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The spherical transform associated with the generalized Gelfand pair $(U(p,q), H_n)$, $p+q=n$

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Abstract. We denote by H_n the $2n+1$-dimensional Heisenberg group. In this work, we study the spherical transform associated with the generalized Gelfand pair $(U(p,q) \rtimes H_n, U(p,q))$, $p+q=n$, which is defined on the space of Schwartz functions on H_n, and we characterize its image.

In order to do that, since the spectrum associated to this pair can be identified with a subset Σ of the plane, we introduce a space $\mathcal{H}_n$ of functions defined on $\mathbb{R}^2$ and we prove that a function defined on Σ lies in the image if and only if it can be extended to a function in $\mathcal{H}_n$.

In particular, the spherical transform of a Schwartz function f on H_n admits a Schwartz extension on the plane if and only if its restriction to the vertical axis lies in $\mathcal{S}(\mathbb{R})$.
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1. Introduction

Let N be a connected and simply connected nilpotent Lie group and K a compact subgroup of automorphisms of N.

Then $(K \rtimes N, N)$, also denoted by (K, N), is called a Gelfand pair when any of the following equivalent conditions hold:

(i) $L^1_K(N) = \{f \in L^1(G) : f(kx) = f(x), \forall x \in N, k \in K\}$ is a commutative convolution algebra,

(ii) the algebra $\mathcal{U}_K(N)$ of left invariant and K-invariant differential operators on N is commutative,

(iii) for any irreducible unitary representation of the semidirect product $K \rtimes N$, the space of vectors fixed by K is at most one dimensional.

In this case, N is at most two step nilpotent(see [3]) and we denote by $\Delta(K,N)$ the Gelfand spectrum of $L^1_K(N)$, which can be identified with the set of bounded spherical functions.

For $f \in L^1_K(N)$, the Gelfand transform $\widehat{f}$ is given by

$$\widehat{f}(\varphi) = \int_N f\overline{\varphi}, \qquad \varphi \in \Delta(K,N).$$

Assume now that N is the Heisenberg group H_n and (K, H_n) is a Gelfand pair. It was proved in [5] that the spectrum $\Delta(K, H_n)$ can be identified with a subset of $\mathbb{R}^{d+1}$ for some natural d, in the following way:

Theorem 1.1. *Let $\{L_1, ..., L_d, T\}$ a set of generators of the algebra $\mathcal{U}_K(H_n)$, where T is the derivation in the central direction of H_n. The map $E : \Delta(K, H_n) \to \mathbb{R} \times (\mathbb{R}^+)^d$ defined by*

$$E(\varphi) = (i\hat{T}(\varphi), |\hat{L}_1(\varphi)|, ..., |\hat{L}_d(\varphi)|)$$

is a homeomorphism on its image, where $\hat{L}_j(\varphi)$ and $\hat{T}(\varphi)$ denote the eigenvalues of L_j and T respectively associated with φ.

It was given in [2] the following characterization of the image of the spherical transform associated with the Gelfand pair (K, H_n).

Let $\mathcal{S}(H_n)$ be the space of Schwartz functions on H_n and let $S_K(H_n)$ be the subspace of K-invariant functions. Let

$$\mathcal{S}(\Delta(K, H_n)) = \{F : \Delta(K, H_n) \to \mathbb{C} : \exists\ \varphi \in \mathcal{S}(\mathbb{R}^{d+1}),\ \varphi|_{\Delta(K,H_n)} = F\}.$$

It is equipped with the quotient topology induced by the topology of $\mathcal{S}(\mathbb{R}^{d+1})$.

Theorem 1.2. *The Gelfand transform $^\wedge : \mathcal{S}_K(H_n) \to \mathcal{S}(\Delta(K, H_n))$ is a topological isomorphism between $\mathcal{S}_K(H_n)$ and $\mathcal{S}(\Delta(K, H_n))$.*

In successive works [10], [11] and [12] it was proved that Theorem 1.1 can be generalized by selecting a suitable set of generators of $\mathcal{U}_K(N)$ that yields an embedding of $\Delta(K, N)$ in some $\mathbb{R}^d$, for some natural number d, and Theorem 1.2 was extended to Gelfand pairs (K, N) where N is in the class of nilpotent Lie groups that satisfy the so called Vinberg condition.

If K is no longer assumed to be compact, $L^1_K(N)$ is trivial and a pair $(K \rtimes N, K)$, also denoted by (K, N), is called a generalized Gelfand pair if for any irreducible unitary representation of $K \rtimes N$, the space of distribution vectors fixed by K is at most one dimensional.

It is known that $(U(p, q), H_n)$ is a generalized Gelfand pair (see [8]), and it is natural to introduce the notions of the spectrum and Gelfand transform for it (see [13] and [15]).

In this paper we define the **normalized Gelfand transform** defined on $\mathcal{S}(H_n)$, we characterize its image and obtain a similar result to Theorem 1.2.

In order to introduce the notion of spectrum associated with the pair $(U(p, q), H_n)$, we recall that if (K, H_n) is a Gelfand pair then every bounded spherical function is of positive type (see [4]), in contrast with the semisimple case.

If $\mathcal{P}$ denotes the cone of the K-invariant functions of positive type, then $\Delta(K, H_n)$ is precisely the set of extremal points of $\mathcal{P}$. If K is not compact it is natural to define the spectrum $\Delta(K, H_n)$ associated with the pair (K, H_n) by the set of K-invariant, of positive type, extremal distributions on H_n, which in turn, are in correspondence with the spherical irreducible unitary representations of $K \rtimes H_n$. Moreover, every extremal distribution is spherical, that is, is an eigendistribution of $\mathcal{U}_K(H_n)$ (see [9]).

If $K = U(p,q)$, the algebra $\mathcal{U}_{U(p,q)}(H_n)$ is generated by

$$\mathcal{D} = \sum_{j=0}^{p}(X_j^2 + Y_j^2) - \sum_{j=p+1}^{n}(X_j^2 + Y_j^2) \text{ and } T = \frac{\partial}{\partial t},$$

where $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$ is the standard basis of the Heisenberg Lie algebra, $[X_i, Y_j] = \delta_{i,j}\, T$ and all other brackets are zero.

Also the spherical irreducible unitary representations of $U(p,q) \rtimes H_n$ were determined in [21] and are parameterized by $\{\pi_{\lambda,k} : \lambda \neq 0,\ k \in \mathbb{N}\} \cup \{\pi_\sigma : \sigma \in \mathbb{R}\}$ and the trivial representation. Also, in this case, the corresponding spherical distributions are tempered (see [8]).

A family of spherical distributions were explicitly computed in [8], [13] and [15], and they satisfy

$$iT(S_{\lambda,k}) = \lambda\, S_{\lambda,k}, \qquad -\mathcal{D}(S_{\lambda,k}) = |\lambda|(2k+p-q)\, S_{\lambda,k}, \tag{1.1}$$

$$iT(S_\sigma) = 0, \qquad -\mathcal{D}(S_\sigma) = \sigma\, S_\sigma. \tag{1.2}$$

Motivated by Theorem 1.1, the following result for the case $U(p,q)$ has been proved in [15].

Theorem 1.3. *The map $\mathcal{E} : \Delta(U(p,q), H(n)) \setminus \{1\} \to \mathbb{R}^2$ defined by*

$$\mathcal{E}(\varphi) = (i\hat{T}(\varphi), -\hat{\mathcal{D}}(\varphi))$$

is a homeomorphism onto its image, where $\hat{\mathcal{D}}(\varphi)$ and $\hat{T}(\varphi)$ denote the eigenvalues of $\mathcal{D}$ and T associated with φ respectively.

So, from now on we will identify the spectrum associated with the generalized Gelfand pair $(U(p,q), H_n)$ with

$$\Sigma = \{(\lambda, (2k+p-q)|\lambda|) : \lambda \neq 0,\ k \in \mathbb{Z}\} \cup \{(0,\sigma) : \sigma \in \mathbb{R}\}$$

equipped with the relative topology of $\mathbb{R}^2$.

To prove Theorem 1.3, the authors showed that

$$\langle S_\sigma, f\rangle = \lim_{\substack{(\lambda,(2k+p-q)|\lambda|)\to(0,\sigma)\\ \sigma>0}} |\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle, \tag{1.3}$$

$$\langle S_\sigma, f\rangle = \lim_{\substack{(\lambda,(2k+p-q)|\lambda|)\to(0,\sigma)\\ \sigma<0}} (-1)^{n-2}|\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle. \tag{1.4}$$

This result gives rise to the following

Definition 1.4. Let $f \in \mathcal{S}(H_n)$. Then the normalized spherical transform of f is the function $\mathcal{F}(f)$ defined on Σ by

$$\mathcal{F}(f)(\lambda, |\lambda|(2k+p-q)) = \begin{cases} |\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle, & k \geq 0, \\ (-1)^{n-2}|\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle, & k < 0, \end{cases} \tag{1.5}$$

and by

$$\mathcal{F}(f)(0,\sigma) = \langle S_\sigma, f\rangle. \tag{1.6}$$

Let Σ^+ and Σ^- the following subsets of Σ

$$\Sigma^+ = \{(\lambda, (2k+p-q)|\lambda|) : \lambda \neq 0,\ k \geq -p+1\} \cup \{(0,\sigma) : \sigma \geq 0\} \tag{1.7}$$

and

$$\Sigma^- = \{(\lambda, (2k+p-q)|\lambda|) : \lambda \neq 0, k \leq q-1\} \cup \{(0,\sigma) : \sigma \leq 0\}. \tag{1.8}$$

In section 3 of this work we prove the following

Theorem 1.5. *Let F be a function defined on Σ. If there exist $\varphi, \psi \in \mathcal{S}(\mathbb{R}^2)$ such that*

$$F|_{\Sigma^+} = \varphi|_{\Sigma^+} \quad \text{and} \quad F|_{\Sigma^-} = \psi|_{\Sigma^-},$$

then there exists $f \in \mathcal{S}(H_n)$ such that $F = \mathcal{F}(f)$.

Thus, we have a similar result to that shown by Veneruso in [19].

Corollary 1.6. *If $\varphi \in \mathcal{S}(\mathbb{R}^2)$ then there exists $f \in \mathcal{S}(H_n)$ such that $\mathcal{F}(f) = \varphi|_\Sigma$.*

In order to characterize the image of the spherical transform, we introduce the space $\mathcal{H}_n$.

Definition 1.7. Let $\mathcal{H}_n$ be the space of functions defined on $\mathbb{R}^2$ of the form

$$\varphi(\lambda, s) = \varphi_1(\lambda, s) + \prod_{k=-p+1}^{q-1} (s - (2k+p-q)|\lambda|)\ \varphi_2(\lambda, s)\ H(s),$$

where $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^2)$ and H is the Heaviside function.

We remark that

$$\prod_{k=-p+1}^{q-1} (s - (2k+p-q)|\lambda|) = \begin{cases} s \prod_{k=0}^{\frac{n-4}{2}} (s^2 - (n-2-2k)^2\lambda^2), & \text{if } n \text{ is even} \\ \prod_{k=0}^{\frac{n-3}{2}} (s^2 - (n-2-2k)^2\lambda^2), & \text{if } n \text{ is odd.} \end{cases}$$

So, the map $(\lambda, s) \mapsto \prod_{k=-p+1}^{q-1} (s - (2k+p-q)|\lambda|)\ \varphi_2(\lambda, s)$ is in $\mathcal{S}(\mathbb{R}^2)$.

As a consequence from Theorem 1.5 we have the following

Corollary 1.8. *If $\varphi \in \mathcal{H}_n$ then there exists $f \in \mathcal{S}(H_n)$ such that $\mathcal{F}(f) = \varphi|_\Sigma$.*

The main result of this paper is the following

Theorem 1.9. *A function F defined on the spectrum Σ is in the image of the normalized spherical transform if and only if there exists $\varphi \in \mathcal{H}_n$ such that $F = \varphi|_\Sigma$.*

We recall that the spectrum of the Gelfand pair $(U(n), H_n)$ can be identified with the set $\Delta(U(n), H_n) = \{(\lambda, |\lambda|(2k+n)) : \lambda \neq 0, n \in \mathbb{N}_0\} \cup \{(0,s) : s \in \mathbb{R}\}$. For $f \in \mathcal{S}_{U(n)}(H_n)$, we denote by $\widehat{f}$ its spherical transform, and by $s \to \widehat{f}(0,s)$ the restriction of $\widehat{f}$ to the vertical axis.

The proof of Theorem 1.9 follows the ideas developed in [1] where it was proved that $\widehat{f}$ can be extended to a Schwartz function on $\mathbb{R}^2$. In our case, the fundamental difference is that for $f \in \mathcal{S}(H_n)$ the map

$$s \mapsto \mathcal{F}(f)(0,s), \quad \forall s \in \mathbb{R}$$

is of class C^{n-2} at the origin (see Proposition 2.5 in Preliminaries) and therefore it can be extended to a function in $\mathcal{H}_n$, while for $f \in \mathcal{S}_{U(n)}(H_n)$ the map

$$s \mapsto \widehat{f}(0,s), \quad \forall s \geq 0,$$

can be extended to a function in $\mathcal{S}(\mathbb{R}^2)$.

With the same arguments we prove the following result.

Proposition 1.10. *Let $f \in \mathcal{S}(H_n)$. If n is even and the map $s \mapsto \mathcal{F}(f)(0,s)$ lies in $C^{n-2+k}(\mathbb{R})$ then there exists $\varphi \in \mathcal{H}_n$, k times differentiable on $\mathbb{R}^2$ such that $\mathcal{F}(f) = \varphi|_\Sigma$.*

If n is odd and the map $s \mapsto \mathcal{F}(f)(0,s)$ lies in $C^{n-1+k}(\mathbb{R})$ then there exists $\varphi \in \mathcal{H}_n$, k times differentiable on $\mathbb{R}^2$ such that $\mathcal{F}(f) = \varphi|_\Sigma$.

The following result states a necessary and sufficient condition for the spherical transform $\mathcal{F}(f)$ of a function $f \in \mathcal{S}(H_n)$ to admit a Schwartz extension on $\mathbb{R}^2$. This result is similar to the case $q = 0$ that was studied by Astengo, Di Blasio and Ricci in [1].

Theorem 1.11. *Let $f \in \mathcal{S}(H_n)$. Then $\mathcal{F}(f)$ admits a Schwartz extension on $\mathbb{R}^2$ if and only if the map $\sigma \mapsto \mathcal{F}(f)(0,\sigma)$ is a Schwartz function on $\mathbb{R}$.*

Finally, in the last section of this work, we relate the differentiability of the function $s \to \mathcal{F}(f)(0,s)$ with the differentiability of some extension of $\mathcal{F}(f)$ in $\mathcal{H}_n$.

Proposition 1.12. *Let $f \in \mathcal{S}(H_n)$. If the map $s \mapsto \mathcal{F}(f)(0,s)$ lies in $C^{k+n-2}(\mathbb{R})$ then $\mathcal{F}(f)$ admits an extension in $\mathcal{H}_n$ of the form*

$$\varphi(\lambda,s) = \varphi_1(\lambda,s) + s^k \left(\prod_{k=-p+1}^{q-1} (s - (2k+p-q)|\lambda|) \right) \varphi_2(\lambda,s) H(s).$$

So, $\varphi \in C^{k-1}(\mathbb{R}^2)$ if n is odd or $\varphi \in C^k(\mathbb{R}^2)$ if n is even.

Theorem 1.13. *Let $f \in \mathcal{S}(H_n)$. Let us suppose that $\mathcal{F}(f)$ admits an extension φ such that satisfies:*

(i) *φ is a C^∞ and rapidly decreasing function on $\{(\lambda, s) : s > 0\}$,*

(ii) *is a C^∞ and rapidly decreasing function on $\{(\lambda, s) : s < 0\}$ and*

(iii) *is a C^k function on $\mathbb{R}^2$.*

Then, $s \mapsto \mathcal{F}(f)(0, s)$ is a C^{k+n-2} function on $\mathbb{R}$ if n is even and it is a C^{k+n-1} function on $\mathbb{R}$ if n is odd. Even more, every extension φ of $\mathcal{F}(f)$ that satisfies **(i)**, **(ii)** *and* **(iii)** *is as follows:*

$$\varphi(\lambda, s) = \varphi_1(\lambda, s) + s^{k+1} \prod_{\substack{k=-p+1 \\ 2k+p-q \neq 0}}^{q-1} (s - (2k + p - q)|\lambda|)\ \varphi_2(\lambda, s)\ H(s),$$

where $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^2)$, this is, $\varphi \in \mathcal{H}_n$.

2. Preliminaries

Let $n \geq 2$ and let p, q be natural numbers such that $p + q = n$. Let H_n be the $2n + 1$-dimensional Heisenberg group defined by $H_n = \mathbb{C}^n \times \mathbb{R}$ with law group

$$(z, t)(w, s) = (z + w, t + s - \frac{1}{2}\ ImB(z, w))$$

where

$$B(z, w) = \sum_{j=1}^{p} z_j \bar{w}_j - \sum_{p+1}^{n} z_j \bar{w}_j.$$

Let $U(p, q) = \{g \in Gl(n, \mathbb{C}) :\ B(gz, gw) = B(z, w)\ \forall (z, w) \in \mathbb{C}^n\}$. Then, $U(p, q)$ acts by automorphisms on H_n via

$$g.(z, t) = (gz, t) \text{ for } (z, t) \in H_n.$$

In order to introduce the definition of the spectrum associated with the generalized Gelfand pair $(U(p, q), H_n)$, we recall some definitions.

Definition 2.1. A distribution T on $G = U(p, q) \rtimes H_n$ is of positive type if the map

$$\begin{aligned} \Theta : \mathcal{D}(G) \times \mathcal{D}(G) &\to \mathbb{C} \\ (\varphi, \psi) &\mapsto T(\tilde{\psi} * \varphi) \end{aligned}$$

is hermitian, continuous and it satisfies $\Theta(\varphi, \varphi) \geq 0$ for all $\varphi \in \mathcal{D}(G)$, where $\tilde{\psi}(g) = \overline{\psi(g^{-1})}$.

Let $\mathcal{P}$ be the cone the distributions of positive type, $U(p,q)$-biinvariant on $U(p,q) \rtimes H_n$. We say that $T \in \mathcal{P}$ is extremal in $\mathcal{P}$ if and only if $S \in \mathcal{P}$ and $S - T \in \mathcal{P}$ implies $S = \alpha T$ for some $\alpha \in \mathbb{R}$. For $S, S' \in \mathcal{P}$ we write $S \sim S'$ if and only if $S = \alpha S'$ for some $\alpha > 0$. Thus, $\sim$ is an equivalence relation on $\mathcal{P}$. For $S \in \mathcal{P}$ we put $[S]$ for its equivalence class.

By general theory (see [7] and pag. 374 in [9]) one knows that there exists a one to one correspondence between the set of equivalence classes of unitary representations (π, V) of $U(p,q) \rtimes H_n$ that admits a cyclic distribution vector fixed by $U(p,q)$ (spherical representations), and the set of the equivalence classes of the $U(p,q)$-biinvariant distributions of positive type.

More precisely, the correspondence is given by

$$T_\pi(\varphi) = \langle \xi_\pi, \pi(\varphi)\xi_\pi \rangle,$$

where ξ_π denotes the distribution vector and $\varphi \in C^\infty(U(p,q) \rtimes H_n)$. One says that T_π is the reproducing distribution of the representation π.

We recall also that π is irreducible if and only if T_π is extremal in $\mathcal{P}$. As usual, we identify the $U(p,q)$-biinvariant distributions on $U(p,q) \rtimes H_n$ with the $U(p,q)$-invariant distributions on H_n. A extremal distribution of $\mathcal{P}$ is spherical (see [9]), but the reciprocal is not true as we can be see for the case $(U(p,q), H_n)$.

Let E be the set of extremal points in $\mathcal{P}$. Motivated by the results of the compact case, we define

Definition 2.2. $\Delta(U(p,q), H_n) = E/\sim$, equipped with the quotient of the pointwise convergence topology of $\mathcal{S}'(H_n)$.

Let

$$\{\pi_{\lambda,k} : \lambda \neq 0,\ k \in \mathbb{Z}\} \cup \{\pi_\sigma : \sigma \in \mathbb{R}\} \cup \{\pi^1\}$$

the set of spherical irreducible unitary representations of $U(p,q) \rtimes H_n$ that are given in [21], and let

$$\{S_{\lambda,k} : \lambda \neq 0,\ k \in \mathbb{Z}\} \cup \{S_\sigma : \sigma \in \mathbb{R}\} \cup \{1\}$$

be the set of associated reproducing distributions. We recall that these distributions happen to be tempered.

Furthermore, we have

$$iT(S_{\lambda,k}) = \lambda\ S_{\lambda,k}, \qquad -\mathcal{D}(S_{\lambda,k}) = |\lambda|(2k + p - q)\ S_{\lambda,k}, \tag{2.1}$$

$$iT(S_\sigma) = 0, \qquad -\mathcal{D}(S_\sigma) = \sigma\ S_\sigma, \tag{2.2}$$

where $\mathcal{D}$ and T are given in the introduction.

Following [5], in [15] we consider the map $\mathcal{E} : \Delta(U(p,q), H_n) \to \mathbb{R}^2$ given by

$$\mathcal{E}([\psi]) = (-\hat{\mathcal{D}}(\psi), i\hat{T}(\psi)),$$

where $\hat{\mathcal{D}}(\psi)$ and $\hat{T}(\psi)$ denote the eigenvalues of $\mathcal{D}$ and T associated with ψ respectively. Let Σ denote the image of $\mathcal{E}$. Equipped with the relative topology of $\mathbb{R}^2$ it is called the Heisenberg fan of the generalized Gelfand pair $(U(p,q),H_n)$ and it is given by

$$\Sigma=\{(\lambda,(2k+p-q)|\lambda|):\lambda\neq 0,\ k\in\mathbb{Z}\}\cup\{(0,\sigma):\sigma\in\mathbb{R}\}.$$

Then the following has been proved

Theorem 2.3. (see [15]) *The map $\mathcal{E}:\Delta(U(p,q),H(n))\setminus\{1\}\to\Sigma$ is a homeomorphism.*

To prove the previous theorem the authors proved (1.3) and (1.4).
This result leads to introduce the normalized spherical transform as given in 1.4.

We recall some known facts in order to present explicitly the spherical distributions $S_{\lambda,k}$ for $\lambda\neq 0$, $k\in\mathbb{Z}$ and S_σ for $\sigma\in\mathbb{R}$. Let H be the Heaviside function (this is, $H=\chi_{(0,\infty)}$) and let

$$\mathcal{H}=\{\varphi:\mathbb{R}\mapsto\mathbb{C}:\varphi(\tau)=\varphi_1(\tau)+\tau^{n-1}\varphi_2(\tau)\ H(\tau),\quad \varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R})\}. \qquad (2.3)$$

It was proved in [18] that $\mathcal{H}$, equipped with a suitable topology, is a Fréchet space. For $p+q=n$, $p,q\in\mathbb{N}$, in [18] there is also given a linear, continuous and surjective map $N:\mathcal{S}(\mathbb{R}^n)\to\mathcal{H}$ whose adjoint $N':\mathcal{H}'\to\mathcal{S}'(\mathbb{R}^n)^{O(p,q)}$ is a linear homeomorphism onto the space of the $O(p,q)$-invariant, tempered distributions on $\mathbb{R}^n$. Of course, this construction also works out as well to describe the space $\mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$, this is, there exists a linear, continuous and surjective map, still denoted by $N:\mathcal{S}(\mathbb{C}^n)\to\mathcal{H}$ whose adjoint map $N':\mathcal{H}'\to\mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$ is a homeomorphism.

We introduce new coordinates in $\mathbb{C}^n$. Given $u=(u_1,\dots,u_p,u_{p+1},\dots,u_n)\in\mathbb{C}^n$ let $\rho=|u_1|^2+\cdots+|u_n|^2$ and $\tau=|u_1|^2+\cdots+|u_p|^2-(|u_{p+1}|^2+\cdots+|u_n|^2)$. It is clear that

$$\|(u_1,\dots,u_p)\|=(\frac{\rho+\tau}{2})^{1/2},\quad \|(u_{p+1},\dots,u_n)\|=(\frac{\rho-\tau}{2})^{1/2},$$

and let also

$$w_1=(\frac{\rho+\tau}{2})^{-1/2}(u_1,\dots,u_p)\in S^{2p-1}, w_2=(\frac{\rho-\tau}{2})^{-1/2}(u_{p+1},\dots,u_n)\in S^{2q-1}.$$

We denote by $\mathcal{H}^{\#}$ the space of the functions φ defined on $\mathbb{R}^2$ of the form

$$\varphi(\tau,t)=\varphi_1(\tau,t)+\tau^{n-1}\varphi_2(\tau,t)\ H(\tau),\quad \varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^2), \qquad (2.4)$$

where H is the Heaviside function. A straightforward adaptation of the Tengstrand map in [18] shows that the map $N:\mathcal{S}(H_n)\to\mathcal{H}^{\#}$ defined by

$$Nf(\tau,t)=\int\limits_{\rho>|\tau|}\ \int\limits_{S^{2p-1}\times S^{2q-1}} f((\frac{\rho+\tau}{2})^{1/2}\omega_u,(\frac{\rho-\tau}{2})^{1/2}\omega_v,t)d\omega_u d\omega_v(\rho+\tau)^{p-1}(\rho-\tau)^{q-1}d\rho, \qquad (2.5)$$

is linear, continuous and surjective, and its adjoint map $N':(\mathcal{H}^{\#})'\to\mathcal{S}'(H_n)^{U(p,q)}$ is a homeomorphism.

Remark 2.4. We observe that, unlike the case where K is compact, the normalized spherical transform is defined for all Schwartz functions on H_n. Moreover, since it was proved in [15] that $\mathcal{F}(f) = \mathcal{F}(g)$ if and only if $Nf = Ng$, we may assume that it is defined on $\mathcal{H}^{\#}$.

On the other hand, we recall the definition of the Laguerre polynomials

$$L_m^{(0)}(\tau) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{\tau^j}{j!}, \qquad L_{m-1}^{\alpha+1}(\tau) = -\frac{d}{d\tau} L_m^{\alpha}(\tau),$$

for $m, \alpha \in \mathbb{N}_0$, according to [17]. Therefore, $L_m^{\alpha}(0) = \binom{\alpha+m}{m}$.

Then, the distributions $S_{\lambda,k}$ calculated in [13] are given by

$$S_{\lambda,k} = F_{\lambda,k} \otimes e^{-i\lambda t},$$

with $F_{\lambda,k} \in \mathcal{S}'(\mathbb{C}^n)$ defined by

$$\langle F_{\lambda,k}, f(\cdot,t)\rangle = \langle (L_{k-q+n-1}^{(0)} H)^{n-1}, \tau \mapsto 2|\lambda|^{-1} e^{-\tau/2} Nf(2|\lambda|^{-1}\tau, t)\rangle, \text{ for } k \geq 0,\ \lambda \neq 0$$

and by

$$\langle F_{\lambda,k}, f(\cdot,t)\rangle = \langle (L_{-k-p+n-1}^{(0)} H)^{n-1}, \tau \mapsto 2|\lambda|^{-1} e^{-\tau/2} Nf(-2|\lambda|^{-1}\tau, t)\rangle, \text{ for } k < 0,\ \lambda \neq 0.$$

Therefore,

$$\langle S_{\lambda,k}, f\rangle = \langle (L_{k-q+n-1}^{(0)} H)^{n-1}, \tau \mapsto 2|\lambda|^{-1} e^{-\tau/2} Nf(2|\lambda|^{-1}\tau, \hat{\lambda})\rangle \text{ for } k \geq 0, \lambda \neq 0, \quad (2.6)$$

and

$$\langle S_{\lambda,k}, f\rangle = \langle (L_{-k-p+n-1}^{(0)} H)^{n-1}, \tau \mapsto 2|\lambda|^{-1} e^{-\tau/2} Nf(-2|\lambda|^{-1}\tau, \hat{\lambda})\rangle \text{ for } k < 0, \lambda \neq 0, \quad (2.7)$$

where $Nf(\tau, \hat{\lambda})$ denotes the Fourier transform of $Nf(\tau,\cdot)$ in λ.

Moreover, the distributions S_σ calculated in [14] are given by

$$\langle S_\sigma, f\rangle = (-1)^{n-1} \int_{\mathbb{R}} \int_0^\infty J_0((\sigma\tau)^{1/2}) (Nf(\cdot,t))^{(n-1)}(\tau)\, d\tau\, dt \quad (2.8)$$

for $\sigma \geq 0$ and by

$$\langle S_\sigma, f\rangle = (-1)^{n-2} \int_{\mathbb{R}} \int_0^\infty J_0((-\sigma\tau)^{1/2}) (Nf(\cdot,t))^{(n-1)}(-\tau)\, d\tau\, dt \quad (2.9)$$

for $\sigma < 0$, where $J_m(\tau) = (\frac{\tau}{2})^m \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+m)!} (\frac{\tau}{2})^{2k}$ is the Bessel function of order m of the first kind.

Proposition 2.5. *Let $f \in \mathcal{S}(H_n)$. Then the map $\sigma \mapsto \mathcal{F}(f)(0,\sigma)$ lies in the Tensgtrand space $\mathcal{H}$ defined by* (2.3).

Proof. For $f \in \mathcal{S}(H_n)$ we can see in [15] and [7] that

$$\langle S_\sigma, f\rangle = \int_{B(u,u)=-\sigma} \int_{H_n} e^{iReB(u,z)} f(z,t)dz\ dt\ d\mu_\sigma(u),$$

where $d\mu_\sigma$ is the measure such that $Nf(\sigma,t) = \int_{B(z,z)=\tau} f(z,t)d\mu_\sigma(z)$.

Given $u = (u_1, \ldots, u_{p.}, u_{p+1}, \ldots, u_n) \in \mathbb{C}^n$, we have that

$$\begin{aligned}&\{u \in \mathbb{C}^n : B(u,u) = -\sigma\}\\ &= \{((\frac{\rho-\sigma}{2})^{1/2} w_1, (\frac{\rho+\sigma}{2})^{1/2} w_2) : w_1 \in S^{2p-1}, w_2 \in S^{2q-1}, \rho \geq |\sigma|\}.\end{aligned}$$

An easy computation shows that

$$\begin{aligned}&\int_{B(u,u)=-\sigma} \int_{H_n} e^{iRe(u,z)} f(z,t)dz\ dt\ d\mu_\sigma(u)\\ &= \int\limits_{\rho>|\sigma|} \int\limits_{S^{2p-1}\times S^{2q-1}} \tilde{f}((\frac{\rho-\sigma}{2})^{1/2} w_1, (\frac{\rho+\sigma}{2})^{1/2} w_2, 0)(\rho-\sigma)^{p-1}(\rho+\sigma)^{q-1} dw_1 dw_2 d\rho\\ &= N\tilde{f}(\sigma, \hat{0})\end{aligned}$$

where the last equality is a consequence of (2.5) with

$$\tilde{f}(u,0) = \int_{H_n} e^{iReB(u,z)} f(z,t)dz\ dt.$$

We know that $N\tilde{f} \in \mathcal{H}^\#$, then by (2.4) the proof is complete. ■

As a consequence of Theorem 3.1 proved in [14] and by definition of the normalized spherical transform we obtain the following result:

Theorem 2.6. *For $f \in \mathcal{S}(H_n)$ and $k \in \mathbb{Z}$, the derivatives $\partial^j(\mathcal{F}(f)(\lambda,k))/\partial\lambda^j$ exist for all $j \in \mathbb{N}$ and $\lambda \neq 0$. Moreover, for each $j, N \in \mathbb{N}_0$ there exists a positive constant c independent of λ and k such that*

$$\left|\frac{\partial^j(\mathcal{F}(f)(\lambda,k))}{\partial\lambda^j}\right| \leq c\left(|k|^{n-1} + \frac{1}{|\lambda|^{n-1}}\right)\frac{1}{|\lambda|^{N+j}(|k|+1)^N}. \tag{2.10}$$

Definition 2.7. For $m : (\mathbb{R}\setminus\{0\}) \times \mathbb{Z} \to \mathbb{C}$ and $(\lambda,k) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{Z}$ we define

$$m^*(\lambda,k) = \begin{cases} m(\lambda,k), & \text{if } k \geq 0,\\ (-1)^{n-2} m(\lambda,k), & \text{if } k < 0.\end{cases}$$

$$m^{**}(\lambda,k) = \begin{cases} m(\lambda,k), & \text{if } k < 0,\\ (-1)^{n-2} m(\lambda,k), & \text{if } k \geq 0.\end{cases}$$

Let

$$E(m)(\lambda,k) = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} m(\lambda, k-l),$$

$$\tilde{E}(m)(\lambda,k) = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} m(\lambda, k+l),$$

Let $\Delta_1 = \{(\lambda, (2k+1)|\lambda|) : \lambda \in \mathbb{R} \setminus \{0\},\ k \in \mathbb{N}_0\}$. Then, by Theorem 1.1 proved in [14] and Veneruso's result in [19] we have the following

Theorem 2.8. *We assume that $p, q \geq 1$ with $p + q = n$. Then, for a function $m : (\mathbb{R} \setminus \{0\}) \times \mathbb{Z} \to \mathbb{C}$ there exists $f \in \mathcal{S}(H_n)$ such that $m(\lambda, k) = \langle S_{\lambda,k}, f\rangle$ if and only if m satisfies the following conditions:*

(i) *for all $N \in \mathbb{N}$ there exists c_N such that*

$$|m(\lambda, k)| \leq c_N \left(|k|^{n-1} + \frac{1}{|\lambda|^{n-1}}\right) \frac{1}{|\lambda|^N (|k|+1)^N},\ k \in \mathbb{Z}, \tag{2.11}$$

(ii) *the functions defined on Δ_1 by*

$$(\lambda, (2k+1)|\lambda|) \mapsto E(m^*)(\lambda, k+q),\ (\lambda, (2k+1)|\lambda|) \mapsto \tilde{E}(m^{**})(\lambda, -k-p) \tag{2.12}$$

admit Schwartz extensions on $\mathbb{R}^2$.

3. The spaces $\mathcal{H}_n$

Our aim here is to prove that the restriction to Σ of a function in $\mathcal{H}_n$ is in the image of the normalized spherical transform. As a consequence we obtain a similar result to the one showed by Veneruso in [19], which states that the restriction to the spectrum $\Delta(U(n), H_n)$ of a Schwartz function on $\mathbb{R}^2$ is in the image of the spherical transform associated to the Gelfand pair $(U(n), H_n)$.

In fact, let $\Omega = \{(\lambda, s)/\lambda \neq 0\}$. We start with some propositions that will be used later.

Proposition 3.1. *For $\varphi \in \mathcal{S}(\mathbb{R}^2)$ the function*

$$\begin{aligned} F_\varphi : \Omega &\to \mathbb{R} \\ (\lambda, s) &\mapsto \frac{\varphi(\lambda, s+|\lambda|) - \varphi(\lambda, s-|\lambda|)}{|\lambda|}, \end{aligned} \tag{3.1}$$

admits an extension in $\mathcal{S}(\mathbb{R}^2)$.

Proof. Note that for $(\lambda, s) \in \Omega$,

$$F_\varphi(\lambda, s) = \frac{1}{|\lambda|} \int_{s-|\lambda|}^{s+|\lambda|} \frac{\partial \varphi}{\partial u}(\lambda, u)\, du = \int_{-1}^{1} \frac{\partial \varphi}{\partial s}(\lambda, |\lambda|t + s)\, dt,$$

where in the last equality we have used the change of variables $u = |\lambda|t + s$.

It is easy to see that $\bar{F}_\varphi$ defined by

$$\bar{F}_\varphi(\lambda, s) = \int_{-1}^{1} \frac{\partial \varphi}{\partial s}(\lambda, |\lambda|t + s)\, dt, \quad \forall (\lambda, s) \in \mathbb{R}^2,$$

is a continuous extension of F_φ, since we can derive it under the integral symbol by Dominated Convergence Theorem.

We will prove that $\bar{F}_\varphi$ lies in $\mathcal{S}(\mathbb{R}^2)$.

(i) For $i, j \in \mathbb{N}_0$, it is easy to show that

$$\frac{\partial^{i+j} F_\varphi}{\partial \lambda^i \partial s^j}(\lambda, s)=\sum_{l=0}^{i} \binom{i}{l} \int_{-1}^{1} \frac{\partial^{i+j+1}\varphi}{\partial \lambda^{i-l} \partial s^{j+l+1}}(\lambda, |\lambda| t + s)(sg(\lambda)t)^l \, dt, \ \forall (\lambda, s) \in \Omega.$$

Since if l is odd then $\int_{-1}^{1} t^l \, dt = 0$, given $s_0 \in \mathbb{R}$ we have that

$$\lim_{\substack{(\lambda,s)\to(0,s_0)\\(\lambda,s)\in\Omega}} \frac{\partial^{i+j} F_\varphi}{\partial \lambda^i \partial s^j}(\lambda, s) = \sum_{\substack{l=0\\ l \text{ even}}}^{i} \binom{i}{l} \frac{\partial^{i+j+1}\varphi}{\partial \lambda^{i-l} \partial s^{j+l+1}}(0, s_0) \int_{-1}^{1} t^l \, dt,$$

Then, $\bar{F}_\varphi \in C^\infty(\mathbb{R}^2)$ since $\varphi \in C^\infty(\mathbb{R}^2)$.

(ii) $\bar{F}_\varphi$ is a rapidly decreasing function since so is φ.

∎

Lemma 3.2. *Let $n \geq 2$. If $\varphi \in \mathcal{S}(\mathbb{R}^2)$ then there exists $\varphi_{n-2} \in \mathcal{S}(\mathbb{R}^2)$ such that*

$$F_{\varphi_{n-2}}(\lambda, s) = \frac{1}{|\lambda|^{n-1}} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \varphi(\lambda, s + |\lambda|(n-1-2l)), \ \forall (\lambda, s) \in \Omega. \quad (3.2)$$

Proof. For $n = 2$, it is clear that $\varphi_0 = \varphi$ satisfies the claim.

By inductive hypothesis, we suppose for $n \geq 2$ that $\varphi \in \mathcal{S}(\mathbb{R}^2)$ implies that there is $\varphi_{n-2} \in \mathcal{S}(\mathbb{R}^2)$ such that (3.2) holds.

Then, for $n+1$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$ we have

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\varphi(\lambda, s + |\lambda|(n-2l))}{|\lambda|^n} = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \frac{F_\varphi(\lambda, s + |\lambda|(n-1-2l))}{|\lambda|^{n-1}},$$

since

$$\begin{aligned}
&\frac{1}{|\lambda|^{n-1}} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} F_\varphi(\lambda, s + |\lambda|(n-1-2l)) \\
&= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \frac{[\varphi(\lambda, s+|\lambda|(n-1-2l)+|\lambda|) - \varphi(\lambda, s+|\lambda|(n-1-2l)-|\lambda|)]}{|\lambda|^{n-1}|\lambda|} \\
&= \frac{1}{|\lambda|^n} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} [\varphi(\lambda, s + |\lambda|(n-2l)) - \varphi(\lambda, s + |\lambda|(n-2(l+1)))] \\
&= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \frac{\varphi(\lambda, s+|\lambda|(n-2l))}{|\lambda|^n} + \sum_{l=0}^{n-1} (-1)^{l+1} \binom{n-1}{l} \frac{\varphi(\lambda, s+|\lambda|(n-2(l+1)))}{|\lambda|^n} \\
&= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \frac{\varphi(\lambda, s+|\lambda|(n-2l))}{|\lambda|^n} + \sum_{l=1}^{n} (-1)^l \binom{n-1}{l-1} \frac{\varphi(\lambda, s+|\lambda|(n-2l))}{|\lambda|^n} \\
&= \frac{1}{|\lambda|^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \varphi(\lambda, s + |\lambda|(n-2l)),
\end{aligned}$$

where in the last equality we have used $\binom{n-1}{l}+\binom{n-1}{l-1}=\binom{n}{l}$.

Due to Proposition 3.1 we know that F_φ defined on Ω can be extended to a function $\bar{F}_\varphi \in \mathcal{S}(\mathbb{R}^2)$ and, by inductive hypothesis, there is $(\bar{F}_\varphi)_{n-2} \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}\frac{F_\varphi(\lambda, s+|\lambda|(n-1-2l))}{|\lambda|^{n-1}} = \frac{(\bar{F}_\varphi)_{n-2}(\lambda, s+|\lambda|)-(\bar{F}_\varphi)_{n-2}(\lambda, s-|\lambda|)}{|\lambda|}.$$

We take $\varphi_{n-1}=(\bar{F}_\varphi)_{n-2}$ and the Proposition follows. ∎

Proof. (*of Theorem 1.5*) Let m be the map defined on $(\mathbb{R}\setminus\{0\})\times\mathbb{Z}$ by

$$m(\lambda,k)=\begin{cases}\frac{1}{|\lambda|^{n-1}}F(\lambda,|\lambda|(2k+p-q)), & k\geq 0\\ \\ \frac{(-1)^{n-2}}{|\lambda|^{n-1}}F(\lambda,|\lambda|(2k+p-q)), & k<0.\end{cases}$$

We will prove that m satisfies the conditions **(i)** and **(ii)** of Theorem 2.8. In fact, **(i)** follows from the fact that $\varphi,\psi\in\mathcal{S}(\mathbb{R}^2)$.

To show **(ii)**, by Definition 2.7 we have that

$$\begin{aligned}E(m^*)(\lambda,k+q) &= \sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}m^*(\lambda,k+q-l)\\ &= \sum_{l=0}^{\min\{k+q,n-l\}}(-1)^l\binom{n-1}{l}m(\lambda,k+q-l)\\ &\quad+\sum_{l=k+q+1}^{n-1}(-1)^l\binom{n-1}{l}(-1)^{n-2}m(\lambda,k+q-l)\\ &= \frac{1}{|\lambda|^{n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}F(\lambda,|\lambda|(2(k-l)+n))\\ &= \frac{1}{|\lambda|^{n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}\varphi(\lambda,|\lambda|(2(k-l)+n)),\end{aligned}$$

since $2(k-l)+n\geq -n+2$ if $0\leq k$, $0\leq l\leq n-1$, and $F|_{\Sigma^+}=\varphi|_{\Sigma^+}$. Therefore, the first map of (2.12) agrees on Δ_1 with the map defined by

$$(\lambda,s)\mapsto F_{\varphi_{n-2}}(\lambda,s),\quad \forall(\lambda,s)\in\Omega,$$

where $\varphi_{n-2}\in\mathcal{S}(\mathbb{R}^2)$ is given by Lemma 3.2. Then, by Proposition 3.1, this map admits an extension in $\mathcal{S}(\mathbb{R}^2)$.

Similarly, for the second map in (2.12) we have that

$$\begin{aligned}
\tilde{E}(m^{**})(\lambda,-k-p) &= \sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}m^{**}(\lambda,-k-p+l)\\
&= \sum_{l=0}^{\min\{k+p-1,n-1\}}(-1)^l\binom{n-1}{l}m(\lambda,-k-p+l)\\
&\quad+\sum_{l=k+p}^{n-1}(-1)^l\binom{n-1}{l}(-1)^{n-2}m(\lambda,-k-p+l)\\
&= (-1)^{n-2}\frac{1}{|\lambda|^{n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}F(\lambda,|\lambda|(-2(k-l)-n))\\
&= (-1)^{n-2}\frac{1}{|\lambda|^{n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}\psi(\lambda,|\lambda|(-2(k-l)-n)),
\end{aligned}$$

since $-2(k-l)-n\leq n-2$ if $0\leq k,\ 0\leq l\leq n-1$ and by hypothesis $F|_{\Sigma^-}=\psi|_{\Sigma^-}$.

Therefore, $\tilde{E}(m^{**})$ agrees on Δ_1 with the fucntion defined on Ω by

$$(\lambda,s)\mapsto\frac{1}{|\lambda|^{n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}\bar{\psi}(\lambda,s+|\lambda|(n-1-2l)),$$

where $\bar{\psi}(\lambda,s)=(-1)^{n-2}\psi(\lambda,-s)$. Since obviously $\bar{\psi}\in\mathcal{S}(\mathbb{R}^2)$, then by Lemma 3.2 and Proposition 3.1, this map can be extended by a function in $\mathcal{S}(\mathbb{R}^2)$. ■

Proof. (*of Corollary 1.6*) If $\varphi\in\mathcal{S}(\mathbb{R}^2)$ then $F=\varphi|_\Sigma$ satisfies the hypothesis of the previous Theorem, so the Corollary follows. ■

Proof. (*of Corollary 1.8*) Given $\varphi\in\mathcal{H}_n$ there exist $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^2)$ such that

$$\varphi(\lambda,s)=\varphi_1(\lambda,s)+\prod_{k=-p+1}^{q-1}(s-(2k+p-q)|\lambda|)\ \varphi_2(\lambda,s)\ H(s).$$

Let $\overline{\varphi}_2$ be the map defined by

$$\overline{\varphi}_2(\lambda,s)=\varphi_1(\lambda,s)+\prod_{k=-p+1}^{q-1}(s-(2k+p-q)|\lambda|)\ \varphi_2(\lambda,s).$$

So, $\overline{\varphi}_2|_{\Sigma^+}=\varphi|_{\Sigma^+}$ and $\varphi_1|_{\Sigma^-}=\varphi|_{\Sigma^-}$ since $\prod_{k=-p+1}^{q-1}(s-(2k+p-q)|\lambda|)\ \varphi_2(\lambda,s)=0$ for all $(\lambda,s)\in\Sigma^+\cap\Sigma^-$. Then, by Theorem 1.5 this Corollary follows. ■

4. Characterization of the image of the normalized spherical transform

It is clear that we cannot hope that given any function $f\in\mathcal{S}(H_n)$ its normalized spherical transform $\mathcal{F}(f)$ can be extended to a function in $\mathcal{S}(\mathbb{R}^2)$ (see Proposition 2.5 in Preliminaries). In this section we will prove that for $f\in\mathcal{S}(H_n)$ we can extend $\mathcal{F}(f)$ to a function in $\mathcal{H}_n$.

Definition 4.1. Let $\phi \in \mathcal{S}(\mathbb{R})$ such that $\phi(\lambda) = 1$ if $|\lambda| < 1$ and $\phi(\lambda) = 0$ if $|\lambda| \geq 2$.

Lemma 4.2. *Given $f \in \mathcal{S}(H_n)$, let $\theta_1, \theta_2 \in \mathcal{S}(\mathbb{R})$ such that*

$$\mathcal{F}(f)(0,s) = \theta_1(s) + s^{n-1}\ \theta_2(s)\ H(s),\ \forall s \in \mathbb{R}.$$

Then, the map

$$\varphi(\lambda, s) = \theta_1(s)\ \phi(\lambda) + \prod_{k=-p+1}^{q-1} (s - (2k+p-q)|\lambda|)\ \theta_2(s)\ \phi(\lambda)\ H(s)$$

is in $\mathcal{H}_n$ and $\varphi(0,s) = \mathcal{F}(f)(0,s)$ for all $s \in \mathbb{R}$.

Proof. It is immediate. ∎

The following two propositions allow us to develop the normalized spherical transform $\mathcal{F}(f)$ in a Taylor expansion similar to Geller's lemma proved by Astengo, Di Blasio and Ricci in [1].

Proposition 4.3. *If $f \in \mathcal{S}(H_n)$ satisfies $\mathcal{F}(f)(0,s) = 0$ if $s \in \mathbb{R}$ then, there exists $h \in \mathcal{S}(H_n)$ such that*

$$\mathcal{F}(f)(\lambda, |\lambda|(2k+p-q)) = \lambda\ \mathcal{F}(h)(\lambda, |\lambda|(2k+p-q)),\ \forall \lambda \neq 0, k \in \mathbb{Z}.$$

Proof. By hypothesis and the definition of S_σ, see (2.8) and (2.9), we have that

$$0 = \langle S_\sigma, f\rangle = (-1)^{n-1} \int_0^\infty J_0((\sigma\tau)^{1/2})(Nf)^{n-1}(\tau, \hat{0})\ d\tau,\ \forall \sigma \geq 0, \tag{4.1}$$

$$0 = \langle S_\sigma, f\rangle = (-1)^{n-2} \int_0^\infty J_0((-\sigma\tau)^{1/2})(Nf)^{n-1}(-\tau, \hat{0})\ d\tau,\ \forall \sigma < 0. \tag{4.2}$$

Let $f_1 \in \mathcal{S}_{U(1)}(H_1)$ defined by $f_1(z,t) = (Nf)^{n-1}(|z|^2, t)$ for all $(z,t) \in \mathbb{C} \times \mathbb{R}$. Let us denote by $\widehat{f_1}$ the spherical transform of f_1 associated with the Gelfand pair $(U(1), H_1)$. We will show that $\widehat{f_1}(0,\sigma) = 0$ for all $\sigma \geq 0$, so $f_1(x, \widehat{0}) = 0$ for all $x \in \mathbb{R}^2$ (see [1] pag. 789).

In fact, for $\sigma \geq 0$ we have

$$\begin{aligned}
\widehat{f_1}(0,\sigma) &= \int_{\mathbb{C}} J_0(\sigma|z|) f_1(z, \widehat{0})\ dz \\
&= \int_{\mathbb{C}} J_0(\sigma|z|)(Nf)^{n-1}(|z|^2, \widehat{0})\ dz \\
&= 2\pi \int_0^\infty J_0(\sigma r)(Nf)^{n-1}(r^2, \widehat{0}) r\ dr \\
&= \pi \int_0^\infty J_0(\sigma\tau^{1/2})(Nf)^{n-1}(\tau, \widehat{0})\ d\tau \\
&= (-1)^{n-1}\pi\langle S_{\sigma^2}, f\rangle \\
&= 0
\end{aligned}$$

where in the last equality we have used (4.1). Then $(Nf)^{n-1}(|z|^2,\widehat{0})=0$ for all $z\in\mathbb{C}$, that is,

$$(Nf)^{n-1}(\tau,\widehat{0})=0 \quad \forall\tau\geq 0. \tag{4.3}$$

Now, let $f_2\in\mathcal{S}_{U(1)}(H_1)$ defined by $f_2(z,t)=(Nf)^{n-1}(-|z|^2,t)$ for all $(z,t)\in\mathbb{C}\times\mathbb{R}$. In the similar way, for $\sigma>0$ we have that

$$\begin{aligned}
\widehat{f_2}(0,\sigma) &= \int_{\mathbb{C}} J_0(\sigma|z|)f_2(z,\widehat{0})\;dz\\
&= \int_{\mathbb{C}} J_0(\sigma|z|)(Nf)^{n-1}(-|z|^2,\widehat{0})\;dz\\
&= 2\pi\int_0^{\infty} J_0(\sigma r)(Nf)^{n-1}(-r^2,\widehat{0})r\;dr\\
&= \pi\int_0^{\infty} J_0((-(-\sigma^2)\tau)^{1/2})(Nf)^{n-1}(-\tau,\widehat{0})\;d\tau\\
&= (-1)^{n-2}\pi\langle S_{-\sigma^2},Nf\rangle\\
&= 0,
\end{aligned}$$

where we have used (4.2). Then $(Nf)^{n-1}(-|z|^2,\widehat{0})=0$ for all $z\in\mathbb{C}$, that is,

$$(Nf)^{n-1}(\tau,\widehat{0})=0 \quad \forall\tau\leq 0. \tag{4.4}$$

By (4.3) and (4.4) we obtain $(Nf)^{n-1}(\tau,\widehat{0})=0$ for all $\tau\in\mathbb{R}$. So, $Nf(\tau,\widehat{0})$ is a polynomial of degree $n-2$ in τ and moreover it is a rapidly decreasing function, therefore $Nf(\tau,\widehat{0})=0$ for all $\tau\in\mathbb{R}$, that is,

$$\int_{-\infty}^{\infty} Nf(\tau,t)\;dt=0. \tag{4.5}$$

Let φ be defined on $\mathbb{R}^2$ by

$$\varphi(\tau,x)=\int_{-\infty}^{x} Nf(\tau,t)\;dt. \tag{4.6}$$

It is clear that $x\mapsto\varphi(\tau,x)$ is in $C^\infty(\mathbb{R})$ and it is not difficult to see that also, it is a rapidly decreasing function by using (4.5) and that the map $t\mapsto Nf(\tau,t)$ is in $\mathcal{S}(\mathbb{R})$.

By (4.6) we obtain

$$Nf(\tau,\widehat{x})=(\frac{\partial\varphi}{\partial x})(\tau,\widehat{x})=ix\;\;\varphi(\tau,\widehat{x})=x\;\;i\varphi(\tau,\widehat{x}).$$

Moreover, as the map $N:\mathcal{S}(H_n)\to\mathcal{H}^{\#}$ is surjective there is $h\in\mathcal{S}(H_n)$ such that $Nh=i\varphi$, then

$$Nf(\tau,\widehat{\lambda})=\lambda\;Nh(\tau,\widehat{\lambda}),\quad \forall(\tau,\lambda)\in\mathbb{R}^2.$$

Finally,

$$\langle S_{\lambda,k},f\rangle=\langle F_{\lambda,k},Nf(\cdot,\widehat{\lambda})\rangle=\langle F_{\lambda,k},\lambda\;Nh(\cdot,\widehat{\lambda})\rangle=\lambda\;\langle S_{\lambda,k},h\rangle,$$

implies $\mathcal{F}(f)(\lambda,(2k+p-q)|\lambda|)=\lambda\;\mathcal{F}(h)(\lambda,(2k+p-q)|\lambda|)$ for all $\lambda\neq 0,\;k\in\mathbb{Z}$. ■

Proposition 4.4. *Let $N \in \mathbb{N}$. Given $f \in \mathcal{S}(H_n)$ there exist $f_j \in \mathcal{S}(H_n)$ for all $j = 0, 1, \dots, N$ and $\varphi_j \in \mathcal{H}_n$ for all $j = 0, 1, \dots, N-1$ such that*

$$\mathcal{F}(f)(\lambda, |\lambda|(2k+p-q)) = \sum_{j=0}^{N-1} \lambda^j \varphi_j(\lambda, |\lambda|(2k+p-q)) + \lambda^N \mathcal{F}(f_N)(\lambda, |\lambda|(2k+p-q)), \quad (4.7)$$

where φ_j is the function of Lemma 4.2 associated to f_j.

Proof. Let ϕ be as in Definition 4.1 and $f \in \mathcal{S}(H_n)$. We will do the proof by induction on N.

Let $\theta_1, \theta_2 \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(f)(0, s) = \theta_1(s) + s^{n-1}\theta_2(s)\ H(s)$, and let

$$\varphi_0(\lambda, s) = \theta_1(s)\ \phi(\lambda) + \prod_{k=-p+1}^{q-1} (s - (2k+p-q)|\lambda|)\ \theta_2(s)\ \phi(\lambda)\ H(s).$$

Then, by Theorem 1.8 there exists $f_0 \in \mathcal{S}(H_n)$ such that $\mathcal{F}(f_0) = \varphi_0|_\Sigma$. So,

$$\mathcal{F}(f - f_0)(0, s) = \mathcal{F}(f)(0, s) - \varphi_0(0, s) = 0, \ \forall s \in \mathbb{R}.$$

By Proposition 4.3, there exists $f_1 \in \mathcal{S}(H_n)$ such that

$$\mathcal{F}(f - f_0)(\lambda, |\lambda|(2k+p-q)) = \lambda\ \mathcal{F}(f_1)(\lambda, |\lambda|(2k+p-q)).$$

Therefore,

$$\begin{aligned}\mathcal{F}(f)(\lambda, |\lambda|(2k+p-q)) &= \mathcal{F}(f_0)(\lambda, |\lambda|(2k+p-q)) + \lambda\ \mathcal{F}(f_1)(\lambda, |\lambda|(2k+p-q)) \\ &= \varphi_0(\lambda, |\lambda|(2k+p-q)) + \lambda\ \mathcal{F}(f_1)(\lambda, |\lambda|(2k+p-q)).\end{aligned}$$

Now, we suppose that for $N \geq 1$ there exist $f_j \in \mathcal{S}(H_n)$ for all $j = 0, \dots, N$ and $\varphi_j \in \mathcal{H}_n$ for all $j = 0, \dots, N-1$ such that

$$\mathcal{F}(f)(\lambda, |\lambda|(2k+p-q)) = \sum_{j=0}^{N-1} \lambda^j \varphi_j(\lambda, |\lambda|(2k+p-q)) + \lambda^N \mathcal{F}(f_N)(\lambda, |\lambda|(2k+p-q)). \quad (4.8)$$

Then, for the first part of this proof with f_N instead of f, there exist $\varphi_N \in \mathcal{H}_n$ and $f_{N+1} \in \mathcal{S}(H_n)$ such that

$$\mathcal{F}(f_N)(\lambda, |\lambda|(2k+p-q)) = \varphi_N(\lambda, |\lambda|(2k+p-q)) + \lambda\ \mathcal{F}(f_{N+1})(\lambda, |\lambda|(2k+p-q)). \quad (4.9)$$

So by (4.8) and (4.9) we get (4.7) for $N+1$. ∎

The following Definition 4.5, Proposition 4.6 and Corollary 4.7 are straightforward adaptations of Lemma 3.1 proved by A., Di B. and R. in [1].

Definition 4.5. For a function h defined on Σ, let $E(h)$ be the function defined on $\mathbb{R}^2$ by

$$E(h)(\lambda, s) = \begin{cases} \sum_{k \in \mathbb{Z}} h(\lambda, |\lambda|(2k+p-q))\omega(\frac{s - |\lambda|(2k+p-q)}{|\lambda|}), & \lambda \neq 0, \\ 0, & \lambda = 0, \end{cases} \quad (4.10)$$

where ω is a function in $C_c^\infty(\mathbb{R})$ such that $\omega(t) = 1$ if $|t| \leq \frac{1}{2}$ and $\omega(t) = 0$ if $|t| \geq \frac{3}{4}$.

Proposition 4.6. *For $N \in \mathbb{N}$ let $f \in \mathcal{S}(H_n)$ satisfying*

$$\mathcal{F}(f_j)(0,s) = 0 \ \forall s \in \mathbb{R}, \ \forall j = 0, ..., 2N.$$

Then, the function $E(\mathcal{F}(f))$ as in (4.10) lies in $C^N(\mathbb{R}^2)$ and

(i) $E(\mathcal{F}(f))(\lambda, |\lambda|(2k+p-q)) = \mathcal{F}(f)(\lambda, |\lambda|(2k+p-q)) \ \forall \lambda \neq 0, \ k \in \mathbb{Z}$.

(ii) $\frac{\partial^i E(\mathcal{F}(f))}{\partial \lambda^i}(0,s) = 0 \ \forall s \in \mathbb{R}, \ 0 \leq i \leq N$.

(iii) *For any $i, j, M \in \mathbb{N}_0$ there exists a positive constant $C_{M,N}$ such that*

$$\sup_{(\lambda,s)\in\mathbb{R}^2} \left| (\lambda^2 + s^2)^M \frac{\partial^{i+j} E(\mathcal{F}(f))}{\partial \lambda^i \partial s^j}(\lambda, s) \right| \leq C_{M,N}, \quad \forall i + j \leq N.$$

Proof. It follows the same lines as in the proof of Lemma 3.1 in [1]. But here, we have to use Theorems 2.6 and 2.8 in [14]. ■

As a consequence of this Proposition we have the following (see Proposition 7.5 in [2])

Corollary 4.7. *We suppose that f belongs to $\mathcal{S}(H_n)$ satisfies*

$$\mathcal{F}(f_j)(0,s) = 0, \ \forall s \in \mathbb{R}, \ \forall j \in \mathbb{N}_0.$$

Then, the map $E(\mathcal{F}(f))$ defined by (4.10) is in $C^\infty(\mathbb{R}^2)$ and

(i) $E(\mathcal{F}(f))(\lambda, |\lambda|(2k+p-q)) = \mathcal{F}(f)(\lambda, |\lambda|(2k+p-q)) \ \forall \lambda \neq 0, \ k \in \mathbb{Z}$,

(ii) $\frac{\partial^i E(\mathcal{F}(f))}{\partial \lambda^i}(0,s) = 0 \ \forall s \in \mathbb{R}, \ \forall i \in \mathbb{N}_0$,

(iii) $E(\mathcal{F}(f)) \in \mathcal{S}(\mathbb{R}^2)$.

From now on, our purpose is to remove the restrictive condition $\mathcal{F}(f_j)(0,s)=0$ for all $s \in \mathbb{R}$, $j \in \mathbb{N}_0$.

Lemma 4.8. *Let $\{\theta_j(s) = \theta_{1,j}(s) + s^{n-1}\theta_{2,j}(s) \ H(s)\}_{j=0}^\infty$ be a sequence of functions as in (2.3). Then, there exists a sequence $\{\nu_j\}_{j=0}^\infty$ of positive numbers greater than 1 such that the function G defined on $\mathbb{R}^2$ by*

$$G(\lambda,s) = \sum_{j=0}^\infty \frac{\lambda^j}{j!} \ \theta_{1,j}(s) \ \omega(\nu_j\lambda) + \prod_{k=-p+1}^{q-1} (s - (2k+p-q)|\lambda|) \sum_{j=0}^\infty \frac{\lambda^j}{j!} \ \theta_{2,j}(s) \ \omega(\nu_j\lambda) \ H(s),$$

lies in $\mathcal{H}_n$.

Proof. Let ω be as in Definition 4.5. To prove Lemma 4.8, it is enough to show that for an appropriate sequence $\{\nu_j\}$ of positive numbers, if

$$f_j(\lambda,s) := \frac{\theta_{1,j}(s)}{j!}\lambda^j\omega(\nu_j\lambda), \quad g_j(\lambda,s) := \frac{\theta_{2,j}(s)}{j!}\lambda^j\omega(\nu_j\lambda),$$

then $f=\sum_{j=0}^{\infty} f_j$ and $g=\sum_{j=0}^{\infty} g_j$ are in $\mathcal{S}(\mathbb{R}^2)$.

To see this, we will take a sequence $\{\nu_j\}_{j=0}^{\infty}$ such that

$$\sup_{(\lambda,s)\in\mathbb{R}^2}\left|s^k\frac{\partial^{i+l}f_j}{\partial s^i\partial\lambda^l}(\lambda,s)\right|\le\frac{1}{2^j}\text{ and }\sup_{(\lambda,s)\in\mathbb{R}^2}\left|s^k\frac{\partial^{i+l}g_j}{\partial s^i\partial\lambda^l}(\lambda,s)\right|\le\frac{1}{2^j}\tag{4.11}$$

for all $0\le i,l,k\le j$. Since in this case, given $k',k,N\in\mathbb{N}_0$ we obtain

$$\left|\lambda^{k'}s^k\frac{\partial^{i+l}f_j}{\partial s^i\partial\lambda^l}(\lambda,s)\right|\le\frac{1}{2^j}|\lambda|^{k'},\ \forall j\ge\max\{N,k\},\forall i+l=N.$$

Moreover, $f_j(s,\lambda)=0$ for all $(\lambda,s)\in\mathbb{R}^2$ such that $|\lambda|\ge 1$. Then,

$$\sup_{(\lambda,s)\in\mathbb{R}^2}\left|\lambda^{k'}s^k\frac{\partial^{i+l}f_j}{\partial s^i\partial\lambda^l}(\lambda,s)\right|\le\sum_{j=0}^{\bar{M}}\sup_{(\lambda,s)\in\mathbb{R}^2}\left|\lambda^{k'}s^k\frac{\partial^{i+l}f_j}{\partial s^i\partial\lambda^l}(\lambda,s)\right|+\sum_{j=\bar{M}+1}^{\infty}\frac{1}{2^j}\le C_{k,k',N}.$$

Consider (4.11). By the Leibniz's rule we obtain

$$\begin{aligned}\frac{\partial^{i+l}f_j}{\partial s^i\partial\lambda^l}(\lambda,s)&=\frac{\partial^i\theta_{1,j}}{\partial s^i}(s)\frac{\partial^l(\lambda^j\omega(\nu_j\lambda))}{\partial\lambda^l}\\&=\frac{\partial^i\theta_{1,j}}{\partial s^i}(s)\sum_{r=0}^{l}\binom{l}{r}\frac{j!}{(j-r)!}\lambda^{j-r}\nu_j^{l-r}\omega^{l-r}(\nu_j\lambda)\\&=\frac{\partial^i\theta_{1,j}}{\partial s^i}(s)\ \frac{1}{\nu_j^{j-l}}\sum_{r=0}^{l}\binom{l}{r}\frac{j!}{(j-r)!}(\lambda\nu_j)^{j-r}\omega^{l-r}(\nu_j\lambda).\end{aligned}$$

Then, since $\omega^{(r)}(\nu_j\lambda)=0$ for $|\nu_j\lambda|<1$, we may find positive constants c_j and d_j such that

$$\left|s^k\frac{\partial^{i+l}f_j}{\partial s^i\partial\lambda^l}\right|\le\frac{c_j}{\nu_j j!}\left|s^k\frac{\partial^i\theta_{1,j}}{\partial s^i}\right|\le\frac{c_j}{\nu_j}\sum_{k,i=0}^{j}\left\|s^k\frac{\partial^i\theta_{1,j}}{\partial s^i}\right\|_\infty$$

$$\left|s^k\frac{\partial^{i+l}g_j}{\partial s^i\partial\lambda^l}\right|\le\frac{d_j}{\nu_j j!}\left|s^k\frac{\partial^i\theta_{2,j}}{\partial s^i}\right|\le\frac{d_j}{\nu_j}\sum_{k,i=0}^{j}\left\|s^k\frac{\partial^i\theta_{2,j}}{\partial s^i}\right\|_\infty.$$

Then, we take $\nu_j\ge 1$ such that

$$\nu_j\ge\max\left\{2^jc_j\sum_{k,i=0}^{j}\left\|s^k\frac{\partial^i\theta_{1,j}}{\partial s^i}\right\|_\infty,2^jd_j\sum_{k,i=0}^{j}\left\|s^k\frac{\partial^i\theta_{2,j}}{\partial s^i}\right\|_\infty\right\}.$$

Thus, the series $\sum_{j=0}^{\infty} f_j(\lambda,s)$ and $\sum_{j=0}^{\infty} g_j(\lambda,s)$ lie in $\mathcal{S}(\mathbb{R}^2)$ and G satisfies the claim. ■

Corollary 4.9. *Let $\{b_j(s)\}_{j=0}^{\infty}$ be a sequence of functions in $\mathcal{S}(\mathbb{R})$. Then, there exists a sequence $\{\nu_j\}_{j=0}^{\infty}$ of numbers greater than 1, such that the map H defined by*

$$H(\lambda,s)=\sum_{j=0}^{\infty}\lambda^j\ b_j(s)\ \frac{\omega(\nu_j\lambda)}{j!}$$

is a Schwartz function on $\mathbb{R}^2$.

For $g \in \mathcal{S}(H_n)$, let $\{g_j\}_{j=0}^{\infty}$ be the sequence of functions in $\mathcal{S}(H_n)$ associated to g given by Proposition 4.4.

Proposition 4.10. *Let $f \in \mathcal{S}(H_n)$. Then, there exist $G \in \mathcal{H}_n$ and $g \in \mathcal{S}(H_n)$ such that*

(i) $G|_{\Sigma} = \mathcal{F}(g)$ *and,*

(ii) $\mathcal{F}(f_j)(0,s) - \mathcal{F}(g_j)(0,s) = 0$ *for all* $s \in \mathbb{R},\ j \in \mathbb{N}_0$.

Proof. The sequence of functions

$$\{j!\ \mathcal{F}(f_j)(0,s)\}_{j=0}^{\infty} = \{\theta_{1,j}(s) + s^{n-1}\theta_{2,j}(s)\ H(s)\}_{j=0}^{\infty}$$

satisfies the hypothesis of Lemma 4.8, hence there exists a sequence $\{\nu_j\}_{j=0}^{\infty}$ of positive numbers greater than 1 such that

$$G(\lambda,s)=\sum_{j=0}^{\infty}\lambda^j\theta_{1,j}(s)\omega(\nu_j\lambda) + H(s)\prod_{\bar{k}=-p+1}^{q-1}(s-(2\bar{k}+p-q)|\lambda|)\sum_{j=0}^{\infty}\lambda^j\theta_{2,j}(s)\omega(\nu_j\lambda) \quad (4.12)$$

lies in $\mathcal{H}_n$. Then, by Theorem 1.8 there exists $g \in \mathcal{S}(H_n)$ such that

$$\mathcal{F}(g) = G|_{\Sigma}. \quad (4.13)$$

Moreover, given $N \in \mathbb{N}$, by Proposition 4.4 there exist $g_j \in \mathcal{S}(H_n)$ and

$$\psi_j(\lambda,s) = \theta_{1,j}(s)\phi(\lambda) + H(s)\left(\prod_{k=-p+1}^{q-1}(s-(2k+p-q)|\lambda|)\right)\theta_{2,j}(s)\phi(\lambda)$$

such that

$$\mathcal{F}(g)(\lambda,(2k+p-q)|\lambda|)=\sum_{j=0}^{N}\lambda^j\psi_j(\lambda,(2k+p-q)+\lambda^{N+1}\mathcal{F}(g_{N+1})(\lambda,(2k+p-q), \quad (4.14)$$

for $\lambda \neq 0$ and $k \in \mathbb{Z}$. Then, by (4.12), (4.13) and (4.14) we have

$$\sum_{j=0}^{\infty}\lambda^j\theta_{1,j}((2k+p-q)|\lambda|)\ \omega(\nu_j\lambda) + H(s)\prod_{\bar{k}=-p+1}^{q-1}2|\lambda|(k-\bar{k})\ \sum_{j=0}^{\infty}\lambda^j\theta_{2,j}((2k+p-q)|\lambda|)\ \omega(\nu_j\lambda)$$
$$= \sum_{j=0}^{N}\lambda^j\ \psi_j(\lambda,(2k+p-q)|\lambda|) + \lambda^{N+1}\mathcal{F}(g_{N+1})(\lambda,(2k+p-q)|\lambda|), \quad \forall\lambda \neq 0,\ k \in \mathbb{Z}.$$

Taking limits as $(\lambda, |\lambda|(2k+p-q))$ goes to $(0,s)$ on both sides of the last equality we obtain $\mathcal{F}(f_0)(0,s) = \psi_0(0,s) = \mathcal{F}(g_0)(0,s)$ for all $s \in \mathbb{R}$. Then,

$$\mathcal{F}(f_0)(0,|\lambda|(2k+p-q))\omega(\nu_0\lambda)= \mathcal{F}(g_0)(0,|\lambda|(2k+p-q))\phi(\lambda)= \psi_0(\lambda,|\lambda|(2k+p-q))$$

for all $|\lambda| < \frac{1}{\nu_0}$ and

$$\sum_{j=1}^{\infty}\lambda^j\theta_{1,j}((2k+p-q)|\lambda|)\;\omega(\nu_j\lambda) + H(s)\prod_{\bar{k}=-p+1}^{q-1} 2|\lambda|(k-\bar{k})\;\sum_{j=1}^{\infty}\lambda^j\theta_{2,j}((2k+p-q)|\lambda|)\;\omega(\nu_j\lambda)$$

$$= \sum_{j=1}^{N}\lambda^j\;\psi_j(\lambda,(2k+p-q)|\lambda|) + \lambda^{N+1}\mathcal{F}(g_{N+1})(\lambda,(2k+p-q)|\lambda|),\quad \forall|\lambda| < \frac{1}{\nu_0},\; k\in\mathbb{Z}.$$

Taking again limits as $(\lambda, |\lambda|(2k+p-q))$ goes to $(0,s)$ on both sides of the equality we obtain $\mathcal{F}(f_1)(0,s) = \psi_1(0,s) = \mathcal{F}(g_1)(0,s)$. Then,

$$\mathcal{F}(f_1)(0,|\lambda|(2k+p-q))\;\omega(\nu_1\lambda) = \mathcal{F}(g_1)(0,|\lambda|(2k+p-q))\;\phi(\lambda) = \psi_1(\lambda,|\lambda|(2k+p-q))$$

for all $|\lambda| < \frac{1}{\nu_1}$ and thus

$$\sum_{j=2}^{\infty}\lambda^j\theta_{1,j}((2k+p-q)|\lambda|)\;\omega(\nu_j\lambda) + H(s)\prod_{\bar{k}=-p+1}^{q-1} 2|\lambda|(k-\bar{k})\;\sum_{j=2}^{\infty}\lambda^j\theta_{2,j}((2k+p-q)|\lambda|)\;\omega(\nu_j\lambda)$$

$$= \sum_{j=2}^{N}\lambda^j\;\psi_j(\lambda,(2k+p-q)|\lambda|) + \lambda^{N+1}\mathcal{F}(g_{N+1})(\lambda,(2k+p-q)|\lambda|),\quad \forall|\lambda| < \frac{1}{\nu_1},\; k\in\mathbb{Z}.$$

Iterating this argument we obtain $\mathcal{F}(f_j)(0,s) = \mathcal{F}(g_j)(0,s)$ for all $j = 0,\dots,N$.

Since N is arbitrary we have that $\mathcal{F}(f_j)(0,s) = \mathcal{F}(g_j)(0,s)$ for all $j \in \mathbb{N}_0$. ■

Theorem 4.11. *We suppose that f belongs to $\mathcal{S}(H_n)$. Then, there exists $\varphi \in \mathcal{H}_n$ such that $\mathcal{F}(f) = \varphi|_\Sigma$.*

Proof. Given $f \in \mathcal{S}(H_n)$, let $G \in \mathcal{H}_n$ and $g \in \mathcal{S}(H_n)$ as in Proposition 4.10. Then $h = f - g \in \mathcal{S}(H_n)$ and it satisfies the hypothesis of Corollary 4.7. Therefore, $E(\mathcal{F}(h)) \in \mathcal{S}(\mathbb{R}^2)$. Let

$$\varphi = E(\mathcal{F}(h)) + G.$$

Then $\varphi \in \mathcal{H}_n$ and $\varphi|_\Sigma = \mathcal{F}(f)$. ■

Proof. (*of Theorem 1.9*) It follows immediately from Theorems 1.8 and 4.11. ■

Corollary 4.12. *Given $f \in \mathcal{S}(H_n)$ there exist $\varphi, \psi \in \mathcal{S}(\mathbb{R}^2)$ such that $\mathcal{F}(f)|_{\Sigma^+} = \varphi|_{\Sigma^+}$ and $\mathcal{F}(f)|_{\Sigma^-} = \psi|_{\Sigma^-}$.*

Proof. By Theorem 4.11 there exists $\bar{\varphi} \in \mathcal{H}_n$ such that $\mathcal{F}(f) = \bar{\varphi}|_\Sigma$. Let $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\bar{\varphi}(\lambda,s) = \varphi_1(\lambda,s) + \prod_{k=-p+1}^{q-1}(s-(2k+p-q)|\lambda|)\varphi_2\;H(s).$$

Then, φ defined by $\varphi(\lambda,s) = \varphi_1(\lambda,s) + \prod_{k=-p+1}^{q-1}(s-(2k+p-q)|\lambda|)\varphi_2(\lambda,s)$ and $\psi = \varphi_1$ satisfy the claim. ■

Corollary 4.13. *Let F be a function defined on Σ. Then, F is in the image of the normalized spherical transform if and only if there exist $\varphi, \psi \in \mathcal{S}(\mathbb{R}^2)$ such that $F|_{\Sigma^+} = \varphi|_{\Sigma^+}$ and $F|_{\Sigma^-} = \varphi|_{\Sigma^-}$.*

Proof. It follows from Proposition 1.5 and the previous Corollary. ■

Corollary 4.14. *Let F be a function defined on Σ. The following statements are equivalent:*

(i) *there exists $f \in \mathcal{S}(H_n)$ such that $F = \mathcal{F}(f)$,*

(ii) *there exists $\varphi \in \mathcal{H}_n$ such that $F = \varphi|_{\Sigma}$,*

(iii) *there exist $\varphi, \psi \in \mathcal{S}(\mathbb{R}^2)$ such that $F|_{\Sigma^+} = \varphi|_{\Sigma^+}$ and $F|_{\Sigma^-} = \psi|_{\Sigma^-}$.*

Proof. It follows from Theorem 1.9 and Theorem 4.13. ■

Now, we look for a sufficient condition that $\mathcal{F}(f)$ must satisfy to admit an extension in $\mathcal{S}(\mathbb{R}^2)$. We know that for any f in $\mathcal{S}(H_n)$ the map

$$s \mapsto \mathcal{F}(f)(0,s)$$

lies in $C^{n-2}(\mathbb{R})$. Therefore we ask whether the fact that the map $s \to \mathcal{F}(f)(0,s)$ lies in $C^{\infty}(\mathbb{R})$ is sufficient for $\mathcal{F}(f)$ to admit extension in $\mathcal{S}(\mathbb{R}^2)$.

Theorem 4.15. *Let $f \in \mathcal{S}(H_n)$ such that $s \mapsto \mathcal{F}(f)(0,s)$ is in $\mathcal{S}(\mathbb{R})$. Then, there exists $\varphi \in \mathcal{S}(\mathbb{R}^2)$ such that $\mathcal{F}(f) = \varphi|_{\Sigma}$.*

Proof. By hypothesis the map $s \mapsto \mathcal{F}(f_j)(0,s)$ lies in $\mathcal{S}(\mathbb{R})$. Thus, $H(\lambda, s) = \sum_{j=0}^{\infty} \lambda^j \ \mathcal{F}(f_j)(0,s) \ \omega(\nu_j \lambda)$ lies in $\mathcal{S}(\mathbb{R}^2)$. Now we follow the lines of Proposition 4.10 and we obtain $h \in \mathcal{S}(H_n)$ such that $\mathcal{F}(h) = H|_{\Sigma}$ and $\mathcal{F}(h_j)(0,s) = \mathcal{F}(f_j)(0,s) \quad \forall s \in \mathbb{R},\ \forall j \in \mathbb{N}_0$.

Let $g = f - h \in \mathcal{S}(H_n)$ and let

$$\varphi = E(\mathcal{F}(g)) + H.$$

Then, $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and $\varphi|_{\Sigma} = \mathcal{F}(f)$. ■

Proof. (*of Theorem 1.11*) It follows immediately from the previous theorem. ■

5. Looking for an extension of $\mathcal{F}(f)$ with better differentiability properties

In this section we will show that the functions of the space $\mathcal{H}_n$ which we used to characterize the image of the spherical transform, are as smooth as we can expect.

Moreover, we will show that there is a close relation between the differentiability of $s \to \mathcal{F}(f)(0,s)$, and the differentiability of some extension of $\mathcal{F}(f)$ in $\mathcal{H}_n$.

Proof. (*of Proposition 1.12*) Assume that $s \mapsto \mathcal{F}(f)(0,s)$ lies in $C^{k+n-2}(\mathbb{R})$. So, we note that the functions f_j associated to f according to Proposition 4.4 satisfy that the map $s \mapsto \mathcal{F}(f_j)(0,s)$ lies in $C^{k+n-2}(\mathbb{R})$. Therefore, there are $\theta_{1,j}, \theta_{2,j} \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(f_j)(0,s) = \theta_{1,j}(s) + s^{k+n-1}\theta_{2,j}(s)\ H(s)$.

Then, the map G defined by

$$G(\lambda,s) = \sum_{j=0}^{\infty} \lambda^j \theta_{1,j}(s)\omega(\nu_j\lambda) + s^k \prod_{k=-p+1}^{q-1} (s-(2k+p-q)|\lambda|) \sum_{j=0}^{\infty} \lambda^j \theta_{2,j}(s)\omega(\nu_j\lambda)\ H(s),$$

lies in $\mathcal{H}_n$. Moreover, such as we did in the proof of Proposition 4.10 we can see that $\mathcal{F}(g_j)(0,s) = \mathcal{F}(f_j)(0,s)$ for all $s \in \mathbb{R}$. Then, the map $\varphi = E(\mathcal{F}(f-g)) + G$ extends to $\mathcal{F}(f)$ and the Proposition follows. ∎

In order to prove the main result of this section we need the following

Proposition 5.1. *Let $f \in \mathcal{S}(H_n)$.*

(i) *If $\mathcal{F}(f)(\lambda,(2k+p-q)|\lambda|) = 0$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$, $2k+p-q \geq 0$ then, $\mathcal{F}(f)|_{\Sigma^+} \equiv 0$.*

(ii) *If $\mathcal{F}(f)(\lambda,(2k+p-q)|\lambda|) = 0$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$, $2k+p-q \leq 0$ then, $\mathcal{F}(f)|_{\Sigma^-} \equiv 0$.*

Proof.

(i) By Definition 2.7 we have that:

$$E(m^*)(\lambda,k+q) = \frac{1}{|\lambda|^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \mathcal{F}(f)(\lambda,|\lambda|(2(k-l)+n)).$$

We can suppose that n is odd and the proof is similar when n is even. Under the initial hypothesis we have that

$$|\lambda|^{n-1}E(m^*)(\lambda,k+q) = \begin{cases} \sum\limits_{l=\frac{n+1}{2}+k}^{n-1} (-1)^l \binom{n-1}{l} \mathcal{F}(f)(\lambda,(2(k-l)+n)|\lambda|), & 0 \leq k \leq \frac{n-3}{2}, \\ 0, & \frac{n-3}{2} < k. \end{cases}$$

On the other hand, by using the Inversion formula (Theorem 4.7 in [14]) we know that

$$Nf(\tau,t) = (-1)^{n-1} \int_{\mathbb{R}} \frac{|\lambda|}{2} \sum_{k \geq 0} E(m^*)(\lambda,k+q)\ L_k^0(|\lambda|\tau/2)\ e^{-|\lambda|\tau/4} e^{-i\lambda t}\ d\lambda$$

for $(\tau,t)\in[0,\infty)\times\mathbb{R}$. So,

$$\frac{\partial^j Nf}{\partial\tau^j}(0,\widehat{\lambda})=(-1)^{n-1}\frac{|\lambda|^{j+1}}{2}\frac{(-1)^j}{4^j}\sum_{k\geq 0}E(m^*)(\lambda,k+q)\sum_{l=0}^{j}\tbinom{j}{l}(-2)^l(L_k^0)^{(l)}(0) \tag{5.1}$$

Then, by definition of $\mathcal{F}(f)(\lambda,(2k'+p-q)|\lambda|)$ for $-p<k'<q$, and using (5.1) we get

$$\mathcal{F}(f)(\lambda,(2k'+p-q)|\lambda|)=\frac{(-|\lambda|)^{n-1}}{2}\sum_{j=0}^{n-2}\frac{1}{4^j}c_{j,k'}\sum_{k\geq 0}E(m^*)(\lambda,k+q)\sum_{l=0}^{j}\tbinom{j}{l}(-2)^l(L_k^0)^{(l)}(0),$$

and an easy computation shows that

$$t_k(j):=\sum_{l=0}^{j}\tbinom{j}{l}(-1)^l2^l(L_k^0)^{(l)}(0)=a_kt^k+\cdots+a_0 \tag{5.2}$$

is a polynomial in the variable j of degree k where $a_k=\frac{2^k}{k!}$.

Therefore,

$$\begin{aligned}
&|\lambda|^{n-1}\sum_{k\geq 0}E(m^*)(\lambda,k+q)\sum_{l=0}^{j}\tbinom{j}{l}(-1)^l2^l(L_k^0)^{(l)}(0)\\
&=\sum_{k=0}^{\frac{n-3}{2}}\sum_{l=\frac{n+1}{2}+k}^{n-1}(-1)^l\binom{n-1}{l}\mathcal{F}(f)(\lambda,(2(k-l)+n)|\lambda|)\ t_k(j)\\
&=\sum_{i=1}^{\frac{n-1}{2}}\left(\sum_{l=n-i}^{n-1}(-1)^l\binom{n-1}{l}t_{i-n+l}(j)\right)\mathcal{F}(f)(\lambda,(2i-n)|\lambda|)\\
&=\sum_{i=1}^{\frac{n-1}{2}}B_{i-1}(j)\mathcal{F}(f)(\lambda,(2i-n)|\lambda|)
\end{aligned}$$

where $B_{i-1}(j)=\sum\limits_{l=n-i}^{n-1}(-1)^l\binom{n-1}{l}t_{i-n+l}(j)$ is a polynomial in the variable j of degree $i-1$.

Thus, as $2k'+p-q=2(k'+p)-n$ we set $l=k'+p$ and for $1\leq l\leq\frac{n-1}{2}$ we have that

$$\mathcal{F}(f)(\lambda,(2l-n)|\lambda|)=\frac{(-1)^{n-1}}{2}\sum_{r=1}^{\frac{n-1}{2}}\left(\sum_{j=0}^{n-2}\frac{c_{j,l-p}}{4^j}\ B_{r-1}(j)\right)\mathcal{F}(f)(\lambda,(2r-n)|\lambda|)$$

Let $1\leq r,l\leq\frac{n-1}{2}$. An easy computation shows that for $l<p$ we have

$$\sum_{j=0}^{n-2}c_{j,l-p}\frac{j^r}{4^j}=\begin{cases}0, & l-1>r,\\ \sum\limits_{s=l-1}^{r}(-1)^s\frac{s!}{2^s}a_{s,r}(L^0_{-l+n-1})^{(n-2-s)}(0), & l-1\leq r,\end{cases} \tag{5.3}$$

and

$$\sum_{j=0}^{n-2} c_{j,l-p}\frac{j^r}{4^j} = \begin{cases} 0, & l-1>r, \\ \sum\limits_{i=n-l-1}^{n-2}\left(\sum\limits_{s=1}^{r} i(i-1)..(i-s+1)2^{-s}a_{s,r}\right)(L_{l-1}^0)^{(n-2-i)}(0), & l-1\leq r, \end{cases} \tag{5.4}$$

for $l \geq p$, where $a_{r,r}=1$.

Therefore, if $r<l$ we have that $\sum\limits_{j=0}^{n-2}\frac{c_{j,l-p}}{4^j}B_{r-1}(j)=0$ then,

$$\mathcal{F}(f)(\lambda,(2l-n)|\lambda|) = \frac{(-1)^{n-2}}{2}\sum_{r=l}^{\frac{n-1}{2}}\left(\sum_{j=0}^{n-2}\frac{c_{j,l-p}}{4^j}B_{r-1}(j)\right)\mathcal{F}(f)(\lambda,(2r-n)|\lambda|)$$

hence for each $\lambda\in\mathbb{R}$ we have a homogenous linear system to solve whose associated matrix is upper triangular. By definition of the polynomial B_{l-1} and using (5.3) and (5.4) the diagonal elements of this matrix are given by:

$$a_{l,l} = \begin{cases} \frac{(-1)^{n-2}}{2}-1, & l<p, \\ \frac{1}{2}-1, & l\geq p, \end{cases}$$

so, the only solution is the trivial.

(ii) Let $\varphi\in\mathcal{S}(\mathbb{R}^2)$ such that $\varphi|_{\Sigma^-}=\mathcal{F}(f)|_{\Sigma^-}$ and let $\psi\in\mathcal{S}(\mathbb{R}^2)$ defined by $\psi(\lambda,s)=\varphi(\lambda,-s)$ then there exist $g\in\mathcal{S}(H_n)$ such that $\mathcal{F}(g)|_{\Sigma^+}=\psi|_{\Sigma^+}$. So, $\mathcal{F}(g)$ satisfies the hypothesis of (i) then $\mathcal{F}(g)|_{\Sigma^+}\equiv 0$. ■

Proposition 5.2. *Let $\varphi\in C^\infty(\mathbb{R}^2)$ and let $k_1,k_2,\ldots,k_n$ be integer numbers non zero. If $\varphi(\lambda,\pm k_i|\lambda|)=0$ for all $i=1,\ldots,n$ and $\lambda\in\mathbb{R}$ then, there exists $\psi\in C^\infty(\mathbb{R}^2)$ such that*

$$\varphi(\lambda,s)=\prod_{i=1}^{n}(s^2-k_i^2\lambda^2)\ \psi(\lambda,s),\ \forall(\lambda,s)\in\mathbb{R}^2.$$

Proof. We will do this proof by induction. In fact, let $k\in\mathbb{Z}\setminus\{0\}$ such that

$\varphi(\lambda, \pm k|\lambda|) = 0\ \forall \lambda \in \mathbb{R}$. Then $\varphi(\lambda, \pm k\lambda) = 0, \forall \lambda \in \mathbb{R}$ and we have that

$$\begin{aligned}
&\frac{\varphi(\lambda, s)}{s^2 - (k\lambda)^2} \\
&= \frac{1}{s + k\lambda}\ \frac{\varphi(\lambda, s) - \varphi(\lambda, k\lambda)}{s - k\lambda} \\
&= \frac{1}{s + k\lambda}\ \frac{1}{s - k\lambda} \int_{k\lambda}^{s} \frac{\partial \varphi}{\partial s}(\lambda, t)\ dt \\
&= \frac{1}{s + k\lambda} \int_0^1 \frac{\partial \varphi}{\partial s}(\lambda, k\lambda + t(s - k\lambda))\ dt \\
&= \frac{1}{s + k\lambda} \int_0^1 \frac{\partial \varphi}{\partial s}(\lambda, k\lambda + t(s - k\lambda)) - \frac{\partial \varphi}{\partial s}(\lambda, -s + t(s - k\lambda))\ dt \\
&+ \frac{1}{s + k\lambda} \int_0^1 \frac{\partial \varphi}{\partial s}(\lambda, -s + t(s - k\lambda))\ dt \\
&= \frac{1}{s + k\lambda} \int_0^1 \int_{-s+t(s-k\lambda)}^{k\lambda + t(s-k\lambda)} \frac{\partial^2 \varphi}{\partial s^2}(\lambda, u) du\ dt + \frac{1}{s + k\lambda} \int_0^1 \frac{\partial \varphi}{\partial s}(\lambda, -s + t(s - k\lambda))\ dt \\
&= \int_0^1 \int_0^1 \frac{\partial^2 \varphi}{\partial s^2}(\lambda, -s + t(s - k\lambda) + u(s + k\lambda))\ du\ dt + \frac{\varphi(\lambda, -k\lambda) - \varphi(\lambda, -s)}{s + k\lambda} \\
&= \int_0^1 \int_0^1 \frac{\partial^2 \varphi}{\partial s^2}(\lambda, -s + t(s - k\lambda) + u(s + k\lambda))\ du\ dt + \frac{\varphi(\lambda, k\lambda) - \varphi(\lambda, -s)}{s + k\lambda} \\
&= \int_0^1 \int_0^1 \frac{\partial^2 \varphi}{\partial s^2}(\lambda, -s + t(s - k\lambda) + u(s + k\lambda))\ du\ dt + \int_0^1 \frac{\partial \varphi}{\partial s}(\lambda, -s + t(s + k\lambda))\ dt \\
&= \psi(\lambda, s).
\end{aligned}$$

It is clear that $\psi \in C^\infty(\mathbb{R}^2)$ since $\varphi \in C^\infty(\mathbb{R}^2)$.

Let us suppose, by inductive hypothesis, that there exists $\psi_1 \in C^\infty(\mathbb{R}^2)$ such that

$$\varphi(\lambda, s) = \prod_{i=1}^{j} (s^2 - k_i^2 \lambda^2)\ \psi_1(\lambda, s).$$

Then $\psi_1(\lambda, \pm k_{j+1}|\lambda|) = 0$ for all $\lambda \in \mathbb{R}$. By the first part of this proof, there exists $\psi \in C^\infty(\mathbb{R}^2)$ such that

$$\psi_1(\lambda, s) = (s^2 - k_{j+1}^2 \lambda^2) \psi(\lambda, s)$$

Thus,

$$\varphi(\lambda, s) = \prod_{i=1}^{j+1} (s^2 - k_i^2 \lambda^2)\ \psi(\lambda, s).$$

■

Corollary 5.3. *Let $\varphi \in C^\infty(\mathbb{R}^2)$ such that $\varphi(\lambda, (2k+p-q)|\lambda|) = 0\ \forall \lambda \in \mathbb{R}$ and for all $k \in \{-p+1, \dots, q-1\}$ and $2k+p-q \neq 0$. Then, there exists $\psi \in C^\infty(\mathbb{R}^2)$ such that*

$$\varphi(\lambda, s) = \prod_{\substack{k=-p+1 \\ 2k+p-q \neq 0}}^{q-1} (s - (2k + p - q)|\lambda|)\ \psi(\lambda, s), \quad \forall (\lambda, s) \in \mathbb{R}^2.$$

Proof. (*of Theorem 1.13*) By the differentiability properties of φ it is not difficult to show that there are $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\varphi(\lambda, s) = \varphi_1(\lambda, s) + s^{k+1}\varphi_2(\lambda, s)\ H(s), \quad \forall(\lambda, s) \in \mathbb{R}^2.$$

As $\varphi|_\Sigma$ and $\varphi_1|_\Sigma$ lies in the image of the normalized spherical transform then, the restriction to Σ of the maps defined on $\mathbb{R}^2$ by

$$(\lambda, s) \mapsto s^{k+1}\varphi_2(\lambda, s)\ H(s)$$

and

$$(\lambda, s) \mapsto s^{k+1}\varphi_2(\lambda, s)\ (1 - H(s))$$

lies in the image of the normalized spherical transform. Moreover, by Proposition 5.1 we get

$$((2k + p - q)|\lambda|)^{k+1}\varphi_2(\lambda, (2k + p - q)|\lambda|) = 0$$

for all $-p + 1 \le k \le q - 1$ and $\lambda \in \mathbb{R} \smallsetminus \{0\}$. Then, by the previous Corollary there exists $\psi \in C^\infty(\mathbb{R}^2)$ such that

$$\varphi_2(\lambda, s) = \prod_{\substack{k=-p+1 \\ 2k+p-q \neq 0}}^{q-1} (s - (2k + p - q)|\lambda|)\ \psi(\lambda, s), \ \forall(\lambda, s) \in \mathbb{R}^2.$$

Finally,

$$\varphi(\lambda, s) = \varphi_1(\lambda, s) + s^{k+1} \prod_{\substack{k=-p+1 \\ 2k+p-q \neq 0}}^{q-1} (s - (2k + p - q)|\lambda|)\psi(\lambda, s)\ H(s)$$

for $(\lambda, s) \in \mathbb{R}^2$, hence

$$\mathcal{F}(f)(0, s) = \varphi(0, s) = \begin{cases} \varphi_1(0, s) + s^{k+n-1}\ \psi(0, s)\ H(s), & \text{if } n \text{ is even}, \\ \varphi_1(0, s) + s^{k+n}\ \psi(0, s)\ H(s), & \text{if } n \text{ is odd}. \end{cases}$$

Thus, $s \mapsto \mathcal{F}(f)(0, s)$ is $k + n - 1$ times differentiable at the origin if n is odd and is $k + n - 2$ times differentiable at the origin if n is even. ■

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