Journal of Algebra 397 (2014) 379–406

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Finite-dimensional pointed or copointed Hopf algebras over affine racks ☆

Agustín García Iglesias *, Cristian Vay *

FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, Argentina

ARTICLE INFO

Article history:
Received 27 November 2012
Available online 2 October 2013
Communicated by Michel Broué

MSC:
16T05

Keywords:
Pointed Hopf algebras
Copointed Hopf algebras
Affine racks
Nichols algebras

ABSTRACT

We study the pointed or copointed liftings of Nichols algebras associated to affine racks and constant cocycles for any finite group admitting a principal YD-realization of these racks. In the copointed case we complete the classification for the six affine racks whose Nichols algebra is known to be of finite dimension. In the pointed case our method allows us to finish four of them. In all of the cases the Hopf algebras obtained turn out to be cocycle deformations of their associated graded Hopf algebras. All of them are new examples of finite-dimensional copointed or pointed Hopf algebras over non-abelian groups.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic zero and let H be a semisimple Hopf algebra over $\Bbbk$. This work is in the framework of the classification of finite-dimensional Hopf algebras whose coradical is a Hopf subalgebra isomorphic to H. Let $\mathfrak{F}_H$ be the family of such Hopf algebras. This problem has two interrelated sub-problems:

- To determine all $V \in {}^H_H\mathcal{YD}$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional and give a presentation of $\mathfrak{B}(V)$.
- To classify the lifting Hopf algebras of $\mathfrak{B}(V)$ over H.

☆ This work was partially supported by ANPCyT-FONCyT, CONICET, MinCyT (Córdoba) and Secyt (UNC). Part of the work of A.G.I. was done as a fellow of the Erasmus Mundus programme of the EU in the Università degli Studi di Padova.

* Corresponding author.
E-mail addresses: aigarcia@famaf.unc.edu.ar (A. García Iglesias), vay@famaf.unc.edu.ar (C. Vay).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.09.009

If $A \in \mathfrak{F}_H$ is generated in degree one, then A is a lifting of a Nichols algebra over its coradical. It was conjectured that this holds when H is a group algebra [9]. These steps compose the Lifting Method of [11].

First defined by Nichols, and also called quantum symmetric algebras, Nichols algebras are determined by a profound combinatorial behavior which is no yet fully understood. They are not Hopf algebras in the usual sense, but rather Hopf algebras in the category of Yetter–Drinfeld modules ${}^H_H\mathcal{YD}$.

Let G be a finite group. If G is abelian, all $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with $\dim \mathfrak{B}(V) < \infty$ have been determined in [31] and the presentation of $\mathfrak{B}(V)$ together with a positive answer to the conjecture in [9] were given in [14,15]. If G is non-abelian, it has been shown that for many (simple) groups most $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ yield Nichols algebras of infinite dimension [2,3]. Furthermore, only a few examples of finite-dimensional Nichols algebras are known, see below. Up to date, it is very complicated to find the relations defining the Nichols algebras and to compute their dimension, even using the computer, see [27]. Notice that Nichols algebras in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ and ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ coincide since these categories are braided equivalent, see Section 3.

Recall that the Hopf algebras in $\mathfrak{F}_{\Bbbk G}$ are called *pointed*, while those in $\mathfrak{F}_{\Bbbk^G}$ are called *copointed*, cf. [12].

The most prominent result in the classification of Hopf algebras is in [11] where the pointed Hopf algebras over an abelian group of order coprime with 210 are classified. The classification of nontrivial, *i.e.* different from group algebras, pointed Hopf algebras over non-abelian group is known for: $\mathbb{S}_3$ [7], $\mathbb{S}_4$ [25] and $\mathbb{D}_{4t}$ [18]. Also they have been classified the cases $\mathbb{A}_n$, $n \geqslant 5$, and most simple sporadic groups but all turn out to be group algebras [2,3,19]. In the copointed case the classification is known only for $\mathbb{S}_3$ [12]. The Hopf algebras obtained in the above results are all liftings of Nichols algebras over their coradical and cocycle deformations of each other [37,18,13,22,21].

Also, in [16] the liftings of the quantum line over four families of nontrivial semisimple Hopf algebras are classified, and in [8] another approach for the lifting problem is proposed.

Nichols algebras of finite dimension over non-abelian groups appear associated to racks and 2-cocycles, see [5]. It is worth mentioning that racks appear also in the calculus of knot invariants [29]. Next, we list all pairs of non-abelian indecomposable racks and cocycles whose associated Nichols algebras are known to be finite-dimensional, see for instance [27].

(1) Racks of the conjugacy classes $\mathcal{O}^n_m$ of m-cycles in $\mathbb{S}_n$:

- The rack $\mathcal{O}^n_2$ and constant 2-cocycle -1, $n = 3, 4, 5$.
- The rack $\mathcal{O}^n_2$ and a non-constant 2-cocycle χ, $n = 4, 5$.
- The rack $\mathcal{O}^4_4$ and constant 2-cocycle -1.

Their Nichols algebras were studied in [38,20,5,25]. In [40] it is shown that the Nichols algebras associated to $\mathcal{O}^n_2$ with constant and non-constant 2-cocycle are twist equivalent. All of these racks can be realized over the symmetric groups and their duals. The families $\mathfrak{F}_{\mathbb{S}_3}$ and $\mathfrak{F}_{\mathbb{S}_4}$ were classified in [7,25] respectively, and $\mathfrak{F}_{\Bbbk^{\mathbb{S}_3}}$ in [12].

(2) The affine racks:

- $(\mathbb{F}_3, 2)$, $(\mathbb{F}_4, \omega)$, $(\mathbb{F}_5, 2)$, $(\mathbb{F}_5, 3)$, $(\mathbb{F}_7, 3)$ and $(\mathbb{F}_7, 5)$ with constant 2-cocycle -1 [38,26,5].[1]
- $(\mathbb{F}_4, \omega)$ and a non-constant 2-cocycle ζ [33] with ζ_{ii} a third root of 1 for $i \in \mathbb{F}_4$.

The aim of this work is to study both the pointed and copointed lifting of the Nichols algebras associated to these affine racks $(\mathbb{F}_b, N)$ with constant 2-cocycle -1. In this case no liftings are known, apart from the case $(\mathbb{F}_3, 2)$, see [6, Theorem 3.8]. In [1] a general strategy to classify the family $\mathfrak{F}_H$ is developed showing at the same time that they are cocycle deformations of the bosonization $\mathfrak{B}(V) \# H$. We adapt the ideas there to compute the pointed, and copointed, liftings of these Nichols algebras over any group G. We also give results which apply to other racks.

The classification in the pointed case is given by the next theorem.

[1] As racks $(\mathbb{F}_3, 2) \simeq \mathcal{O}^3_2$, $(\mathbb{F}_5, 2)^* \simeq (\mathbb{F}_5, 3)$ and $(\mathbb{F}_7, 3)^* \simeq (\mathbb{F}_7, 5)$.

Main Theorem 1. *Let G be a finite group. The pointed Hopf algebras over $\Bbbk G$ whose infinitesimal braiding arises from a principal YD-realization of an affine rack X with the constant 2-cocycle $q \equiv -1$ are classified in*

(i) *Theorem* 6.2*, if* $X = (\mathbb{F}_3, 2)$.
(ii) *Theorem* 6.3*, if* $X = (\mathbb{F}_4, \omega)$.
(iii) *Theorem* 6.4*, if* $X = (\mathbb{F}_5, 2)$.
(iv) *Theorem* 6.5*, if* $X = (\mathbb{F}_5, 3)$.

All of these liftings are cocycle deformations of $\mathfrak{B}(X, -1) \# \Bbbk G$.

The first item is already in [6, Theorem 3.8], without the statement about cocycle deformations. It is important to remark that in some of these new examples, some of the relations are not only deformed by elements in the coradical, but also by elements in higher terms of the coradical filtration. This phenomenon is quite new, and was only present previously in some deformations in [32] for the abelian case.

The classification in the copointed case is given by the next theorem.

Main Theorem 2. *Let G be a finite group. The copointed Hopf algebras over $\Bbbk^G$ whose infinitesimal braiding arises from a principal YD-realization of an affine rack X with the constant 2-cocycle $q \equiv -1$ are classified in*

(i) *Theorem* 7.3*, if* $X = (\mathbb{F}_3, 2)$.
(ii) *Theorem* 7.5*, if* $X = (\mathbb{F}_4, \omega), (\mathbb{F}_5, 2), (\mathbb{F}_5, 3), (\mathbb{F}_7, 3)$ *or* $(\mathbb{F}_7, 5)$.

All of these liftings are cocycle deformations of $\mathfrak{B}(X, -1) \# \Bbbk^G$.

We explicitly define biGalois objects to prove the last assertion. These liftings are new examples of Hopf algebras.

The Hopf algebras found are presented as quotients of bosonizations of tensor algebras. Hence the greatest obstacle to achieve our principal results is to show that these quotients have the right dimension, or just to show that they are nonzero. The same issue is present in the rest of the works cited above. We are able to avoid this obstacle by showing that the quotient is a cocycle deformation, as proposed in [1]. However, some very complicated computations are necessary at an intermediate step and we are forced to appeal to computer program [23]. However, we find that the computer is not always enough and some examples cannot be attacked with this method. The same computational impediment is present in the calculation of Nichols algebras themselves. Hence, new tools are required to attack these problems, such as representation theory, see for instance [13,25,18].

The paper is organized as follows: In Section 2 we give some conventions and notations. In Section 3 we give the correspondence between Nichols algebras in braided equivalent categories of Yetter–Drinfeld modules. We recall the notions of rack and Yetter–Drinfeld realization of a rack over a group. In Section 4, we introduce the known examples of finite-dimensional Nichols algebras attached to an affine rack and give some properties of these which will be useful for us. In Section 5 we go through the ideas in [1] and adapt them to prove new results that apply in our setting. In Sections 6 and 7 we use these results to prove our main theorems. We also include Appendix A with the ideas behind some of the computations.

2. Preliminaries

We work over an algebraically closed field $\Bbbk$ of characteristic zero; $\Bbbk^* := \Bbbk \setminus \{0\}$. If X is a set, then $\Bbbk X$ denotes the free vector space over X. If A is an algebra and $X \subset A$, then $\langle X \rangle$ is the two-sided ideal generated by X.

Let G be a finite group. We denote by e the identity element of G, by $\Bbbk G$ its group algebra and by $\Bbbk^G$ the function algebra on G. The usual basis of $\Bbbk G$ is $\{g\colon\ g \in G\}$ and $\{\delta_g\colon\ g \in G\}$ is its dual basis in $\Bbbk^G$, *i.e.* $\delta_g(h) = \delta_{g,h}$ for all $g, h \in G$. If M is a $\Bbbk^G$-module and $g \in G$, the isotypic component of weight g is $M[g] = \delta_g \cdot M$. We write $\operatorname{supp} M = \{g \in G : M[g] \neq 0\}$ and $M^\times = \bigoplus_{g \neq e} M[g]$. The symmetric group in n letters is denoted by $\mathbb{S}_n$ and $\operatorname{sgn} : \mathbb{S}_n \to \mathbb{Z}_2$ denotes the morphism given by the sign.

Let H be a Hopf algebra. Then Δ, ε, $\mathcal{S}$ denote respectively the comultiplication, the counit and the antipode. We use Sweedler's notation for comultiplication and coaction but dropping the summation symbol. We denote by $\{H_{[i]}\}_{i \geqslant 0}$ the coradical filtration of H and by $\operatorname{gr} H = \bigoplus_{n \geqslant 0} \operatorname{gr}^n H = \bigoplus_{n \geqslant 0} H_{[n]}/H_{[n-1]}$ the associated graded Hopf algebra of H with $H_{[-1]} = 0$.

Assume $\mathcal{S}$ is bijective and let ${}^H_H\mathcal{YD}$ be the category of Yetter–Drinfeld modules over H. If $V \in {}^H_H\mathcal{YD}$, then the dual object $V^* \in {}^H_H\mathcal{YD}$ is defined by $\langle h \cdot f, v \rangle = \langle f, \mathcal{S}(h) \cdot v \rangle$ and $f_{(-1)} \langle f_{(0)}, v \rangle = \mathcal{S}^{-1}(v_{(-1)}) \langle f, v_{(0)} \rangle$ for all $v \in V$, $f \in V^*$ and $h \in H$, where $\langle\, ,\rangle$ denotes the standard evaluation.

2.1. *Galois objects*

Let H be a Hopf algebra with bijective antipode and A be a right H-comodule algebra with right H-coinvariants $A^{\operatorname{co} H} = \Bbbk$.

If there exist a convolution-invertible H-colinear map $\gamma : H \to A$, then A is called a right *cleft object*. The map γ can be chosen so that $\gamma(1) = 1$, in which case it is called a *section*. In turn, A is called a right *H-Galois object* if the following linear map is bijective:

$$\operatorname{can} : A \otimes A \mapsto A \otimes H, \quad a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}.$$

Analogously, left H-Galois objects are defined. Let L be another Hopf algebra. An (L, H)-bicomodule algebra is an (L, H)-biGalois object if it is simultaneously a left L-Galois object and a right H-Galois object.

Assume A is a right H-Galois object. There is an associated Hopf algebra $L(H, A)$ such that A is an $(L(A, H), H)$-biGalois object, see [39, Section 3]. $L(A, H)$ is a subalgebra of $A \otimes A^{\mathrm{op}}$. Moreover, if L is a Hopf algebra such that A is (L, H)-biGalois then $L \cong L(A, H)$. More precisely, if δ, δ_L stand for the coactions of $L(A, H)$ and L in A, there is a Hopf algebra isomorphism $F : L(A, H) \to L$ such that $\delta_L = (F \otimes \mathrm{id})\delta$ and

$$F\Big(\sum a_i \otimes b_i\Big) \otimes 1_A = \sum \lambda_L(a_i)(1 \otimes b_i), \quad \sum a_i \otimes b_i \in L(A, H). \tag{1}$$

Thus, one can use Galois objects to find new examples of Hopf algebras. Furthermore, $L(H, A)$ is a cocycle deformation of H [39, Theorem 3.9].

3. Nichols algebras and racks

From now on $\mathcal{C}$ denotes a category of (left, right or left–right) Yetter–Drinfeld modules over a finite-dimensional Hopf algebra H. Then $\mathcal{C}$ is a braided monoidal category. Let c be the canonical braiding of $\mathcal{C}$. See *e.g.* [35] for details about braided monoidal categories.

Let $V \in \mathcal{C}$. The tensor algebra $T(V)$ is an algebra in $\mathcal{C}$. Also, $T(V) \otimes T(V)$ is an algebra with multiplication $(m \otimes m) \circ (\mathrm{id} \otimes c \otimes \mathrm{id})$. Hence $T(V)$ becomes a Hopf algebra in $\mathcal{C}$ extending by the universal property the following maps

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \qquad \varepsilon(v) = 0 \quad \text{and} \quad \mathcal{S}(v) = -v, \quad v \in V.$$

Let $\mathcal{J}(V)$ be the largest Hopf ideal of $T(V)$ generated as an ideal by homogeneous elements of degree $\geqslant 2$.

Definition 3.1. (See [10, Proposition 2.2].) The *Nichols algebra* of V (in $\mathcal{C}$) is $\mathfrak{B}(V) = T(V)/\mathcal{J}(V)$.

See [10] for details about Nichols algebras. Let $n \in \mathbb{N}$; we denote by $\mathcal{J}^n(V)$, respectively $\mathfrak{B}^n(V)$, the homogeneous component of degree n of $\mathcal{J}(V)$, respectively of $\mathfrak{B}(V)$. We set $\mathcal{J}_n(V) = \langle \bigoplus_{l=2}^{n} \mathcal{J}^l(V) \rangle$ and $\widehat{\mathfrak{B}}_n(V) = T(V)/\mathcal{J}_n(V)$.

Let A be Hopf algebra such that $\operatorname{gr} A$ is isomorphic to $\mathfrak{B}(V) \# H$. Then A is called a *lifting of* $\mathfrak{B}(V)$ *over* H. The *infinitesimal braiding of* A is $V \in {}^H_H\mathcal{YD}$ with the braiding of ${}^H_H\mathcal{YD}$. Recall from [12, Proposition 2.4] that there exists a *lifting map* $\phi : T(V) \# H \to A$, that is an epimorphism of Hopf algebras such that

$$\phi_{|H} = \mathrm{id}, \qquad \phi_{|V \# H} \text{ is injective and } \phi\big((\Bbbk \oplus V) \# H\big) = A_{[1]}. \tag{2}$$

We recall another characterization of $\mathcal{J}(V)$, see *e.g.* [4,10]. Fix $n \in \mathbb{N}$. Let $\mathbb{B}_n$ be the *Braid group*: It is generated by $\{\sigma_i\colon 1 \leqslant i < n\}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $1 \leqslant i, j < n$ such that $|i-j| > 1$. The projection $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$, $\sigma_i \mapsto (i\,i+1)$, $1 \leqslant i < n$, admits a set-theoretical section $s : \mathbb{S}_n \to \mathbb{B}_n$ defined by $s(i\,i+1) = \sigma_i$, $1 \leqslant i < n$, and $s(\tau) = \sigma_{i_1} \cdots \sigma_{i_\ell}$, if $\tau = (i_1 i_1 + 1) \cdots (i_\ell\, i_\ell + 1)$ with ℓ minimum; this is the *Matsumoto section*. The *quantum symmetrizer* is:

$$\mathbf{S}_n = \sum_{\tau \in \mathbb{S}_n} s(\tau) \in \Bbbk\mathbb{B}_n.$$

The group $\mathbb{B}_n$ acts on $V^{\otimes n}$ via the assignment $\sigma_i \mapsto c_{i,i+1}$, $1 \leqslant i < n$, where $c_{i,i+1} : V^{\otimes n} \to V^{\otimes n}$ is the morphism

$$\mathrm{id} \otimes c \otimes \mathrm{id} : V^{\otimes i-1} \otimes V^{\otimes 2} \otimes V^{\otimes n-i-1} \to V^{\otimes i-1} \otimes V^{\otimes 2} \otimes V^{\otimes n-i-1}.$$

Then the homogeneous components of $\mathcal{J}(V)$ are given by

$$\mathcal{J}^k(V) = \ker \mathbf{S}_k, \quad k \in \mathbb{N}.$$

3.1. *Correspondence between Nichols algebras in braided equivalent categories*

Let H, $\mathcal{C}$ be as above. Let H' be a finite-dimensional Hopf algebra, $\mathcal{C}'$ be a category of Yetter–Drinfeld modules over H'. Assume there is a functor $(F, \eta) : \mathcal{C} \to \mathcal{C}'$ of braided monoidal categories, *i.e.* $F : \mathcal{C} \to \mathcal{C}'$ is a functor and $\eta : \otimes \circ F^2 \to F \circ \otimes$ is a natural isomorphism such that the diagrams

$$\begin{array}{ccc} F(U) \otimes F(V) \otimes F(W) & \xrightarrow{\ \eta \otimes \mathrm{id}\ } & F(U \otimes V) \otimes F(W) \\ {\scriptstyle \mathrm{id} \otimes \eta}\downarrow & & \downarrow{\scriptstyle \eta} \\ F(U) \otimes F(V \otimes W) & \xrightarrow[\ \eta\ ]{} & F(U \otimes V \otimes W), \end{array} \tag{3}$$

$$\begin{array}{ccc} F(U) \otimes F(V) & \xrightarrow{\ c_{F(U),F(V)}\ } & F(V) \otimes F(U) \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta} \\ F(U \otimes V) & \xrightarrow[\ F(c_{U,V})\ ]{} & F(V \otimes U), \end{array} \tag{4}$$

commute for each $U, V, W \in \mathcal{C}$.

Fix $V \in \mathcal{C}$. For $m, n \in \mathbb{N}$, set $\eta_{m,n} = \eta_{V^{\otimes m}, V^{\otimes n}}$ and

$$\eta_n = \eta_{n-1,1}(\eta_{n-2,1} \otimes \mathrm{id}) \cdots (\eta_{2,1} \otimes \mathrm{id})(\eta \otimes \mathrm{id}) : F(V)^{\otimes n} \to F\big(V^{\otimes n}\big).$$

By abuse of notation, we still write $\eta = \eta_{1,1} = \eta_2$. By (3), it holds that

$$\eta_{m+n+k} = \eta_{m,n+k}\,(\mathrm{id} \otimes \eta_{n,k})(\eta_m \otimes \eta_n \otimes \eta_k), \quad m, n, k \in \mathbb{N}. \tag{5}$$

Note that $\mathbb{B}_n$ acts on $F(V^{\otimes n})$ via $\sigma_i \mapsto F(c_{i,i+1})$. Then the commutative diagram (4) implies that η is an isomorphism of $\mathbb{B}_2$-modules. Moreover, combining (3) and (4) with the fact that η is a natural isomorphism, we obtain that $\eta_n : F(V)^{\otimes n} \to F(V^{\otimes n})$ is an isomorphism of $\mathbb{B}_n$-modules in $\mathcal{C}'$. As a consequence we have the next lemma.

Lemma 3.2. *Assume $(F, \eta) : \mathcal{C} \to \mathcal{C}'$ is exact. Let $V \in \mathcal{C}$ with $\dim V < \infty$. The ideals defining the Nichols algebras $\mathfrak{B}(V)$ and $\mathfrak{B}(F(V))$ are related by*

$$\mathcal{J}^n\big(F(V)\big) = \eta_n^{-1} F\big(\mathcal{J}^n(V)\big) \quad \text{for all } n \in \mathbb{N}.$$

If F preserves dimensions, then $\dim \mathfrak{B}^n(V) = \dim \mathfrak{B}^n(F(V))$ for all $n \in \mathbb{N}$.

Proof. Recall that $\mathcal{J}^n(F(V))$ is the kernel of $\mathbf{S}_n$ acting on $F(V)^{\otimes n}$, $n \in \mathbb{N}$. Since F is exact and η_n is an isomorphism, the theorem follows. $\square$

We can apply the above result to the categories ${}^H_H\mathcal{YD}$ and ${}^{H^*}_{H^*}\mathcal{YD}$. In fact, by [4, Proposition 2.2.1] they are braided equivalent monoidal categories via the functor (F, η) defined as follows: $F(V) = V$ as a vector space,

$$\begin{aligned} &f \cdot v = \big\langle f, \mathcal{S}(v_{(-1)})\big\rangle v_{(0)}, \qquad \delta(v) = f_i \otimes \mathcal{S}^{-1}(h_i)v \quad \text{and} \\ &\eta : F(V) \otimes F(W) \mapsto F(V \otimes W), \qquad v \otimes w \mapsto w_{(-1)}v \otimes w_{(0)} \end{aligned} \tag{6}$$

for every $V, W \in {}^H_H\mathcal{YD}$, $f \in H^*$, $v \in V$, $w \in W$. Here $\{h_i\}$ and $\{f_i\}$ are dual bases of H and H^*.

Lemma 3.3. *Let $V \in {}^H_H\mathcal{YD}$ of finite dimension and $M \subset V^{\otimes n}$ in ${}^H_H\mathcal{YD}$. Let $N = \bigoplus_{m \in \mathbb{N}} N^m$ with $N^m \subset V^{\otimes m}$ in ${}^H_H\mathcal{YD}$, $m \in \mathbb{N}$. Then*

(a) $F(V)^{\otimes m} \otimes \eta_n^{-1} F(M) \otimes F(V)^{\otimes k} = (\eta_{m+n+k})^{-1} F(V^{\otimes m} \otimes M \otimes V^{\otimes k})$.
(b) $\langle \eta_n^{-1} F(M)\rangle = \sum_{m,k} (\eta_{m+n+k})^{-1} F(V^{\otimes m} \otimes M \otimes V^{\otimes k})$.
(c) *Let $\overline{M} \subset T(V)/\langle N\rangle$. In $T(F(V))/\langle \bigoplus_m \eta_m^{-1} F(N^m)\rangle$ it holds that $\overline{\eta_n^{-1} F(M)} = \eta_n^{-1} F(\overline{M})$.*

Proof. (a) Let $x \in V^{\otimes m}$, $r \in M$ and $y \in V^{\otimes k}$. By (5), there exist $x' \in V^{\otimes m}$, $r' \in M$ and $y' \in V^{\otimes k}$ such that $(\eta_{m+n+k})^{-1}(x \otimes r \otimes y) = \eta_m^{-1}(x') \otimes \eta_n^{-1}(r') \otimes \eta_k^{-1}(y')$. Also by (5), there exist $x'' \in V^{\otimes m}$, $r'' \in M$ and $y'' \in V^{\otimes k}$ such that $\eta_{m+n+k}(x \otimes r \otimes y) = x'' \otimes r'' \otimes y''$. Since $(\eta_{m+n+k})^{\pm 1}_{|V^{\otimes m} \otimes M \otimes V^{\otimes k}}$ are injective morphisms the statement follows. (b) and (c) are straightforward. $\square$

Lemma 3.3c is useful to find deformations of Nichols algebras. Next lemma is a consequence of Lemma 3.3(a).

Lemma 3.4. *Let $M = \bigoplus_{m \in \mathbb{N}} M^m$ with $M^m \subset V^{\otimes m}$ in ${}^H_H\mathcal{YD}$, $m \in \mathbb{N}$. Assume that M generates $\mathcal{J}(V)$ as an ideal. Then*

(a) $\bigoplus_{m\in\mathbb{N}} \eta_m^{-1} F(M^m) \in {}^{H^*}_{H^*}\mathcal{YD}$ *generates* $\mathcal{J}(F(V))$ *as an ideal.*
(b) $\mathcal{J}_k(F(V)) = \langle \bigoplus_{l=2}^{k} \eta_l^{-1} F(\mathcal{J}^l(V))\rangle$ *for all* $k \in \mathbb{N}$. □

3.2. Racks

A *rack* is a nonempty set X with an operation $\triangleright : X \times X \to X$ such that

$$\phi_i : X \mapsto X, \quad j \mapsto i \triangleright j,$$

is a bijective map and $\phi_i(j \triangleright k) = \phi_i(j) \triangleright \phi_i(k)$ for all $i, j, k \in X$. The subgroup of $\mathbb{S}_X$ generated by $\{\phi_i\}_{i\in X}$ is denoted $\mathrm{Inn}_\triangleright X$, it is a subgroup of the group of rack automorphisms $\mathrm{Aut}_\triangleright X$.

A function $q : X \times X \to \Bbbk^*$, $(i, j) \mapsto q_{ij}$, is a (*rack*) *2-cocycle* if $q_{i,j\triangleright k} q_{j,k} = q_{i\triangleright j, i\triangleright k} q_{i,k}$ for all $i, j, k \in X$. We refer to [5] for details about racks.

Definition 3.5. (See [6, Definition 3.2], [38, Subsection 5].) Let X be a rack and q be a 2-cocycle on X. A *principal YD-realization* of (X, q) over a finite group G is a collection $(\cdot, g, \{\chi_i\}_{i\in X})$ where

- $\cdot$ is an action of G on X;
- $g : X \mapsto G$, $i \mapsto g_i$, is a function such that $g_{h\cdot i} = h g_i h^{-1}$ and $g_i \cdot j = i \triangleright j$ for all $i, j \in X$ and $h \in G$;
- $\{\chi_i\}_{i\in X}$ is a *1-cocycle* – that is a family of maps $\chi_i : G \to \Bbbk^*$ such that $\chi_i(ht) = \chi_{t\cdot i}(h)\chi_i(t)$ for all $i \in X$, $h, t \in G$ – satisfying $\chi_i(g_j) = q_{ji}$ for all $i, j \in X$.

We will assume that all realizations are *faithful*, that is g is injective.

These data define an object $V(X, q) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ [6]. Namely, as a vector space $V(X, q) = \Bbbk\{x_i\}_{i\in X}$, the action and coaction are

$$t \cdot x_i = \chi_i(t) x_{t\cdot i} \quad \text{and} \quad \lambda(x_i) = g_i \otimes x_i, \quad t \in G,\ i \in X. \tag{7}$$

We denote by $T(X, q)$ the tensor algebra of $V(X, q)$, its Nichols algebra is denoted by $\mathfrak{B}(X, q)$ and the defining ideal is $\mathcal{J}(X, q)$.

Let $W(q, X)$ be the object in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ obtained by applying the functor (6) to the above Yetter–Drinfeld module $V(X, q)$ over $\Bbbk G$. Then

$$\delta_t \cdot x_i = \delta_{t, g_i^{-1}} x_i \quad \text{and} \quad \lambda(x_i) = \sum_{t\in G} \chi_i\big(t^{-1}\big)\delta_t \otimes x_{t^{-1}\cdot i}, \quad t \in G,\ i \in X. \tag{8}$$

We denote by $T(q, X)$ the tensor algebra of $W(q, X)$, its Nichols algebra is denoted by $\mathfrak{B}(q, X)$ and the defining ideal is $\mathcal{J}(q, X)$.

Note that the smash product Hopf algebra $T(X, q) \# \Bbbk G$ satisfies

$$t x_i = \chi_i(t) x_{t\cdot i} t \quad \text{and} \quad \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad t \in G,\ i \in X. \tag{9}$$

The smash product Hopf algebra $T(q, X) \# \Bbbk^G$ satisfies for all $t \in G$, $i \in X$

$$\delta_t x_i = x_i \delta_{g_i t} \quad \text{and} \quad \Delta(x_i) = x_i \otimes 1 + \sum_{t\in G} \chi_i\big(t^{-1}\big)\delta_t \otimes x_{t^{-1}\cdot i}. \tag{10}$$

To find all the groups G supporting a principal YD-realization of (X,q) presents hard computational aspects [6, Section 3], see *e.g.* Lemma 3.6(c) below. A possible approach is the following. Let $F(X)$ be the free group generated by $\{g_i\}_{i\in X}$. The *enveloping group* G_X *of* X, see [17,34], is

$$G_X = F(X)/\langle g_i g_j - g_{i\rhd j} g_i\colon i,j\in X\rangle. \tag{11}$$

If X is finite and indecomposable, then the order n of ϕ_i does not depend on $i\in X$ and is called the *degree* of the rack, see [33, Definition 2.18], also [28]. Thus, there is a series of finite versions of G_X, given by

$$G_X^k = G_X/\langle g_i^{kn}, i\in X\rangle, \quad k\in\mathbb{N}.$$

G_X^1 is denoted by $\overline{G_X}$ and called the finite enveloping group of X in [33].

Lemma 3.6. *Let X be a faithful and indecomposable rack of degree r with a 2-cocycle q. Let $(\cdot, g, \{\chi_i\}_{i\in X})$ be a principal YD-realization of (X,q) over a finite group G and $K\subset G$ be the subgroup generated by $\{g_i\colon i\in X\}$. Then*

(a) *K is normal and a quotient of G_X^r.*
(b) *(See [6, Lemma 3.3(c)].) G acts by rack automorphisms on X.*
(c) *(See [6, Lemma 3.3(d)].) If q is constant, then there exists a multiplicative character $\chi_G: G\to\Bbbk^*$ such that $\chi_G=\chi_i$ for all $i\in X$.*

Proof. (a) Clearly K is normal. As X is faithful, the map $g: X\to G$ is injective and thus we have an epimorphism $F(X)\to K$. Since the relations defining G_X^r are satisfied in K, the epimorphism factorizes through G_X^r. □

Lemma 3.7. *Let (X,q), $(\cdot, g, \{\chi_i\}_{i\in X})$ and K be as in the above lemma.*

(a) *If $i\rhd j\neq j$, then $g_i^\ell\neq g_j$ for all $\ell\in\mathbb{Z}$. In particular, $g_ig_j\neq e$.*
(b) *Let $i,j\in X$ and $\ell\in\mathbb{Z}$ be such that $\phi_i^\ell(j)\neq j$. Then $g_i^\ell\neq e$.*

Assume $q\equiv\xi$ is constant, for an nth root of unity ξ.

(c) *If $n_1+\cdots+n_a\not\equiv m_1+\cdots+m_b$ mod n, then $g_{i_1}^{n_1}\cdots g_{i_a}^{n_a}\neq g_{j_1}^{n_1}\cdots g_{j_b}^{n_b}$.*
(d) $(\chi_{G|K})^n=\varepsilon$.

Proof. (a)–(b) We show that if the equality holds then $g_i=g_j$. Notice that $g_ig_j=g_{i\rhd j}g_i$ for all $i,j\in X$. If $g_j=g_i^\ell$, then $g_j=g_i(g_i^\ell)g_i^{-1}=g_ig_jg_i^{-1}=g_{i\rhd j}$ but $j\neq i\rhd j$. In particular, $g_i^{-1}\neq g_j$ and hence $e\neq g_ig_j$. If $e=g_i^\ell$, then $g_j=g_i^\ell g_j=g_{\phi_i^\ell(j)}g_i^\ell=g_{\phi_i^\ell(j)}$ but $j\neq\phi_i^\ell(j)$. (c) Apply the multiplicative character χ_G. (d) is immediate. □

3.3. The dual rack

Fix a finite rack $(X,\rhd)$. The *dual rack* X^* is the pair $(X,\rhd^{-1})$ where

$$i\rhd^{-1} j=\phi_i^{-1}(j)\quad\text{for all } i,j\in X.$$

Fix a 2-cocycle q on X and a principal YD-realization $(\cdot, g, \{\chi_i\}_{i\in X})$ of (X,q) over a finite group G. Let $q^*: X\times X\to\Bbbk^*$ be the 2-cocycle on X^* given by

$$q^*_{i,j}=q_{i,i\rhd^{-1}j}\quad\text{for all } i,j\in X.$$

Then the dual object to $V(X,q)$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ (resp. $W(q,X)$ in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$) is isomorphic to the object $V(X^*,q^*)$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ (resp. $W(q^*,X^*)$ in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$) attached to the principal YD-realization $(\cdot, g^{-1}, \{\chi_i^{-1}\}_{i\in X})$ over G, see for example [25, Eq. (1)].

We set $q^{-*} := (q^*)^{-1}$. It is easy to see that q^{-*} is a 2-cocycle on X^* and that $(\cdot, g^{-1}, \{\chi_i\}_{i\in X})$ is a principal YD-realization of (X^*, q^{-*}) over G.

Let $V(X,q), V(X^*,q^{-*}) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be defined by (7) for $(\cdot, g, \{\chi_i\}_{i\in X})$ and $(\cdot, g^{-1}, \{\chi_i\}_{i\in X})$, respectively. We denote by $\{y_i\}_{i\in X}$ the basis of $V(X^*,q^{-*})$. We define the linear map $\mathfrak{c}: T(X,q) \to T(X^*,q^{-*})$ as follows: $\mathfrak{c}(1) = 1$,

$$\mathfrak{c}(x_i) = y_i \qquad \text{if } i \in X \quad \text{and}$$
$$\mathfrak{c}(mr) = \mathfrak{c}(m_{(0)})\big(m_{(-1)} \cdot \mathfrak{c}(r)\big) \qquad \text{if } m, r \in T(X,q).$$

It is easy to see that $\mathfrak{c}$ is well defined.

Proposition 3.8. *Let S be a set of generators of the defining ideal $\mathcal{J}(X,q)$ of $\mathfrak{B}(X,q) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. Then the defining ideal of $\mathfrak{B}(X^*,q^{-*}) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ satisfies $\mathcal{J}(X^*,q^{-*}) = \mathfrak{c}(\mathcal{J}(X,q))$ and it is generated by $\mathfrak{c}(S)$.*

Proof. We consider the co-opposite Hopf algebra $(\mathfrak{B}(X,q)\#\Bbbk G)^{\mathrm{cop}}$. As $\Bbbk G$ is cocommutative, $(\mathfrak{B}(X,q)\#\Bbbk G)^{\mathrm{cop}} \simeq R\#\Bbbk G$ for some graded braided Hopf algebra $R \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. Moreover, R is the Nichols algebra of $\mathcal{P}(R) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ because $(\mathfrak{B}(X,q)\#\Bbbk G)^{\mathrm{cop}}$ is generated as an algebra by the first term of its coradical filtration which is $(\mathfrak{B}(X,q)\#\Bbbk G)_{[1]}$.

Now, $\mathcal{P}(R) = \Bbbk\{x_i \# g_i^{-1}\}_{i\in X}$ with coaction $\lambda(x_i \# g_i^{-1}) = g_i^{-1} \otimes x_i \# g_i^{-1}$ and action $g \cdot (x_i \# g_i^{-1}) = g x_i \# g_i^{-1} g^{-1} = \chi_i(g) x_{g\cdot i} \# g_{g\cdot i}^{-1}$ for all $i \in X$, $g \in G$. Then $\mathcal{P}(R) \simeq V(X^*,q^{-*})$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ via the assignment $x_i \# g_i^{-1} \mapsto y_i$ for all $i \in X$. Therefore

$$\vartheta : \big(\mathfrak{B}(X,q)\#\Bbbk G\big)^{\mathrm{cop}} \to \mathfrak{B}\big(X^*,q^{-*}\big)\#\Bbbk G, \quad x_i \# g \mapsto y_i \# g_i g,\ i \in X,\ g \in G$$

is a Hopf algebra isomorphism. Let $m \in \mathcal{J}(X,q)$ be such that $m_{(-1)} \otimes m_{(0)} = g_m \otimes m$. Then $0 = \vartheta(m) = \mathfrak{c}(m) \# g_m$ and hence $\mathfrak{c}(m) \in \mathcal{J}(X^*,q^{-*})$. This shows that $\mathfrak{c}(\mathcal{J}(X,q)) \subseteq \mathcal{J}(X^*,q^{-*})$ and the other inclusion is proved in a similar way. The definition of $\mathfrak{c}$ implies the last statement. □

Now, we consider $W(q,X), W(q^{-*},X^*) \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ according to (8). Let $()^{\mathrm{op}} : T(q,X) \to T(q^{-*},X^*)^{\mathrm{op}}$ be the algebra map given by $x_i^{\mathrm{op}} = y_i$ for all $i \in X$, here $T(q^{-*},X^*)^{\mathrm{op}}$ is the opposite algebra of $T(q^{-*},X^*)$.

Proposition 3.9. *Let S be a set of generators of the defining ideal $\mathcal{J}(q,X)$ of $\mathfrak{B}(q,X) \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$. Then the defining ideal of $\mathfrak{B}(q^{-*},X^*) \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ satisfies $\mathcal{J}(q^{-*},X^*) = (\mathcal{J}(q,X))^{\mathrm{op}}$ and is generated by S^{op}.*

Proof. We consider the opposite Hopf algebra $(\mathfrak{B}(q,X)\#\Bbbk^G)^{\mathrm{op}}$. As $\Bbbk^G$ is commutative, we can see that

$$\vartheta : \big(\mathfrak{B}(q,X)\#\Bbbk^G\big)^{\mathrm{op}} \to \mathfrak{B}\big(q^{-*},X^*\big)\#\Bbbk^G, \quad x_i \# \delta_g \mapsto y_i \# \delta_g,\ i \in X,\ g \in G.$$

is a Hopf algebra isomorphism. If $m \in \mathcal{J}(q,X)$, then $0 = \vartheta(m) = m^{\mathrm{op}}$ and hence $(\mathcal{J}(q,X))^{\mathrm{op}} \subseteq \mathcal{J}(q^{-*},X^*)$. The other inclusion is proved in a similar way and the definition of $()^{\mathrm{op}}$ implies the last statement. □

Proposition 3.10. *The following maps are bijective correspondences.*

$$\{\textit{Liftings of } \mathfrak{B}(X,q) \textit{ over } \Bbbk G\} \mapsto \{\textit{Liftings of } \mathfrak{B}(X^*,q^{-*}) \textit{ over } \Bbbk G\}$$
$$A \mapsto A^{\mathrm{cop}},$$
$$\{\textit{Liftings of } \mathfrak{B}(q,X) \textit{ over } \Bbbk^G\} \mapsto \{\textit{Liftings of } \mathfrak{B}(q^{-*},X^*) \textit{ over } \Bbbk^G\}$$
$$A \mapsto A^{\mathrm{op}}.$$

Proof. We only prove the pointed case. The copointed case is similar.

Let A be a lifting of $\mathfrak{B}(X,q)$ over $\Bbbk G$. It is enough to prove that A^{cop} is a lifting of $\mathfrak{B}(X^*,q^{-*})$ over $\Bbbk G$ since $(A^{\mathrm{cop}})^{\mathrm{cop}} = A$. Clearly A^{cop} is generated as an algebra by $A_{[1]}$ and $\mathrm{gr}(A^{\mathrm{cop}}) = (\mathrm{gr}\, A)^{\mathrm{cop}}$. Then A^{cop} is a lifting of a Nichols algebra $\mathfrak{B}(V)$ for some $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. As in Proposition 3.8, we can see that $V \simeq V(X^*,q^{-*}) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. □

4. Nichols algebras attached to affine racks

Let A be an abelian group and $T \in \mathrm{Aut}\, A$. The *affine rack* $\mathrm{Aff}(A,T)$ is the set A with operation

$$a \triangleright b = T(b) + (\mathrm{id} - T)(a) \quad \text{for all } a, b \in A,$$

see [5]. The dual rack $\mathrm{Aff}(A,T)^*$ is the affine rack $\mathrm{Aff}(A,T^{-1})$.

We define a family of principal YD-realizations for $\mathrm{Aff}(A,T)$ and a constant 2-cocycle. Let C_n be the cyclic group of order $n \in \mathbb{N}$ generated by t. If $\mathrm{ord}\, T$ divides n, then $A \rtimes_T C_n$ is the semidirect product of A and C_n with respect to T where $t \cdot a = T(a)$ for $a \in A$. Let ξ be a primitive root of 1 and $\ell = [\mathrm{ord}\, T, \mathrm{ord}(\xi)]$ be the minimum common multiple of $\mathrm{ord}\, T$ and $\mathrm{ord}\, \xi$.

Proposition 4.1. *Let $k, m \in \mathbb{N}$ with $0 \leqslant k < m$. Consider the affine rack $X = \mathrm{Aff}(A,T)$ with constant 2-cocycle ξ. Let*

- *$g: A \mapsto A \rtimes_T C_{m\ell}$ be the map $a \mapsto g_a = a \times t^{k\ell+1}$;*
- *$\cdot : (A \rtimes_T C_{m\ell}) \times A \to A$ be the assignment $h \cdot a = b$, if $hg_ah^{-1} = g_b$;*
- *$\chi_a : A \rtimes_T C_{m\ell} \to \Bbbk^*$ be the map $\chi_a(b \times t^s) = \xi^s$, for $a, b \in A$, $s \in \mathbb{N}$.*

Then $(g, \cdot, \{\chi_a\}_{a\in A})$ is a faithful Yetter–Drinfeld realization of (X,ξ) over $A \rtimes_T C_{m\ell}$.

A realization $(g, \cdot, \{\chi_a\}_{a\in A})$ as in Proposition 4.1 is called an (m,k)-*affine realization* of $(\mathrm{Aff}(A,T),\xi)$.

Proof. Clearly, g is injective. If $h = a \times t^s \in A \rtimes_T C_{m\ell}$ and $b \in A$, then $hg_bh^{-1} = ((\mathrm{id} - T)(a) + T^s(b)) \times t^{k\ell+1}$. Thus the action $\cdot$ is well defined since the image of g is a conjugacy class and $g_a \cdot b = a \triangleright b$. Also $\chi_a(g_b) = \xi$ and $\chi_a = \chi_b$ is a group morphism for all $a, b \in A$. Then $\{\chi_a\}$ is a 1-cocycle. □

We denote by $\mathbb{F}_b$ the finite field of b elements. The multiplication by $N \in \mathbb{F}_b^*$ is an automorphism which we also denote by N. Then $\mathrm{Aff}(\mathbb{F}_b,N)$ is faithful and indecomposable and satisfies

$$\mathrm{Inn}_\triangleright \mathrm{Aff}(\mathbb{F}_b,N) = \mathbb{F}_b \rtimes_N C_{\mathrm{ord}\, N} = \mathrm{Aut}_\triangleright \mathrm{Aff}(\mathbb{F}_b,N), \tag{12}$$

the first equality is easy; the second one is by [5, Corollary 1.25].

Let q be a 2-cocycle on $\mathrm{Aff}(\mathbb{F}_b,N)$ and let $(\cdot, g, \{\chi_i\}_{i\in X})$ be a principal YD-realization of $(\mathrm{Aff}(\mathbb{F}_b,N),q)$ over a finite group G. If q is constant, pick $i \in X$ and set $\chi_G = \chi_i$, cf. Lemma 3.6(c). From now on, we denote $V(b,N,q) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ the corresponding Yetter–Drinfeld module as in (7). Also, $T(b,N,q)$ and $\mathfrak{B}(b,N,q)$ denote respectively its tensor algebra and the Nichols algebra with ideal of relations $\mathcal{J}(b,N,q)$.

4.1. The Nichols algebras $\mathfrak{B}(b,N,q)\in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$

We list all the known finite-dimensional Nichols algebras attached to an affine rack $\mathrm{Aff}(\mathbb{F}_b,N)$ and a 2-cocycle q, see *e.g.* [27].

4.1.1. The Nichols algebra $\mathfrak{B}(3,2,-1)$

Its ideal $\mathcal{J}(3,2,-1)$ is generated by

$$x_i^2,\quad x_ix_j+x_{2j-i}x_i+x_jx_{2j-i},\quad i,j\in\mathbb{F}_3. \tag{13}$$

This Nichols algebra has dimension 12 and was computed in [38,20].

4.1.2. The Nichols algebra $\mathfrak{B}(4,\omega,-1)$

Let $\omega\in\mathbb{F}_4$ be such that $\omega^2+\omega+1=0$. The ideal $\mathcal{J}(4,\omega,-1)$ is generated by $z_{(4,\omega,-1)}:=(x_\omega x_1x_0)^2+(x_1x_0x_\omega)^2+(x_0x_\omega x_1)^2$ and

$$x_i^2,\quad x_ix_j+x_{(\omega+1)i+\omega j}x_i+x_jx_{(\omega+1)i+\omega j}\quad \forall i,j\in\mathbb{F}_4. \tag{14}$$

This Nichols algebra was computed in [26]; $\dim\mathfrak{B}(4,\omega,-1)=72$.

4.1.3. The Nichols algebra $\mathfrak{B}(5,2,-1)$

The ideal $\mathcal{J}(5,2,-1)$ is generated by $z_{(5,2,-1)}:=(x_1x_0)^2+(x_0x_1)^2$ and

$$x_i^2,\qquad x_ix_j+x_{-i+2j}x_i+x_{3i-2j}x_{-i+2j}+x_jx_{3i-2j}\quad \forall i,j\in\mathbb{F}_5. \tag{15}$$

This Nichols algebra was computed in [5]; $\dim\mathfrak{B}(5,2,-1)=1280$.

4.1.4. The Nichols algebra $\mathfrak{B}(5,3,-1)$

Since $\mathrm{Aff}(\mathbb{F}_5,3)$ is the dual rack of $\mathrm{Aff}(\mathbb{F}_5,2)$ and the 2-cocycle is -1 we can apply Proposition 3.8. Then the ideal $\mathcal{J}(5,3,-1)$ is generated by $z_{(5,3,-1)}:=(x_1x_0)^2+(x_0x_1)^2$ and

$$x_i^2,\qquad x_jx_i+x_ix_{-i+2j}+x_{-i+2j}x_{3i-2j}+x_{3i-2j}x_j\quad \forall i,j\in\mathbb{F}_5. \tag{16}$$

4.1.5. The Nichols algebra $\mathfrak{B}(7,3,-1)$

The ideal $\mathcal{J}(7,3,-1)$ is generated by $z_{(7,3,-1)}:=(x_2x_1x_0)^2+(x_1x_0x_2)^2+(x_0x_2x_1)^2$ and

$$x_i^2,\qquad x_ix_j+x_{-2i+3j}x_i+x_jx_{-2i+3j}\quad \forall i,j\in\mathbb{F}_7. \tag{17}$$

This Nichols algebra was computed in [27]; $\dim\mathfrak{B}(7,3,-1)=326592$.

4.1.6. The Nichols algebra $\mathfrak{B}(7,5,-1)$

As in 4.1.4 we apply Proposition 3.8 since $\mathrm{Aff}(\mathbb{F}_7^5)$ is the dual rack of $\mathrm{Aff}(\mathbb{F}_7,3)$. Then the ideal $\mathcal{J}(7,5,-1)$ is generated by $z_{(7,5,-1)}:=x_2x_4x_0x_5x_3x_0+x_1x_3x_4x_5x_3x_2+x_0x_3x_6x_0x_4x_1$ and

$$x_i^2,\qquad x_jx_i+x_ix_{-2i+3j}+x_{-2i+3j}x_j\quad \forall i,j\in\mathbb{F}_7. \tag{18}$$

4.1.7. The Nichols algebra $\mathfrak{B}(4,\omega,\zeta)$

Let $\xi\in\Bbbk$ be a root of unity of order 3. Then $\mathrm{Aff}(\mathbb{F}_4,\omega)$ admits the 2-cocycle

$$\zeta=(\zeta_{ij})_{i,j\in\mathbb{F}_4}=\begin{bmatrix}\xi & \xi & \xi & \xi\\ \xi & \xi & -\xi & -\xi\\ \xi & -\xi & \xi & -\xi\\ \xi & -\xi & -\xi & \xi\end{bmatrix}. \tag{19}$$

The Nichols algebra $\mathfrak{B}(4,\omega,\zeta)$, see [33, Proposition 7.9], has dimension 5184 and its ideal of relations is generated by x_0^3, x_1^3, x_ω^3, $x_{\omega^2}^3$,

$$\xi^2x_0x_1+\xi x_1x_\omega-x_\omega x_0,\qquad \xi^2x_0x_\omega+\xi x_\omega x_{\omega^2}-x_{\omega^2}x_0,$$
$$\xi^2x_1x_0-\xi x_0x_{\omega^2}-x_{\omega^2}x_1,\qquad \xi^2x_\omega x_1+\xi x_1x_{\omega^2}+x_{\omega^2}x_\omega,$$

plus an extra degree six relation

$$\begin{aligned}z_{(4,\omega,\zeta)}:={}&x_0^2x_1x_\omega x_1^2+x_0x_1x_\omega x_1^2x_0+x_1x_\omega x_1^2x_0^2+x_\omega x_1^2x_0^2x_1+x_1^2x_0^2x_1x_\omega\\&+x_1x_0^2x_1x_\omega x_1+x_1x_\omega x_1x_0^2x_\omega+x_\omega x_1x_0x_1x_0x_\omega+x_\omega x_1^2x_0x_\omega x_0.\end{aligned}$$

4.2. About the top degree relation $z_{(b,N,q)}$

In the rest of the section, the pair (X,q) denotes one of the followings

$$\big(\mathrm{Aff}(\mathbb{F}_3,2),-1\big),\quad \big(\mathrm{Aff}(\mathbb{F}_4,\omega),-1\big),\quad \big(\mathrm{Aff}(\mathbb{F}_5,2),-1\big),\quad \big(\mathrm{Aff}(\mathbb{F}_5,3),-1\big),$$
$$\big(\mathrm{Aff}(\mathbb{F}_7,3),-1\big),\quad \big(\mathrm{Aff}(\mathbb{F}_7,5),-1\big)\quad\text{or}\quad \big(\mathrm{Aff}(\mathbb{F}_4,\omega),\zeta\big).$$

We fix $n=2$ for the first six pairs and $n=3$ for the last one. We set $\pi_n:T(X,q)\twoheadrightarrow\widehat{\mathfrak{B}_n}(X,q)$ the natural projection.

Let $z=z_{(b,N,q)}$ be the top degree defining relation of $\mathfrak{B}(X,q)$. Since $\mathcal{J}(X,q)$ is generated by z and elements of degree $<\deg z$, $\Bbbk\pi_n(z)\in{}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ via a central $t_z\in G$ and a multiplicative character $\chi_z:G\to\Bbbk^*$, that is

$$\pi_n(z)_{(-1)}\otimes\pi_n(z)_{(0)}=t_z\otimes\pi_n(z)\quad\text{and}\quad g\cdot\pi_n(z)=\chi_z(g)\pi_n(z) \tag{20}$$

for all $g\in G$. Moreover, $\pi_n(z)$ is primitive in $\widehat{\mathfrak{B}_n}(X,q)$ and therefore

$$\Delta\big(\pi_n(z)\big)=\pi_n(z)\otimes 1+t_z\otimes\pi_n(z)\quad\text{in}\quad\widehat{\mathfrak{B}_n}(X,q)\#\Bbbk G. \tag{21}$$

Lemma 4.2. *For all $i\in X$, $\chi_z(g_i)=1$. If q is constant, then $\chi_z=\chi_G^{\deg z}$.*

Proof. By Lemma 3.6(b), G acts by rack automorphisms on $\mathrm{Aff}(\mathbb{F}_b,N)$. Let $\bar t$ be the automorphism defined by $t\in G$. Let $K\subset G$ be the subgroup generated by $\{g_i:\ i\in X\}$ and $\mathcal{Z}(K)$ be its center. By [5, Lemma 1.9 (2)] and (12), $K/\mathcal{Z}(K)=\mathrm{Inn}_\triangleright\mathrm{Aff}(\mathbb{F}_b,N)=\mathbb{F}_b\rtimes_N C_{\mathrm{ord}\,N}=\mathrm{Aut}_\triangleright\mathrm{Aff}(\mathbb{F}_b,N)$. Thus there is a multiplicative character $\lambda:\mathrm{Aut}_\triangleright\mathrm{Aff}(\mathbb{F}_b,N)\to\Bbbk^*$ such that

$$\chi_z(t)\pi_n(z)=t\cdot\pi_n(z)=\widetilde{\chi}_z(t)\lambda(\bar t)\pi_n(z),$$

where $\widetilde{\chi}_z$ is given by the 1-cocycle $\{\chi_i\}_{i\in X}$. If q is constant, then $\widetilde{\chi}_z = \chi_G^{\deg z}$. If q is not constant, then it is easy to check that $\widetilde{\chi}_{z|K} = \varepsilon$. Therefore, to finish we have to prove that $\lambda = \varepsilon$. Let $M = \Bbbk G \cdot z \subset T(X,q)$.

Case $\mathrm{Aff}(\mathbb{F}_4,\omega)$. Let $\mathcal{O} = \{(\omega\,1\,0), (1\,\omega\,\omega^2), (0\,1\,\omega^2), (0\,\omega^2\,\omega)\} \subset \mathbb{F}_4^3$ and

$$X_{(abc)} := (x_a x_b x_c)^2 + (x_b x_c x_a)^2 + (x_c x_a x_b)^2, \quad (a\,b\,c) \in \mathcal{O}.$$

Then $z = X_{(\omega 10)}$ and $M = \Bbbk\{X_\sigma\}_{\sigma\in\mathcal{O}}$. Let $Y = \sum_{\sigma\in\mathcal{O}} X_\sigma$ and $C = \Bbbk\{X_\sigma - X_\tau : \sigma, \tau \in \mathcal{O}\}$. Then $M = C \oplus \Bbbk Y$ is a sum of simple $\mathbb{F}_4 \rtimes_\omega C_3$-submodules. Thus $\pi_2(C) = 0$ and $\pi_2(M) \simeq \Bbbk Y$ as $\mathbb{F}_4 \rtimes_\omega C_3$-modules.

Case $\mathrm{Aff}(\mathbb{F}_5,2)$. Here $z = (x_1x_0)^2 + (x_0x_1)^2$. By (15), it holds that

$$\pi_2\big((x_0x_2)^2\big) = \pi_2\big(-x_0(x_3x_2 + x_1x_3 + x_0x_1)x_2\big) = \pi_2(-x_0x_1x_3x_2) \quad \text{and}$$
$$\pi_2\big((x_2x_0)^2\big) = \pi_2\big((x_1x_0)^2 + (x_0x_1)^2 + x_0x_1x_3x_2\big).$$

Hence $\pi_2((x_2x_0)^2 + (x_0x_2)^2) = \pi_2(z)$ and thus $(0\times t)\cdot\pi_2(z) = \pi_2(z)$. Since $(0\times t)(1\times 1) = (2\times 1)(0\times t)$ in $\mathbb{F}_5 \rtimes_2 C_4 = \langle (0\times t), (1\times 1)\rangle$, $\pi_2(M)$ is the trivial $\mathbb{F}_5 \rtimes_2 C_4$-module.

Case $\mathrm{Aff}(\mathbb{F}_5,3)$. As in 3.3, we denote by $\{y_i\}_{i\in\mathbb{F}_5}$ the basis of $V(5,3,-1)$ and recall that $\mathfrak{c}(z_{(5,2,-1)}) = z_{(5,3,-1)}$. Let $\vartheta : (\widehat{\mathfrak{B}_2}(5,2,-1)\#\Bbbk G)^{\mathrm{cop}} \to \widehat{\mathfrak{B}_2}(5,3,-1)\#\Bbbk G$ be the Hopf algebra map given by $\vartheta(x_i\# g) = y_i \# g_i g$ for all $i\in X$, $g\in G$. Then $\vartheta(\overline{g_i}\cdot\pi_2(z_{(5,2,-1)})) = \overline{g_i}\cdot\pi_2(z_{(5,3,-1)})\# t_{z_{(5,2,-1)}}$ since the action is induced by the adjoint action. Hence $\lambda = \varepsilon$ because before we proved that

$$\vartheta\big(\overline{g_i}\cdot\pi_2(z_{(5,2,-1)})\big) = \vartheta\big(\pi_2(z_{(5,2,-1)})\big) = \pi_2(z_{(5,3,-1)})\# t_{z_{(5,2,-1)}}.$$

Cases $\mathrm{Aff}(\mathbb{F}_7,3)$ and $(\mathrm{Aff}(\mathbb{F}_4,\omega),\zeta)$. In both cases, $(0\times t)\cdot\pi_2(z) = \pi_2(z)$, using [23,24]. Then we proceed as for $\mathrm{Aff}(\mathbb{F}_5,2)$.

Case $\mathrm{Aff}(\mathbb{F}_7,5)$ is similar to $\mathrm{Aff}(\mathbb{F}_5,3)$ since $\mathrm{Aff}(\mathbb{F}_7,5)^* \simeq \mathrm{Aff}(\mathbb{F}_7,3)$. □

In the following, $\widehat{\mathfrak{B}_n}(X,q)\#\Bbbk G$ is a right $\mathfrak{B}(X,q)\#\Bbbk G$-comodule via the natural projection.

Lemma 4.3. *It holds that $\pi_n(z)$ is central in $\widehat{\mathfrak{B}_n}(X,q)$ and the subalgebra of right $\mathfrak{B}(X,q)\#\Bbbk G$-coinvariants is the polynomial algebra $\Bbbk[\pi_n(z)t_z^{-1}]$.*

Proof. We check that $\pi_n(z)$ is central using [23] together with the package [24] in all the cases except for $\mathrm{Aff}(\mathbb{F}_5,3)$ and $\mathrm{Aff}(\mathbb{F}_7,2)$. For these we keep the notation of the previous proof and proceed as follows. If $i\in\mathbb{F}_b$,

$$\begin{aligned} 0 &= \vartheta\big(x_i\pi_2(z_{(b,N,-1)}) - \pi_2(z_{(b,N,-1)})x_i\big) \\ &= (y_i\# g_i)\big(\pi_2(z_{(b,N^{-1},-1)})\# t_{z_{(b,N,-1)}}\big) - \big(\pi_2(z_{(b,N^{-1},-1)})\# t_{z_{(b,N,-1)}}\big)(y_i\# g_i) \\ &= \big(y_i\pi_2(z_{(b,N^{-1},-1)}) - \pi_2(z_{(b,N^{-1},-1)})y_i\big)\# t_{z_{(b,N,-1)}} g_i \end{aligned}$$

here we use the above lemma and that $t_{z_{(b,N,-1)}}$ is central. Thus the first part of the lemma is proved. Then $\pi_2(z)t_z^{-1}$ generates a normal subalgebra which forms the coinvariants by [1, Remark 5.5]. It is a polynomial algebra by [1, Lemma 5.13]. □

4.3. The Nichols algebras $\mathfrak{B}(q,b,N) \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$

For each (X,q) as above, consider the object $W(q,X) \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ as in (8). From now on, $T(q,b,N)$ and $\mathfrak{B}(q,b,N)$ denote respectively its tensor algebra and the Nichols algebra with ideal of relations $\mathcal{J}(q,b,N)$. Let $\pi_n : T(q,X) \twoheadrightarrow \widehat{\mathfrak{B}_n}(q,X)$ be the natural projection.

Proposition 4.4.

(a) *The ideal* $\mathcal{J}(-1,3,2)$ *is generated by* (13).
(b) *The ideal* $\mathcal{J}(-1,4,\omega)$ *is generated by*

$$x_i^2, \quad x_jx_i + x_ix_{(\omega+1)i+\omega j} + x_{(\omega+1)i+\omega j}x_j, \quad i,j \in \mathbb{F}_4 \tag{14'}$$

and $z'_{(-1,4,\omega)} := (x_\omega x_{\omega^2}x_0)^2 + (x_1x_{\omega^2}x_\omega)^2 + (x_0x_{\omega^2}x_1)^2$.
(c) *The ideal* $\mathcal{J}(-1,5,2)$ *is generated by*

$$x_i^2, \quad x_jx_i + x_ix_{-i+2j} + x_{-i+2j}x_{3i-2j} + x_{3i-2j}x_j, \quad i,j \in \mathbb{F}_5 \tag{15'}$$

and $z'_{(-1,5,2)} := x_0x_2x_3x_1 + x_1x_4x_3x_0$.
(d) *The ideal* $\mathcal{J}(-1,5,3)$ *is generated by*

$$x_i^2, \quad x_ix_j + x_{-i+2j}x_i + x_{3i-2j}x_{-i+2j} + x_jx_{3i-2j}, \quad i,j \in \mathbb{F}_5 \tag{16'}$$

and $z'_{(-1,5,3)} := x_1x_3x_2x_0 + x_0x_3x_4x_1$.
(e) *The ideal* $\mathcal{J}(-1,7,3)$ *is generated by*

$$x_i^2, \quad x_jx_i + x_ix_{-2i+3j} + x_{-2i+3j}x_j, \quad i,j \in \mathbb{F}_7 \tag{17'}$$

and $z'_{(-1,7,3)} := x_2x_6x_4x_2x_5x_0 + x_1x_5x_2x_3x_6x_2 + x_0x_6x_4x_5x_6x_1$.
(f) *The ideal* $\mathcal{J}(-1,7,5)$ *is generated by*

$$x_i^2, \quad x_ix_j + x_{-2i+3j}x_i + x_jx_{-2i+3j}, \quad i,j \in \mathbb{F}_7 \tag{18'}$$

and $z'_{(-1,7,5)} := x_0x_5x_2x_4x_6x_2 + x_2x_6x_3x_2x_5x_1 + x_1x_6x_5x_4x_6x_0$.
(g) *The ideal* $\mathcal{J}(\zeta,4,\omega)$ *is generated by* $x_0^3, x_1^3, x_\omega^3, x_{\omega^2}^3$,

$$\begin{gathered}
\xi x_\omega x_1 - x_0x_\omega - \xi^2x_1x_0, \qquad \xi x_{\omega^2}x_\omega - x_0x_{\omega^2} - \xi^2x_\omega x_0, \\
\xi x_{\omega^2}x_0 - x_1x_{\omega^2} + \xi^2x_0x_1, \qquad \xi x_{\omega^2}x_1 + x_\omega x_{\omega^2} + \xi^2x_1x_\omega \quad \text{and} \\
z'_{(\zeta,4,\omega)} := x_0^2x_{\omega^2}x_0x_1^2 + x_0x_\omega x_0x_{\omega^2}^2x_0 + x_1x_0x_\omega^2x_0^2 + x_\omega x_{\omega^2}^2x_\omega^2x_1 - x_1^2x_\omega^2x_0x_\omega \\
+ x_1x_{\omega^2}^2x_\omega x_{\omega^2}x_1 + x_1x_0x_\omega x_{\omega^2}^2x_\omega - x_\omega x_{\omega^2}x_0x_1x_{\omega^2}x_\omega + x_\omega x_{\omega^2}^2x_\omega x_1x_0.
\end{gathered}$$

Proof. In (a), (b), (c), (e) and (g) we apply the functor (F,η) given by (6) and use Lemma 3.4. In (d), respectively (f), we apply Proposition 3.9 since it corresponds to the dual case of (c), respectively (e), and the 2-cocycle is -1. $\square$

Set $z' = z'_{(q,b,N)} = \eta^{-1}_{\deg z_{(b,N,q)}}(z_{(b,N,q)})$. Then $\Bbbk\pi_n(z') \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ as follows

$$\pi_n(z')_{(-1)} \otimes \pi_n(z')_{(0)} = \chi_z^{-1} \otimes \pi_n(z') \quad \text{and} \quad \delta_g \cdot \pi_n(z) = \delta_{g,t_z^{-1}}\pi_n(z) \tag{22}$$

for all $g \in G$ by Lemma 3.3(c) and Lemma 4.2. Also, $\pi_n(z')$ is primitive in $\widehat{\mathfrak{B}_n}(q, X)$ and therefore

$$\Delta(\pi_n(z')) = \pi_n(z') \otimes 1 + \chi_z^{-1} \otimes \pi_n(z') \quad \text{in } \widehat{\mathfrak{B}_n}(q, X) \# \Bbbk^G. \tag{23}$$

In the following, $\widehat{\mathfrak{B}_n}(q, X) \# \Bbbk^G$ is a right $\mathfrak{B}(q, X) \# \Bbbk^G$-comodule via the natural projection.

Lemma 4.5. *It holds that $\pi_n(z')$ is central in $\widehat{\mathfrak{B}_n}(q, X)$ and the subalgebra of right $\mathfrak{B}(q, X) \# \Bbbk^G$-coinvariants is the polynomial algebra $\Bbbk[\pi_n(z')\chi_z]$.*

Proof. If $i \in \mathbb{F}_b$, then $\pi_n \eta_7^{-1}(x_i z - z x_i) = 0$ by Lemma 4.3 and Lemma 3.4(b). By (5), $\eta_7^{-1}(x_i z - z x_i) = x_{t_z^{-1} \cdot i} z' - z' x_i = x_i z' - z' x_i$, here we use Lemma 4.2 and that t_z is central. Hence $\pi_n(z')$ is central in $\widehat{\mathfrak{B}_n}(q, X)$. The lemma follows using [1, Remark 5.5, Lemma 5.13] as in Lemma 4.3. □

5. Lifting via cocycle deformation

Let H be a semisimple Hopf algebra and $V \in {}^H_H\mathcal{YD}$, $\dim V < \infty$. Assume that the ideal $\mathcal{J}(V)$ defining the Nichols algebra $\mathfrak{B}(V)$ is finitely generated and let $\mathcal{G}$ be a minimal set of homogeneous generators of $\mathcal{J}(V)$. In [1] a strategy was developed to compute all the liftings of $\mathfrak{B}(V)$ over H as cocycle deformations of $\mathfrak{B}(V) \# H$. We briefly recall this strategy, see [1, Section 5] for details.

Set $\mathcal{T}(V) = T(V) \# H$ and $\mathcal{H} = \mathfrak{B}(V) \# H$. Let $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_N$ be an *adapted stratification* of $\mathcal{G}$ [1, 5.1]. Among other things, this ensures that

$$\mathfrak{B}_k = \mathfrak{B}_{k-1}/\langle \mathcal{G}_{k-1} \rangle, \quad 1 \leqslant k \leqslant N + 1,$$

are braided Hopf algebras in ${}^H_H\mathcal{YD}$ where $\mathfrak{B}_0 = T(V)$. Then we have a chain of subsequent quotients of Hopf algebras

$$\mathcal{T}(V) \twoheadrightarrow \mathfrak{B}_1 \# H \twoheadrightarrow \cdots \twoheadrightarrow \mathfrak{B}_N \# H \twoheadrightarrow \mathcal{H} = \mathfrak{B}_{N+1} \# H.$$

The strategy basically consists in the following two steps:

(1) To compute at each level a family of cleft objects of $\mathfrak{B}_k \# H$ as quotients of cleft objects of $\mathfrak{B}_{k-1} \# H$, following the results in [30].

To do this, we start with the trivial cleft object for $\mathcal{T}(V)$. In the final level, we have a set Λ of cleft objects of $\mathcal{H}$ and hence a list of cocycle deformations L, which arise as $L \simeq L(\mathcal{A}, \mathcal{H})$, for $\mathcal{A} \in \Lambda$ as in [39].

(2) To check that any lifting of $\mathfrak{B}(V)$ over H is obtained as one of these deformations.

In [39] a series of tools to deduce this was developed. In particular, it was studied in [1, Section 4] the shape of all the possible liftings. We refine the results there for copointed liftings in Subsection 5.2.

We use the Strategy to prove the main theorems. In that order, we carry out the Strategy in the next subsection under certain general conditions which are satisfied in our case.

5.1. Pointed Lifting of Nichols algebras with a single top degree relation

Let X be an indecomposable rack with a 2-cocycle q. Let G be a finite group and $(\cdot, g, \{\chi_i\}_{i \in X})$ be a principal YD-realization of (X, q). Let $V = \Bbbk\{x_i\}_{i \in X}$ be the corresponding Yetter–Drinfeld module over G, see (7). Assume that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional.

Let $n \in \mathbb{N}$ be such that $\operatorname{ord} q_{ii} = n \geqslant 2$. Then $x_i^n \in \mathcal{J}(V)$ for all $i \in X$.

Recall from [25] that the space of quadratic relations in $\mathcal{J}(V)$ is spanned by $\{b_C\}_{C\in\mathcal{R}'}$ where $\mathcal{R}'$ is a subset of the set $\mathcal{R}=X\times X/\sim$ of classes of the equivalence relation generated by $(i,j)\sim(i\triangleright j,i)$. More precisely, $C=\{(i_2,i_1),\dots,(i_{n(C)},i_1)\}\in\mathcal{R}'$ iff $\prod_{h=1}^{n(C)} q_{i_{h+1},i_h}=(-1)^{n(C)}$ and then

$$b_C:=\sum_{h=1}^{n(C)}\eta_h(C)x_{i_{h+1}}x_{i_h}, \tag{24}$$

where $\eta_1(C)=1$ and $\eta_h(C)=(-1)^{h+1}q_{i_2i_1}q_{i_3i_2}\dots q_{i_hi_{h-1}}$, $h\geqslant 2$.

Set $\mathcal{T}(V)=T(V)\#\Bbbk G$ and $\pi_n:T(V)\to\widehat{\mathfrak{B}_n}(V)$. We assume that there is a generator $z\in\mathcal{J}(V)$ with $\deg z>n$ such that

- $\Bbbk\pi_n(z)\in{}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, that is, there exist a central $t_z\in G$ and a multiplicative character $\chi_z:G\to\Bbbk^*$ such that (20) holds;
- $\pi_n(z)$ is primitive in $\widehat{\mathfrak{B}_n}(V)$ and hence (21) is satisfied in $\widehat{\mathfrak{B}_n}(V)\#\Bbbk G$;
- the following holds in $\widehat{\mathfrak{B}_n}(V)\#\Bbbk G$:

$$x_i\pi_n(z)=\pi_n(z)x_i,\quad i\in X\quad\text{and}\quad t_z\pi_n(z)=\pi_n(z)t_z. \tag{25}$$

We assume that the ideal $\mathcal{J}(V)$ admits an adapted stratification:

$$\mathcal{G}_0=\{x_i^n\colon i\in X\},\qquad \mathcal{G}_1=\{b_C\colon C\in\mathcal{R}'\}\setminus\{x_i^2\colon i\in X\},\qquad \mathcal{G}_2=\{z\} \tag{26}$$

and apply the Strategy in this setting. Set $\mathcal{H}_i=\mathfrak{B}_{i-1}/\langle\mathcal{G}_{i-1}\rangle\#\Bbbk G$ for $i=1,2,3$ with $\mathfrak{B}_0=T(V)$. We also assume that

$$g_i^n\neq g_j\quad\text{and}\quad g_k\neq g_ig_j,\qquad\text{for every } i,j,k\in X, \tag{27}$$

$$t_z\neq g_i,\qquad\text{for every } i\in X. \tag{28}$$

Notice that (27) is not a relevant restriction by Lemma 3.7. In particular, this lemma applies to affine racks.

We shall consider scalars $\lambda_1,\lambda_2,\lambda_3\in\Bbbk$ subject to the following conditions

$$\lambda_1=0\quad\text{if } \chi_i^n\neq\varepsilon\ \forall i,\qquad \lambda_2=0\quad\text{if } \chi_i\chi_j\neq\varepsilon\ \forall i,j,\qquad \lambda_3=0\quad\text{if } \chi_z\neq\varepsilon. \tag{29}$$

Let $\lambda_1,\lambda_2\in\Bbbk$ subject to (29) and let b_C be as in (24). Set

$$\mathcal{A}_1(\lambda_1)=\mathcal{T}(V)/\langle x_i^n-\lambda_1\colon i\in X\rangle, \tag{30}$$

$$\mathcal{A}_2(\lambda_1,\lambda_2)=\mathcal{A}_1(\lambda_1)/\langle b_C-\lambda_2\colon C\in\mathcal{R}'\rangle. \tag{31}$$

Note that $\mathcal{A}_1(\lambda_1)\neq 0$. In fact, $\mathcal{A}_1(\lambda_1)\simeq T(V)/\langle x_i^n-\lambda_1\colon i\in X\rangle\otimes\Bbbk G$ as vector spaces by the choice of λ_1 in (29) and we can define a nonzero algebra map $F:T(V)/\langle x_i^n-\lambda_1\colon i\in X\rangle\to\Bbbk$ by $F(x_i)=\lambda_1^{1/n}$ for all $i\in X$.

Set also $\mathcal{L}_1(\lambda_1)=\mathcal{T}(V)/\langle x_i^n-\lambda_1(1-g_i^n)\colon i\in X\rangle$ and

$$\mathcal{L}_2(\lambda_1,\lambda_2)=\mathcal{L}_1(\lambda_1)/\langle b_C-\lambda_2(1-g_ig_j)\colon C\in\mathcal{R}',\ (i,j)\in C\rangle.$$

It is straightforward to see that $\mathcal{L}_k$ is a Hopf algebra quotient of $\mathcal{T}(V)$ and $\mathcal{A}_k$ is naturally an $(\mathcal{L}_k,\mathcal{H}_k)$-bicomodule algebra with coactions δ_L^k, δ_R^k induced by the comultiplication in $\mathcal{T}(V)$.

Proposition 5.1. *Let $\mathcal{A}_1 = \mathcal{A}_1(\lambda_1)$, $\mathcal{A}_2 = \mathcal{A}_2(\lambda_1, \lambda_2)$, $\mathcal{L}_1 = \mathcal{L}_1(\lambda_1)$ and $\mathcal{L}_2 = \mathcal{L}_2(\lambda_1, \lambda_2)$. Assume $\mathcal{A}_2 \neq 0$. Then*

(a) *$\mathcal{A}_k$ is a right Galois object of $\mathcal{H}_k$.*
(b) *There is a section $\gamma_k : \mathcal{H}_k \to \mathcal{A}_k$ with $\gamma_{k|\Bbbk G} = \mathrm{id}$.*
(c) *$L(\mathcal{A}_k, \mathcal{H}_k) \cong \mathcal{L}_k$. Hence $\mathcal{L}_k$ is a cocycle deformation of $\mathcal{H}_k$.*

Proof. (a) follows by [30, Theorem 8] applied to a suitable right coideal subalgebra Y_k. If $k = 1$, we take Y_1 to be generated by $x_i^n g_i^{-n}$ for some $i \in X$. If $k = 2$, this is done in several steps, one for each $C \in \mathcal{R}'$, up to conjugacy, taking $Y_{2,C}$ as the subalgebra generated by $b_C g_j^{-1} g_i^{-1}$ for $(i, j) \in C$.

(b) This is [1, Lemma 5.8 (b)].

(c) follows by applying [1, Proposition 5.10]. □

It is possible to use [23,24] in specific examples to check that $\mathcal{A}_2(\lambda_1, \lambda_2) \neq 0$. We do this in the next section to prove Main Theorem 1. We now compute Galois objects for $\mathcal{H} = \mathcal{H}_3 = \mathfrak{B}(V)\#\Bbbk G$.

Proposition 5.2. *Assume that $\mathcal{A}_2(\lambda_1, \lambda_2) \neq 0$ for some λ_1, λ_2.*

(a) *There exists $a_X \in \mathcal{A}_2(\lambda_1, \lambda_2)$ and $\lambda_3 \in \Bbbk$ subject to (29) such that*

$$\mathcal{A} = \mathcal{A}(\lambda_1, \lambda_2, \lambda_3) = \mathcal{A}_2(\lambda_1, \lambda_2)/\langle z - a_X - \lambda_3\rangle.$$

is a Galois object of $\mathcal{H}_3$.

(b) *$L(\mathcal{A}, \mathcal{H}_3) \cong \mathcal{L}_3(\lambda_1, \lambda_2, \lambda_3)$ where*

$$\mathcal{L}_3(\lambda_1, \lambda_2, \lambda_3) = \mathcal{L}_2(\lambda_1, \lambda_2)/\big\langle z - s_X - \lambda_3(1 - t_z)\big\rangle$$

and $s_X \in \mathcal{L}_2(\lambda_1, \lambda_2)$ is such that

$$(z - s_X) \otimes 1 = \delta_L\big(\gamma(z)\big) - t_z \otimes \gamma(z). \tag{32}$$

Proof. Set $\mathcal{H}' = \mathcal{H}_2$, $\mathcal{A}' = \mathcal{A}_2(\lambda_1, \lambda_2)$, $\mathcal{L}' = \mathcal{L}_2(\lambda_1, \lambda_2)$, $\gamma = \gamma_2 : \mathcal{H}' \to \mathcal{A}'$. We consider $\mathcal{H}'$ as a right $\mathcal{H}$-comodule via the natural projection. We use [30, Theorem 4] to find cleft objects of $\mathcal{H}$. For that, we have to compute the subalgebra $\mathcal{H}'^{\,\mathrm{co}\,\mathcal{H}}$ of right $\mathcal{H}$-coinvariants and the set $\mathrm{Alg}^{\mathcal{H}'}_{\mathcal{H}'}(\mathcal{H}'^{\,\mathrm{co}\,\mathcal{H}}, \mathcal{A}')$ of algebra maps from $\mathcal{H}'^{\,\mathrm{co}\,\mathcal{H}}$ to $\mathcal{A}'$ in ${}_{\mathcal{H}'}\mathcal{YD}^{\mathcal{H}'}$.

Let Y be the subalgebra of $\mathcal{H}'$ generated by zt_z^{-1}. Then Y is normal, by (25), and a polynomial algebra, by [1, Lemma 5.13] and (25). Hence $Y = \mathcal{H}'^{\,\mathrm{co}\,\mathcal{H}}$ by [1, Remark 5.5]. By [1, Remark 5.11] $f \in \mathrm{Alg}^{\mathcal{H}'}_{\mathcal{H}'}(Y, \mathcal{A}')$ if and only if $f(zt_z^{-1}) = \gamma(zt_z^{-1}) - \lambda_3 t_z^{-1}$ for some $\lambda_3 \in \Bbbk$.

Therefore (a) follows with $a_X = z - \gamma(z)$ by [30, Theorem 4]. Now (b) follows by [1, Corollary 5.12]. □

We need to compute $\gamma_2(z)$, $\delta_R^2(z)$ and $\delta_L^2(\gamma_2(z))$ to apply the above proposition. We explain in Appendix A how we can do this using [23,24].

The pointed liftings of $\mathfrak{B}(V)$ are given by the next theorem.

Theorem 5.3. *Let L be a lifting of $\mathfrak{B}(V)$ over $\Bbbk G$. Then there exist scalars $\lambda_1, \lambda_2, \lambda_3 \in \Bbbk$ such that $L \cong \mathcal{L}_3(\lambda_1, \lambda_2, \lambda_3)$ and hence L is a cocycle deformation of $\mathfrak{B}(V)\#\Bbbk G$.*

Proof. Consider the lifting map $\phi : \mathcal{T}(V) \to L$ defined by (2). If $r \in \mathcal{G}_0 \cup \mathcal{G}_1$, then r is a skew-primitive in $\mathcal{T}(V)$ and thus $\phi(r) \in L_{[1]}$. Moreover, $\phi(r) \in L_{[0]}$ by (27). Hence ϕ induces $\phi' : \mathcal{L}_2(\lambda_1, \lambda_2) \twoheadrightarrow L$ for some $\lambda_1, \lambda_2 \in \Bbbk$.

It follows that $\bar{z} = z - s_X$ is a $(1, t_z)$-primitive in $\mathcal{L}_2(\lambda_1, \lambda_2)$ and thus $\phi'(\bar{z}) \in L_{[1]}$. By (28) we see that $\phi'(\bar{z}) \in L_{[0]}$ and therefore there is $\lambda_3 \in \Bbbk$ such that $\phi'(\bar{z}) = \lambda_3(1 - t_z)$. Therefore ϕ' induces $\phi'' : \mathcal{L}(\lambda_1, \lambda_2, \lambda_3) \twoheadrightarrow L$ and this is an isomorphism since both algebras have dimension $\dim \mathfrak{B}(V)|G|$. □

To avoid repetitions, we further normalize the scalars $\lambda_1, \lambda_2, \lambda_3$ by

$$\lambda_1 = 0 \quad \text{if } g_i^n = 1, \qquad \lambda_2 = 0 \quad \text{if } g_i g_j = 1, \qquad \lambda_3 = 0 \quad \text{if } t_z = 1, \tag{33}$$

and consider the set

$$\mathcal{S}_X = \{(\lambda_1, \lambda_2, \lambda_3) \in \Bbbk^3 \mid \text{satisfying (29) and (33)}\}. \tag{34}$$

Proposition 5.4. *If $(\lambda_1, \lambda_2, \lambda_3), (\lambda_1', \lambda_2', \lambda_3') \in \mathcal{S}_X$ then $\mathcal{L}_3(\lambda_1, \lambda_2, \lambda_3) \cong \mathcal{L}_3(\lambda_1', \lambda_2', \lambda_3')$ if and only if $(\lambda_1, \lambda_2, \lambda_3) = \mu(\lambda_1', \lambda_2', \lambda_3')$ for some $\mu \in \Bbbk$.*

Proof. Follows as [25, Lemma 6.1]. □

The results above restrict to the case in which there is no relation z as in (26), that is when $\mathcal{J}(V)$ admits an adapted stratification $\mathcal{G}_0 \cup \mathcal{G}_1$. We collect this information in the following corollary. In this case we also denote

$$\mathcal{S}_X = \{(\lambda_1, \lambda_2) \in \Bbbk^2 \mid \text{satisfying (29) and (33)}\}. \tag{34'}$$

Corollary 5.5. *Let $\mathcal{J}(V)$ be as above. Let L be a lifting of $\mathfrak{B}(V)$ over $\Bbbk G$.*

(a) *There exist $(\lambda_1, \lambda_2) \in \mathcal{S}_X$ such that $L \cong \mathcal{L}_2(\lambda_1, \lambda_2)$.*
(b) *If $(\lambda_1, \lambda_2), (\lambda_1', \lambda_2') \in \mathcal{S}_X$, then $\mathcal{L}_2(\lambda_1, \lambda_2) \cong \mathcal{L}_2(\lambda_1', \lambda_2')$ if and only if $(\lambda_1, \lambda_2) = \mu(\lambda_1', \lambda_2')$ for some $\mu \in \Bbbk$.*
(c) *If $\mathcal{A}_2(\lambda_1, \lambda_2) \neq 0$, then L is a cocycle deformation of $\mathfrak{B}(V) \# \Bbbk G$.*

5.2. The shape of copointed liftings

Let G be a finite group and $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, $\dim V < \infty$. If $\{v_i\}$, $\{\alpha_i\}$ are dual bases of V and V^*, set $f_{ji} : \Bbbk G \to \Bbbk$, $h \mapsto \langle \alpha_j, h \cdot v_i \rangle$. By (6), $V \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ via

$$f \cdot v = \langle \mathcal{S}(f), v_{(-1)} \rangle v_{(0)} \quad \text{and} \quad \lambda(v_i) = \sum_j \mathcal{S}^{-1}(f_{ji}) \otimes v_j \tag{35}$$

for all $f \in \Bbbk^G$, $v \in V$. This definition is independent of the basis $\{v_i\}$. We say that $\{e_{ij} := \mathcal{S}^{-1}(f_{ji})\}$ is the *comatrix basis associated to* V and $\{v_i\}$.

In particular, let $\cdot$ be an action of G on a set X and let $\{\chi_i\colon G \to \Bbbk\}_{i \in X}$ be a 1-cocycle, see page 385. Then $\Bbbk X$ with basis $\{m_i\}_{i \in X}$ is a G-module via

$$g \cdot m_i = \chi_i(g) m_{g \cdot i} \quad \text{for all } i \in X,\ g \in G \tag{36}$$

and the comatrix basis $\{e_{ij}\}$ associated to $\Bbbk X$ and $\{m_i\}_{i \in X}$ is

$$e_{ij} = \sum_{g \in G} \delta_{j, g \cdot i}\, \chi_i(g) \delta_{g^{-1}} \quad \text{for all } i, j \in X.$$

Let A be a lifting of $\mathfrak{B}(V)$ over $\Bbbk^G$ with a lifting map $\phi : T(V)\#\Bbbk^G \to A$, recall (2). We consider the first term of the coradical filtration $A_{[1]} \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ in such a way that $\phi_{|(\Bbbk\oplus V)\#\Bbbk^G} : (\Bbbk \oplus V)\#\Bbbk^G \to A_{[1]}$ is an isomorphism in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$, cf. [1, Section 4]. Then we identify both modules.

The following lemma is a particular case of [1, Lemma 4.8]. It helps us to describe the image by ϕ of a submodule M of $T(V)$ in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ *compatible with* ϕ [1, Definition 4.7], that is

$$\Delta\big(\phi(m)\big) = \phi(m) \otimes 1 + m_{(-1)} \otimes \phi(m_{(0)}) \quad \text{for all } m \in M.$$

Then $\phi(m) \in (\Bbbk \oplus V)\#\Bbbk^G$. We define the ideal of $T(V)$

$$\mathcal{I}_M = \big\langle m - \phi(m) : m \in M\big\rangle. \tag{37}$$

Note that if $M \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$, then $M[e]$ and $M^\times$ are submodules of M in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ such that $M = M[e] \oplus M^\times$. In fact, ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ is a semisimple category and the supports of the simple objects are conjugacy classes of G [4, Proposition 3.1.2].

Lemma 5.6. *Let G, V, A and ϕ be as above. Let $M \subset T(V)$ be compatible with ϕ and $\{e_{ij}\}$ be the comatrix basis associated to $M[e]$ and $\{m_i\}_{i=1}^r$. Then*

(a) *$\phi_{|M^\times} : M^\times \to V$ is a morphism in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$.*

(b) *Assume that $M = M^\times$ is a simple object in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ and $V \simeq M^m \oplus P$ with m maximum. Then there exist $\lambda_1, \dots, \lambda_m \in \Bbbk$ such that*

$$\phi_{|M} = \lambda_1 \operatorname{id}_M \oplus \dots \oplus \lambda_m \operatorname{id}_M \oplus 0.$$

In particular, $\phi_{|M} = 0$ if $\operatorname{supp} M \cap \operatorname{supp} V = \emptyset$.

(c) *If $e \notin \operatorname{supp} V$, then there exist $a_1, \dots, a_r \in \Bbbk$ such that*

$$\phi(m_i) = a_i - \sum_{j=1}^{r} a_j c_{ij} \quad \text{for all } i = 1, \dots, r.$$

(d) *If $e \notin \operatorname{supp} V$ and $M = M[e]$ with the G-action on M satisfying (36), then there exist $(a_i)_{i\in X} \in \Bbbk^X$ such that*

$$\phi(m_i) = \sum_{g\in G} \big(a_i - \chi_i(g) a_{g\cdot i}\big)\delta_{g^{-1}} \quad \text{for all } i \in X.$$

(e) *Let $\phi' : T(V)\#\Bbbk^G \to A'$ be a lifting map and $\Theta : A \to A'$ be an isomorphism of Hopf algebras. If $e \notin \operatorname{supp} V$, then $\Theta\phi(V) = \phi'(V)$.* □

Proof. The lemma follows from [1, Lemma 4.8] since $\phi(M^\times) \subset \phi(V\#\Bbbk^G)$, $\phi(M[e]) \subset A_1[e]$ and $A_1[e] = \Bbbk^G$ if $e \notin \operatorname{supp} V$. □

Under certain conditions, it is showed in [1, Section 4] that $\mathcal{I}_M$ defines the lifting A. We recall this in our case.

A *good module of relations* [1, Definition 4.10] is a graded submodule $M = \bigoplus_{i=1}^t M^{n_i}$ of $T(V)$ in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$, $M^{n_i} \subset V^{\otimes n_i}$, such that it generates $\mathcal{J}(V)$ and for all $s = 1, \dots, t-1$ and $m \in M^{n_{s+1}}$: $n_s < n_{s+1}$, $M^{n_s} \neq 0$ and

$$\Delta(m) - m \otimes 1 - m_{(-1)} \otimes m_{(0)} \in I_N \otimes T(V)\#\Bbbk^G + T(V)\#\Bbbk^G \otimes I_N$$

where $N = \bigoplus_{i=1}^{s} M^{n_i}$ and M turn out to be compatible with ϕ by [1, Lemma 4.9]. The next result is [1, Theorem 4.11]. Recall (37).

Theorem 5.7. *Let A be a lifting of $\mathfrak{B}(V)$ over $\Bbbk^G$ with lifting map ϕ. Let M be a good module of relations for $\mathfrak{B}(V)$. Then $A \simeq T(V)\#\Bbbk^G/\mathcal{I}_M$.* $\square$

6. Pointed Hopf algebras over affine racks

Let $\mathrm{Aff}(\mathbb{F}_b, N)$ be one of the affine racks $\mathrm{Aff}(\mathbb{F}_3, 2)$, $\mathrm{Aff}(\mathbb{F}_4, \omega)$, $\mathrm{Aff}(\mathbb{F}_5, 2)$ or $\mathrm{Aff}(\mathbb{F}_5, 3)$. Through this section, we fix a finite group G together with a principal YD-realization $(\cdot, g, \{\chi_i\}_{i\in X})$ of $(\mathrm{Aff}(\mathbb{F}_b, N), -1)$. Let $\mathfrak{B}(b, N, -1)$ be the Nichols algebra of $V = \Bbbk\{x_i\}_{i\in\mathbb{F}_b}$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ given by (7). In this section we prove Main Theorem 1 using the Strategy of [1] described in 5.1.

Recall from Subsection 4.1 a set of generators of the ideal $\mathcal{J}(b, N, -1)$ and set $z = z_{(b,N,-1)}$ the top degree generator. Then the hypotheses of Subsection 5.1 hold for these Nichols algebras. Namely,

- $\mathcal{J}(b, N, -1)$ admits a stratification as in (26).
- z satisfies (21) and also (25) by Lemmas 4.2 and 4.3.
- Eqs. (27) and (28) hold by Lemma 3.7.

Therefore we can apply Theorem 5.3 to compute the liftings of $\mathfrak{B}(b, N, -1)$ over $\Bbbk G$ once we have proved that

- The algebras in (31) are nonzero.

In the next subsections, we do this using [23,24]. We stick to the notation in Subsection 5.1. Recall the definition of the sets $\mathcal{S}_X$ in (34), (34').

6.1. *Pointed Hopf algebras over* $\mathrm{Aff}(\mathbb{F}_3, 2)$

Let $(\lambda_1, \lambda_2) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_3,2)}$. Let $A(\lambda_1, \lambda_2)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2 - \lambda_1 \quad \text{and} \quad x_0x_1 + x_1x_2 + x_2x_0 - \lambda_2.$$

Let $H(\lambda_1, \lambda_2)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2 - \lambda_1\left(1 - g_0^2\right) \quad \text{and} \quad x_0x_1 + x_1x_2 + x_2x_0 - \lambda_2(1 - g_0g_1).$$

Remark 6.1. The pointed Hopf algebras over $\mathbb{S}_3$ were classified in [7,6]. These are isomorphic either to $\mathbb{S}_3$ or to some $H(\lambda_1, \lambda_2)$. In [22] it was shown that the nontrivial liftings are cocycle deformations of the bosonization $\mathfrak{B}(3, 2, -1)\#\Bbbk\mathbb{S}_3$. We give a different proof of this facts in Theorem 6.2. Also, items (a) and (d) of this theorem are already in [6, Theorem 3.8], by different methods.

Theorem 6.2. *Let H be a lifting of $\mathfrak{B}(3, 2, -1)$ over $\Bbbk G$.*

(a) *There exists $(\lambda_1, \lambda_2) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_3,2)}$ such that $H \cong H(\lambda_1, \lambda_2)$.*
(b) *$A(\lambda_1, \lambda_2)$ is an $(H(\lambda_1, \lambda_2), \mathfrak{B}(3, 2, -1)\#\Bbbk G)$-biGalois object for every $(\lambda_1, \lambda_2) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_3,2)}$.*
(c) *H is a cocycle deformation of $\mathfrak{B}(3, 2, -1)\#\Bbbk G$.*
(d) *$H(\lambda_1, \lambda_2)$ is a lifting of $\mathfrak{B}(3, 2, -1)$ over $\Bbbk G$ for every $(\lambda_1, \lambda_2) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_3,2)}$.*
(e) *$H(\lambda_1, \lambda_2) \cong H(\lambda_1', \lambda_2')$ iff $(\lambda_1, \lambda_2) = \mu(\lambda_1', \lambda_2')$ for some $\mu \in \Bbbk$.*

Proof. Follows by Corollary 5.5. We consider the stratification of $\mathcal{J}(3,2,-1)$ given by $\mathcal{G}_0=\{x_i^2\colon i\in\mathbb{F}_3\}$ and $\mathcal{G}_1=\{x_ix_j+x_{-i+2j}x_i+x_jx_{-i+2j}\colon\ i,j\in\mathbb{F}_3\}$ and then we use Diamond Lemma to see that $A(\lambda_1,\lambda_2)\neq 0$. □

6.2. Pointed Hopf algebras over $\mathrm{Aff}(\mathbb{F}_4,\omega)$

Let $(\lambda_1,\lambda_2,\lambda_3)\in\mathcal{S}_{\mathrm{Aff}(\mathbb{F}_4,\omega)}$. Let $A(\lambda_1,\lambda_2,\lambda_3)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2-\lambda_1,\qquad x_0x_1+x_1x_2+x_2x_0-\lambda_2\quad\text{and}$$
$$(x_0x_1x_2)^2+(x_1x_2x_0)^2+(x_2x_0x_1)^2-a_X-\lambda_3$$

for $a_X=\lambda_2(x_1x_0x_2x_1+x_0x_2x_1x_0+x_2x_1x_0x_2)+\lambda_2(\lambda_2-\lambda_1)(x_2x_1+x_1x_0+x_0x_2)$.

Let $H(\lambda_1,\lambda_2,\lambda_3)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2-\lambda_1\big(1-g_0^2\big),\qquad x_0x_1+x_1x_2+x_2x_0-\lambda_2(1-g_0g_1)\quad\text{and}$$
$$x_2x_1x_0x_2x_1x_0+x_1x_0x_2x_1x_0x_2+x_0x_2x_1x_0x_2x_1-s_X-\lambda_3\big(1-g_0^3g_1^3\big)$$

where

$$\begin{aligned}
s_X={}&\lambda_2(x_2x_1x_0x_2+x_1x_0x_2x_1+x_0x_2x_1x_0)-\lambda_2^3\big(g_0g_1-g_0^3g_1^3\big)\\
&+\lambda_1^2g_0^2\big(g_3^2(x_2x_3+x_0x_2)+g_1g_3(x_2x_1+x_1x_3)+g_1^2(x_1x_0+x_0x_3)\big)\\
&-2\lambda_1^2g_0^2(x_0x_3-x_2x_3-x_1x_2+x_1x_0)-2\lambda_1^2g_2^2(x_2x_3-x_1x_3+x_0x_2-x_0x_1)\\
&-2\lambda_1^2g_1^2(x_2x_1+x_1x_3+x_1x_2-x_0x_3+x_0x_1)\\
&+\lambda_2\lambda_1\big(g_2^2x_0x_3+g_1^2x_2x_3+g_0^2x_1x_3\big)+\lambda_2^2g_0g_1(x_2x_1+x_1x_0+x_0x_2-\lambda_1)\\
&-\lambda_2\lambda_1^2\big(3g_0^3g_3-2g_0g_1^3-g_0^2g_2^2-2g_0^3g_1+g_2^2-g_1^2+g_0^2\big)\\
&-\lambda_2(\lambda_1-\lambda_2)\big(\lambda_1g_0^2\big(g_3^2+g_1g_3+g_1^2+2g_0g_1^3\big)+x_2x_1+x_1x_0+x_0x_2\big).
\end{aligned}$$

Theorem 6.3. *Let H be a lifting of $\mathfrak{B}(4,\omega,-1)$ over $\Bbbk G$.*

(a) *There exists $(\lambda_1,\lambda_2,\lambda_3)\in\mathcal{S}_{\mathrm{Aff}(\mathbb{F}_4,\omega)}$ such that $H\cong H(\lambda_1,\lambda_2,\lambda_3)$.*
(b) *$A(\lambda_1,\lambda_2,\lambda_3)$ is an $(H(\lambda_1,\lambda_2,\lambda_3),\mathfrak{B}(4,\omega,-1)\#\Bbbk G)$-biGalois object for every $(\lambda_1,\lambda_2,\lambda_3)\in\mathcal{S}_{\mathrm{Aff}(\mathbb{F}_4,\omega)}$.*
(c) *H is a cocycle deformation of $\mathfrak{B}(4,\omega,-1)\#\Bbbk G$.*
(d) *$H(\lambda_1,\lambda_2,\lambda_3)$ is a lifting of $\mathfrak{B}(4,\omega,-1)\#\Bbbk G$ for all $(\lambda_1,\lambda_2,\lambda_3)\in\mathcal{S}_{\mathrm{Aff}(\mathbb{F}_4,\omega)}$.*
(e) *$H(\lambda_1,\lambda_2,\lambda_3)\cong H(\lambda_1',\lambda_2',\lambda_3')$ iff $(\lambda_1,\lambda_2,\lambda_3)=\mu(\lambda_1',\lambda_2',\lambda_3')$ for some $\mu\in\Bbbk$.*

Proof. The algebras $H(\lambda_1,\lambda_2,\lambda_3)$ are found following the strategy described in Subsection 5.1. We check that the algebras $\mathcal{A}_2(\lambda_1,\lambda_2)$ are nonzero using [23,24]. We compute $\gamma_2(z)$, for $\gamma_2:\mathcal{H}_2\to\mathcal{A}_2(\lambda_1,\lambda_2)$ as in Proposition 5.1 (b), again using [23,24], as explained in Appendix A. We end up with the liftings $H(\lambda_1,\lambda_2,\lambda_3)$ using Proposition 5.2, which states (b) and (d), consequently (c) and (e). Now (a) follows from Theorem 5.3. □

6.3. *Pointed Hopf algebras over* $\mathrm{Aff}(\mathbb{F}_5, 2)$

Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,2)}$. Let $A(\lambda_1, \lambda_2, \lambda_3)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2 - \lambda_1, \qquad x_0x_1 + x_2x_0 + x_3x_2 + x_1x_3 - \lambda_2,$$
$$(x_0x_1)^2 + (x_1x_0)^2 - \lambda_2(x_1x_0 + x_0x_1) - \lambda_3.$$

Let $H(\lambda_1, \lambda_2, \lambda_3)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2 - \lambda_1\big(1 - g_0^2\big), \qquad x_0x_1 + x_2x_0 + x_3x_2 + x_1x_3 - \lambda_2(1 - g_0g_1) \quad \text{and}$$
$$x_1x_0x_1x_0 + x_0x_1x_0x_1 - s_X - \lambda_3\big(1 - g_0^2g_1g_2\big),$$

for $s_X = \lambda_2(x_1x_0 + x_0x_1) + \lambda_1 g_1^2(x_3x_0 + x_2x_3) - \lambda_1 g_0^2(x_2x_4 + x_1x_2) + \lambda_2\lambda_1 g_0^2(1 - g_1g_2)$.

Theorem 6.4. *Let H be a lifting of $\mathfrak{B}(5, 2, -1)$ over $\Bbbk G$.*

(a) *There exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,2)}$ such that $H \cong H(\lambda_1, \lambda_2, \lambda_3)$.*
(b) *$A(\lambda_1, \lambda_2, \lambda_3)$ is an $(H(\lambda_1, \lambda_2, \lambda_3), \mathfrak{B}(5, 2, -1)\#\Bbbk G)$-biGalois object for every $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,2)}$.*
(c) *H is a cocycle deformation of $\mathfrak{B}(5, 2, -1)\#\Bbbk G$.*
(d) *$H(\lambda_1, \lambda_2, \lambda_3)$ is a lifting of $\mathfrak{B}(5, 2, -1)\#\Bbbk G$ for every $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,2)}$.*
(e) *$H(\lambda_1, \lambda_2, \lambda_3) \cong H(\lambda_1', \lambda_2', \lambda_3')$ iff $(\lambda_1, \lambda_2, \lambda_3) = \mu(\lambda_1', \lambda_2', \lambda_3')$ for some $\mu \in \Bbbk$.*

Proof. Set $z = (x_0x_1)^2 + (x_1x_0)^2 \in \mathcal{A}' = \mathcal{A}_2(\lambda_1, \lambda_2)$, $t_z = g_0^2g_1g_2 \in G$. Using [23,24],[2] the coaction of z in $\mathcal{A}'$ is $\delta_R^2(z) = z \otimes 1 + t_z \otimes z$ plus:

$$\lambda_2 g_0g_3 \otimes x_1x_0 + \lambda_2 g_0g_1 \otimes x_0x_1 - \lambda_2 g_1x_3 \otimes x_1 + \lambda_2 g_1x_0 \otimes x_1 - \lambda_2 g_0x_3 \otimes x_0 + \lambda_2 g_0x_1 \otimes x_0.$$

If $z' = x_1x_0 + x_0x_1$ we get $\delta_R^2(z - \lambda_2 z') = (z - \lambda_2 z') \otimes 1 + t_z \otimes z$. Thus $\gamma_2(z) = z - \lambda_2 z'$ and the theorem follows as Theorem 6.3. □

6.4. *Pointed Hopf algebras over* $\mathrm{Aff}(\mathbb{F}_5, 3)$

Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,3)}$. Let $A(\lambda_1, \lambda_2, \lambda_3)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2 - \lambda_1, \qquad x_1x_0 + x_0x_2 + x_2x_3 + x_3x_1 - \lambda_2,$$
$$(x_0x_1)^2 + (x_1x_0)^2 - \lambda_2(x_0x_1 + x_1x_0) - \lambda_3.$$

Let $H(\lambda_1, \lambda_2, \lambda_3)$ be the quotient of $T(V)\#\Bbbk G$ by the ideal generated by

$$x_0^2 - \lambda_1\big(1 - g_0^2\big), \qquad x_1x_0 + x_0x_2 + x_2x_3 + x_3x_1 - \lambda_2(1 - g_0g_1) \quad \text{and}$$
$$x_0x_2x_3x_1 + x_1x_4x_3x_0 - s_X - \lambda_3\big(1 - g_0^2g_1g_3\big),$$

for $s_X = \lambda_2(x_0x_1 + x_1x_0) - \lambda_1 g_1^2(x_3x_2 + x_0x_3) - \lambda_1 g_0^2(x_3x_4 + x_1x_3) + \lambda_1\lambda_2(g_1^2 + g_0^2 - 2g_0^2g_1g_3)$.

[2] See log files in http://www.mate.uncor.edu/~aigarcia/publicaciones.htm.

Theorem 6.5. *Let H be a lifting of $\mathfrak{B}(5,3,-1)$ over $\Bbbk G$.*

(a) *There exists $(\lambda_1,\lambda_2,\lambda_3)\in\mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,3)}$ such that $H\cong H(\lambda_1,\lambda_2,\lambda_3)$.*
(b) *$A(\lambda_1,\lambda_2,\lambda_3)$ is an $(H(\lambda_1,\lambda_2,\lambda_3),\mathfrak{B}(5,3,-1)\#\Bbbk G)$-biGalois object for every $(\lambda_1,\lambda_2,\lambda_3)\in\mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,3)}$.*
(c) *H is a cocycle deformation of $\mathfrak{B}(5,3-1)\#\Bbbk G$.*
(d) *$H(\lambda_1,\lambda_2,\lambda_3)$ is a lifting of $\mathfrak{B}(5,3,-1)\#\Bbbk G$ for every $(\lambda_1,\lambda_2,\lambda_3)\in\mathcal{S}_{\mathrm{Aff}(\mathbb{F}_5,3)}$.*
(e) *$H(\lambda_1,\lambda_2,\lambda_3)\cong H(\lambda'_1,\lambda'_2,\lambda'_3)$ iff $(\lambda_1,\lambda_2,\lambda_3)=\mu(\lambda'_1,\lambda'_2,\lambda'_3)$ for some $\mu\in\Bbbk$.*

Proof. Analogous to Theorem 6.5 *mutatis mutandis*. □

6.5. Proof of Main Theorem 1

Assume $X=\mathrm{Aff}(\mathbb{F}_3,2)$. Let H be a pointed Hopf algebra over G whose infinitesimal braiding is given by a principal YD-realization $V\in{}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of $(X,-1)$. Then H is generated in degree one by [6, Theorem 2.1]. Therefore H is a lifting of $\mathfrak{B}(V)$ over $\Bbbk G$ and Main Theorem 1(i) follows by Theorem 6.2.

The proof of items (ii), (iii), (iv) is analogous, again using the fact that any such H is generated in degree one by [6, Theorem 2.1] and Theorems 6.3, 6.4 or 6.5, depending on each case. □

7. Copointed Hopf algebras over affine racks

Through this section, we consider the affine racks $\mathrm{Aff}(\mathbb{F}_b,N)$ with constant 2-cocycle -1. We fix a finite group G and a principal YD-realization $(\cdot,g,\{\chi_i\}_{i\in X})$ of $(\mathrm{Aff}(\mathbb{F}_b,N),-1)$ over G. Let $\mathfrak{B}(-1,b,N)$ be the Nichols algebra of $W(-1,b,N)=\Bbbk\{x_i\}_{i\in\mathbb{F}_b}$ in ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ given by (8). We give the classification of the lifting Hopf algebras of $\mathfrak{B}(-1,b,N)$ over $\Bbbk^G$ and therefore the proof of Main Theorem 2.

7.1. Copointed Hopf algebras over $\mathrm{Aff}(\mathbb{F}_3,2)$

This subsection is inspired by [12,13] where the case $G=\mathbb{S}_3$ was considered. Recall that $\mathrm{Inn}_\triangleright\mathrm{Aff}(\mathbb{F}_3,2)=\mathbb{S}_3=\mathrm{Aut}_\triangleright\mathrm{Aff}(\mathbb{F}_3,2)$ by (12). Let $G\to\mathbb{S}_3$, $t\mapsto\bar t$ be the epimorphism given by Lemma 3.6(b). We consider the group $\Gamma=\Bbbk^*\times\mathbb{S}_3$ acting on

$$\mathfrak{A}=\left\{\mathbf{a}=(a_0,a_1,a_2)\in\Bbbk^{\mathbb{F}_3}\colon a_0+a_1+a_2=0\right\}$$

via $(\mu,\theta)\triangleright\mathbf{a}=\mu(a_{\theta 0},a_{\theta 1},a_{\theta 2})$. The equivalence class of $\mathbf{a}$ under this action is denoted by $[\mathbf{a}]$. Given $\mathbf{a}\in\mathfrak{A}$, we define

$$f_i=\sum_{t\in G}(a_i-a_{t^{-1}.i})\delta_t\in\Bbbk^G,\quad i\in\mathbb{F}_3.$$

Definition 7.1. Set $\mathcal{A}_{G,[0]}=\mathfrak{B}(-1,3,2)\#\Bbbk^G$. Let $\mathbf{a}\in\mathfrak{A}$ and assume that $g_i^2=e$ $\forall i\in\mathbb{F}_3$. We define the Hopf algebra $\mathcal{A}_{G,[\mathbf{a}]}=T(-1,3,2)\#\Bbbk^G/\mathcal{J}_{\mathbf{a}}$ where $\mathcal{J}_{\mathbf{a}}$ is the ideal generated by

$$x_i^2-f_i,\qquad x_ix_j+x_{-i+2j}x_i+x_jx_{-i+2j},\quad i,j\in\mathbb{F}_3,$$

and the algebra $\mathcal{K}_{G,[\mathbf{a}]}=T(-1,3,2)\#\Bbbk^G/\mathcal{I}_{\mathbf{a}}$ where $\mathcal{I}_{\mathbf{a}}$ is generated by

$$x_i^2+f_i-a_i,\qquad x_ix_j+x_{-i+2j}x_i+x_jx_{-i+2j},\quad i,j\in\mathbb{F}_3.$$

The algebras $\mathcal{A}_{G,[\mathbf{a}]}$ and $\mathcal{K}_{G,[\mathbf{a}]}$ are nonzero by the next lemma.

Lemma 7.2. *Consider the $\Bbbk^G$-module $M = \Bbbk\{m_t\}_{t\in G}$, $m_t \in M[t]$. Then for all $\mathbf{a} \in \mathfrak{A}$, M is an $\mathcal{A}_{G,[\mathbf{a}]}$-module and a $\mathcal{K}_{G,[\mathbf{a}]}$-module via*

$$x_i \cdot m_t = \begin{cases} m_{g_i^{-1}t} & \text{if } \operatorname{sgn} \bar{t} = -1, \\ \lambda_{i,t} m_{g_i^{-1}t} & \text{if } \operatorname{sgn} \bar{t} = 1, \end{cases}$$

where $\lambda_{i,t} = (a_i - a_{t^{-1}\cdot i})$ for $\mathcal{A}_{G,[\mathbf{a}]}$ and $\lambda_{i,t} = -a_{t^{-1}\cdot i}$ for $\mathcal{K}_{G,[\mathbf{a}]}$.

Proof. We check that the action of $\mathcal{K}_{G,[\mathbf{a}]}$ is well defined; for $\mathcal{A}_{G,[\mathbf{a}]}$ it is similar. Notice that $\operatorname{sgn}(\overline{g_i}) = -1$. We start by $\delta_h x_i = x_i \delta_{g_i h}$, cf. (10):

$$\delta_h(x_i \cdot m_t) = \delta_h(\lambda m_{g_i^{-1}t}) = \lambda \delta_{g_i h}(t) m_{g_i^{-1}t} = x_i \cdot (\delta_{g_i h} \cdot m_t)$$

for a certain $\lambda \in \Bbbk$. Clearly $x_i \cdot (x_i \cdot m_t) = \lambda_{i,t} m_t$. Since $\mathbf{a} \in \mathfrak{A}$, then $(x_i x_j + x_{-i+2j} x_i + x_j x_{-i+2j}) \cdot m_t = -(a_0 + a_1 + a_2) m_{g_i^{-1} g_j^{-1} t} = 0$. □

The following theorem presents all the liftings of $\mathfrak{B}(-1, 3, 2)$ over $\Bbbk^G$.

Theorem 7.3. *Let H be a lifting of $\mathfrak{B}(-1, 3, 2)$ over $\Bbbk^G$.*

(a) *If $g_i^2 \neq e$ for some (and thus all) $i \in \mathbb{F}_3$, then $H \simeq \mathcal{A}_{G,[0]}$.*
(b) *If $g_i^2 = e$ for some (and thus all) $i \in \mathbb{F}_3$, then there is $\mathbf{a} \in \mathfrak{A}$ such that $H \simeq \mathcal{A}_{G,[\mathbf{a}]}$.*
(c) *$\mathcal{K}_{G,[\mathbf{a}]}$ is an $(\mathcal{A}_{G,[0]}, \mathcal{A}_{G,[\mathbf{a}]})$-biGalois object for all $\mathbf{a} \in \mathfrak{A}$.*
(d) *$\mathcal{A}_{G,[\mathbf{a}]}$ is a cocycle deformation of $\mathcal{A}_{G,[\mathbf{b}]}$ for all $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$.*
(e) *$\mathcal{A}_{G,[\mathbf{a}]}$ is a lifting of $\mathfrak{B}(-1, 3, 2)$ over $\Bbbk^G$ for all $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$.*
(f) *$\mathcal{A}_{G,[\mathbf{a}]} \simeq \mathcal{A}_{G,[\mathbf{b}]}$ if and only if $[\mathbf{a}] = [\mathbf{b}]$.*

Proof. Let $\phi : T(-1, 3, 2) \# \Bbbk^G \to H$ be a lifting map and let

$$W = \{x_i^2,\ x_i x_j + x_{-i+2j} x_i + x_j x_{-i+2j}\colon\ i, j \in \mathbb{F}_3\}$$

be the set of quadratic relations defining the Nichols algebra $\mathfrak{B}(-1, 3, 2)$, see Proposition 4.4. Let $M \subset T(-1, 3, 2)$ be the Yetter–Drinfeld submodule generated by W. Then $\phi(M[g_i^{-1} g_j^{-1}]) = 0$ by Lemma 5.6(b) using Lemma 3.7(a) and (c). Hence:

(a) follows from Lemma 5.6(b) and Theorem 5.7 using Lemma 3.7(a).

(b) follows from Lemma 5.6(d) and Theorem 5.7.

(c) follows from [36, Theorem 2]. In fact, fix $\mathbf{a} \in \mathfrak{A}$ and let K be the braided Hopf subalgebra of $T(-1, 3, 2)$ generated by W. Then $K \# \Bbbk^G$ is a Hopf subalgebra of $T(-1, 3, 2) \# \Bbbk^G$. By [13, Lemma 28], we can define an algebra map $\psi = \psi_K \otimes \epsilon : K \# \Bbbk^G \to \Bbbk$ where

$$\psi_K(x_i^2) = -a_i \quad \text{and} \quad \psi_K(x_i x_j + x_{-i+2j} x_i + x_j x_{-i+2j}) = 0 \quad \forall i, j \in \mathbb{F}_3.$$

If $J = \langle W \rangle \subset K \# \Bbbk^G$, then $\psi^{-1} \rightharpoonup J \leftharpoonup \psi = \mathcal{J}_{\mathbf{a}}$ and $\psi^{-1} \rightharpoonup J = \mathcal{I}_{\mathbf{a}}$. By Lemma 7.2, $\mathcal{K}_{G,[\mathbf{a}]} \neq 0$ and [36, Theorem 2] asserts (c); hence (d) and (e).

(f) Fix $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$. Let $\phi_{\mathbf{a}}$ and $\phi_{\mathbf{b}}$ be lifting maps of $\mathcal{A}_{G,[\mathbf{a}]}$ and $\mathcal{A}_{G,[\mathbf{b}]}$. Let $\Theta : \mathcal{A}_{G,[\mathbf{a}]} \to \mathcal{A}_{G,[\mathbf{b}]}$ be an isomorphism of Hopf algebras. Then $(\Theta_{|\Bbbk^G})^*$ induces a group automorphism θ of G. By Lemma 5.6(e) and using the adjoint action of $\Bbbk^G$, we see that θ is a rack automorphism of $\operatorname{Aff}(\mathbb{F}_3, 2)$ and $\Theta\phi_{\mathbf{a}}(x_i) = \mu_i \phi_{\mathbf{b}}(x_{\theta i})$ with $\mu_i \in \Bbbk^*$ for all $i \in \mathbb{F}_3$. Since Θ is a coalgebra map, using (10) we obtain that $\mu_i = \mu$ for all $i \in \mathbb{F}_3$. Therefore $\mathbf{a} = (\mu^2, \theta) \triangleright \mathbf{b}$. The proof of the converse statement is easy, recall that $\operatorname{Inn}_\triangleright \operatorname{Aff}(\mathbb{F}_3, 2) = \mathbb{S}_3$. □

7.2. Copointed Hopf algebras over $\mathrm{Aff}(\mathbb{F}_b, N)$

Here $\mathrm{Aff}(\mathbb{F}_b, N)$ denotes one the racks $\mathrm{Aff}(\mathbb{F}_4, \omega)$, $\mathrm{Aff}(\mathbb{F}_5, 2)$, $\mathrm{Aff}(\mathbb{F}_5, 3)$, $\mathrm{Aff}(\mathbb{F}_7, 3)$ or $\mathrm{Aff}(\mathbb{F}_7, 5)$. Recall that $\chi_G = \chi_i$ is a multiplicative character for all $i \in X$ by Lemma 3.6(c). Let $\pi_2 : T(-1, b, N) \twoheadrightarrow \widehat{\mathfrak{B}_2}(-1, b, N)$ be the natural projection. Set $z' = z'_{(-1,b,N)}$ and $\chi_z = \chi_G^{\deg z}$, recall (22).

Definition 7.4. Set $\mathcal{A}_{G,0} = \mathfrak{B}(-1, b, N) \# \Bbbk^G$. If $z' \in T(-1, b, N)[e]$ and $\lambda \in \Bbbk^*$, then we define the Hopf algebra

$$\mathcal{A}_{G,\lambda} = \widehat{\mathfrak{B}_2}(-1, b, N) \# \Bbbk^G / \langle \pi_2(z') - \lambda(1 - \chi_z^{-1}) \rangle$$

and the algebra $\mathcal{K}_{G,\lambda} = \widehat{\mathfrak{B}_2}(-1, b, N) \# \Bbbk^G / \langle \pi_2(z') - \lambda \rangle$.

The following theorem presents all the liftings of $\mathfrak{B}(-1, b, N)$ over $\Bbbk^G$.

Theorem 7.5. *Let H be a lifting of $\mathfrak{B}(-1, b, N)$ over $\Bbbk^G$.*

(a) *If G is generated by $\{g_i^{-1}: i \in \mathbb{F}_b\}$ or $\chi_z = \varepsilon$, then $H \simeq \mathcal{A}_{G,0}$.*
(b) *If $z' \in T(-1, b, N)^{\times}$, $H \simeq \mathcal{A}_{G,0}$.*
(c) *If $z' \in T(-1, b, N)[e]$, then $H \simeq \mathcal{A}_{G,\lambda}$ for some $\lambda \in \Bbbk$.*
(d) *$\mathcal{K}_{G,\lambda}$ is an $(\mathcal{A}_{G,0}, \mathcal{A}_{G,\lambda})$-biGalois object for all $\lambda \in \Bbbk$.*
(e) *$\mathcal{A}_{G,\lambda}$ is a cocycle deformation of $\mathcal{A}_{G,\lambda'}$, for all $\lambda, \lambda' \in \Bbbk$.*
(f) *$\mathcal{A}_{G,\lambda}$ is a lifting of $\mathfrak{B}(-1, b, N)$ over $\Bbbk^G$ for all $\lambda, \lambda' \in \Bbbk$.*
(g) *$\mathcal{A}_{G,\lambda} \simeq \mathcal{A}_{G,1} \not\simeq \mathcal{A}_{G,0}$ for all $\lambda \in \Bbbk$.*

Proof. Let $\phi : T(-1, b, N) \# \Bbbk^G \to H$ be a lifting map and $M \subset T(-1, b, N)$ be the Yetter–Drinfeld submodule generated by the quadratic relations defining $\mathfrak{B}(-1, b, N)$, see Proposition 4.4. Then

$$M = M^{\times} = \bigoplus_{i,j \in \mathbb{F}_b} M\left[(g_i g_j)^{-1}\right] \oplus \bigoplus_{i \in \mathbb{F}_b} M\left[g_i^{-2}\right]$$

by Lemma 3.7. Moreover, $\phi(M^{\times}) = 0$ by Lemma 5.6(b) using Lemma 3.7. Therefore ϕ factorizes through $\widehat{\mathfrak{B}_2}(-1, b, N) \# \Bbbk^G$ and the Yetter–Drinfeld module $M_{z'}$ generated by z' is compatible with ϕ. Therefore:

(c) follows from Lemma 5.6(d) and Theorem 5.7 by (23) . (a) follows from (c) since $\chi_z = \chi_G^{\deg z} = \varepsilon$ by Lemma 4.2. (b) follows from Lemma 5.6(b) and Theorem 5.7 since $1 = \dim \pi_2(M_{z'}) < \dim W(-1, b, N)$; the equality holds by Eq. (22).

(d) Let $w = \pi_2(z')\chi_z$. By Lemma 4.5 $\Bbbk[w]$ is the subalgebra of right $\mathfrak{B}(-1, b, N) \# \Bbbk^G$-coinvariants. By [1, Corollaries 3.7 and 3.8] we can apply [30, Theorem 4] to the Yetter–Drinfeld algebra map

$$\Bbbk[w] \to \widehat{\mathfrak{B}_2}(-1, b, N) \# \Bbbk^G, \quad w \mapsto w - \lambda\chi_z.$$

Hence $\mathcal{K}_{G,\lambda}$ is an $(\mathcal{A}_{G,\lambda}, \mathcal{A}_{G,0})$-biGalois object. (e) and (f) are consequences of (d). For (g), the map $F : \mathcal{A}_{G,\lambda} \to \mathcal{A}_{G,1}$ given by $F(x_i) = \lambda^{1/\deg z} x_i$ and $F_{|\Bbbk^G} = \mathrm{id}_{|\Bbbk^G}$ is an isomorphism of Hopf algebras. □

Example 7.6. There are nontrivial liftings of $\mathfrak{B}(-1, b, N)$ isomorphic to $\mathcal{A}_{G,\lambda}$. In fact, suppose that $m \mid \ell k + 1$ and consider the (m, k)-affine realization of $(\mathrm{Aff}(\mathbb{F}_b, N), -1)$; note that $z' \in T(-1, b, N)[e]$. Let G' be a finite group with a multiplicative character $\chi_{G'} : G' \to \Bbbk^*$ such that $\chi_{G'}^{\deg z} \neq \varepsilon$. Then $G = (\mathbb{F}_b \rtimes C_{m\ell}) \times G'$ acts on $W(-1, b, N)$ via $(h \times g') \cdot x_i = \chi_{G'}(g')h \cdot x_i$ and thus the (m, k)-affine realization induces a principal YD-realization of $(\mathrm{Aff}(\mathbb{F}_b, N), -1)$ over G such that $z' \in T(-1, b, N)[e]$ and $\chi_z = \chi_G^{\deg z} \neq \varepsilon$.

7.3. Proof of Main Theorem 2

Let H be a copointed Hopf algebra over $\Bbbk^G$ whose infinitesimal braiding is given by a principal YD-realization $W(-1,b,N) \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$. Then H is generated in degree one. Indeed, we can repeat the proof of [6, Theorem 2.1], *mutatis mutandis*, using the results of Subsection 4.3. Hence H is a lifting of $\mathfrak{B}(-1,b,N)$ over $\Bbbk^G$ and Main Theorem 2 follows by Theorem 7.3 for $\mathrm{Aff}(\mathbb{F}_3,2)$ or else by Theorem 7.5. $\square$

Acknowledgments

We thank Nicolás Andruskiewitsch for suggesting us this problem and for the many useful comments he shared with us in previous versions of this work. We also thank Iván Angiono and Leandro Vendramin for very useful discussions. In particular, the idea for computing coproducts with [23] came up in conversations with L.V. Also, A.G.I. specially thanks Giovanna Carnovale from the Università degli Studi di Padova where part of his work was done and for her warm hospitality and help. We are also indebted to Martín Mombelli for sharing his knowledge about braided categories with us.

Appendix A. On computations

Through Appendix A we keep the hypotheses and notation in Subsection 5.1. We explain how we can compute using [23,24] the left and right coactions of the top degree relation of a Nichols algebra as there.

Set $\mathcal{H}' = \mathcal{H}_2$, $\mathcal{A}' = \mathcal{A}_2(\lambda,\mu)$ and $\mathcal{L}' = \mathcal{L}(\lambda,\mu)$; these are quotients of $T(V)\#\Bbbk G$. Let δ_R and δ_L be the coactions on $\mathcal{A}'$ over $\mathcal{H}'$ and $\mathcal{L}'$, respectively; these are induced by the comultiplication of $T(V)\#\Bbbk G$.

Let z be the top degree generator of $\mathcal{J}(V)$, set $\ell = \deg z$. It is an element of $T(V)$ but we still denote by z its class in $\mathcal{H}'$, $\mathcal{A}'$ or $\mathcal{L}'$. We compute

(i) The coaction $\delta_R(z) \in \mathcal{A}' \otimes \mathcal{H}'$.
(ii) The section $\gamma_2(z) \in \mathcal{A}'$.
(iii) The coaction $\delta_L(\gamma_2(z)) \in \mathcal{L}' \otimes \mathcal{A}'$.

For item (i) we proceed as follows. Let $\theta = \dim V$ and denote by y_i, $i = 1,\dots,\theta$ the generators of $\mathcal{A}'$. We work with $3\theta + 2$ variables $G_1,\dots,G_\theta$, $X_1,\dots,X_\theta$, $Y_1,\dots,Y_\theta$, U, V. The variables G_i stand for the elements $g_i \otimes 1$ in $\mathcal{A}' \otimes \mathcal{H}'$. The variables Y_i and X_i stand for $y_i \otimes 1$ and $1 \otimes x_i$, respectively. The variables U and V are included to have a homogeneous system of generators. In most cases one of them is enough (for instance if $n = 2$) and there are cases where we can omit them. We fix two indeterminate elements $s, t \in \Bbbk$, corresponding to λ, μ.

We define an ideal K of relations in the algebra generated by these variables whose generators are, for every $1 \leqslant i, j \leqslant \theta$ and for every class $C \in \mathcal{R}'$:

$$Y_iX_j - X_jY_i, \qquad X_iG_j - G_jX_i, \qquad G_iG_j - G_{i\triangleright j}G_i, \qquad G_iY_j + Y_{i\triangleright j}G_i,$$
$$X_i^n, \qquad Y_i^n - tU, \qquad b_C\big(\{X_i\}_{i\in X}\big), \qquad b_C\big(\{Y_i\}_{i\in X}\big) + sV.$$

Also, U and V commute with all X, Y, G. Recall that $b_C(\cdot)$ stands for a generator of the space of quadratic relations, see (24). This ideal is homogeneous if we declare all X, Y, G of degree 1, U of degree n and V of degree 2. We compute the (truncated, up to degree ℓ) Gröbner basis of K.

We define $d_i = Y_i + G_iX_i$, $i = 1,\dots,\theta$. These elements stand for the coaction of $y_i \in \mathcal{A}'$. We can now compute the coaction $\delta_R(z)$ by adding and multiplying the d_i's in a suitable way. For U, V we consider it as 1.

We now explain how we get $\gamma_2(z)$ in item (ii). Let $\delta_R(z) - t_z \otimes z = \sum A_i \otimes B_i$ and let z' be the sum of the terms B_i of greatest degree with $A_i \in \Bbbk G$. Re-write the B_i's in the variables Y_i and consider $z_1 = z - z'$. We calculate $\delta_R(z_1)$ and repeat the proceeding: in the examples considered, the order of the elements we subtract decreases. When $\delta_R(z_m) - t_z \otimes z \in \mathcal{A}' \otimes \Bbbk G$, for some m, we get that $\delta_R(z_m) = z_m \otimes 1 + t_z \otimes z$ and thus $\gamma_2(z) = z_m$.

Finally, we find $\delta_L(\gamma_2(z))$ in (iii) in a similar way as we did for $\delta_R(z)$.

Example A.1. Let $X = (\mathbb{F}_4, \omega)$. We use [23,24] to see that the algebras $\mathcal{A}' = \mathcal{A}_2(\lambda_1, \lambda_2)$ are nonzero. See the log files in http://www.mate.uncor.edu/~aigarcia/publicaciones.htm. Set $z = (y_0y_1y_2)^2 + (y_1y_2y_0)^2 + (y_2y_0y_1)^2 \in \mathcal{A}'$, $t_z = g_0^3g_1^3 \in G$. Using [23,24], the coaction of z in $\mathcal{A}'$ is $\delta_R(z) = z \otimes 1 + t_z \otimes z$ plus:

$$\begin{aligned}
&\lambda_2 g_0^2 g_3^2 \otimes x_1x_0x_2x_1 + \lambda_2 g_0^2 g_1 g_3 \otimes x_0x_2x_1x_0 + \lambda_2 g_0^2 g_1^2 \otimes x_2x_1x_0x_2 \\
&\quad + \lambda_2 g_0 g_1 g_3 (y_2 - y_0) \otimes x_2x_1x_0 + \lambda_2 g_0 g_1 g_3 (y_1 - y_2) \otimes x_1x_0x_2 \\
&\quad + \lambda_2 g_0 g_1 g_3 (y_0 - y_1) \otimes x_0x_2x_1 + \lambda_2 g_0^2 g_1 (y_3 - y_1) \otimes x_0x_1x_2 \\
&\quad + \lambda_2 g_0 g_1^2 (y_0 - y_3) \otimes x_1x_2x_1 + \lambda_2 g_0^2 g_3 (y_2 - y_3) \otimes x_0x_1x_0 \\
&\quad + \lambda_2 g_1 g_3 (y_0y_2 - y_1y_2 - y_0y_1) \otimes x_2x_1 + 2\lambda_2\lambda_1 g_1 g_3 \otimes x_2x_1 \\
&\quad + \lambda_2 g_0 g_3 (y_2y_1 + y_1y_2) \otimes x_1x_0 + \lambda_2 g_0 g_2 (y_1y_0 + y_0y_1) \otimes x_0x_2 \\
&\quad + \lambda_2 g_0 g_1 (y_2y_3 - y_2y_1 - 2y_1y_3 + y_1y_0 + y_0y_3) \otimes x_1x_2 \\
&\quad + \lambda_2 g_0 g_1 (y_0y_2 + 2y_2y_3 - y_2y_1 - y_1y_3 - y_0y_3) \otimes x_0x_1 \\
&\quad + \lambda_2 g_0 (y_2y_1y_0 - y_1y_2y_1 - y_1y_0y_3 - y_0y_1y_3) \otimes x_0 \\
&\quad + \lambda_2 g_1 (y_0y_2y_1 + y_0y_1y_2 - y_2y_1y_3 - y_1y_2y_3) \otimes x_1 \\
&\quad + \lambda_2 g_2 (y_1y_2y_3 + y_1y_0y_2 - y_0y_2y_3 + y_0y_1y_3 - y_0y_1y_0) \otimes x_2 \\
&\quad + \lambda_2(2\lambda_1 - \lambda_2) g_0 g_3 \otimes x_1x_0 + \lambda_2(2\lambda_1 - \lambda_2) g_0 g_2 \otimes x_0x_2 \\
&\quad + \lambda_2(\lambda_2 - 2\lambda_1) g_0 (y_3 - y_2) \otimes x_0 + \lambda_2 g_0 (\lambda_2 y_1 - \lambda_1 y_0) \otimes x_0 \\
&\quad + \lambda_2(2\lambda_1 - \lambda_2) g_1 (y_0 - y_3) \otimes x_1 - \lambda_1\lambda_2 g_2 (y_2 + y_3) \otimes x_2 \\
&\quad + \lambda_2 g_2 (2\lambda_1 - \lambda_2) y_1 \otimes x_2 + \lambda_2 g_2 (\lambda_2 y_0 - \lambda_1 y_3) \otimes x_2.
\end{aligned}$$

If $z_1 = z - \lambda_2 z' + \lambda_2(\lambda_2 - \lambda_1)z''$ where $z' = y_1y_0y_2y_1 + y_0y_2y_1y_0 + y_2y_1y_0y_2$, $z'' = y_2y_1 + y_1y_0 + y_0y_2$, we get $\delta_R^2(z_1t_z^{-1}) = z_1t_z^{-1} \otimes t_z^{-1} + 1 \otimes z$. Thus, $\gamma_2(zt_z^{-1}) = z_1t_z^{-1}$ and $\gamma_2(z) = z_1$. The computation of $\delta_L^2(\gamma_2(z)) - 1 \otimes \gamma_2(z)$ yields s_X in Subsection 6.2.

References

[1] N. Andruskiewitsch, I. Angiono, A. García Iglesias, A. Masuoka, C. Vay, Lifting via cocycle deformation, J. Pure Appl. Algebra (2013), http://dx.doi.org/10.1016/j.jpaa.2013.08.008, in press, arXiv:1212.5279v2.

[2] N. Andruskiewitsch, F. Fantino, L. Graña Vendramin, Finite-dimensional pointed Hopf algebras with alternating groups are trivial, Ann. Mat. Pura Appl. (4) 190 (2) (2011) 225–245.

[3] N. Andruskiewitsch, F. Fantino, L. Graña Vendramin, Pointed Hopf algebras over the sporadic simple groups, J. Algebra 325 (2011) 305–320.

[4] N. Andruskiewitsch, M. Graña, Braided Hopf algebras over non abelian finite groups, Bol. Acad. Cienc. (Córdoba) 63 (1999) 45–78.

[5] N. Andruskiewitsch, M. Graña, From racks to pointed Hopf algebras, Adv. Math. 178 (2003) 177–243.

[6] N. Andruskiewitsch, M. Graña, Examples of liftings of Nichols algebras over racks, AMA Algebra Montp. Announc. 1 (2003) (electronic).

[7] N. Andruskiewitsch, I. Heckenberger, H.J. Schneider, The Nichols algebra of a semisimple Yetter–Drinfeld module, Amer. J. Math. 132 (2010) 1493–1547.

[8] A. Ardizzoni, C. Menini, D. Ştefan, A monoidal approach to splitting morphisms of bialgebras, Trans. Amer. Math. Soc. 359 (2007) 991–1044.

[9] N. Andruskiewitsch, H.J. Schneider, Finite quantum groups and Cartan matrices, Adv. Math. 154 (2000) 1–45.
[10] N. Andruskiewitsch, H.J. Schneider, Pointed Hopf algebras, in: New Directions in Hopf Algebras, in: Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, 2002, pp. 1–68.
[11] N. Andruskiewitsch, H.J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, Ann. of Math. 171 (2010) 375–417.
[12] N. Andruskiewitsch, C. Vay, Finite dimensional Hopf algebras over the dual group algebra of the symmetric group in three letters, Comm. Algebra 39 (2011) 4507–4517.
[13] N. Andruskiewitsch, On a family of Hopf algebras of dimension 72, Bull. Belg. Math. Soc. Simon Stevin 19 (2012) 415–443.
[14] I. Angiono, A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems, J. Eur. Math. Soc. (JEMS) (2013), in press, arXiv:1008.4144.
[15] I. Angiono, On Nichols algebras of diagonal type, J. Reine Angew. Math. (2013), http://dx.doi.org/10.1515/crelle-2011-0008, in press, arXiv:1104.0268.
[16] C. Călinescu, S. Dăscălescu, A. Masuoka, C. Menini, Quantum lines over non-cocommutative cosemisimple Hopf algebras, J. Algebra 273 (2004) 753–779.
[17] P. Etingof, M. Graña, On rack cohomology, J. Pure Appl. Algebra 177 (2003) 49–59.
[18] F. Fantino, G.A. García, On pointed Hopf algebras over dihedral groups, Pacific J. Math. 252 (2011) 69–91.
[19] F. Fantino, L. Vendramin, On twisted conjugacy classes of type D in sporadic simple groups, in: Hopf Algebras and Tensor Categories, in: Contemp. Math., vol. 585, 2013, pp. 247–259.
[20] S. Fomin, A.N. Kirilov, Quadratic algebras, Dunkl elements and Schubert calculus, Progr. Math. 172 (1999) 146–182.
[21] G.A. García, M. Mastnak, Deformation by cocycles of pointed Hopf algebras over non-abelian groups, arXiv:1203.0957v1.
[22] A. García Iglesias, M. Mombelli, Representations of the category of modules over pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$, Pacific J. Math. 252 (2011) 343–378.
[23] The GAP Group, GAP — Groups, Algorithms and Programming, http://www.gap-system.org, 2008, Version 4.4.12.
[24] A.M. Cohen, D.A.H. Gijsbers, GBNP 0.9.5 (Non-commutative Gröbner bases), http://www.win.tue.nl/~amc.
[25] G.A. García, A. García Iglesias, Pointed Hopf algebras over $\mathbb{S}_4$, Israel J. Math. 183 (2011) 417–444.
[26] M. Graña, On Nichols algebras of low dimension, in: New Trends on Hopf Algebra Theory, La Falda, 1999, in: Contemp. Math., vol. 267, 2000, pp. 111–134.
[27] M. Graña, Zoo of finite-dimensional Nichols algebras of non-abelian group type, available at, http://mate.dm.uba.ar/~matiasg/zoo.html.
[28] M. Graña, I. Heckenberger, L. Vendramín, Nichols algebras of group type with many quadratic relations, Adv. Math. 227 (2011) 1956–1989.
[29] M. Graña, V. Turaev, Knot theory for self-indexed graphs, Trans. Amer. Math. Soc. 357 (2005) 535–553.
[30] R. Gunther, Crossed products for pointed Hopf algebras, Comm. Algebra 27 (1999) 4389–4410.
[31] I. Heckenberger, Classification of arithmetic root systems, Adv. Math. 220 (2009) 59–124.
[32] M. Helbig, On the lifting of Nichols algebras, arXiv:1003.5882v1.
[33] I. Heckenberger, A. Lochmann, L. Vendramin, Braided racks, Hurwitz actions and Nichols algebras with many cubic relations, Transform. Groups 17 (1) (2012) 157–194.
[34] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Algebra 23 (1982) 37–65.
[35] C. Kassel, Quantum Groups, Springer, Berlin, 1995.
[36] A. Masuoka, Defending the negated Kaplansky conjecture, Proc. Amer. Math. Soc. 129 (2001) 3185–3192.
[37] A. Masuoka, Abelian and non-abelian second cohomologies of quantized enveloping algebras, J. Algebra 320 (2008) 1–47.
[38] A. Milinski, H.J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, in: Contemp. Math., vol. 267, 2000, pp. 215–236.
[39] A. Milinski, H.J. Schneider, Hopf Bigalois extentions, Comm. Algebra 24 (1996) 3797–3825.
[40] L. Vendramin, Nichols algebras associated to the transpositions of the symmetric group are twist-equivalent, Proc. Amer. Math. Soc. 140 (2012) 3715–3723.