. (6)–(8) as:

$$\langle t_e\rangle = \frac{1}{2\mathcal{A}_{non-A}}\left(\ln\frac{N_{non-A}^2\mathcal{A}_{non-A}}{\theta} - \psi\left(\frac{1}{2}\right) + \cdots\right), \quad \lambda > 0, \tag{13}$$

here N_{non-A} and $\mathcal{A}_{non-A}$ are explicit functions of the delay parameter λ the threshold value N_f and noise intensity θ, see Ref. [5] for its derivation

$$N_{non-A} = \frac{2\Xi N_f}{1+\Xi} + \sqrt{\frac{\theta}{\lambda(1+\Xi)}\frac{1-\Xi}{1+\Xi}} \tag{14}$$
$$\mathcal{A}_{non-A} = \left|\frac{\lambda}{2}(1-\Xi)\right|,$$

where

$$\Xi = \sqrt{1+4r/\lambda}.$$

Compare the result (13) with the Markov approximation given in Eq. (12). In particular from Eq. (13) and in the limit of large $\lambda \gg 1$ (small mean time-delay $\epsilon = 1/\lambda$) the SPPA predicts an effective rate

$$\mathcal{A}_{non-A} \to r(1 - r\epsilon + \cdots), \quad \epsilon \to 0, \tag{15}$$

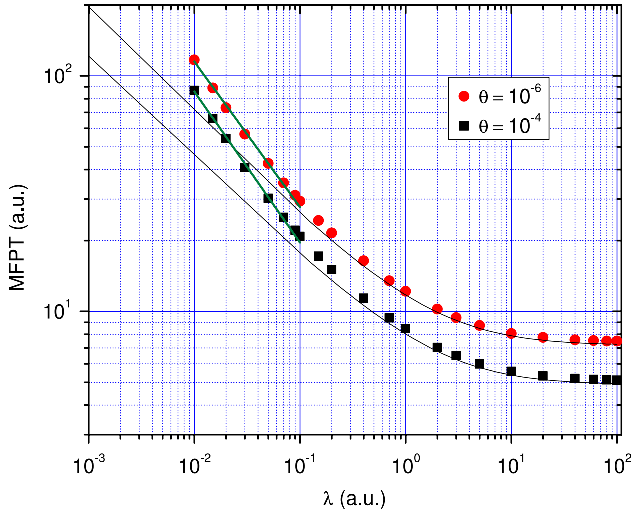

Fig. 1. The MFPT (in arbitrary units) for $N_f = 1/2$ and $r = K = 1$ for the exponential distributed time-delay equation (1) using (8), as a function of the delay parameter λ for the noise intensity $\theta = \{10^{-4}, 10^{-6}\}$. The asymptotic value of the MFPT $\langle t_e \rangle_{MC}$ for $\lambda \gg 1$ are $4.93\cdots$ and $7.25\cdots$ for the noise intensity $\theta = \{10^{-4}, 10^{-6}\}$ respectively, and are in excellent agreement with the Monte Carlo simulations of Ref. [4]. The crossover at $\lambda \sim 1$ as well as the power-law behavior ($\lambda \ll 1$) for the MFPT $\langle t_e \rangle_{MC} \sim \lambda^{-\nu}$ can be seen from the simulations. Thick green lines, for $\lambda \ll 1$, correspond to the fitting ($\nu \simeq 0.6$) with the Monte Carlo simulation for $\lambda \in (10^{-2} - 10^{-1})$. The theoretical prediction in the adiabatic regime, i.e., Eq. (13) using the asymptotic expressions (15) and (16) shows an excellent agreement against the simulations. The SPPA scaling prediction $\propto 1/\sqrt{\lambda}$ (for $\lambda \ll 1$) is in agreement with the present Monte Carlo simulations (within the error bars) at the highly non-adiabatic regime.

which is in agreement with the adiabatic Markov approach to $\mathcal{O}(\epsilon)$, see Eq. (12). In addition the threshold N_{non-A}, in the same limit, gives

$$N_{non-A} \to N_f\,(1 + r\epsilon + \cdots) - \sqrt{\theta}\left(\frac{r}{8}\epsilon^{3/2} + \cdots\right), \quad \text{for } \epsilon \to 0. \tag{16}$$

Note that, this last correction cannot be obtained from the Markov approximation. For example, the second term in Eq. (16) is a small perturbation which comes from the fluctuations of N_G induced from the dynamics of $N(t)$. In the non-delay case ($\epsilon = 0$) this result has been tested for the particular noisy Logistic equation and we have found excellent agreement against the numerical solutions of the associated SDE [4]. The comparison of the SPPA against delay Monte Carlo simulations for any λ is shown in Fig. 1.

In the opposite limit $\lambda \ll 1$ (highly non-adiabatic regime, i.e., large mean time-delay $\epsilon = 1/\lambda$) it has been shown [5] from Eq. (13) that the dominant term (for fixed θ) is given by

$$\langle t_e \rangle \approx \frac{1}{2\sqrt{\lambda r}}\left[\ln(1/2) - \psi(1/2) + \ln\left(1 - 2N_f\sqrt{\frac{2\sqrt{\lambda r}}{\theta}}\right)^2\right], \quad \text{for } \lambda \ll 1, \tag{17}$$

precluding the behavior $\langle t_e \rangle \propto 1/\sqrt{\lambda}$ for $\lambda \ll 1$. A similar power-law behavior for the MFPT: $\langle t_e \rangle_{MC} \sim \lambda^{-\nu}$ was found from the present numerical simulations.

5. Monte Carlo simulations of the MFPT

Delay Monte Carlo simulations of the stochastic escape times from the unstable state, associated to Eqs. (6)–(8) can be carried out by solving numerically the following set of coupled equations

$$\begin{aligned} N(i+1) &= N(i) + \frac{\Delta}{2} r\left\{\left[N_G(i) - N^2(i)/K\right] + \left[N_G(i) - \hat{N}^2(i+1)/K\right]\right\} + \sqrt{\Delta\theta}\,\eta_i \\ \hat{N}(i+1) &= N(i) + r\Delta\left[N_G(i) - N^2(i)/K\right] + \sqrt{\Delta\theta}\,\eta_i \\ N_G(i+1) &= N_G(i) + \lambda\Delta\left[N(i) - N_G(i)\right], \end{aligned} \tag{18}$$

here $\hat{N}(i+1)$ is the predicator step that improves the algorithm [19] and η_i are statistical independent Gaussian random numbers with mean zero and second moment $\langle \eta_i^2 \rangle = 1$, this random numbers can be generated using the Box–Muller formula:

$$\eta_i = \sqrt{-2\ln u_{i1}}\cos 2\pi u_{i2},$$

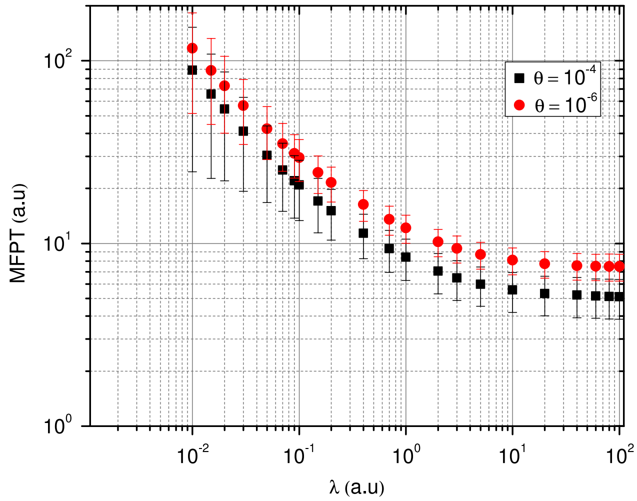

Fig. 2. Data points with Monte Carlo error bars as a function of λ for two values of θ.

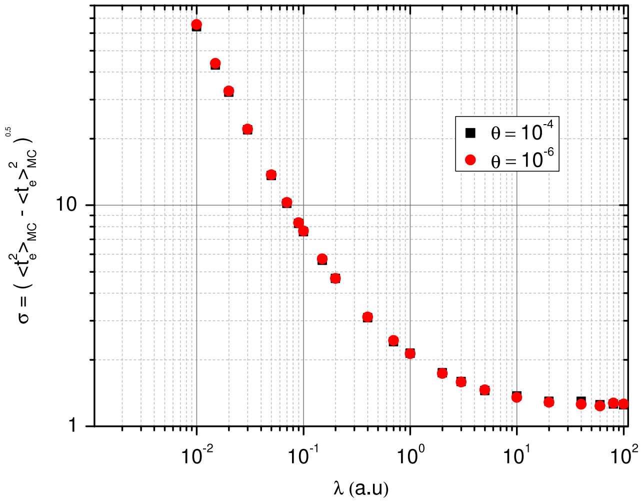

Fig. 3. Standard deviation σ as a function of λ for two values of the noise parameter θ.

where $u_{i1,i2}$ are statistical independent uniformly distributed number in $(0, 1)$ and $t_i - t_{i-1} = \Delta$ is the time interval in the numerical approach.

We have solved the SDE (18) and calculated the histogram $\mathcal{P}(t_e)$ at the random times t_e when the stochastic process $N(t_i)$ reach for the first time the threshold value $N_f = K/2$. In Fig. 1 we show the MFPT calculated numerically from our histogram as

$$\langle t_e \rangle_{MC} = \int_0^\infty t_e \mathcal{P}(t_e) \mathrm{d}t_e.$$

To calculate $\langle t_e \rangle_{MC}$ from the present Monte Carlo simulations we have run 10000 realizations for $\theta = 10^{-4}$ and $\theta = 10^{-6}$, with a time discretization of $\Delta = 0.01$.

In Fig. 1 we show as a function of λ, the Monte Carlo data and the theoretical prediction (lines) for two values of noise intensity θ. We have also performed a fitting with data points in the interval of $\lambda \in \left(10^{-2}, 10^{-1}\right)$, and we got $\nu \sim 0.6$ with and adjusted R-Square of $R \sim 0.9$. These fitting-lines can also been seen as thick lines in the same figure. In Fig. 2 we show data points with the error bars for the two values of θ we have used. The error bars correspond to the Monte Carlo Standard Deviation σ. As one can notice from this graph, Standard Deviation increases for low values of λ. This means that the variation about mean values μ of the first passage times increases as $\lambda \to 0$. Thus we can conclude that the theoretical prediction falls within the interval $\mu \pm \sigma$. It is interesting to see that Standard Deviation does not depend on the noise parameter, see Fig. 3 where we show $\sigma = \sqrt{\langle t_e^2 \rangle_{MC} - \langle t_e \rangle_{MC}^2}$ as a function of λ.

6. Conclusions

In order to characterize the stochastic dynamics toward the final attractor in an exponential distributed time-delay nonlinear model, numerical simulations of the (delay) escape process have been presented. The passage time statistics can then be studied in the small noise approximation. The first passage time statistics (from the linear unstable state $N = 0$) associated to this non-Markovian type of system have been calculated numerically. We have compared our results against recent theoretical predictions [5] and we have found agreement between them. We have also found (within the numerical errors of our simulations and for fixed noise intensity θ) a power-law behavior for small λ (large mean time-delay $\epsilon = \lambda^{-1}$) which goes asymptotically as $\langle t_e \rangle_{MC} \sim \lambda^{-\nu}$ for $\lambda \ll 1$, in agreement with the theoretical prediction $\propto 1/\sqrt{\lambda}$. We have also shown the existence of a crossover for $\lambda \sim 1$ in the mean first passage time $\langle t_e \rangle_{MC}$ as a function of the delay parameter λ.

The critical situation when the instability is nonlinear and/or when the noise is multiplicative can also be worked out in a similar way by using the Stochastic Path Perturbation Approach [16–18], and the work along this line is in progress.

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Appendix A. Corrections to the MFPT in the Markov approximation

In the adiabatic approach the MFPT from $0 \to N_f$ for a $1D$ continuous Markov process with reflecting BC at $N = 0$ is given by Ref. [6,7,11,20]

$$\tau(N_f) = \frac{2}{\theta} \int_0^{N_f} dy \int_0^y \exp\left(\frac{2}{\theta}\left[U(y) - U(z)\right]\right) dz. \tag{19}$$

This is the solution of Dynkin's equation associated to the Fokker–Planck equation (9), which is consistent to the SDE (11).

Using the perturbed delay-dependent adiabatic Logistic potential (10) up to the linear order in ϵ, we can write $U(N) \simeq -\frac{1}{2}\mathcal{A}N^2 + \frac{1}{3}\mathcal{B}N^3$ where $\mathcal{A} = r(1 - \epsilon r)$ and $\mathcal{B} = \mathcal{A}/K$. Then from Eq. (19) the MFPT is given by

$$\tau(N_f) = \frac{2}{\theta} \int_0^{N_f} dy \int_0^y \exp\left(\frac{-\mathcal{A}}{\theta}\left[y^2 - z^2\right]\right) f_{\mathcal{B}}(y, z)\, dz \tag{20}$$
$$f_{\mathcal{B}}(y, z) = \exp\left(\frac{2\mathcal{A}}{3\theta K}\left[y^3 - z^3\right]\right).$$

Noting that $N_f \ll K$ and introducing a suitable series expansion for $f_{\mathcal{B}}(y, z)$ we can write a perturbation formula for $\tau(N_f)$ in terms of the saturation contribution $f_{\mathcal{B}}(y, z)$ as follows. Denoting $\alpha \equiv \theta/\mathcal{A}$ and introducing the change of variable $u = \left(y^2 - z^2\right)/\alpha$ we get

$$\tau(N_f) = \frac{2\alpha}{\theta} \int_0^{N_f} dy \int_0^{y/\sqrt{\alpha}} \frac{u e^{-u^2}}{\sqrt{y^2 - \alpha u^2}} \exp\left(\frac{2}{3\alpha K}\left[y^3 - \left(y^2 - \alpha u^2\right)^{3/2}\right]\right) du. \tag{21}$$

For small α we can take $\left[y^3 - \left(y^2 - \alpha u^2\right)^{3/2}\right] = \frac{3}{2}\alpha y u^2 + \mathcal{O}\left(\alpha^2\right)$, then we can introduce a perturbation from the saturation term (the term $1/\sqrt{y^2 - \alpha u^2}$ should not be expanded because this function is responsible of the singular character of the integrand), so we finally get a perturbation expansion in the small quantity y/K

$$\tau(N_f) \simeq \frac{2\alpha}{\theta} \int_0^{N_f} dy \int_0^{y/\sqrt{\alpha}} \frac{u}{\sqrt{y^2 - \alpha u^2}} \exp\left(-u^2\left[1 - \frac{y}{K}\right]\right) du$$
$$= \frac{2\alpha}{\theta} \int_0^{N_f} \frac{D\left[\left(\frac{\sqrt{1 - \frac{y}{K}}}{\sqrt{\alpha}}\right) y\right]}{\sqrt{\alpha}\sqrt{1 - \frac{y}{K}}} dy, \tag{22}$$

here we have used the definition of the Dawson function $D[x]$ [21]. Noting that $y \le N_f \ll K$ we can approximate the function $\left(1 - \frac{y}{K}\right)^{-1/2}$, then the dominant contribution to the integrand of (22) looks like

$$\frac{D\left[\left(\frac{\sqrt{1 - \frac{y}{K}}}{\sqrt{\alpha}}\right) y\right]}{\sqrt{\alpha}\sqrt{1 - \frac{y}{K}}} = D\left[\frac{y}{\sqrt{\alpha}}\right]\left(1 + \frac{y}{2K}\right)\frac{1}{\sqrt{\alpha}} + \mathcal{O}\left(\left(\frac{y}{K}\right)^2\right). \tag{23}$$

Introducing this expression in the integral we get

$$\tau(N_f) \simeq \frac{2\alpha}{\theta}\int_0^{N_f}\frac{D\left[\frac{y}{\sqrt{\alpha}}\right]}{\sqrt{\alpha}}\mathrm{d}y + \frac{2\alpha}{\theta}\int_0^{N_f} D\left[\frac{y}{\sqrt{\alpha}}\right]\frac{y}{2K\sqrt{\alpha}}\mathrm{d}y. \tag{24}$$

Using that $\alpha \equiv \theta/\mathcal{A}$ the first integral gives the dominant contribution

$$\frac{2\alpha}{\theta}\int_0^{N_f}\frac{D\left[\frac{y}{\sqrt{\alpha}}\right]}{\sqrt{\alpha}}\mathrm{d}y = \frac{N_f^2}{\theta}F_{pq}\left[\{1,1\},\left\{\frac{3}{2},2\right\},-\frac{N_f^2\mathcal{A}}{\theta}\right], \tag{25}$$

where $F\left[\{a_1,\ldots,a_p\},\{a_1,\ldots,a_q\},z\right] = F_{pq}[a,b,z]$ is a generalized hypergeometric function [21]. The second integral in Eq. (24) gives the correction due to the saturation term

$$\frac{2\alpha}{\theta}\int_0^{N_f} D\left[\frac{y}{\sqrt{\alpha}}\right]\frac{y}{2K\sqrt{\alpha}}\mathrm{d}y = \frac{N_f}{2K\mathcal{A}} - \frac{\sqrt{\theta/\mathcal{A}}}{2K\mathcal{A}}D\left[N_f\sqrt{\mathcal{A}/\theta}\right], \tag{26}$$

here in order to do the integral we have used the relation: $D'[x] = 1 - 2xD[x]$. Calling $\eta = \frac{N_f^2\mathcal{A}}{\theta}$ and noting that

$$\lim_{\eta\to\infty} 2\eta\, F_{pq}\left[\{1,1\},\left\{\frac{3}{2},2\right\},-\eta\right] \to \left(\ln\eta - \psi\left(\frac{1}{2}\right)\right),$$

where $\psi(x)$ is the Digamma function, we can write, in the small noise limit, the final expression for the MFPT as

$$\tau(N_f) \to \frac{1}{2\mathcal{A}}\left(\ln\frac{N_f^2\mathcal{A}}{\theta} - \psi\left(\frac{1}{2}\right) + \frac{N_f}{K} - \frac{\sqrt{\theta/\mathcal{A}}}{K}D\left[\frac{N_f}{\sqrt{\theta/\mathcal{A}}}\right] + \cdots\right). \tag{27}$$

This means that asymptotically for small noise Suzuki's scaling-time is equivalent to the dominant term in the MFPT. Using that $\lim_{x\to\infty} D[x] \to \frac{1}{2x} + \frac{1}{4x^3} + \cdots$ we see that the first correction from the nonlinear saturation term is just the constant N_f/K.

It is meaningful to comment here that Suzuki's scaling-time $\tau_S = \frac{1}{2\mathcal{A}}\left(\ln\frac{N_f^2\mathcal{A}}{\theta} - \psi\left(\frac{1}{2}\right)\right)$ can also be obtained from the SPPA working out the SDE $\mathrm{d}N = \mathcal{A}N\mathrm{d}t + \sqrt{\theta}\mathrm{d}W(t)$ (without saturation) and solving the random time t_e at the threshold value $N(t_e) = N_f$ [4], compare also with the next appendix. In addition, a similar approach can be used when the noise is Gaussian but non-white [1], or when the coefficient $\mathcal{A} = \mathcal{A}(t)$ is a function of time (i.e.: a time dependent potential) [2,22].

Appendix B. Revisiting Suzuki's scaling time for a noisy bistable flux

Let a dynamical flux be represented by the SDE

$$\frac{\mathrm{d}X}{\mathrm{d}t} = aX(t) - bX(t)^3 + \sqrt{\theta}\xi(t); \qquad X(0) = 0, \tag{28}$$

here, as before, $\xi(t)\mathrm{d}t = \mathrm{d}W(t)$ is a Wiener differential. At short time any realization is dominated by the linear instability term, then it can be approximated to $\mathcal{O}(\sqrt{\theta})$ by

$$X(t) \simeq \sqrt{\theta}\mathrm{e}^{at}\int_0^t \mathrm{e}^{-as}\mathrm{d}W(s) \equiv \sqrt{\theta}\mathrm{e}^{at}h(t), \quad t \geq 0. \tag{29}$$

Where we have defined a new stochastic process $h(t) \equiv \int_0^t \mathrm{e}^{-as}dW(s) \geq 0$, which is the solution of the SDE

$$\dot{h}(t) = \mathrm{e}^{-at}\xi(t), \quad h(0) = 0,\ t \geq 0.$$

It is important to note that even when the correlation function of the process $\xi(t)$ is white, the stochastic process $h(t)$ saturates at long time ($t \gg a^{-1}$). In particular it is possible to prove that $\langle h(\infty)^2\rangle = 1/2a$, then the stationary pdf is

$$P(h,\infty) \equiv P(\Omega) = \frac{\mathrm{e}^{-\Omega^2/a}}{\sqrt{\pi/a}}, \quad \Omega \in (-\infty,+\infty). \tag{30}$$

Therefore we can approximate the paths (29) in the form:

$$X(t) \simeq \sqrt{\theta}\Omega\exp(at). \tag{31}$$

Formula (31) gives the stochastic paths we were looking for as a mapping from the random number Ω. These realizations give an accurate representation of the paths for short and intermediate times, except for the small fluctuations around the final steady state $X(\infty)^2 = a/b$, i.e., in the long-time limit $t \to \infty$. At *intermediate* times, the random scape times t_e (when the stochastic paths leave the initial domain $\mathcal{O}(\sqrt{\theta})$ and fall into the attractor of the saturation valley) can be obtained by inverting t_e from (31) in the form $X(t_e)^2 \simeq \theta\Omega^2 \exp(2at_e) = X_f^2$. Therefore to $\mathcal{O}(\sqrt{\theta})$ we get the transformation law

$$t_e = \frac{1}{2a}\log\left(\frac{X_f^2}{\theta\Omega^2}\right). \tag{32}$$

This formula teaches us that the MFPT is just given by the mean value $\langle t_e\rangle$ over the distribution of the random variable Ω, using (30) we get

$$\begin{aligned}\langle t_e\rangle &= \frac{1}{2a}\int_{-\infty}^{\infty}\log\left(\frac{X_f^2}{\theta\Omega^2}\right)P(\Omega)\,\mathrm{d}\Omega \\ &= \frac{1}{2a}\left(\ln\frac{X_f^2 a}{\theta} - \psi\left(\frac{1}{2}\right)\right),\end{aligned} \tag{33}$$

where $\psi\left(\frac{1}{2}\right) \simeq -1.9351$ is the Digamma function, compare this formula with the previous result (27). As expected by the symmetry of the problem, Suzuki's scaling-time for a bistable flux [1,2,22] is the same as the one obtained for the noisy Logistic equation with reflecting BC at the origin [4].

In order to obtain the FPTD $\mathcal{P}(t_e)$ (i.e., the probability that amplitude $X^2(t)$ reaches a given threshold $\mathcal{O}(a/b)$ between t_e and $t_e + \mathrm{d}t_e$), we begin with the relation between Ω and t_e expressed in Eq. (32). Then the FPTD $\mathcal{P}(t_e)$ follows as:

$$\mathcal{P}(t_e) = \int \delta\left(t_e - t_e[\Omega]\right)P(\Omega)\mathrm{d}\Omega.$$

Using the Jacobian of the transformation $\left|\frac{\mathrm{d}\Omega}{\mathrm{d}t_e}\right| = a\,|\Omega|$, we arrive to

$$\mathcal{P}(t_e) = \mathcal{N}\frac{aX_f}{\sqrt{\theta}}\exp(-at_e)\,P\left(\Omega = \frac{X_f}{\sqrt{\theta}}\exp(-at_e)\right), \quad t_e \in (0,\infty), \tag{34}$$

where $P(\Omega)$ is given in Eq. (30) and the normalization constant is $\mathcal{N}^{-1} = \frac{1}{2}\mathrm{erf}\left(aX_f/\sqrt{\theta}\right)$.

B.1. Anomalous fluctuations and the distributed time-delay stochastic bistable flux

From the FPTD (34) it is possible to study the dynamics of the nonlinear process by introducing the instanton-like approximation

$$X(t) \simeq \pm\Theta\left(t - t_e\right), \tag{35}$$

here $\Theta(z)$ is the step function and t_e the random escape times characterized by $\mathcal{P}(t_e)$, this approximation gives a good description for the analysis of the anomalous fluctuations of the process [3,4,15–18,22,23]. To calculate *analytically* the anomalous fluctuation, we approximate the transient toward the global attracting solution by using (35) which is for $t > t_e$ the $\mathcal{O}(1)$ macroscopic amplitude size (characterizing the attractor valley). Then the transient anomalous fluctuation is given by

$$\sigma_X^2(t) = \langle\Theta(t - t_e)\rangle - \langle\Theta(t - t_e)\rangle^2, \quad t \geq 0, \tag{36}$$

where

$$\begin{aligned}\langle\Theta(t - t_e)\rangle &= \int_0^{\infty}\Theta(t - t_e)\mathcal{P}(t_e)\,\mathrm{d}t_e \\ &= \int_0^{t}\mathcal{P}(t_e)\,\mathrm{d}t_e,\end{aligned}$$

here we have used, invoking symmetry arguments, that $\langle X(t)\rangle = \langle\pm\Theta(t - t_e)\rangle = \langle\Theta(t - t_e)\rangle$. In this instanton-like approximation the maximum of the function $\sigma_X^2(t)$ is at the most probable escape value. The function $\sigma_X^2(t)$ depicts the qualitative behavior of the anomalous fluctuation we were looking for. From the behavior of $\sigma_X^2(t)$ we see that in the transient regime the initial fluctuations are amplified and give rise to the transient anomalous fluctuations of $\mathcal{O}(1)$ as compared with the initial or final fluctuations of $\mathcal{O}(\sqrt{\epsilon})$.

It is interesting to note that the FPTD $\mathcal{P}(t_e)$ for the escape process from $X(0) \rightarrow \pm X_f$ for an exponential distributed time-delay stochastic bistable flux:

$$\dot{X} = rX_G - X^3 + \sqrt{\theta}\xi(t)$$
$$X_G(t) = \lambda \int_0^\infty e^{-\lambda s} X(t-s)\mathrm{d}s, \qquad (37)$$

can be read from Eq. (34) replacing $X_f \rightarrow N_{non-A}$ and $a \rightarrow \mathcal{A}_{non-A}$ from Eq. (14).

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