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Passage Time Statistics in Exponential Distributed Time-Delay Models: Noisy Asymptotic Dynamics

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Abstract The stochastic dynamics toward the final attractor in exponential distributed time-delay non-linear models is presented, then the passage time statistic is studied analytically in the small noise approximation. The problem is worked out by going to the associated two-dimensional system. The mean first passage time $\langle t_e \rangle$ from the unstable state for this non-Markovian type of system has been worked out using two different approaches: firstly, by a rigorous adiabatic Markovian approximation (in the small mean delay-time $\epsilon = \lambda^{-1}$); secondly, by introducing the stochastic path perturbation approach to get a non-adiabatic theory for any λ. This first passage time distribution can be written in terms of the important parameters of the models. We have compared both approaches and we have found excellent agreement between them in the adiabatic limit. In addition, using our non-adiabatic approach we predict a crossover and a novel behavior for the relaxation scaling-time as a function of the delay parameter which for $\lambda \ll 1$ goes as $\langle t_e \rangle \sim 1/\sqrt{\lambda}$.

Keywords Distributed time-delay · Non-linear population models · Non-adiabatic approach · Non-Markov process · Relaxation from unstable states · First passage time statistics

1 Introduction

Fluctuations are ubiquitous in nature and relaxation in a far-from-equilibrium system has been studied extensively by many authors [1,2]. In particular, the relaxation process from the initial unstable state has been one of the challenging problems in non-equilibrium statistical mechanics. In fact, this problem is closely connected to the stochastic evolution of a population starting from a situation far from equilibrium [3]. The relaxation analysis associated with problems of distributed time-delay are under continuous investigation and will be the

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subject of our present program. The first obvious complication in the stochastic analysis is the lack of a mathematical theory to tackle the first passage time (escape times) for a general non-linear non-Markovian problem [4–6]. To our knowledge, this is the first time that a systematic approach has been presented to solve the characterization of the escape time from an initial unstable state toward the attractor in distributed time-delay non-linear models.

In the present paper we are going to focus on the study of the stochastic dynamics and the escape time from an unstable point in distributed time-delay nonlinear models. In order to carry out this research we first introduce an adiabatic perturbation to describe the deterministic dynamics of the system. Following this, we introduce noise into the system to study its fluctuations. To tackle the problem of the escape time analytically we use two approaches which are valid in the small noise limit. Firstly, the introduction of an adiabatic Markov approximation and secondly, the use of the stochastic path perturbation approach (SPPA) [7] which will turn out to be, in the present case, a non-adiabatic theory to get the mean first passage time (MFPT) from the unstable point. This second approach leads us to an analytic (non perturbative in the mean delay-time) formula for the first passage time statistics toward the attractor in non-linear systems. Both approaches will be rigorously shown to be perturbation approaches in the small noise approximation. We have compared the non-adiabatic result vs. the adiabatic (Markov) approximation showing very good agreement for small mean delay-time, and also against previous non-delay simulations [3]. The full theoretical prediction, for any value of the mean delay-time, has been compared against delay Monte Carlo simulations.

Another important point in presenting the SPPA to solve distributed delay models is that this approach can also be extended to tackle many different normal forms, or when the unstable point is for example non-linear (as may happen in non-biological models), allowing us to characterize the general mechanism of relaxation in delay non-linear models. In the present paper we are concerned with linear instabilities in the presence of an exponential distributed delay, other cases will be the subject of future research.

2 Distributed Time-Delay in Non-Linear Population Models

Many species exhibit complex behavior far more wide-ranging than can be described by the simple birth-death and logistic equation [8,9]. In order to advance the understanding of this issue mathematicians have proposed time-delay population models. Concerning this delay, the most general Verhults's model, is the one that allows both a reproductive time-lag τ_B and a logistic reaction time-lag τ_D [10] . In general, if the duration of the delay is longer than the "natural period" of the system (usually as $1/r$ where r denotes population growth rate in the absence of regulation) then large amplitude oscillations will be the result. Indeed, as the time-delay increases it can have an increasingly destabilizing effect on the population order parameter $N(t)$, giving rise first to damped oscillations and then possibly to divergent oscillations in the system of interest. Time-delay in the reaction rate can contribute to destabilizing effects in the attractor $N = K$ (where K is the carrying capacity of the model); on the other hand, delayed birth rates are important to estimate perturbations in the pattern formation. The above models involve idealizing the time-delay as a fixed single value τ_D or τ_B [10–12]. Different deterministic situations have been studied considering distributed delay models to affect the birth and/or death rate; all these cases have been proposed in order to enlarge the description of a structured population model [13].

Consider the following distributed time-delay non-linear problem:

$$\frac{dN}{dt} = F(N, N_G), \quad N(t) \geq 0, \forall t \geq 0 \tag{1}$$

$$N_G(t) = \int_0^\infty G(s)N(t-s)ds, \quad G(t) \geq 0, \int_0^\infty G(t)dt = 1. \tag{2}$$

The occurrence of the weighted average N_G may happen in the linear term, representing a reproductive time-lag which may be measured by the gestation time; therefore, in the early stage of the population growth this time-lag may be important in slowing down the rate of population increase. In contrast, if N_G occurs in the reaction term, this time-lag represents modifications in the death rate due to maturation effects. In general, from any $G(s)$ putting $N(t) = N^*(1 + n(t))$ and $N_G(t) = N^*(1 + n_G(t))$, where $F(N^*, N^*) = 0$, and calling $a = -\partial_N F|_{N=N_G=N^*}$; $b = -\partial_{N_G} F\big|_{N=N_G=N^*}$ the linear stability analysis around the attractor, $N \neq 0$, for a perturbation like $n(t) \propto \exp(-ct + i\omega t)$ leads to the conclusion that the rate of chance dn/dt is affected by $n(t)$ and the weighted average $n_G(t)$ in the form: $\dot{n} = -an - bn_G$, where the damped rate and the frequency of oscillations are given by

$$c = a + b \int_0^\infty G(s)e^{cs} \cos(\omega s)\, ds \tag{3}$$

$$\omega = b \int_0^\infty G(s)e^{cs} \sin(\omega s)\, ds. \tag{4}$$

For further details we present the stability analysis in Appendix 1.

In the present paper we will adopt

$$G(s) = \lambda e^{-\lambda s}, \tag{5}$$

as the probability modeling the distributed time-delay in the population model, but other distribution could also be studied (see Appendix 2 for a generalization of the exponential model). The mean time characterizing the time-lag in an exponential distributed time-delay model is $\epsilon = \int_0^\infty tG(t)dt = \lambda^{-1}$. In particular, when $\lambda \to \infty$ we recover the (usual) non-delay case. Using explicitly the fact that the distribution time-delay is exponential we get (see Appendix 1)

$$c + i\omega = \frac{\lambda}{2}\left[\left(1 + \frac{a}{\lambda}\right) \pm \sqrt{\left(1 + \frac{a}{\lambda}\right)^2 - \frac{4}{\lambda}(a + b)}\right], \tag{6}$$

therefore oscillations may occurs ($\omega \neq 0$) if and only if:

$$\left(1 + \frac{a}{\lambda}\right)^2 < \frac{4}{\lambda}(a + b) \tag{7}$$

2.1 Case $a = 0$

In this particular situation we get $\dot{n} = -bn_G$ thus if $b > 0$ the system is stable. In addition, oscillations may happen if $1 < 4b/\lambda$ and we get $\omega = \frac{\lambda}{2}\sqrt{\frac{4b}{\lambda} - 1}$. In the case $1 > 4b/\lambda$ there are no oscillations. Therefore, in the case $a = 0$ an exponentially distributed time-delay causes only damped oscillations about the steady state value, but it cannot completely destabilize the system. However, this restriction does not hold in the general case $a \neq 0$.

A typical example of this case is the logistic model with distributed time-delay in the reaction term:

$$\frac{dN}{dt} = r\left[N - \frac{N}{K}N_G\right], \quad r > 0. \tag{8}$$

In this case $a = -\,\partial_N F|_{N=N_G=N^*} = 0$, $b = -\,\partial_{N_G} F\big|_{N=N_G=N^*} = r > 0$. Therefore for any λ the system is stable, but may show damped oscillations if $r/\lambda > 1/4$. This is in direct contrast to the fixed reaction time-delay model (leading to $\dot{n} = -rn(t - \tau_D)$) for which increasing τ_D affects both stability and the type of damping, i.e., if $r\tau_D > \pi/2$ the steady state K is unstable.

2.2 Case $b = 0$

In this particular situation we get $\dot{n} = -an$ and the stability analysis is trivial ($c + i\omega = \frac{\lambda}{2}\left[\left(1 + \frac{a}{\lambda}\right) \pm \left(1 - \frac{a}{\lambda}\right)\right]$). There are no oscillations and the system is stable if $a > 0$.

A typical example of this case is the Gompertz model with distributed time-delay in the birth rate:

$$\frac{dN}{dt} = -kN_G \ln\frac{N}{K}, \quad k > 0. \tag{9}$$

In this case $a = -\,\partial_N F|_{N=N_G=N^*} = k$, $b = -\,\partial_{N_G} F\big|_{N=N_G=N^*} = 0$. Therefore for any λ there are no change in the steady state structure.

2.3 Case $(a + b) > 0$

In this situation the system $\dot{n} = -an - bn_G$ is always stable ($c > 0$). In addition, if $\left(1 + \frac{a}{\lambda}\right)^2 > \frac{4}{\lambda}(a + b)$ there are no oscillations. In the opposite case when $\left(1 + \frac{a}{\lambda}\right)^2 < \frac{4}{\lambda}(a + b)$ there are damped oscillations with frequency $\omega = \frac{\lambda}{2}\sqrt{\frac{4}{\lambda}(a + b) - \left(1 + \frac{a}{\lambda}\right)^2}$.

A typical example of this case is the logistic model with linear distributed time-delay:

$$\frac{dN}{dt} = r\left[N_G - \frac{NN}{K}\right], \quad r > 0. \tag{10}$$

In this case $a = -\,\partial_N F|_{N=N_G=N^*} = 2r$, $b = -\,\partial_{N_G} F\big|_{N=N_G=N^*} = -r$. Therefore $(a + b) = r > 0$ the system is stable, but does not present oscillations for any value of λ. This result is in direct contrast to the fixed birth time-delay model (leading to $\dot{n} = -2rn(t) + rn(t - \tau_D)$) for which increasing τ_D affects both stability and the type of damping [14].

2.4 Case $(a + b) < 0$

In this case the dynamic system $\dot{n} = -an - bn_G$ has no oscillations and is always unstable.

2.5 Case $(a + b) = 0$

To close this section let me comment that there are other models that could also be included in the class presented in Eq. (1). Consider, for example, an open-ended logistic-based growth function [15]; in this case the proposed dynamics in suitable dimensionless variables can be written in the form

$$\frac{dN}{dt} = \left[N - \frac{NN}{K}\right] \tag{11}$$

$$\frac{dK}{dt} = \gamma\left[N - K\right], \quad \gamma > 0. \tag{12}$$

In this model the logistic equation is coupled to an equation for the carrying capacity, a model proposed in Ref. [15] to try to link the direct dependency of the population to its carrying capacity, where γ is a ratio between the environmental development rate and the population growth rate. First, note that if we write

$$K(t) = \int_0^\infty \gamma e^{-\gamma s} N(t-s)ds = \int_{-\infty}^t \gamma e^{-\gamma(t-s)} N(s)ds,$$

we can prove that $K(t)$ fulfills the differential equation (12). Therefore we can study its stability analysis as we presented previously by denoting $K(t) \to N_G(t)$ and $\gamma \to \lambda$. From (11) it is simple to see that $a = -\partial_N F|_{N=K} = 1$, $b = -\partial_K F|_{N=K} = -1$. Therefore $(a+b) = 0$ the system is stable and does not present oscillations for any value of γ. In fact, the line $N = K \neq 0$ is a critical line with eigenvalues $\{0; -(\gamma+1)\}$ of the Jacobian matrix, and $K = 0$ is singular in the differential equation (11). It is also possible to see that the trajectories of the system (11) and (12) are hyperbolic-like: $N \propto K^{-1/\gamma}$ [16]. Although this model may be of importance in biology, in the remainder of this paper we will not work on this because its linear term does not have a delay.

3 Elimination of Fast Variables for Exponential Distributed Time-Delay Models

In numerous biophysical systems one is confronted with interplay between mechanisms that evolve on vastly different time scales. An example is given by Eqs. (1), (2) and (5) when $\lambda \gg 1$. In this case it is possible to see that N_G is governed by a fast dissipative mechanism. Then the fast variable N_G is driven to a partial equilibrium, conditioned by the slow variable N; subsequently this partial equilibrium moves on the slow time scale.

In order to state these facts with more rigor, let us rewrite Eqs. (1), (2) and (5) in the form

$$\frac{dN}{dt} = F(N, N_G), \quad N(t) \geq 0, \forall t \geq 0 \tag{13}$$

$$\frac{dN_G}{dt} = \frac{1}{\epsilon}\left(N - N_G\right), \quad \frac{1}{\epsilon} = \lambda > 0, \tag{14}$$

where ϵ is the mean delay-time. Because the distribution $G(s)$ is exponential the variable $N_G(t)$ fulfills a closed differential equation in terms of N and N_G. To prove this, we first write Eq. (2) in the alternative way

$$N_G(t) = \int_{-\infty}^t G(t-s)N(s)ds, \tag{15}$$

then taking the time derivative in Eq. (15) we get

$$\frac{dN_G}{dt} = G(0)N(t) + \int_{-\infty}^t \left(\frac{d}{dt}G(t-s)\right) N(s)ds.$$

Now using the fact that $G(t)$ fulfills $\dot{G}(t) = -\lambda G(t)$ we finally get $\dot{N}_G = \lambda(N - N_G)$, see also Appendix 2 for a more general case than the simple exponential distribution. The initial condition for $N_G(0)$ is related to the pre-function $\varphi(t)$ for $t \in (-\infty, 0)$. This is simple to see from the fact that Eq. (2) can also be written in the form (15) then $N_G(0) = \int_{-\infty}^{0} G(-s)N(s)ds = \int_0^{\infty} G(s)\varphi(-s)ds$.

3.1 Adiabatic Approximation from Eqs. (13) and (14)

Substitute for N_G a power series in the small parameter ϵ

$$N_G = N_G^{(0)} + \epsilon N_G^{(1)} + \epsilon^2 N_G^{(2)} + \cdots \tag{16}$$

and require the various orders of ϵ vanish separately. First there is a term of order ϵ^{-1},

$$\left(N - N_G^{(0)}\right) = 0 \tag{17}$$

then $N_G^{(0)} = N$. Thus, substituting this result in Eq. (13) gives

$$\frac{dN}{dt} = F(N, N) + \mathcal{O}(\epsilon), \tag{18}$$

which is the trivial approximation in the adiabatic elimination approach. The next order consists of the terms in Eq. (14) proportional to ϵ^0: $dN_G^{(0)}/dt = -N_G^{(1)}$. Finally, using Eq. (17) we get

$$\frac{dN}{dt} = -N_G^{(1)}.$$

Now using the previous result, Eq. (18), we have

$$N_G^{(1)} = -F(N, N), \tag{19}$$

this solution can be introduced in Eq. (13) to obtain the first correction to Eq. (18) in the form

$$\frac{dN}{dt} = F\left(N, N_G^{(0)} + \epsilon N_G^{(1)}\right) + \mathcal{O}\left(\epsilon^2\right). \tag{20}$$

Note that this method can also be generalized to the multidimensional case when there are y_ν slow variables N and z_κ fast variables N_G [17]; see also Appendix 2 to tackle the generalized exponential case.

3.2 Example 1: Death Distributed Time-Delay in the Logistic Equation

Consider the situation where the distributed time-delay appears in the reaction term, in which case we have

$$\begin{aligned} \frac{dN}{dt} &= F(N, N_G) \\ &= r\left[N - \frac{NN_G}{K}\right]. \end{aligned} \tag{21}$$

Then using Eqs. (17), (19) and (20) for this particular form of $F(N, N_G)$, we get up to $\mathcal{O}\left(\epsilon^2\right)$.

$$\begin{aligned}\frac{dN}{dt} &= r\left[N - \frac{N}{K}\left(N - \epsilon F(N, N)\right)\right] \\ &= r\left[N - \frac{NN}{K} + \epsilon\frac{N}{K}F(N, N)\right] \\ &= r\left[N - (1-\epsilon r)\frac{NN}{K} - \epsilon r\frac{NNN}{K^2}\right] + \mathcal{O}\left(\epsilon^2\right). \end{aligned} \tag{22}$$

We can now rewrite this deterministic equation in the form

$$\frac{dN}{dt} = -U'(N), \tag{23}$$

where the perturbed delay-dependent adiabatic potential is

$$U(N) = -rN^2/2 + r\left(1-\epsilon r\right)N^3/3K + \epsilon r^2 N^4/4K^2 + \mathcal{O}\left(\epsilon^2\right). \tag{24}$$

As expected, one of the important effects in a death distributed time-delay model is that there is a change of $\mathcal{O}\left(\epsilon\right)$ in the curvature around the attractor: $N = K$.

3.3 Example 2: Birth Distributed Time-Delay in the Logistic Equation

In this case we have

$$\begin{aligned}\frac{dN}{dt} &= F(N, N_G) \\ &= r\left[N_G - \frac{NN}{K}\right]. \end{aligned} \tag{25}$$

Then using Eqs. (17), (19) and (20) for the present $F(N, N_G)$, we get up to $\mathcal{O}\left(\epsilon^3\right)$.

$$\begin{aligned}\frac{dN}{dt} &= r\left[\left(N_G^{(0)} + \epsilon N_G^{(1)} + \epsilon^2 N_G^{(2)}\right) - \frac{NN}{K}\right] \\ &= rN\left[\left(1 - \epsilon r + \epsilon^2 r^2\right) - \frac{N}{K}\left(1 - \epsilon r + 3\epsilon^2 r^2\right) + 2\epsilon^2 r^2\frac{N^2}{K^2}\right] + \mathcal{O}\left(\epsilon^3\right). \end{aligned} \tag{26}$$

As expected, in this case the relaxation from the unstable state $N = 0$ is affected to $\mathcal{O}\left(\epsilon\right)$ by the distributed time-delay. We can also see from the perturbed delay-dependent adiabatic potential, that the first change in the curvature is now $\mathcal{O}\left(\epsilon^2\right)$

$$U(N) = -\frac{r\left(1 - \epsilon r + \epsilon^2 r^2\right)}{2}N^2 + \frac{r\left(1 - \epsilon r + 3\epsilon^2 r^2\right)}{3K}N^3 - \frac{2\epsilon^2 r^3}{4K^2}N^4 + \mathcal{O}\left(\epsilon^3\right), \tag{27}$$

a situation which is different from the previous death distributed time-delay example, see Eq. (24).

Remark In conclusion, from these two simple examples we see that if we had a non-perturbative approach we should expect important changes in the relaxation from $N = 0$. Therefore, in the presence of noise the characteristic time-scale for the relaxation from $N = 0$ (escape process) would be affected by the distributed time-delay. In the present paper we would like to show how the analysis of this scaling-time can be tackled in the small noise approximation. We will present two different approaches: firstly, introducing an adiabatic approximation (valid for $\lambda \gg 1$) and secondly, introducing a non-perturbative approach valid for any value of $\lambda > 0$.

4 Stochastic Approach for Exponential Distributed Time-Delay Models

Fluctuations are ubiquitous in biophysical systems, and we are often forced to consider stochastic perturbation in the dynamic system because some parameters may fluctuate in time or because there are degrees of freedom which are not considered in the dynamic model [4–6, 18, 19]. A general way to introduce these fluctuations into a dynamic system of type (1) may therefore be written in the form

$$\frac{dN}{dt} = F(N, N_G) + \sqrt{\theta} g(N, N_G)\xi(t), \quad \forall t \geq 0 \tag{28}$$

$$\frac{dN_G}{dt} = \frac{1}{\epsilon}(N - N_G), \quad \epsilon^{-1} = \lambda > 0 \tag{29}$$

where $\xi(t)$ is a zero mean Gaussian white noise characterized by the correlation $\langle \xi(t)\xi(t')\rangle = \delta(t - t')$, and $\sqrt{\theta}$ is the noise parameter. Here $g(N, N_G)$ represents a possible noise-multiplicative character in the stochastic dynamics. Note that we have used Eq. (14) to take into account the (exponential) distributed time-delay effects present in Eq. (2). In this form now the processes $N(t)$ and $N_G(t)$ conform a two-dimensional Markov process. Therefore the associated Fokker–Planck equation can be written, which gives the exact time evolution of the conditional probability of the system $P(N, N_G, t| N(0), N_G(0))$. The pdf (probability distribution function) of the process $N(t)$ itself will be given by the marginal distribution:

$$P(N, t| N(0)) = \int\int P(N, N_G, t| N(0), N_G(0))\, P(N_G(0)) dN_G dN_G(0). \tag{30}$$

Here, in agreement with Eq. (2), the initial condition associated with the delay population $N_G(0)$ has been characterized by a general distribution $P(N_G(0))$.

As can be noted, the process $N(t)$ itself is non-Markovian, which means that its time evolution is a non-trivial problem to solve [20]. Nevertheless, the $2D$ representation given by Eqs. (28) and (29) allows us to write down a treatable adiabatic perturbation theory when ϵ is a small parameter. The crucial point in doing this is the way we model the intensity of the noise in terms of the adiabatic small parameter ϵ, and this calculation will depend on the stochastic calculus that we use [18].

In general the $2D$ Fokker–Planck equation associated with the processes $\{N, N_G\}$ is given (in the Stratonovich calculus) by

$$\partial_t P(N, N_G, t| N(0), N_G(0)) = \mathcal{L} P(N, N_G, t| N(0), N_G(0)), \tag{31}$$

where the Fokker–Planck operator is

$$\mathcal{L} \equiv \mathcal{L}(N, N_G, \partial_N, \partial_{N_G}) = -\partial_N F(N, N_G) - \frac{1}{\epsilon}\partial_{N_G}(N - N_G) + \frac{\theta}{2}\partial_N^2 g(N, N_G)^2. \tag{32}$$

The Fokker–Planck equation can be written in the form

$$\partial_t P(N, N_G, t| N(0), N_G(0)) = -\nabla \cdot \mathbf{J}, \tag{33}$$

where $\nabla \equiv (\partial_N, \partial_{N_G})$, and the vector current $\mathbf{J} \equiv (J_N, J_{N_G})$ is given by

$$\mathbf{J}(N, N_G, t| N(0), N_G(0)) = \left[F(N, N_G) - \frac{\theta}{2}\partial_N g(N, N_G)^2, \frac{1}{\epsilon}(N - N_G)\right] P(N, N_G, t| N(0), N_G(0)). \tag{34}$$

The Fokker–Planck evolution (33) must be solved with suitable boundary conditions (BC) [19]; for example to assure the positivity of the process $N(t)$ a reflecting BC would be $(1, 0) \cdot \mathbf{J}|_{N=0} = 0$, (when $N = 0$ is a regular point). Then the continuity equation (33) ensures that the total probability remains constant inside the domain of interest defined by the semi-infinite space $\mathcal{D} \equiv \{N \in (0, \infty), N_G \in (-\infty, \infty)\}$.

In principle, it is possible that the stationary pdf $P_{st}(N, N_G)$ (associated to $\mathcal{L}P(N, N_G) = 0$) does not have a potential structure [21], which is why the asymptotic adiabatic perturbation theory in ϵ is a very useful method to tackle the problem posed in Eqs.(28) and (29).

A $2D$ Fokker–Planck operator can be worked out using eigenfunction techniques [19]; asymptotic techniques to go to the marginal $1D$ problem (associated with the process $N(t)$ itself) can also be used in a similar way as we did with the adiabatic approximation in the previous "deterministic" section [17]. But if $g(N, N_G) \neq 1$ the result depends on the particular stochastic calculus we use [18]. Therefore we now introduce a simplification in the stochastic model and consider additive noise alone, which means that $g(N, N_G) = 1$. In a population model [3], this is a plausible ansatz when the unspecified random contributions are more important at low density, see Appendix 3. Thus, we can use our perturbation approach from Section III and add additive noise at the end of the adiabatic deterministic calculations. In this way we can immediately write down the associated $1D$ Fokker–Planck for the marginal process $N(t)$

$$\partial_t P(N, t | N(0)) = \left[\partial_N U'(N) + \frac{\theta}{2} \partial_N^2 \right] P(N, t | N(0)), \tag{35}$$

where $U(N)$ is the perturbed delay-dependent adiabatic potential (see Section III, and the next subsection). Therefore Eq. (35) represents a genuine Markovian approximation, see Eq. (37).

In Appendix 4 we solve a particular $2D$ problem to show a calculation where a non-Markov problem can be solved analytically, then we can compare this exact result with the stochastic adiabatic (Markovian) approach (35).

4.1 Stochastic Adiabatic (Markov) Approach

The adiabatic expansion given in section III allows us to tackle the noise additive case in a simple way. In general we have proved that, starting with a nonlinear model with an exponential distributed time-delay, and introducing an adiabatic perturbation in $\epsilon = \lambda^{-1}$ we can write in the small parameter ϵ the perturbative dynamics

$$\frac{dN}{dt} = F\left(N, N_G^{(0)} + \epsilon N_G^{(1)} + \epsilon^2 N_G^{(2)} + \cdots\right). \tag{36}$$

Therefore, by adding noise (white and Gaussian) to this deterministic dynamics we may write in terms of Wiener differentials

$$dN = -U'(N)dt + \sqrt{\theta} dW(t), \tag{37}$$

here $U(N)$ is the adiabatic potential calculated from Eq. (20)

$$U(N) = -\int^N F\left(N', N_G^{(0)} + \epsilon N_G^{(1)} + \epsilon^2 N_G^{(2)} + \cdots\right) dN'. \tag{38}$$

The $1D$ system proposed in Eq. (37) is much simpler than the $2D$ problem proposed previously in Eq. (31). In particular, the stationary pdf $P_{st}(N)$ can always be found. In fact, because here we are interested in the case $N(t) \geq 0$ the system is reduced to work out the zero current problem; therefore, we have to solve

$$\left[U'(N) + \frac{\theta}{2}\partial_N\right] P_{st}(N)\bigg|_{N=0} = 0.$$

This equation has the solution

$$P_{st}(N) \propto \exp\left(-\frac{2}{\theta}U(N)\right). \tag{39}$$

From this we can calculate the first passage time to leave the unstable state $N = 0$. A time-scale can therefore be defined as representing the relaxation toward the stationary state (attractor of the problem $N = N^*$). This problem will be presented in the next section.

4.2 The MFPT in the Adiabatic (Markovian) Approximation

To study the first passage time statistics for a non-Markov process is a complex task, and to calculate the passage time statistics for a general $2D$ Markov problem is also a non-trivial problem [4]. Here we are going to use our previous adiabatic result Eq. (37) (a Markovian approximation to any $\mathcal{O}(\epsilon^n)$) to calculate the MFPT. This value is the characteristic time-scale taken to reach the domain of attraction in the noisy version of Eq. (1), i.e., the scaling-time to characterize the stochastic relaxation from $N = 0$ to $N = N^*$ from the dynamics of Eqs. (28) and (29) with $g(N, N_G) = 1$.

It has been proved that from linear instability there is a universal scaling-time which characterizes the transition from microscopic disorder to macroscopic order. The pioneer calculations were carried out using a stochastic self-consistent scaling theory [1]. This onset time has been called Suzuki's scaling-time τ_S. The dominant contribution in terms of the noise intensity $\sqrt{\theta}$ is given by: $\tau_S \propto \ln(1/\theta)$. Nevertheless, if the instability were not linear this scaling-time would follow a different law in terms of noise intensity. This fact can also be seen from the self-consistent scaling theory [1,22], and from the SPPA [7,23,24]. Alternatively, by using the first passage time theory for a Markovian process it is also possible to prove that asymptotically this scaling-time is just given by the MFPT calculated from $N = 0$ to N^*. In order to ascertain this fact we want to introduce this explicit calculation as a complementary approach to be compared later, for full details see Appendix 5.

For small noise the scaling-time for the onset of the macroscopic order is universal and independent of the saturation term [1,22], therefore we can use formula (97) with the potential $U(N) = -\frac{1}{2}\mathcal{A}N^2$ (neglecting the saturation term) from Eq. (38). Thus, we get for the dominant contribution in the small parameter θ

$$\tau(N_f) \simeq \frac{1}{2\mathcal{A}}\left(\ln\frac{N_f^2\mathcal{A}}{\theta} - \psi\left(\frac{1}{2}\right) + \cdots\right), \tag{40}$$

where $\psi\left(\frac{1}{2}\right) \simeq -1.9351$ is the Digamma function $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ and $N_f < \mathcal{O}(N^*)$. Corrections to this expression depend on the neglected saturation term. For example in a Logistic model the first correction coming from the non-linear term would be N_f/K.

Remark In the small noise approximation, the scaling-time for the onset in non-linear models with linear instability (with exponential distributed time-delay structure) is characterized in the Markov approximation, by the curvature of the perturbed delay-dependent adiabatic potential (38) around the unstable point, i.e., $\mathcal{A} \equiv \mathcal{A}(\epsilon) = U''(N = 0)$. Therefore Eq. (40) gives (for small noise) the scaling time in the Markov adiabatic approximation to any order ϵ.

5 Non-Adiabatic Calculation of the Relaxation Scaling-Time

The non-adiabatic analysis of stochastic escape times from the unstable state can be carried out by introducing the SPPA into the stochastic dynamics of (exponential) distributed time-delay models (1), i.e.: the dynamics of Eqs. (28) and (29) with $g(N, N_G) = 1$

$$\frac{dN}{dt} = F(N, N_G) + \sqrt{\theta}\xi(t)$$
$$\frac{dN_G}{dt} = \lambda(N - N_G), \quad \forall\lambda \geq 0, \tag{41}$$

where $\xi(t)$ is a zero mean Gaussian white noise, and as before λ is the delay parameter.

In the small noise approximation the SPPA consists of obtaining information about the first passage time statistics without solving the Fokker–Planck equation. This is done by analyzing the stochastic realizations of the process under study when they are written in terms of Wiener paths [3,7,23–25].

Here we are going to use the SPPA to calculate the characteristic time-scale of the escape process from $N = 0$ associated with the set of equations (41). The universality of the escape process is controlled by the type of instability [7,23,24,26,27]. The non-linear term from the saturation introduces corrections to the dominant small noise calculations [3].

In particular, in the next calculations we will focus in linear unstable points with BC $N(t) > 0$ for $t > 0$; it should be noted that invoking symmetry arguments the present result can also be used to analyze, for example, the distributed time-delay bistable stochastic flux: $\dot{X} = X_G - X^3 + \sqrt{\theta}\xi(t)$ which is an archetypal model for pattern formation in non-biological system.

Therefore, starting from (41) at short time we only need to consider the expansion around $N = N_G = 0$

$$F(N, N_G) \simeq \partial_{N_G} F\big|_{N=N_G=0} N_G + \cdots, \tag{42}$$

defining $q = \partial_{N_G} F\big|_{N=N_G=0}$ we can write near the unstable point $(N, N_G) = (0, 0)$ the $2D$ stochastic differential equation (SDE)

$$\frac{dN}{dt} = qN_G + \sqrt{\theta}\xi(t), \quad \forall q > 0 \tag{43}$$
$$\frac{dN_G}{dt} = \lambda(N - N_G), \qquad \forall\lambda \geq 0. \tag{44}$$

Any saturation term would depend on the proposed non-linear model $F(N, N_G)$, but this term is not important to the dominant contribution of the calculation of the escape process when the instability is linear. For small enough θ this contribution can be neglected. Eqs. (43) and (44) can be written in matrix notation

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = -\mathbf{A}\cdot\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \sqrt{\theta}\begin{pmatrix} \xi_1(t) \\ 0 \end{pmatrix}, \tag{45}$$

where the matrix $\mathbf{A} = \begin{pmatrix} 0 & -q \\ -\lambda & \lambda \end{pmatrix}$, and $X_1 = N$, $X_2 = N_G$. A particular solution (stochastic realization) of this SDE can be written as

$$\mathbf{X}(t) = \exp(-\mathbf{A}t)\,\mathbf{X}(0) + \sqrt{\theta}\int_0^t \exp\left[-\mathbf{A}\left(t - t'\right)\right] d\mathbf{W}\left(t'\right). \tag{46}$$

Considering $\mathbf{X}(0) = 0$ and the Wiener differential $d\mathbf{W}(t') = (dW_1(t'), 0)$ in Eq. (46) the realization of $X_1(t)$ can be written in the form

$$X_1(t) = \int_0^t \exp\left[-\mathbf{A}(t-t')\right]_{11} dW_1(t') \tag{47}$$

$$= \frac{\sqrt{\theta}}{2}\left[1 - \frac{1}{\Xi}\right]\int_0^t e^{-\alpha_+(t-t')} dW_1(t') + \frac{\sqrt{\theta}}{2}\left[1 + \frac{1}{\Xi}\right]\int_0^t e^{-\alpha_-(t-t')} dW_1(t'), \quad t \geq 0 \tag{48}$$

where

$$\Xi \equiv \sqrt{1 + 4q/\lambda} > 1. \tag{49}$$

Note that the eigenvalues of this system can be read from Appendix 4, making the replacement $\beta \to -q$. Therefore, we see that there is one eigenvalue which is negative

$$\begin{aligned} \alpha_+ &= \frac{\lambda}{2}\left(1 + \sqrt{1 + 4q/\lambda}\right) > 0 \\ \alpha_- &= \frac{\lambda}{2}\left(1 - \sqrt{1 + 4q/\lambda}\right) < 0. \end{aligned} \tag{50}$$

For strictly $\lambda > 0$ and for $t \to \infty$ the first integral in Eq. (48) is bounded (in fact considering that $\alpha_+ > 0$ this integral represents a genuine Ornstein-Uhlenbeck process $h_+(t)$). For $t \to \infty$ the second integral diverges exponentially because $\alpha_- < 0$, therefore it is convenient to rewrite this second integral in the form

$$\sqrt{\theta}\frac{e^{|\alpha_-|t}}{2}\left[1 + \frac{1}{\Xi}\right]\int_0^t e^{-|\alpha_-|t'} dW_1(t') = \sqrt{\theta}\frac{e^{|\alpha_-|t}}{2}\left[1 + \frac{1}{\Xi}\right] h_-(t), \ \lambda > 0. \tag{51}$$

Here the process $h_-(t)$ fulfills the SDE

$$\frac{dh_-(t)}{dt} = e^{-|\alpha_-|t}\xi(t), \ h_-(0) = 0,$$

where $\xi(t)$ is a zero mean Gaussian white noise. It is therefore possible to see that the process $h_-(t)$ saturates and for $t \to \infty$ it is a well defined random variable Ω. Due to the fact that in the present calculations we are interested in positive solutions for $X_1(t)$, the stationary pdf for $P_{st}(h_-(\infty))$ adopts the form (see [3])

$$P(\Omega) = 2\sqrt{\frac{|\alpha_-|}{\pi}} e^{-|\alpha_-|\Omega^2}, \quad \Omega \in (0, \infty). \tag{52}$$

Now we introduce the approximation that in the intermediate regime $h_-(t)$ saturates to the random variable Ω and $h_+(t) \to \sqrt{\langle h_+(t)^2 \rangle}$ to a constant, thus we can replace $h_-(t) \to \Omega$ and $h_+(t) \to (2\alpha_+)^{-1/2}$. Taking into account all these considerations we can rewrite Eq. (48) in the form

$$X_1(t) = \frac{\sqrt{\theta}}{2}\left[1 - \frac{1}{\Xi}\right] h_+(\infty) + \sqrt{\theta}\frac{e^{|\alpha_-|t}}{2}\left[1 + \frac{1}{\Xi}\right] h_-(\infty), \quad \lambda > 0, t > 0. \tag{53}$$

Therefore, in this intermediate regime, when $X(t_e) = X_f$, Eq. (53) can be inverted as

$$t_e = \frac{1}{|\alpha_-|} \ln\left(\frac{X_f - \sqrt{\theta} C_+ (2\alpha_+)^{-1/2}}{\sqrt{\theta}\Omega C_-}\right), \tag{54}$$

where

$$C_+ = \frac{1}{2}\left[1 - \frac{1}{\Xi}\right] > 0$$
$$C_- = \frac{1}{2}\left[1 + \frac{1}{\Xi}\right] > 0.$$

Formula (54) gives the escape time t_e to reach a prescribed value X_f as a random quantity whose statistics are determined by those of $h_-(\infty)$. From this transformation law it is possible to get all the moments of the passage times, indeed it is also possible to calculate the pdf of the first passage times [3]. Now we just want to calculate the MFPT and to compare this value with the one obtained from the adiabatic (Markov) perturbation theory presented previously. From Eqs. (54) and (52) the MFPT is given by

$$\begin{aligned}
\langle t_e \rangle &= \frac{1}{|\alpha_-|} \left\langle \ln\left(\frac{X_f - \sqrt{\theta} C_+ (2\alpha_+)^{-1/2}}{\sqrt{\theta}\Omega C_-}\right)\right\rangle \\
&= \frac{1}{|\alpha_-|} \left[\ln\left(\frac{X_f - \sqrt{\theta} C_+ (2\alpha_+)^{-1/2}}{\sqrt{\theta} C_-}\right) - \int_0^\infty \ln(\Omega) P(\Omega) d\Omega\right] \\
&= \frac{1}{|\alpha_-|} \left[\ln\left(\frac{X_f - \sqrt{\theta} C_+ (2\alpha_+)^{-1/2}}{\sqrt{\theta} C_-}\right) + \frac{1}{2}\left(\ln|\alpha_-| - \psi\left(\frac{1}{2}\right)\right)\right] \\
&= \frac{1}{2|\alpha_-|} \left[\ln\left(\left(\frac{X_f - \sqrt{\theta} C_+ (2\alpha_+)^{-1/2}}{C_-}\right)^2 \frac{|\alpha_-|}{\theta}\right) - \psi\left(\frac{1}{2}\right)\right].
\end{aligned} \tag{55}$$

Comparing this result with the one from the adiabatic approximation Eq. (40), we see that as a consequence of the distributed time-delay the non-adiabatic rate $\mathcal{A}_{non-A}$ is different from the adiabatic one $\mathcal{A}(\epsilon) \equiv U''(N=0)$

$$\mathcal{A}_{non-A} = |\alpha_-| = \left|\frac{\lambda}{2}\left(1 - \sqrt{1 + 4q/\lambda}\right)\right|. \tag{56}$$

In addition, the non-adiabatic threshold N_{non-A} is now a function of q, θ and λ, see Eq. (49),

$$N_{non-A} = \frac{X_f - \sqrt{\theta} C_+ (2\alpha_+)^{-1/2}}{C_-} = \frac{2\Xi N_f}{1+\Xi} + \sqrt{\frac{\theta}{2\alpha_+}} \frac{1-\Xi}{1+\Xi}, \tag{57}$$

the first correction is $\mathcal{O}\left(\theta^0\right)$ in the noise intensity and this represents a larger effective threshold $2\Xi N_f/(1+\Xi) \geq N_f$, which will lead to a delay in the MFPT. The second correction in Eq. (57) comes from the coupled fluctuations of N and N_G and is $\mathcal{O}\left(\sqrt{\theta}\right)$. In Fig. 1, for different noise intensities θ we present the Log-Log plot of the MFPT as a function of the delay parameter λ, showing a crossover between the short and long regimen at $\lambda \sim 1$.

In the limit of large $\lambda \gg 1$ (adiabatic regime: $\epsilon = \lambda^{-1} \to 0$) the SPPA predicts a rate $\mathcal{A}_{non-A} = |\alpha_-| \to q(1 - q\epsilon + \cdots)$ which is in agreement with the adiabatic

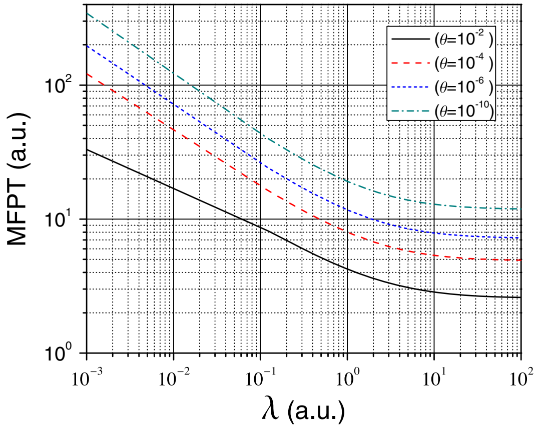

Fig. 1 MFPT from Eq. (55) (in arbitrary units) for $N_f = 1/2$ and $q = 1$ as a function of the delay parameter λ for four values of the noise intensity $\theta (= 10^{-10}, 10^{-6}, 10^{-4}, 10^{-2})$. The corresponding asymptotic values for $\lambda \gg 1$ from top to bottom (11.90; 7.25; 4.93; 2.60) are in excellent agreement with the non-delay Monte Carlo simulations of Ref. [3]. The crossover for $\lambda \sim 1$ and the power law regime for $\lambda \ll 1$ can clearly be seen

Markov approach to $\mathcal{O}(\epsilon)$. Also in this limit the non-adiabatic threshold N_{non-A} gives asymptotically

$$N_{non-A} = \frac{N_f - \sqrt{\theta} C_+ (2\alpha_+)^{-1/2}}{C_-} \to N_f (1 + q\epsilon + \cdots) - \sqrt{\theta}\left(\frac{q}{8}\epsilon^{3/2} - \frac{q^2}{2}\epsilon^{5/2} + \cdots\right), \text{ for } \epsilon = \lambda^{-1} \to 0, \quad (58)$$

note that this correction cannot be obtained from the Markov approximation. The second term in Eq. (58) is a small perturbation which comes from the fluctuations of N_G induced from the dynamics of $N(t)$, see (41). In Fig. 1 it can be seen that the asymptotic values of the MFPT for different noise intensities and large λ are in excellent agreement with the Monte Carlo simulations (for different noise intensities θ) of the stochastic logistic equation of reference [3]. See also Fig. 2 were we present numerical simulations for any λ and noise intensity $\theta = 10^{-4}$.

In the opposite limit $\lambda \ll 1$ (large mean-delay time $\epsilon = \int_0^\infty tG(t)dt = \lambda^{-1}$, i.e., highly non-adiabatic regime), we get from the SPPA, see Eq. (55), the interesting dominant result

$$\langle t_e \rangle \to \frac{1}{2\sqrt{\lambda r}} \left[\ln(1/2) - \psi(1/2) + \ln\left(1 - 2N_f\sqrt{\frac{2\sqrt{\lambda r}}{\theta}}\right)^2 \right], \text{ for } \lambda \ll 1, \quad (59)$$

precluding a power law behavior $\langle t_e \rangle \propto 1/\sqrt{\lambda}$ for $\lambda \ll 1$, this novel behavior can also be seen in Fig. 1.

In Fig. 2 we show a numerical simulation of Eq. (41) with $F(N, N_G) = N_G - N^2$. We have found (within the numerical errors) a power-law behavior $\propto \lambda^{-\nu}$ for small λ in partial agreement with the theoretical prediction $\langle t_e \rangle \propto 1/\sqrt{\lambda}$. Also the existence of the crossover at $\lambda \sim 1$ can be seen. In the same figure we show, as a function of λ, Monte Carlo data against the theoretical prediction (line) for $\theta = 10^{-4}$. A fitting with data points in the interval of $\lambda \in (10^{-2}, 10^{-1})$ would give $\nu \sim 0.6$. In Fig. 2 we also show data points with the error bars. The error bars correspond with the Monte Carlo Standard Deviation σ. As one can notice from this graph Standard Deviation increases for low values of λ. This means that variation about numerical mean values μ of the first passage times increases as $\lambda \to 0$. Thus we can conclude that the theoretical prediction falls within the confidence interval $\mu \pm \sigma$.

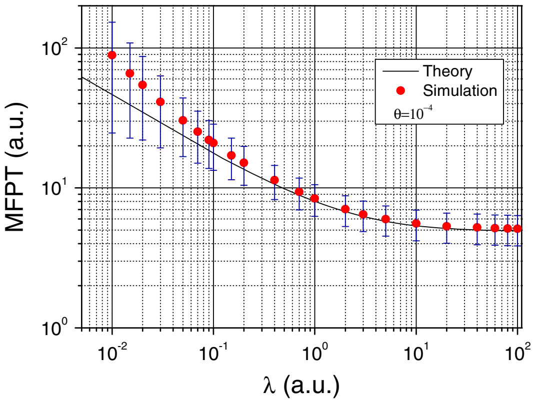

Fig. 2 Monte Carlo simulation of the MFPT (in arbitrary units) for $N_f = 1/2$ for the exponential distributed time-delay Eq. (41) using a logistic model $F(N, N_G) = N_G - N^2$, as a function of the delay parameter λ, and for noise intensity $\theta = 10^{-4}$. The asymptotic value of the MFPT for $\lambda \gg 1$ is $4.93\cdots$ and is in agreement with the non-delay Monte Carlo simulations of Ref. [3]. The crossover at $\lambda \sim 1$ as well as the power-law behavior $\propto \lambda^{-\nu}$ (for $\lambda \ll 1$) can also be seen from the simulations. The SPPA scaling prediction $\langle t_e \rangle \propto 1/\sqrt{\lambda}$ (for $\lambda \ll 1$) is in agreement with the present delay Monte Carlo simulations (within the *error bars*) at the highly non-adiabatic regime

To conclude this section let me note that an analytical expression for the pdf of the first passage time can also be calculated from the transformation law (54):

$$\begin{aligned}\mathcal{P}(t_e) &= P(\Omega(t_e)) \left| \frac{d\Omega}{dt_e} \right| \\ &= P\left(\Omega = \frac{N_{non-A} \exp\left[-|\alpha_-|\, t_e\right]}{\sqrt{\theta}}\right) \frac{N_{non-A}\, |\alpha_-| \exp\left[-|\alpha_-|\, t_e\right]}{\sqrt{\theta}}, \end{aligned} \tag{60}$$

with $P(\Omega)$ given by Eq. (52).

In addition if we wish to have a description for the stochastic dynamics of $N(t)$ we could introduce the instanton-like approximation, by using Eq. (60), in order to have a mean-field description for the stochastic evolution of $N(t)$ [3]

$$N(t) \simeq \Theta(t - t_e),$$

with $\Theta(z)$ the step function and t_e a random variable characterized by $\mathcal{P}(t_e)$.

In the non-delay case ($\epsilon = 0$) the pdf (60) has been tested for the stochastic Logistic equation and we have found excellent agreement against the numerical solutions of the associated SDE [3]. The comparison of our non-adiabatic theory against time-delay Monte Carlo simulations for different non-linear models will be the subject of future contributions. It is interesting to note that more general cases than additive noise, as we have presented in Eq. (41), could also be tackled using the SPPA, as well as the case where the instability is non-linear. In these situations higher order contributions in the noise intensity perturbation have to be considered [26,27]; work along these lines is in progress.

6 Conclusions

In order to characterize the stochastic dynamics toward the final attractor in exponential distributed time-delay non-linear models, the dynamics of the associated $2D$ system is presented. The passage time statistics has been studied analytically in the small noise approximation $\mathcal{O}\left(\sqrt{\theta}\right)$. The first passage time problem (from the linear unstable state) for this non-Markovian type of system has been worked out using two different approaches. Firstly, we calculated the MFPT from a rigorous adiabatic Markovian approximation in the small parameter $\epsilon = \lambda^{-1}$ (the mean delay-time), and secondly, we introduced a non-adiabatic SPPA (valid for any value of $\lambda > 0$) to find an analytic expression for the first passage time distribution of the present non-Markovian problem, see (60). The MFPT, (55), has been written in terms of the important parameters of the (exponential) distributed time-delay model. We have compared both approaches and we have found excellent agreement between them when $\lambda \gg 1$. Using our stochastic path perturbation approach we predicted for small λ (large mean delay-time ϵ) a novel behavior for the relaxation scaling-time which goes as $\langle t_e \rangle \sim 1/\sqrt{\lambda}$ for $\lambda \ll 1$ (for fixed noise intensity), therefore using the present non-adiabatic approach we have proved the existence of an important crossover in the behavior of the MFPT as a function of the delay parameter λ.

Numerical stochastic simulations have been presented to check the validity of our theoretical predictions, see Fig. 2. To end these conclusions let me point out that a generalized biparametric exponential model for the time-delay distribution $G(s)$ can also be studied in a similar way by introducing a second perturbation approach, see Appendix 2, this will be the subject of future contributions.

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Appendix 1: Linear Stability Analysis for Distributed Time-Delay Processes

Here we study the stability around the attractor $N = N_G = N^* > 0$ for distributed time delay models. Expanding $F(N, N_G)$ around N^* and putting $N(t) = N^*(1 + n(t))$; $N_G(t) = N^*(1 + n_G(t))$ in Eq. (1) we get

$$\frac{dN(t)}{dt} \simeq \partial_N F|_{N=N_G=N^*}\left(N - N^*\right) + \partial_{N_G} F\big|_{N=N_G=N^*}\left(N_G - N^*\right), \tag{61}$$

thus defining $a = -\left.\partial_N F\right|_{N=N_G=N^*}$; $b = -\left.\partial_{N_G} F\right|_{N=N_G=N^*}$ we obtain:

$$\dot{n}(t) = -an(t) - bn_G(t), \tag{62}$$

where $n_G(t) = \int_0^\infty G(s)n(t-s)ds$. Introducing the perturbation $n(t) \propto \exp(-ct + i\omega t)$ the stability analysis is characterized by the integral

$$c + i\omega = a + b\int_0^\infty G(s)e^{(c+i\omega)s}ds. \tag{63}$$

Using an exponential distribution: $G(s) = \lambda e^{-\lambda s}$ we get

$$c + i\omega = a - \lambda b\,[c - \lambda + i\omega]^{-1}, \tag{64}$$

here we have used $(c - \lambda) < 0$ in order to have a convergent integral. This quadratic equation can be solved for $(c + i\omega)$ then we get the solution given in Eq. (6).

Appendix 2: Concerning the Exponential Distribution

In order to justify the use of an exponential probability distribution $G(s)$ as a plausible example for our approach, consider a distribution which has a peak around t_D and a width σ around this peak. Choice of the distribution $G(s)$ is dictated both by the nature of the delay and the feasibility of solving the stability problem at the attractor and the calculation of the escape time from the unstable state $N = 0$. Since to our knowledge, there are no experimental results which lead to an automatic choice for $G(s)$, the sensible approach is to select functions which have not only appropriated shapes but which are also sufficiently simple to enable the problem analytically. A plausible biparametric distribution could be

$$G(s) = \mathcal{N}\lambda \exp\left(-\lambda \left|s - t_D\right|\right), \quad s \in (0, \infty), \tag{65}$$

the constant $\mathcal{N}$ is given by normalization condition in the positive domain, i.e., $\int_0^\infty G(s)ds = 1$, then:

$$\mathcal{N} = \left(2 - e^{-\lambda t_D}\right)^{-1}. \tag{66}$$

In the case $t_D = 0$ we get the exponential distribution $G(s) = \lambda e^{-\lambda s}$. From the distribution (65) the mean-delay time and the variance are

$$\langle s \rangle = \mathcal{N}\left(\frac{e^{-\lambda t_D}}{\lambda} + 2t_D\right)$$
$$\sigma^2 \equiv \left\langle s^2 \right\rangle - \langle s \rangle^2 = 2\left(\frac{1}{\lambda^2} + \mathcal{N}t_D^2\right) - \mathcal{N}^2\left(\frac{e^{-\lambda t_D}}{\lambda} + 2t_D\right)^2.$$

So the variance $\sigma^2 \equiv \sigma^2(\lambda, t_D)$ as function of the parameters has interesting behaviors to be explored. For example for fixed λ the variance saturates at large t_D, and for fixed t_D the variance goes to zero for large λ. For $t_D \neq 0$ the limit $\lambda \to \infty$ corresponds to the fixed delay case $G(s) \to \delta(s - t_D)$, so this distribution is a good subject for study.

Note that according to the sign of $(s - t_D)$ the distribution $G(s)$ fulfills the equations

$$\frac{dG^+}{ds} = -\lambda G^+(s), \;\; s > t_D \tag{67}$$
$$\frac{dG^-}{ds} = +\lambda G^-(s), \;\; s < t_D. \tag{68}$$

Using these Green functions $G^\pm(s)$ it is possible to write a systematic evolution equation for the variable $N_G(t)$. To do this we first introduce a change of variable to write Eq. (2) in the alternative way

$$N_G(t) = \int_{-\infty}^{t} G(t - s)N(s)ds. \tag{69}$$

Note the difference between the present definition (with a weighted average) and Mori-Langevin models written in terms of convolution equations [20]. Now, if we take the time derivative in Eq. (69) we get

$$\begin{aligned}
\frac{dN_G}{dt} &= G(0)N(t) + \int_{-\infty}^{t} \left(\frac{d}{dt}G(t-s)\right) N(s)ds \\
&= G(0)N(t) + \int_{-\infty}^{t-t_D} \left(\frac{d}{dt}G(t-s)\right) N(s)ds + \int_{t-t_D}^{t} \left(\frac{d}{dt}G(t-s)\right) N(s)ds \\
&= G(0)N(t) - \lambda \int_{-\infty}^{t-t_D} G^{+}(t-s)N(s)ds + \lambda \int_{t-t_D}^{t} G^{-}(t-s)N(s)ds. \qquad (70)
\end{aligned}$$

In the last line integrals have been simplified by using the Green functions $G^{\pm}(s)$. Now if we take another time derivative in Eq. (70) we get

$$\begin{aligned}
\frac{d^2N_G}{dt^2} &= G(0)\frac{dN(t)}{dt} - 2\lambda G(t_D)N(t-t_D) + \lambda G(0)N(t) \\
&\quad - \lambda \int_{-\infty}^{t-t_D} \left(\frac{d}{dt}G^{+}(t-s)\right) N(s)ds + \lambda \int_{t-t_D}^{t} \left(\frac{d}{dt}G^{-}(t-s)\right) N(s)ds. \\
&= G(0)\frac{dN(t)}{dt} - 2\lambda G(t_D)N(t-t_D) + \lambda G(0)N(t) + \lambda^2 N_G(t). \qquad (71)
\end{aligned}$$

In the last line we have used once again the Green functions and the definition of $N_G(t)$. We see the complex structure induced by the sharp peak of the distribution around t_D.

Equation (71) can be used to find a closed evolution equation for $N_G(t)$ only if we introduce a perturbation in t_D. For example defining a new variable $\dot{N}_G(t)$ and introducing a Taylor expansion $N(t-t_D) = [N(t) - \dot{N}(t)t_D + \ddot{N}(t)t_D^2/2 + \cdots]$ we can find to any order in t_D a set of coupled equations equivalent to the system (1)

$$\begin{aligned}
\frac{dN}{dt} &= F(N, N_G) \\
\frac{dN_G}{dt} &= V_G \qquad (72) \\
\frac{dV_G}{dt} &= G(0)\dot{N}(t) - 2\lambda G(t_D)N(t-t_D) + \lambda G(0)N(t) + \lambda^2 N_G(t).
\end{aligned}$$

We could tackle this problem, but instead of doing this we prefer in this paper to introduce a systematic non-perturbative approach in the delay distribution. For this reason, and in order to get insight into this complex problem as we follow the work, we will be concerned only with the exponential distribution.

In the exponential case, $G(t)$ fulfills $dG(t)/dt = -\lambda G(t)$ then system (1) and (2) are reduced to the set of equations

$$\begin{aligned}
\frac{dN}{dt} &= F(N(t), N_G(t)) \\
\frac{dN_G}{dt} &= \lambda\left(N(t) - N_G(t)\right).
\end{aligned}$$

This result can also be seen from the previous calculation taking $t_D = 0$.

Appendix 3: Concerning the Additive Noise Ansatz

At low density, and in order to justify the use of additive noise, let us start the discussion with the well accepted generalized Volterra equation for n-species

$$\frac{dN_i}{dt} = k_i N_i + \beta_i^{-1} \sum_{j=1}^{n} a_{ij} N_i N_j. \tag{73}$$

The first term describes the behavior of ith species in the absence of others; when $k_i > 0$, the ith species is postulated to grow in an exponential Malthusian manner with k_i as the rate constant. When $k_i < 0$ and all other $N_j = 0$, the population of the ith species would died exponentially. The quadratic terms describe the interaction of the i th species with all other species. The constants a_{ij} might be either positive, negative or zero. A positive (negative) tells us how rapidly encounters will lead to an increase (decrease) in N_i; a zero a_{ij} denotes the fact that ith and jth species do not interact. The positive quantities β_j^{-1} have been named "equivalence" numbers by Volterra, also $a_{ij} = -a_{ji}$ and it is assumed $a_{ii} = 0$. The quantity q_j is defined as the value of N_j in the steady state, i.e., the set of values $\{q_j\}$ is defined by the equation $q_i \left[k_i \beta_i + \sum_{j=1} a_{ij} q_j\right] = 0$, which is also valid under canonical average [28]. When none of the $q's$ vanish, it was proved by Volterra that there exist a constant of motion which depends on the set $\{q_k\}$. Many other important properties concerning canonical average of functions of N_i and $\dot{N}_i$ have also been proved [28], in particular

$$[[N_i]] = q_i$$
$$\left[\left[N_i N_j\right]\right] = q_i q_j,$$

here $[[\cdots]]$ indicates the canonical (or time) average. Defining $v_i = \log N_i/q_i$ it is also possible to prove that the constants a_{ij}/β_i can be related to canonical averages.

Let us now assume that the species of interest is not only influenced by other specific species of the set of n, but also by external random facts. Then the basic equations for population growth might be of the form (73) with the additional term $U_i(t)$ representing random unspecified influences.

$$\frac{dN_i}{dt} = k_i N_i + N_i \left[U_i(t) + \sum_{j=1}^{n} (a_{ij}/\beta_i) N_j \right]. \tag{74}$$

In [28] it was assumed that the combination of $U_i(t)$ (external influence) and the sum in (74) might then be considered as a random function of time, $F_i(t)$ (in particular a zero mean Gaussian white noise).

Considering only the dynamics of species ith (disregarding the dynamics of the remaining species) and based on the canonical average $\left(a_{ij}/\beta_i\right) = \left[\left[\dot{v}_i v_j\right]\right] / \left[\left[v_j^2\right]\right]$, and if the time-scale of the rest of species is much faster, it is plausible to consider the sum as a random function of time, then the sum in Eq. (74) will end up in a contribution like $\sim N_i F_i(t)$ (i.e., a multiplicative noise which may be important at high density).

Now we want to study a new unspecified contribution $U_i(t) \to U_i(t, \{N_k\})$ but considering that this unspecified influence becomes more important at low densities (which may happen for a particular species [29]), the simplest assumption to propose is $U_i(t, \{N_k\}) \sim \xi_i(t)/N_i(t)$ with $\xi_i(t)$ being a Gaussian white noise. Thus, this new unspecified contribution will end up in the form of an additive noise to the dynamics of the ith species. Then at low density the important stochastic contribution will come from the additive noise. We note that the present additive noise is not tempted to describe "extraneous predation".

Appendix 4: The Distributed Time-Delay Ornstein–Uhlenbeck Process

The unidimensional Ornstein-Uhlenbeck process with an exponential distributed time-delay:

$$dX_1 = -\beta X_2 dt + \sqrt{\theta} dW(t) \tag{75}$$

$$X_2(t) = \int_0^\infty \lambda e^{-\lambda s} X_1(t-s) ds, \tag{76}$$

can be written in the following form (I use Einstein's notation for repeated indices's)

$$dX_i = -A_{ij} X_j dt + B_{ij} dW_j(t), \ \{i,j\} = 1, 2, \tag{77}$$

where $dW_j(t)$ is a Wiener differential, and A_{ij}, B_{ij} are constant matrices

$$\mathbf{A} = \begin{pmatrix} 0 & \beta \\ -\lambda & \lambda \end{pmatrix} \tag{78}$$

$$\mathbf{B} = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}. \tag{79}$$

Compare with Eqs. (43) and (44) associating $N \to X_1$ and $N_G \to X_2$.

From Eq. (77) the corresponding $2D$ Fokker–Planck is

$$\partial_t P\left(\mathbf{X}(t) | \mathbf{X}(t_0)\right) = \left[\partial_{X_i} A_{ij} X_j + \frac{1}{2} \partial_{X_i} \partial_{X_j} \left[\mathbf{B} \cdot \mathbf{B}^\top \right]_{ij} \right] P\left(\mathbf{X}(t) | \mathbf{X}(t_0)\right). \tag{80}$$

Due to the linearity of the process, this equation can be solved using different methods [6, 18, 19]. However, because the process is Gaussian, the conditional probability $P\left(\mathbf{X}(t) | \mathbf{X}(t_0)\right)$ is completely characterized by the first $\langle \mathbf{X}(t) \rangle$ and second cumulant $\langle\langle \mathbf{X}(t) \mathbf{X}^\top(s) \rangle\rangle \equiv \boldsymbol{\sigma}_{t-s}$.

It can also be noted that the eigenvalues of A_{ij} have a positive real part (if $\beta > 0$)

$$\begin{aligned} \alpha_+ &= \frac{\lambda}{2} \left(1 + \sqrt{1 - 4\beta/\lambda} \right) \\ \alpha_- &= \frac{\lambda}{2} \left(1 - \sqrt{1 - 4\beta/\lambda} \right), \end{aligned} \tag{81}$$

which means that the process does have a stationary state. Note that if $4\beta/\lambda > 1$ there will be oscillatory behaviors during the transient. Following Gardiner [18] we will focus on the fact that the matrix $\mathbf{A}$ is of dimension 2×2, therefore the stationary covariant matrix $\boldsymbol{\sigma}$ is given by the formula

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{(\det \mathbf{A})\, \mathbf{B} \cdot \mathbf{B}^\top + [\mathbf{A} - (\mathrm{Tr}\mathbf{A})\, \mathbf{1}] \cdot \mathbf{B} \cdot \mathbf{B}^\top \cdot [\mathbf{A} - (\mathrm{Tr}\mathbf{A})\, \mathbf{1}]^\top}{2\, (\mathrm{Tr}\mathbf{A})\, (\det \mathbf{A})} \\ &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \frac{\theta}{2\beta} \begin{pmatrix} \frac{\beta}{\lambda} + 1 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{82}$$

The moments are

$$\langle \mathbf{X}(t) \rangle \equiv \mathbf{M}(t) = \exp(-\mathbf{A}t) \cdot \mathbf{X}(0), \tag{83}$$

and stationary correlations are

$$\left\langle\left\langle \mathbf{X}(t)\mathbf{X}^{\top}(s)\right\rangle\right\rangle \equiv \boldsymbol{\sigma}_{t-s} = \exp\left[-\mathbf{A}\,(t-s)\right]\cdot\boldsymbol{\sigma}. \tag{84}$$

We see that the cumulant is the Green function $\mathbf{G}\,(t) = \exp\left(-\mathbf{A}t\right)\cdot\boldsymbol{\sigma}$. Using right and left eigenvectors of $\mathbf{A}$ the matrix $\exp\left(-\mathbf{A}t\right)$ can be calculated as:

$$\exp\left(-\mathbf{A}t\right) = \begin{pmatrix} \frac{e^{-\alpha_+ t}}{2}\left[1-\frac{1}{\Xi}\right]+\frac{e^{-\alpha_- t}}{2}\left[1+\frac{1}{\Xi}\right] & \frac{\beta}{\lambda\Xi}\left(e^{-\alpha_+ t}-e^{-\alpha_- t}\right) \\ \frac{-1}{\Xi}\left(e^{-\alpha_+ t}-e^{-\alpha_- t}\right) & \frac{e^{-\alpha_+ t}}{2}\left[1+\frac{1}{\Xi}\right]+\frac{e^{-\alpha_- t}}{2}\left[1-\frac{1}{\Xi}\right] \end{pmatrix}, \tag{85}$$

with

$$\Xi \equiv \sqrt{1-4\beta/\lambda}.$$

The $2D$ conditional probability distribution is given by

$$P\left(\mathbf{X}(t)|\,\mathbf{X}(s)\right) = \left(2\pi\,\det\mathbf{A}\right)^{-1/2}\exp\left[\frac{-1}{2}\left(\mathbf{X}-\mathbf{M}\,(t-s)\right)^{\top}\cdot\boldsymbol{\sigma}_{t-s}^{-1}\cdot\left(\mathbf{X}-\mathbf{M}\,(t-s)\right)\right]. \tag{86}$$

Other interesting properties of the system can also be calculated analytically [18]. For example, the spectrum matrix:

$$\begin{aligned}\mathbf{S}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\,\mathbf{G}\,(\tau)\,d\tau \\ &= \frac{1}{2\pi}\left(\mathbf{A}+i\omega\right)^{-1}\cdot\mathbf{B}\cdot\mathbf{B}^{\top}\cdot\left(\mathbf{A}^{\top}-i\omega\right)^{-1}.\end{aligned}$$

Therefore the spectrum of the process $X_1(t)$ is given by

$$\mathbf{S}(\omega)_{11} = \frac{1}{2\pi}\frac{\theta^2\left(\lambda^2+\omega^2\right)\left(\beta^2\lambda^2-2\beta\lambda\omega^2+\lambda^2\omega^2+\omega^4\right)}{\left[\lambda^2\omega^2+\left(\beta\lambda-\omega^2\right)^2\right]^2}.$$

This formula allows us to characterize the spectral properties of the exponential distributed time-delay Orstein-Uhlenbeck process. Note that even when the noise is white the spectrum of $X_1(t)$ has a structure for any value of λ. In the limit $\lambda\to\infty$ (the non-delay case) we reobtain the spectrum of the Orstein-Uhlenbeck process $\mathbf{S}(\omega)_{11}\to\frac{\theta^2}{\beta^2+\omega^2}$. To conclude this section, note that due to the Markovian nature of this $2D$ problem the regression theorem states that the time development $\mathbf{G}\,(t)$ is for $t>0$ governed by the same law of the time development of the mean.

The Non-Markov Marginal Distribution

To calculate the marginal distribution, projected on the variable X_1, it is very simpler to work out with the Fourier transform of the $2D$ conditional probability distribution $\tilde{P}\left(\mathbf{k},t|\,\mathbf{X}(t_0)\right)$, which from Eq. (86) reads

$$\tilde{P}\left(\mathbf{k},t|\,\mathbf{X}(t_0)\right) = \exp\left[-\frac{1}{2}k_jk_i\left(\boldsymbol{\sigma}_{t-t_0}\right)_{ij}+ik_iM_i\left(t-t_0\right)\right], \tag{87}$$

taking $k_2=0$ in this expression we get the Fourier transform of the marginal process X_1

$$\tilde{P}\left(k_1,t|\,\mathbf{X}(0)\right) = \exp\left[-\frac{1}{2}k_1k_1\left(\boldsymbol{\sigma}_t\right)_{11}+ik_1M_1\left(t\right)\right], \tag{88}$$

which can immediately be Fourier anti-transform giving

$$P\left(X_1,t|\,\mathbf{X}(0)\right)=\frac{1}{\sqrt{2\pi\left(\boldsymbol{\sigma}_t\right)_{11}}}\exp\left(\frac{-\left(X_1-M_1\left(t\right)\right)^2}{2\left(\boldsymbol{\sigma}_t\right)_{11}}\right),\tag{89}$$

with

$$\begin{aligned}\left(\boldsymbol{\sigma}_t\right)_{11}&=\left(e^{-\alpha_+t}\left[1-\frac{1}{\Xi}\right]+e^{-\alpha_-t}\left[1+\frac{1}{\Xi}\right]\right)\left(\frac{\theta}{4\lambda}+\frac{\theta}{4\beta}\right)\\&\quad+\frac{\theta}{2\lambda\Xi}\left(e^{-\alpha_+t}-e^{-\alpha_-t}\right)\end{aligned}\tag{90}$$

$$\begin{aligned}M_1\left(t\right)&=\left(e^{-\alpha_+t}\left[1-\frac{1}{\Xi}\right]+e^{-\alpha_-t}\left[1+\frac{1}{\Xi}\right]\right)\frac{X_1(0)}{2}\\&\quad+\frac{\beta}{\lambda\Xi}\left(e^{-\alpha_+t}-e^{-\alpha_-t}\right)X_2(0).\end{aligned}\tag{91}$$

This is an exact non-Markov result. We can see, as expected, that the marginal conditional pdf for the process $X_1(t)$ depends on both initial conditions $X_1(0)$ and $X_2(0)$. At long time, $t\to\infty$, the process $X_1(t)$ reaches a stationary pdf $P_{st}\left(X_1\right)$ which depends on the delay parameter λ. Therefore, we can compare this non-perturbative result with that obtained from the adiabatic perturbation theory.

At long time from Eqs.(90) and (91) we get that $M(t)\to 0$ and $\left(\boldsymbol{\sigma}_t\right)_{11}\to\boldsymbol{\sigma}_{11}=\frac{\theta}{2}\left(\frac{1}{\lambda}+\frac{1}{\beta}\right)$, therefore $P_{st}\left(X_1\right)$ can be written in the form

$$P_{st}\left(X_1\right)=\frac{1}{\sqrt{2\pi\boldsymbol{\sigma}_{11}}}\exp\left(-\frac{2}{\theta}U(X_1)\right),\tag{92}$$

with

$$U(X_1)=\frac{X_1^2}{2\left(\frac{1}{\lambda}+\frac{1}{\beta}\right)}.\tag{93}$$

Physically this non-adiabatic result tells us that for the exponential distributed time-delay Orstein-Uhlenbeck process, the relaxation time is changed to $\beta\to\left(\frac{1}{\lambda}+\frac{1}{\beta}\right)^{-1}$. This result is in agreement with the adiabatic approximation given in Eq. (27) using the replacement $r\to-\beta$ and $K\to\infty$. As can be seen from Eq. (27), making $N\to X_1$, and adding noise of intensity $\sqrt{\theta}$ we can write: $dN=-U'(N)dt+\sqrt{\theta}dW(t)$, where the perturbed delay-dependent adiabatic potential $U(N)\simeq\frac{\beta\left(1-\epsilon\beta+\epsilon^2\beta^2+\mathcal{O}(\epsilon^3)\right)}{2}N^2\to\frac{\beta}{2(1+\epsilon\beta)}X_1^2$ is just the expansion of the exact result given in Eq. (93) with $1/\lambda=\epsilon$.

Appendix 5: MFPT in the Adiabatic (Markov) Approximation

From the $1D$ Fokker–Planck Eq. (35) the general equation to solve the MFPT to reach threshold N^* for the first time is given by the Dynkin equation [6,30]

$$-U'(N_0)\frac{dT(N^*/N_0)}{dN_0}+\frac{\theta}{2}\frac{d^2T(N^*/N_0)}{dN_0^2}=-1,\quad\text{with BC}\quad T(N^*/N^*)=0.\tag{94}$$

The second BC comes from some regular or singular BC at the other extreme of the domain [31]. The solution of this equation can be written in terms of

$$\psi(x) = \exp \frac{2}{\theta} \int_{x_0}^{x} dU(x'),\ x_0 < N^*, \tag{95}$$

compare this function with the stationary pdf in the adiabatic approximation (39). Therefore, the MFPT from $N = 0$ to N^* associated with the Fokker–Planck Eq. (35) with reflecting BC at $N = 0$ will be denoted by $\tau(N^*) \equiv T(N^*/0)$ and is given by the formula [31]

$$\tau(N^*) = \frac{2}{\theta} \int_0^{N^*} \frac{dy}{\psi(y)} \int_0^{y} \psi(z) dz. \tag{96}$$

To get an analytical expression for this MFPT we now introduce an asymptotic calculation in the small noise parameter $\sqrt{\theta}$. Consider now that the adiabatic potential (38) takes the form $U(N) = -\frac{1}{2}\mathcal{A}N^2 + \frac{1}{n}\mathcal{B}N^n + \cdots$; the exponent n depends on the type of nonlinearity occurring in the saturation term of Eq. (1), and the coefficients $\mathcal{A}$, $\mathcal{B}$ are given by its adiabatic expansion (see the examples given in Eqs.(24) and (27) for two different delay cases). A nonlinear saturation model extrapolated from the Logistic to the Gompertz case would give a different exponent n in the saturation of the adiabatic potential and would depend on the model of delay we used. For example, $\frac{rN}{\nu}\left[1 - (N/K)^{\nu}\right]$ reproduces the non-delay Gompertz's or Logistic models in the limit $\nu \to 0$ or $\nu \to 1$ respectively.

The MFPT from $0 \to N_f$ for a $1D$ continuous Markov process with reflecting BC at $N = 0$ is, using (96), given by

$$\tau(N_f) = \frac{2}{\theta} \int_0^{Nf} dy \int_0^{y} \exp\left(\frac{2}{\theta}\left[U(y) - U(z)\right]\right) dz, \tag{97}$$

here we have used the Fokker–Planck Eq. (35) which is consistent to the SDE (37). Using the perturbed delay-dependent adiabatic potential (38) and neglecting the saturation terms we write $U(N) \simeq -\frac{1}{2}\mathcal{A}N^2$, then MFPT from Eq. (97) is (denoting $\alpha \equiv \theta/\mathcal{A}$) given by

$$\begin{aligned}
\tau(N_f) &= \frac{2}{\theta} \int_0^{N_f} dy \int_0^{y} \exp\left(\frac{-\mathcal{A}}{\theta}\left[y^2 - z^2\right]\right) dz \\
&= \frac{2\alpha}{\theta} \int_0^{N_f} dy \int_0^{y/\sqrt{\alpha}} \frac{z e^{-z^2}}{\sqrt{y^2 - \alpha z^2}} dz \\
&= \frac{\sqrt{\pi\alpha}}{\theta} \int_0^{N_f} e^{-y^2/\alpha}\, \mathrm{Erfi}\left[y/\sqrt{\alpha}\right] dy \\
&= \frac{N_f^2}{\theta} F_{pq}\left[\{1, 1\}, \left\{\frac{3}{2}, 2\right\}, -\frac{N_f^2 \mathcal{A}}{\theta}\right],
\end{aligned}$$

where $\mathrm{Erfi}[z] = \mathrm{erf}[iz]/i$, and $F\left[\{a_1, \ldots, a_p\}, \{a_1, \ldots, a_q\}, z\right] = F_{pq}\left[a, b, z\right]$ is a generalized hypergeometric function [32]. Introducing the dimensionless parameter $\eta = \frac{N_f^2 \mathcal{A}}{\theta}$ we can see that asymptotically for $\eta \gg 1$

$$\frac{2\eta\ F_{pq}\left[\{1,1\},\left\{\frac{3}{2},2\right\},-\eta\right]}{\left(\ln\eta-\psi\left(\frac{1}{2}\right)\right)}\to 1.$$

This means that asymptotically for small noise Suzuki's scaling-time is equivalent to the MFPT

$$\tau(N_f)\to\frac{1}{2\mathcal{A}}\left(\ln\frac{N_f^2\mathcal{A}}{\theta}-\psi\left(\frac{1}{2}\right)\right). \tag{98}$$

It is meaningful to comment here that Suzuki's scaling-time $\tau_S=\frac{1}{2\mathcal{A}}\left(\ln\frac{N_f^2\mathcal{A}}{\theta}-\psi\left(\frac{1}{2}\right)\right)$ can also be obtained from the SPPA working out the SDE $dN=\mathcal{A}Ndt+\sqrt{\theta}dW(t)$ to solve the random time t_e at the threshold value $N(t_e)=N_f$ [3]. In addition, this problem can be solved when the noise is Gaussian and non-white [25], or when the coefficient $\mathcal{A}=\mathcal{A}(t)$ is a function of time (i.e.: a time dependent potential) [33,34].

References

1. Suzuki, M.: In: Prigogine, I., Rice, S. (eds) Advances in Chemical Physics, vol. 46, p. 195. Wiley, New York (1981)
2. van Kampen, N.: In: Prigogine, I., Rice, S. (eds) Advances in Chemical Physics, vol. 34, p. 254. Wiley, New York (1976)
3. Cáceres, M.O.: Passage time statistics in a stochastic Verhulst model. J. Stat. Phys. **132**, 487–500 (2008)
4. van Kampen, N.G.: Stochastic Processes in Physics and Chemistry, 2a edn. North Holland, Amsterdam (1992)
5. Cáceres, M.O.: Elementos de estadistica de no equilibrio y sus aplicaciones al transporte en medios desordenados. Reverté S.A, Barcelona (2003). (in Spanish)
6. Stratonovich, R.L.: Topics in the Theory of Random Noise, vol. 1 and 2. Gordon and Breach, New York (1963)
7. San Miguel, M., Hernandez-Garcia, H., Colet, P., Cáceres, M.O., De Pasquale, F.: In: Tirapegui, E., Zeller, W. (eds.) Instabilities and Nonnequilibrium Structures III, pp. 143–155. Kluwer, Boston (1991)
8. Volterra V.: Variazioni e fluttuazioni del numero d'individui in specie animali conciventi (Variations and Fluctuations of the Number of Individuals in Animal Species Living Rogether), Memoria della R. Accademia Nazionale dei Lincei. Series VI, vol. 2, pp. 31–113 (1926). [Translated in Chapman, R.: Animal Ecology, pp. 409–448. McGraw Hill, New York (1931)]
9. Murray, J.D.: Mathematical Biology, vol. 1, 3rd edn. Springer, Berlin (2007)
10. Renshaw, E.: Modelling Biological Populations in Space and Time. Cambridge University Press, Cambridge (1991)
11. Mackey, M.C., Nechaeva, I.G.: Solution moment stability in stochastic differential delay equations. Phys. Rev. E **52**, 3366 (1995)
12. Budini, A.A., Cáceres, M.O.: Functional characterization of linear delay Langevin equations. Phys. Rev. E **70**, 046104 (2004)
13. MacDonald, G.: In: Levine, S. (ed.) Time lags in Biological Models. Lectures Notes in Biomathematics, vol. 27. Springer, Berlin (1978)
14. Beddington, J.R., May, R.M.: Time delays are not necessarily destabilizing. Math. Biosci. **27**, 109–117 (1975)
15. Thornley, J.H.M., France, J.: An open-ended logistic-based growth function. Ecol. Model. **184**, 257–261 (2005)
16. Safuan, H., Towers, I.N., Jovanoski, Z., Sidhu, H.S.: Coupled logistic-carrying capacity model, ANZIAM J. (EMAC2011), **53**, C172–C184 (2012). (http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/4972)
17. van Kampen, N.G.: Elimination of fast variables. Phys. Rep. **124**, 69–160 (1985)
18. Gardiner, C.W.: Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences. Springer, Berlin (1983)

19. Risken, H.: The Fokker–Planck Equation, 2nd edn. Springer, Berlin (1988)
20. Hänggi, P.: Correlation functions and masterequations of generalized (non-Markovian) Langevin equations. Z. Physik B **31**, 407 (1978)
21. Grahan, R., Tél, T.: Weak-noise limit of Fokker–Planck models and nondifferentiable potentials for dissipative dynamical systems. Phys. Rev. A **31**, 1109–1122 (1985)
22. Suzuki, M., Liu, Y., Tsuno, T.: Revisit to the scaling theory of transient phenomena—generalization to correlated noise and singular perturbation expansion up to infinite order. Physica A **138**, 433–455 (1986)
23. Cáceres, M.O., Budde, C.E., Sibona, G.: Stochastic escape processes from a non-symmetric potential normal form. J. Phys. A **28**, 2877 (1995)
24. Cáceres, M.O., Fuentes, M.A.: Stochastic escape processes from a non-symmetric potential normal form III: extended explosive systems. J. Phys. A **32**, 3209 (1999)
25. Sancho, J.M., San Miguel, M.: Passage times for the decay of an unstable state triggered by colored noise. Phys. Rev. A **39**, 2722–2724 (1989)
26. Colet, P., de Pasquale, P., Cáceres, M.O., San Miguel, M.: Theory for relaxation at a subcritical pitchfork bifurcation. Phys. Rev. A **41**, 1901–1911 (1990)
27. Cáceres, M.O., Fuentes, M.A., Budde, C.E.: Stochastic escape processes from a non-symmetric potential normal form II: the marginal case. J. Phys. A **30**, 2287 (1997)
28. Goel, N.S., Maitra, S.C., Montroll, E.W.: On the Volterra and other nonlinear models of interacting populations. Rev. Mod. Phys. **43**, 231–276 (1971)
29. Levins, R.: The strategy of model building in population biology. Am. Sci. **54**, 4 (1966)
30. Dynkin, E.B., Juschkewitz, A.A.: Sätze und Aufgaben über Markoffsche Prozesse. Springer, Berlin (1969)
31. Goel, N.S., Richter-Dyn, N.: Stochastic Models in Biology. Academic Press, New York (1974)
32. Spanier, J., Oldham, K.B.: An Atlas of Functions. Springer, Berlin (1987)
33. Cáceres, M.O., Becker, A.: Passage times for the decay of a time-dependent unstable state. Phys. Rev. A **42**, 696 (1990)
34. Torrent, M.C., San Miguel, M.: Stochastic-dynamics characterization of delayed laser threshold instability with swept control parameter. Phys. Rev. A **38**, 245 (1988)