

ON s -SETS IN SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. Let (X, d, μ) be a space of homogeneous type. In this note we study the relationship between two types of s -sets: relative to a distance and relative to a measure. We find a condition on a closed subset F of X under which we have that F is s -set relative to the measure μ if and only if F is s -set relative to δ . Here δ denotes the quasi-distance defined by Macías and Segovia such that (X, δ, μ) is a normal space. In order to prove this result, we show a covering type lemma and a type of Hausdorff measure based criteria for the s -set condition relative to μ of a given set.

1. INTRODUCTION, NOTATION AND DEFINITIONS

A **quasi-metric** on a set X is a non-negative function d defined on $X \times X$ satisfying the following properties:

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for every $x, y \in X$;
- (3) there exists a constant $K \geq 1$ such that $d(x, y) \leq K(d(x, z) + d(z, y))$, for every $x, y, z \in X$.

We will refer to K as the triangle constant for d . A quasi-distance d on X induces a topology through the neighborhood system given by the family of all subsets of X containing a d -ball $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$ (see [3]). In a quasi-metric space (X, d) the **diameter** of a subset E is defined as

$$\text{diam}(E) = \sup\{d(x, y) : x, y \in E\}.$$

Throughout this paper (X, d) shall be a quasi-metric space such that the d -balls are open sets. Also we shall assume that (X, d) has **finite metric dimension**. This means that there exists a constant $N \in \mathbb{N}$ such that any d -ball $B(x, 2r)$ contains at most N points of any r -disperse subset of X . A set U is said to be **r -disperse** if $d(x, y) \geq r$ for every $x, y \in U$, $x \neq y$. If a quasi-metric space (X, d) has finite metric dimension, every r -disperse subset of X has at most N^m points in each d -ball of radius $2^m r$, for all $m \in \mathbb{N}$ and every $r > 0$ (see [3] and [2]). Also it is well known that every bounded subset F of X is totally bounded, so that for every $r > 0$ there

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exists a finite maximal r -disperse on F , whose cardinal depends on $\text{diam}(F)$ and on r .

We shall say that a closed subset F of X is **s -set in (X, d)** with associated measure ν , if ν is a Borel measure supported on F such that

$$(1.1) \quad c^{-1}r^s \leq \nu(B(x, r)) \leq cr^s,$$

for every $x \in F$ and every $0 < r < \text{diam}(F)$, for some constant $c \geq 1$. When the above conditions hold for every $0 < r < r_0$, where r_0 is a positive number less than $\text{diam}(F)$, we say that F is **locally s -set in (X, d)** . In some references related to problems of harmonic analysis and partial differential equations, see for example [1], this sets are called **(locally) s -Ahlfors**. In the bibliography belonging to geometric measure theory, such as [6], an s -set F is one for which $0 < \mathcal{H}^s(F) < \infty$ where \mathcal{H}^s is the Hausdorff measure of dimension s . Nevertheless, following [10] we shall adopt the expression s -set to name a set that supports a measure ν for which $\nu(B(x, r))$ behaves as r^s for r small.

In [1] is proved that the concepts of s -set and locally s -set coincide when the set F is bounded and (X, d) has finite metric dimension.

We shall now recall the definitions of Hausdorff measure and Hausdorff dimension of a set in a quasi-metric space (X, d) . The basic aspects related to this concepts can be found in [6]. For $\rho > 0$, we say that a sequence $\{B_i = B(x_i, r_i)\}$ of subsets of X is a **ρ -cover** by d -balls of a set F if $F \subseteq \bigcup B_i$ and $r_i \leq \rho$ for every i . Let $F \subseteq X$ and $s \geq 0$ fixed. We define

$$\mathcal{H}_\rho^s(F) = \inf \left\{ \sum_{i=1}^{\infty} r_i^s : \{B_i\} \text{ is a } \rho\text{-cover by } d\text{-balls of } F \right\}.$$

Clearly $\mathcal{H}_\rho^s(F)$ increases when ρ decreases, so that the limit when ρ tends to 0 exists (although it may be infinite). Then we define

$$\mathcal{H}^s(F) = \lim_{\rho \rightarrow 0} \mathcal{H}_\rho^s(F) = \sup_{\rho > 0} \mathcal{H}_\rho^s(F).$$

We shall refer to $\mathcal{H}^s(F)$ as the **Hausdorff measure** of F . The corresponding **Hausdorff dimension** of F is defined as $\dim_{\mathcal{H}}(F) = \inf\{s > 0 : \mathcal{H}^s(F) = 0\}$. It is easy to see that any s -set F in (X, d) satisfies that $\dim_{\mathcal{H}}(F) = s$ (see [10]).

We shall point out that, if (F, d) is (locally) s -set, then there exists essentially only one Borel measure ν satisfying the condition required in the definition. This fact is known in the Euclidean setting (see for instance [11]), and was proved for general quasi-metric spaces in [1]. More precisely, is proved that if (X, d) has finite metric dimension and F is (locally) s -set in

(X, d) with measure ν , then F is (locally) s -set en (X, d) with the restriction of \mathcal{H}^s to F .

A sufficient condition under which a quasi-metric space (X, d) has finite metric dimension is when X supports a doubling measure (see [3]). A Borel measure μ defined on the d -balls is said to be **doubling** if for some constant $A \geq 1$ we have the inequality

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty,$$

for every $x \in X$ and every $r > 0$. When μ is a doubling measure, we say that a point x in (X, d, μ) is an **atom** if $\mu(\{x\}) > 0$. When $\mu(\{x\}) = 0$ for every $x \in X$ we say that μ is a **non-atomic** doubling measure. Macías and Segovia proved in [8] that a point is an atom if and only if it is topologically isolated, and that the set of such points is at most countable. Throughout this paper we shall say that (X, d, μ) is a **space of homogeneous type** if μ is a non-atomic doubling measure on the quasi-metric space (X, d) .

Given a space of homogeneous type (X, d, μ) , the Hausdorff measure and the Hausdorff dimension *relative to μ* is consider in [10]. Precisely, the **Hausdorff measure relative to μ** is defined as $H^s_\rho(F) := \lim_{\rho \rightarrow 0} H^s_\rho(F)$, where

$$H^s_\rho(F) = \inf \left\{ \sum_{i=1}^{\infty} \mu^s(B_i) : F \subseteq \bigcup_i B_i \text{ and } \mu(B_i) \leq \rho \right\},$$

where B_i are d -balls on X . Then the **Hausdorff dimension relative to μ** is defined by

$$\dim_H(F) = \inf \{s > 0 : H^s(F) = 0\}.$$

These concepts conduce to give a definition of s -set relative to the measure μ , compatible with H^s . Given a space of homogeneous type (X, d, μ) , we shall say that a closed subset F of X is **s -set in (X, d, μ)** if there exist a constant $c \geq 1$ and a Borel measure m supported on F such that

$$(1.2) \quad c^{-1}\mu(B(x, r))^s \leq m(B(x, r)) \leq c\mu(B(x, r))^s,$$

for every $x \in F$ and every $0 < r < \text{diam}(F)$. As before, if (1.2) holds for every $0 < r < r_0$, where r_0 is a positive number less than $\text{diam}(F)$, we say that F is **locally s -set in (X, d, μ)** .

It is now easy to see that each s -set F in (X, d, μ) satisfies $\dim_H(F) = s$.

Given a space of homogeneous type (X, d, μ) , in [10] are also considered the concepts of s -sets, Hausdorff measure and Hausdorff dimension relative to a particular quasi-metric δ related to (X, d, μ) . This quasi-metric was

constructed by Macías and Segovia in [8], in such a way that the new structure (X, δ, μ) becomes a normal space (in the sense that every δ -ball in X has μ -measure equivalent to its ratio), and the topologies induced on X by d and δ coincide. This quasi-metric is defined by

$$\delta(x, y) = \inf\{\mu(B) : B \text{ is a } d\text{-ball with } x, y \in B\}$$

if $x \neq y$, and $\delta(x, y) = 0$ if $x = y$. It will be also useful to notice that in the proof of the above mentioned result of Macías and Segovia it is proved that

$$B_\delta(x, r) = \bigcup\{B : B \text{ is a } d\text{-ball with } x \in B \text{ and } \mu(B) < r\},$$

for every $x \in X$ and every $r > 0$, where $B_\delta(x, r) := \{y \in X : \delta(x, y) < r\}$ denotes the ball in X relative to δ . Throughout this paper δ shall denote this quasi-metric.

Then, we can consider the concept of s -set in (X, δ) , the Hausdorff measure relative to δ , and the corresponding Hausdorff dimension. More precisely, we shall denote $G^s(F) := \lim_{\rho \rightarrow 0} G_\rho^s(F)$, where

$$G_\rho^s(F) = \inf \left\{ \sum_{i=1}^{\infty} r_i^s : F \subseteq \bigcup_i B_\delta(x_i, r_i) \text{ and } r_i \leq \rho \right\},$$

and

$$\dim_G(F) = \inf\{s > 0 : G^s(F) = 0\}.$$

In [10, Propo. 1.5] is proved that $H^s(F)$ and $G^s(F)$ are equivalent, and then $\dim_H(F) = \dim_G(F)$ for any subset F of X . In this note we explore the relationship between the concepts of s -set in (X, d, μ) and s -set in (X, δ) .

The paper is organized as follows. Section 2 contains the main results. Theorem 2.1 states that under certain typical conditions, being s -set in (X, δ) is stronger than being s -set in (X, d, μ) . A sufficient condition under which every s -set in (X, d, μ) is an s -set in (X, δ) is contained in Theorem 2.5. We show that every bounded set satisfies this condition, and we give examples of unbounded set satisfying it. In Proposition 2.6 we obtain a criteria to check the s -set condition related to μ of a given set based the Hausdorff measure. Section 3 is devoted to the proof of Proposition 2.6, for which we state and proof a covering type lemma of a bounded set by balls with small measure and controlled overlap (see Lemma 3.1).

2. MAIN RESULTS

Let (X, d, μ) be a given space of homogeneous type, and set δ the quasi-metric defined in previous section. We shall first prove that, under certain condition, being s -set in (X, δ) is stronger than being s -set in (X, d, μ) .

Theorem 2.1.

- (1) If F is an unbounded s -set in (X, δ) with associated measure ν , then F is s -set in (X, d, μ) with the same measure ν .
- (2) If F is locally s -set in (X, δ) with associated measure ν and $\mu(F) = 0$, then F is locally s -set in (X, d, μ) with the same measure ν .

Proof. By hypothesis, there exist $c \geq 1$ and $r_0 > 0$ such that the inequalities

$$c^{-1}r^s \leq \nu(B_\delta(x, r)) \leq cr^s,$$

hold for every $x \in F$ and every $0 < r < r_0$, where ν is a Borel measure supported in F , and $r_0 = \infty$ in case (1).

Fix $x \in F$ and $r > 0$. By definition of δ , we have that $B(x, r) \subseteq B_\delta(x, 2\mu(B(x, r)))$. Then,

$$\nu(B(x, r)) \leq \nu(B_\delta(x, 2\mu(B(x, r)))) \leq c2^s \mu^s(B(x, r))$$

provided that $\mu(B(x, r)) < \frac{r_0}{2}$. On the other hand, fix ℓ such $3K^2 \leq 2^\ell$ where K denotes the triangular constant for d . Then $B_\delta(x, A^{-\ell}\mu(B(x, r))) \subseteq B(x, r)$ (see [8, pag. 262]), where A is the constant for the doubling condition for μ . Hence

$$\nu(B(x, r)) \geq \nu(B_\delta(A^{-\ell}\mu(B(x, r)))) \geq c^{-1}A^{-\ell s} \mu^s(B(x, r)),$$

provided that $\mu(B(x, r)) < A^\ell r_0$.

Since every d -ball has finite μ -measure, (1) is proved. On the other hand, we obtain (2) if we can choose r_1 in such a way that $0 < r < r_1$ implies $\mu(B(x, r)) < \min\{\frac{r_0}{2}, A^\ell r_0\} = \frac{r_0}{2}$, for every $x \in F$. But this is possible from the hypothesis $\mu(F) = 0$. \square

We shall point out that the assumption $\mu(F) = 0$ is natural in many problems related with partial differential equations, in which F plays the role of the boundary of a domain in a metric measure space (X, d, μ) (see for example [4] or [5]).

In order to obtain a sufficient condition under which every locally s -set in (X, d, μ) becomes a locally s -set in (X, δ) , we shall give the following definition.

Definition 2.2. Let F be a closed subset of X . We shall say that F is *consistent with μ* if there exists a positive number R such that

$$\inf_{x \in F} \mu(B(x, R)) > 0.$$

Let us remark that if F is a set consistent with μ , then we have that $\inf_{x \in F} \mu(B(x, r)) > 0$ for every $r > 0$. In fact, the claim is trivial for every $r \geq R$. On the other hand, for a fixed $0 < r < R$, for every $x \in F$ we have that

$$\mu(B(x, r)) = \mu\left(x, \frac{r}{R}R\right) \geq \frac{1}{A^m} \mu(B(x, R)),$$

where m is a positive integer such that $2^m \geq R/r$ and A denotes the doubling constant for μ .

We want also to point out that every bounded subset of X is consistent with μ . In fact, set $R = 2K \text{diam}(F)$, with K the triangular constant for d , and fix $x_0 \in F$. Then $B(x_0, \text{diam}(F)) \subseteq B(x, R)$ for every $x \in F$. Then $\inf_{x \in F} \mu(B(x, R)) \geq \mu(B(x_0, \text{diam}(F))) > 0$, since μ is doubling.

However, there exist also unbounded sets satisfying this condition.

Example 2.3. Consider $X = \mathbb{R}^2$ equipped with the usual distance d and the Lebesgue measure λ . Fix $a > 0$ and set $F = \{(t, 0) : t \geq a\}$. Then $\lambda(B(x, r))$ is equivalent to r^2 for every $x \in F$, thus F is consistent with λ .

Example 2.4. Also we can consider another measure μ defined on (\mathbb{R}^2, d) in such a way that (X, d, μ) is not an Ahlfors space. For example, let us consider the measure μ define by

$$\mu(E) = \int_E |y|^\beta dy,$$

for a fixed $\beta > -2$. Then (X, d, μ) is a space of homogeneous type since $|x|^\beta$ is a Muckenhoupt weight (see [9] or [7]). For the set F considered in the above example, it is easy to see that $\mu(B(x, r))$ is equivalent to $r^2|x|^\beta$ for $x \in F$ and $0 < r \leq a/2$. So that F is consistent with μ provided that $\beta > 0$.

With this terminology, we have the following result.

Theorem 2.5.

- (1) *If F is an unbounded s -set in (X, d, μ) , then F is s -set in (X, δ) .*
- (2) *If F is a locally s -set in (X, d, μ) which is consistent with μ , then F is locally s -set in (X, δ) .*

Let us observe that every bounded s -set in (X, d, μ) satisfies the hypothesis of the above theorem. In order to prove this theorem, we shall need the following three auxiliary results.

The first one states that, as in the case of s -sets relative to a distance, when F is s -set relative to the measure μ , there exists essentially only one Borel measure ν satisfying the required condition. More precisely, we state the following result that we shall prove in Section 3.

Proposition 2.6. *If F is (locally) s -set in (X, d, μ) with measure m , then F is (locally) s -set in (X, d, μ) with the restriction of H^s to F , where H^s denotes the s -dimensional Hausdorff measure relative to μ .*

The following statement is about a characterization of consistent sets, and says that the radii of all the d -balls centering in a set consistent with μ are as small as we want, provided that the ball has sufficiently small measure.

Lemma 2.7. *F is consistent with μ if and only if given $r_0 > 0$, there exists C such that if $x \in F$ and $\mu(B(x, t)) \leq C$, then $t < r_0$.*

Proof. Suppose first that F is consistent with μ but the property is false. Then there exists $r_0 > 0$ such that for every natural number n we can find $x_n \in F$ and $t_n \geq r_0$ with $\mu(B(x_n, t_n)) \leq \frac{1}{n}$. So that $\mu(B(x_n, r_0)) \leq \frac{1}{n}$ for every natural n , which implies that $\inf_{x \in F} \mu(B(x, r_0)) = 0$. But this is a contradiction, since F is consistent with μ . Reciprocally, assume that F is not consistent with μ . Then, for every $r_0 > 0$ we have that $\inf_{x \in F} \mu(B(x, r_0)) = 0$. So that for every natural n there exists $x_n \in F$ such that $\mu(B(x_n, r_0)) < \frac{1}{n}$. Hence, given $C > 0$ we can choose n such that $1/n \leq C$ and obtain $\mu(B(x_n, r_0)) < C$ but $r_0 \not< r_0$. \square

The last result that we shall need is a technical lemma, which is showed in [10], so that we shall omit its proof.

Lemma 2.8. *Given $x \in X$ and $0 < r < 2\mu(X)$, there exist numbers $0 < a \leq b < \infty$ such that*

$$B(x, a) \subseteq B_\delta(x, r) \subseteq B(x, b)$$

and

$$C_1 r \leq \mu(B(x, a)) \leq \mu(B(x, b)) \leq C_2 r,$$

where C_1 and C_2 only depend on X .

Proof of Theorem 2.5. From Proposition 2.6, there exist $c \geq 1$ and $r_0 > 0$ such that

$$c^{-1} \mu(B(x, r))^s \leq H^s(B(x, r) \cap F) \leq c \mu(B(x, r))^s,$$

for every $x \in F$ and every $0 < r < r_0$, where $r_0 = \infty$ in case (1).

Fix $x \in F$ and $0 < r < 2\mu(X)$, and let a and b be as in Lemma 2.8. Then

$$H^s(B_\delta(x, r) \cap F) \leq H^s(B(x, b) \cap F) \leq c\mu^s(B(x, b)) \leq cC_2^s r^s,$$

and

$$H^s(B_\delta(x, r) \cap F) \geq H^s(B(x, a) \cap F) \geq c^{-1}\mu^s(B(x, a)) \geq c^{-1}C_1^s r^s,$$

provided that $a, b < r_0$. Then (1) is proved. On the other hand, (2) is showed if we can choice $r_1 \leq 2\mu(X)$ such that $r < r_1$ implies $a, b < r_0$. In order to do this, let C be such that if $x \in F$ and $\mu(B(x, t)) \leq C$, then $t < r_0$ (see Lemma 2.7). Let us define $r_1 = \min\{2\mu(X), C/C_2\}$, with C_2 the constant that appears in Lemma 2.8. Then $\mu(B(x, a))$ and $\mu(B(x, b))$ are both bounded above by C , so that $a, b < r_0$. \square

Remark 2.9. We want to point out that the condition “ F consistent with μ ” in Theorem 2.5 is sufficient for a locally s -set in (X, d, μ) to be a locally s -set in (X, δ) , but is not necessary. In fact, let us consider (X, d, μ) and F as in Example 2.4. Taking

$$\nu(E) = \int_{E \cap F} |s|^{\beta/2} ds$$

as the Borel measure supported on F we can show that F is locally $\frac{1}{2}$ -set in (X, δ) , and from Theorem 2.1 we have that F is locally $\frac{1}{2}$ -set in (X, d, μ) . Nevertheless, it is easy to see that $F = \{(t, 0) : t \geq a\}$ is not consistent with μ if $\beta < 0$.

3. PROOF OF PROPOSITION 2.6

In order to prove Proposition 2.6, we shall use the following covering type lemma that we shall prove at the end of this section.

Lemma 3.1. *Let G be a bounded subset of X . For a given $\rho > 0$, there exists a finite covering $\{B(x_i, r_i), i = 1, \dots, I_\rho\}$ of G by d -balls with $x_i \in G$ and $\mu(B(x_i, r_i)) < \rho$. Also, each $y \in X$ belongs to at most Λ of such balls, where Λ is a geometric constant which depends only on X .*

Remark 3.2. Notice that if $\rho \leq \mu(G)$, then $r_i \leq \text{diam}(G)$ for every i . In fact, let us assume that $r_i > \text{diam}(G)$ for some i . Then $G \subseteq B(x_i, r_i)$, so that $\mu(G) \leq \mu(B(x_i, r_i)) < \rho \leq \mu(G)$, which is an absurd.

Proof of Proposition 2.6. By hypothesis there exist $r_0 > 0$, a constant $c \geq 1$ and a Borel measure m supported on F such that

$$c^{-1}\mu(B(x, r))^s \leq m(B(x, r)) \leq c\mu(B(x, r))^s,$$

for every $x \in F$ and every $0 < r < r_0$. Here r_0 is infinite if F is an unbounded s -set in (X, d, μ) , and is finite otherwise.

Fix $x \in F$, $0 < r < r_0$ and $\varepsilon > 0$. For each $\rho > 0$, there exists a covering $\{B_i = B(x_i, r_i)\}$ of $B(x, r) \cap F$ by balls such that $\mu(B_i) < \rho$ and

$$\sum_{i \geq 1} \mu^s(B_i) < H_\rho^s(B(x, r) \cap F) + \varepsilon \leq H^s(B(x, r) \cap F) + \varepsilon.$$

Choosing an appropriated value of ρ , we can also obtain $r_i < r_0$ for every i . In fact, take $\rho = \mu(B(x, r))/A^\ell$ with ℓ an integer such that $2^\ell \geq 3K^2$. Then, since we can assume that each $B(x_i, r_i)$ intersects $B(x, r)$, if some $r_i \geq r_0$ then we have that $B(x, r) \subseteq B(x_i, 3K^2r_i)$. Hence $\mu(B(x, r)) \leq A^\ell \mu(B_i) < \mu(B(x, r))$, which is absurd. Then we can assume $r_i < r_0$ for every i , and hence

$$c^{-1} \mu(B(x, r))^s \leq m(B(x, r)) \leq \sum_i m(B_i) \leq c \sum_i \mu(B_i)^s.$$

Hence, $c^{-1} \mu(B(x, r))^s < cH^s(B(x, r) \cap F) + c\varepsilon$ for every $\varepsilon > 0$, which proves that

$$H^s(B(x, r) \cap F) \geq c^{-2} \mu(B(x, r))^s.$$

In order to obtain an upper bound for $H^s(B(x, r) \cap F)$, let us first assume that $r < \frac{r_0}{4K^2}$ and we fix $0 < \rho < \mu(B(x, r) \cap F)$. From Lemma 3.1, there exists a finite covering $\{B(x_i, r_i), i = 1, \dots, I_\rho\}$ of $B(x, r) \cap F$ by d -balls satisfying $\mu(B(x_i, r_i)) < \rho$, $x_i \in F$ and $r_i \leq 2Kr$. Also, each $y \in X$ belongs to at most Λ of such balls, where Λ is a geometric constant which does not depend on ρ , r or x . So, we have that

$$\begin{aligned} H_\rho^s(B(x, r) \cap F) &\leq \sum_{i=1}^{I_\rho} \mu(B(x_i, r_i))^s \\ &\leq c \sum_{i=1}^{I_\rho} m(B(x_i, r_i)) \\ &\leq c\Lambda m\left(\bigcup_{i=1}^{I_\rho} B(x_i, r_i)\right) \\ &\leq c\Lambda m(B(x, 4K^2r)) \\ &\leq c^2\Lambda \mu(B(x, 4K^2r))^s \\ &= \tilde{C} \mu(B(x, r))^s, \end{aligned}$$

with $\tilde{C} = c^2\Lambda A^j$, where j is a positive integer such that $2^{j-2} \geq K^2$. Taking $\rho \rightarrow 0$ we obtain the desired result for this case.

Finally, if r_0 is finite, we shall consider the case $\frac{r_0}{4K^2} \leq r < r_0$. In this case, since $B(x, r)$ is bounded, there exists a finite $r_0(8K^2)^{-1}$ -disperse maximal set in $B(x, r)$, let us say $U = \{x_1, \dots, x_I\}$, with $I \leq N^{2+\log_2 K}$. Then $B(x, r) \cap F \subseteq \bigcup_{i=1}^I B(x_i, \frac{r_0}{8K^2})$, and applying the previous case we obtain

$$H^s(B(x, r) \cap F) \leq \sum_{i=1}^I H^s\left(B\left(x_i, \frac{r_0}{8K^2}\right) \cap F\right) \leq \tilde{C}I\mu(B(x, 2Kr))^s,$$

and the result follows from the doubling property of μ . \square

For the proof of Lemma 3.1, we shall use the next result about the behavior of δ -diameter $\text{diam}_\delta(E) := \sup\{\delta(y, w) : y, w \in E\}$ of a bounded set E .

Lemma 3.3. *Let E be a bounded subset of X . For $B = B(x, \text{diam}(E))$ and $x \in E$ we have*

$$A^{-\ell}\mu(B) \leq \text{diam}_\delta(E) \leq A\mu(B),$$

where A is the doubling constant for μ and ℓ is a positive integer satisfying $\ell \geq \log_2(8K^3)$, with K the triangular constant for d .

Proof. Let us fix $x \in E$, and let y and w any two points in E . Since $y, w \in B(x, 2\text{diam}(E))$, from the definition of δ follows that $\delta(y, w) \leq \mu(B_d(x, 2\text{diam}(E))) \leq A\mu(B)$. Taking supreme the upper bound for $\text{diam}_\delta(E)$ is obtained.

For the lower bound, let $y_0, w_0 \in E$ such that $\text{diam}(E) < 2d(y_0, w_0)$. For a given $\varepsilon > 0$, let $B(x_0, r_0)$ be a ball containing y_0 and w_0 such that $\mu(B(x_0, r_0)) < \delta(y_0, w_0) + \varepsilon$. We claim that $B \subseteq B(x_0, 8K^3r_0)$. Assuming this fact true we have that

$$\text{diam}_\delta(F) \geq \delta(y_0, w_0) > \mu(B(x_0, r_0)) - \varepsilon \geq A^{-\ell}\mu(B) - \varepsilon.$$

By letting ε tends to zero we obtain the result. Only remains to prove the claim, for which fix $z \in B$. Then

$$\begin{aligned} d(z, x_0) &\leq K^2[d(x, x) + d(x, w_0) + d(w_0, x_0)] \\ &< K^2[2\text{diam}(E) + r_0] \\ &< K^2[4d(y_0, w_0) + r_0] \\ &< K^2[4K(d(y_0, x_0) + d(x_0, w_0)) + r_0] \\ &< 8K^3r_0, \end{aligned}$$

and the lemma is proved. \square

Proof of Lemma 3.1. Let us denote \tilde{K} the triangular constant for δ and \tilde{N} the constant for the finite metric dimension of (X, δ, μ) . Given $\rho > 0$, let

$t = \frac{\rho}{4\tilde{K}A^{\ell+1}}$, with ℓ as in Lemma 3.3. Set $U = \{x_1, \dots, x_{I_t}\}$ a finite t -disperse maximal set in G with respect to the quasi-metric δ . So that $\{B_\delta(x_i, t)\}$ is a covering of G . Let us define $B_i = B(x_i, r_i)$, with $r_i = 2\text{diam}(B_\delta(x_i, t))$. Let us first check that $\{B_i\}$ is covering of G . In fact, if $y \in G$ then there exists i such that $y \in B_\delta(x_i, t)$. Then

$$d(x_i, y) \leq \text{diam}(B_\delta(x_i, t)) < 2\text{diam}(B_\delta(x_i, t)),$$

so that $y \in B_i$. In order to estimate the measure of each B_i , using Lemma 3.3 with $E = B_\delta(x_i, t)$ we obtain

$$\mu(B_i) \leq A\mu(B(x_i, \text{diam}(B_\delta(x_i, t)))) \leq A^{\ell+1}\text{diam}_\delta(B_\delta(x_i, t)) \leq A^{\ell+1}2\tilde{K}t.$$

From the choice of t , we have $\mu(B_i) < \rho$. So that it only remains to prove that we can control the overlapping of this balls by a geometric constant Λ . In fact, for a fixed $y \in X$ we have that if $y \in B(x_i, r_i)$, then $B(y, r_i) \subseteq B(x_i, 2Kr_i)$. So that $\mu(B(y, r_i)) \leq A^p\rho$, with p and integer such that $2^{p-1} \geq K$, and then

$$x_i \in B(y, r_i) \subseteq B_\delta(y, 2\mu(B(y, r_i))) \subseteq B_\delta(y, 2A^p\rho) = B_\delta(y, 8\tilde{K}A^{\ell+p+1}t).$$

Hence, the number of balls $B(x_i, r_i)$ to which y belongs is less than or equal to the cardinal of $U \cap B_\delta(y, 2^m t)$, with m a natural number such that $2^m \geq 8\tilde{K}A^{\ell+p+1}$. Since U is t -disperse with respect to δ , we have that $\Lambda \leq \tilde{N}^m$ and the lemma is proved. \square

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