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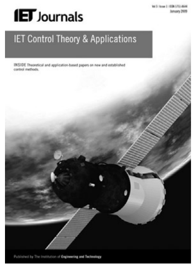

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Observer-based adaptive control using multiple-models switching and tuning

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Abstract: Despite the remarkable theoretical accomplishments and successful applications of adaptive control, this field is not mature enough to solve challenging problems where strict performance and robustness guarantees are required. The needs of an approach that explicitly accounts for robust performance and stability specifications is a critical to the design of practical adaptive control systems. Towards this goal, this study extends the robust adaptive controller using multiple models, switching and tuning to multiple input multiple output and non-linear systems. The use of 'extended superstability', instead of superstability, allows us to establish overall performance guarantees and reduce the conservativeness of the resulting closed-loop system. The authors show that under the proposed framework, the output and states remain bounded for bounded disturbances, as a direct consequence of the passivation properties of superstability. The effectiveness of the proposed algorithm is demonstrated in numerical simulations of a non-linear continuous stirred tank reactor.

1 Introduction

When model uncertainty is 'small', robust linear time invariant (LTI) theories, for example, H_∞, l_1 and μ-synthesis, ensures, when it is possible, the satisfaction of closed-loop objectives specified through meaningful engineering terms (frequency weights on relevant transfer functions, constraints on system time-domain response, constraints on parameters of control law, and so on). However, changes in operating conditions, system dynamics and/or degradation in components may lead to systems with fast time-varying uncertain coefficients, which cannot be stabilised by a single LTI controller [1] or conventional adaptive controllers [2].

Adaptive control copes with large parametric uncertainty by tunning the control law in response to estimated changes in the model. In conventional adaptive control [3] the controller parameters are computed in real time, based on the estimated plant model. The complicated relationship between plant parameters and robust controller gains has been the main obstacle to use conventional adaptive versions of modern robust compensator or the inclusion of any a priori information available. One way of avoiding these issues is through the use of multiple model adaptive control (MMAC) architecture. The MMAC architecture provides an attractive framework for combining adaptive and robust control tools and easily incorporates any information available a priori. It comprises two level of control: (i) an inner loop $\mathbb{C}(S(k))$, called 'multicontroller', that is able to generate finely tuned controls $u(k)$ and (ii) an outer-loop $\mathbb{S}$, called 'supervisor', that adjusts the multicontroller $\mathbb{C}(S(k))$ (see Fig. 1*a*). This task is typically done by selecting, weighting or blending the candidate controllers $C_l \in \mathcal{C}$ (*cover set* $\mathcal{C}$) based on system input/output data. Depending on the mechanism of adjustment employed by the supervisor $\mathbb{S}$, two families of MMAC controllers emerge: (i) switching-based MMAC where the adjustment is performed by switching among the controllers of $\mathcal{C}$ [4] and (ii) mixing-based MMAC where the adjustment is performed by continuously interpolating the controllers of $\mathcal{C}$ [5, 6].

In switching-based MMAC the multicontroller is only able to generate the candidate control laws since the supervisor can only select one of the controllers of $\mathcal{C}$ ($\mathbb{C}(S(k)) \in \mathcal{C}\ \forall k$). Switching is based on the 'certainty equivalence principle', the implementation and analysis of MMAC control is simplified by considering a 'finite cover' set $\mathcal{C}_m = \{C_l\ l \in \mathcal{L} = [1, \ldots, m]\}$ [7]. The compromise between robustness and performance is made off-line when $\mathcal{C}_m$ is designed. The controller selection is made by continuously comparing in real time suitably defined norm of estimation error, known as 'performance signals' $\mu_l(k)$, and the controller C_l with the smallest $\mu_l(k)$ is placed in the loop according to an appropriate switching logic [8]. Alternative approaches to supervisory control use Kalman filters and information measure [5], calibrate forecasts [9], set-valued observers [10], robust observers [11], state-dependent switching functions [12] and ν-gap metric combined with robust design techniques [13]. Stability results for these schemes show that all closed-loop signals are bounded and 'robust performance is only recovered in steady state' (see e.g. [13–15]).

In mixing-based MMAC the multicontroller generates, by continuous interpolation, a stable mixed of the candidate control laws [5, 6]. Hence, mixing-based MMAC generates

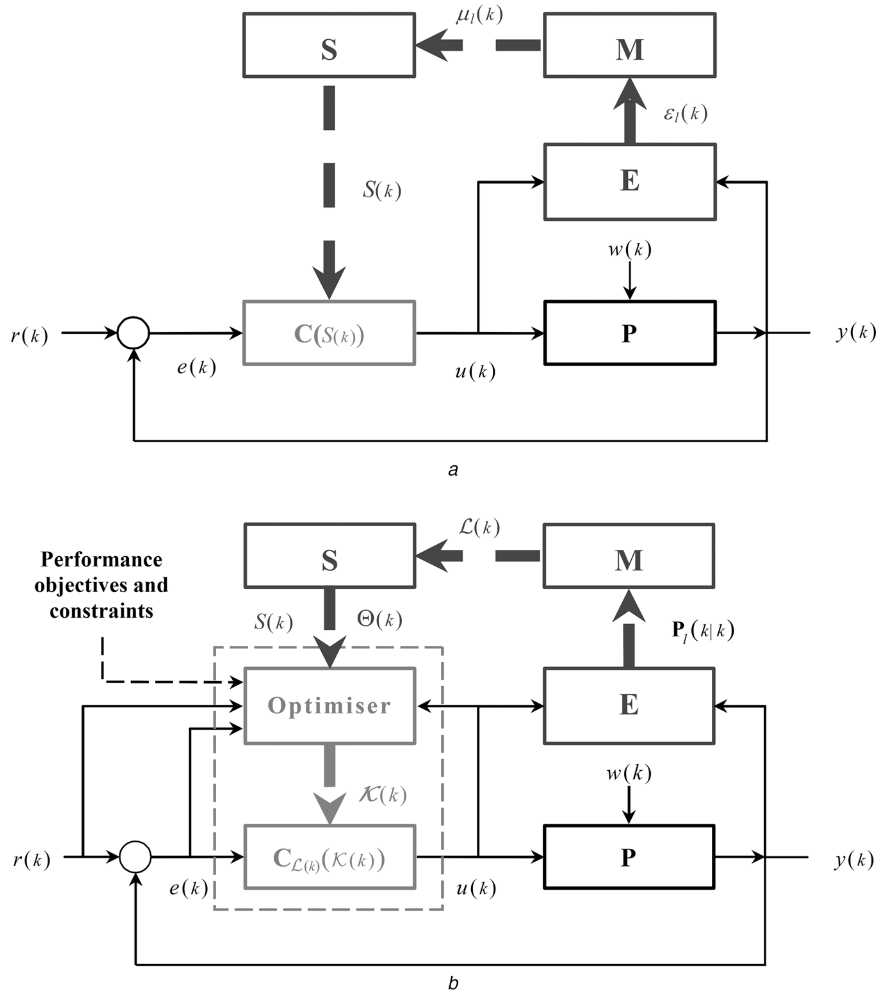

Fig. 1 *Structure of the supervisory controller*

a Multiple model adaptive control architecture

b Switching and tuning control supervisor

an infinite cover set $\mathcal{C}_\infty$ $(m \to \infty)$ such that the multicontroller $\mathbb{C}(S(k))$ evolves from one controller to another in a continuous way, avoiding the switching and its undesirable behaviours. Robust MMAC (RMMAC) uses the conditional probability of each model of $\mathcal{C}_m$, computed by the supervisor $\mathbb{S}$ through a set of Kalman filters, as mixing weights of the controllers outputs [5]. The accurate knowledge of disturbances and noise models, as well as the satisfaction of standard Kalman filter assumptions, are essential requirements to achieve good performance. To overcome this limitation, the RMMAC supervisor has been modified, replacing the Kalman filters by robust observers and employing the output prediction error energy to compute the weights [11]. On the other hand, 'multiple model adaptive control with mixing' uses a supervisor $\mathbb{S}$ based on robust parameter estimation to interpolate the adaptive controller parameters [6]. As the discontinuous switching logic has been replaced with a smooth and stable interpolation, mixing-based schemes may not be able to respond to dramatic changes in plant as fast as switching-based schemes can.

From a practical point of view, these methods suffer from the following drawbacks: (i) the number of models m needed to assure that at least one of the models P_l of the set of admissible models $\mathcal{P}_\mathcal{L}$ is sufficiently close to the plant P in parameter space is large and grows exponentially with the dimension of the unknown parameter vector, (ii) the information provided by every model is not used efficiently since it is only used to locate a model close to the plant, (iii) the switching results in discontinuous control signals that affect the closed-loop stability and (iv) the identification and control are coupled, since the model chosen dictates the choice of the controller, leaving little freedom for new designs to emerge. In spite of these shortcomings, the methods are found to perform satisfactorily when the plant is time-invariant and the number of models is sufficiently large.

To address the first two problems, Narendra and Han [16, 17] proposed a new way of using multiple models for the identification of an unknown LTI plant. The proposed approach provides an estimate which depends on the collective outputs of all the models, and can be viewed as a time-varying convex combination of the estimates. It decomposes the identification problem into two levels: the adaptation of a finite number of individual models (first level of adaptation), using conventional adaptive identifiers, which are used to built a polytopic model that is re-parameterise at each sample, using the information generated by the individual identifiers, to identify the unknown plant (second level of adaptation). This procedure result in an improved stability and faster convergence in the control of time-invariant plants.

The other two problems (the discontinuity of control signals and the coupled of identification and control) have been

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addressed by Giovanini [18]. The architecture proposed, called 'adaptive switching and tuning control' (see Fig. 1*b*), was introduced and analysed only for SISO systems using transfer functions. It differs from previous developments in the following ways:

1. The multicontroller $\mathbb{C}(S(k))$ is designed on-line using a linear optimisation problem and a time-varying set of models $\mathcal{P}_{\mathcal{L}(k)}$ such that the resulting closed-loop systems is dissipative.
2. The set $\mathcal{P}_{\mathcal{L}(k)}$ is built by excluding from $\mathcal{P}_{\mathcal{L}}$ those models $P_l \in \mathcal{P}_{\mathcal{L}}$ that cannot explain the time evolution of the inputs and outputs of the plant P.
3. The switching takes places in the constraints of the optimisation problem and
4. The objective function is built through a time-varying convex combination of the objective function corresponding to $\mathcal{P}_{\mathcal{L}(k)}$.

In this way, this architecture can generate an infinite number of controllers ($\mathbb{C}(S(k)) \in \mathcal{C}_\infty$) with different degrees of robustness, according with the system information available at each sample, combining the advantages of both MMAC architectures: the fast adaptations of switching-based MMAC and the continuous evolution of mixing-based MMAC with the addition of robust stability and performance guarantees for all time. The properties of the resulting closed-loop system are a direct result of the identification process that is able to find a time-varying combination of models that are close to the plant ($\mathcal{P}_{\mathcal{L}(k)}$) and the properties of superstable systems that guaranteed the closed-loop performance.

The immediate motivation of this work is the extension of the adaptive switching and tuning control scheme to MIMO and non-linear systems. This paper is structured as follow: the class of superstable system is recalled for state-space models and its main properties are analysed at the beginning of Section 2. Then, the class of linear parameter varying (LPV) superstable systems is introduced and analysed. These results are extended to the more general class of 'extended superstable systems'. These results will allow us to formulate a linear optimisation problem for LPV systems passivation at the end of this section. The use of extended superstability, instead of superstability, allows us to reduce the conservativeness of the results. In Section 3, the multiple models switching and tuning scheme is revisited and reformulated for state-space models. Stability of the resulting closed-loop system is analysed in Section 4 using the results developed in Section 2. Section 5 presents numerical simulations of an irreversible exothermic reaction that is carried out in a constant volume continuous stirred tank reactor. Concluding remarks and possible extensions of the proposed adaptive control algorithm are presented in Section 6.

2 Preliminaries

2.1 Notation

In the sequel $A \in \mathbb{R}^{n\times m}$ is $n \times m$ matrix and its transpose is denoted by A^{T}. For a vector $x \in \mathbb{R}^n$, $\|x\|_2$ is the Euclidean norm $(x^{\mathrm{T}}x)^{1/2}$ and the corresponding induced matrix norm of A is denoted as $\|A\|_2$. The index set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{I}_n$, the ∞-norm of x is denoted by $\|x\|_\infty = \max_{i\in\mathcal{I}_n} |x_i|$ and the corresponding induced matrix norm of A, denoted as $\|A\|_1 = \max_{j\in\mathcal{I}_m} \sum_{i\in\mathcal{I}_n} |a_{ij}|$. Finally, and l_∞ stands for $\|x(k)\|_\infty \le 1 \ \forall k \ge 0$.

2.2 Superstable systems

System superstability play a key role in the development of the adaptive switching and tuning control algorithm since it provides a simple framework for solving the 'passive control problem' using bounded realness. The aim of this problem is to design a controller such that the resulting closed-loop system satisfied a certain passivity performance index, guaranteeing the closed-loop stability [19]. Since there is a one-to-one relationship between positive realness and passivity [20] and superstability and positive realness, passivity analysis can be converted into a superstability analysis. This relationship was exploited to design the passivation adaptive controller at each sample [18].

For a LTI state-space discrete system

$$x(k+1) = Ax(k) + B_W w(k) \quad x(0) = x_0 \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ is an exogenous input such that $w(k) \in l_\infty$ and A and B_W are matrices of proper size, superstability is defined as $q = \|A\|_1 < 1$ [21]. The main property of a superstable system is the non-asymptotic estimates for arbitrary initial conditions ($\|x_0\|_\infty \le \mu$) and bounded inputs ($w(k) \in l_\infty$)

$$\|x(k)\|_\infty \le \eta + q^k \max\{0, \mu - \eta\} \quad \forall k \ge 0 \tag{2}$$

where $\eta = \|B_W\|_1/(1-q)$. If system (1) becomes autonomous ($w(k) \equiv 0$), the estimates (2) is transformed into

$$\|x(k)\|_\infty \le q^k \mu \quad \forall k \ge 0 \tag{3}$$

which describes the system behaviour with respect to initial conditions.

In the particular case of $\|x_0\|_\infty \le \eta$, the estimate (2) becomes $\|x(k)\|_\infty \le \eta$. This fact implies the existence of an *invariant set* $\mathcal{Q} = \{\|x(k)\|_\infty \le \eta \ \forall k, \ \forall w(k) \in l_\infty\}$ and a Lyapunov function $V(x) = \|x(k)\|_\infty$ with the following properties: (i) It is piecewise-linear, (ii) it grows linearly $V(\delta x) = \delta V(x) \ \forall \delta \ge 0$ and (iii) its time estimate is given by $V(x(k)) \le q^k V(x_0)$.

Remark 1: System superstability implies the positive realness of system (1) since $\rho(A) = \max_{i\in\mathcal{I}_n} |\lambda_i(A)| \le \|A\|_1 < 1$.

The results derived for LTI systems can be extended to LPV systems of the form

$$x(k+1) = A(S(k))x(k) + B_W(S(k))w(k) \tag{4}$$

where

$$A(S(k)) = \sum_{l\in\mathcal{L}} s_l(k)A_l, \quad B_W(S(k)) = \sum_{l\in\mathcal{L}} s_l(k)B_{W\,l} \tag{5}$$

$S(k) = [s_l(k)] \in \mathcal{S}$ is the time-varying parameter, $\mathcal{S} \subset \mathbb{R}^m$ is a compact set, $\mathcal{L}$ is a finite index set and $P_l = (A_l, B_{W\,l})$ $l \in \mathcal{L}$ are the models of polytope $\mathcal{P}_{\mathcal{L}}$.

Definition 1: A LPV system is superstable if $\|A(S(k))\|_1 < 1$ $\forall S(k) \in \mathcal{S}$.

Lemma 1: If the LPV system (4) is superstable and the initial conditions $\|x_0\|_\infty \le \mu$, then the following facts hold for all admissible trajectories of $S(k) \in \mathcal{S}$

- for $w(k) \in l_\infty$ the norm of the LPV system states is bounded by

$$\|x(k)\|_\infty \le \bar{\eta} + \bar{\gamma}^k \max\{0, \mu - \bar{\eta}\} \tag{6}$$

where

$$\bar{\eta} = \frac{\bar{\beta}}{1-\bar{\gamma}}, \quad \bar{\beta} = \sup_{\forall k} \sup_{\forall S(k)\in\mathcal{S}} \|B_W(S(k))\|_1,$$
$$\bar{\gamma} = \sup_{\forall k} \sup_{\forall S(k)\in\mathcal{S}} \|A(S(k))\|_1 \tag{7}$$

- for $w(k) \equiv 0$ the norm of the LPV system states is bounded by

$$\|x(k)\|_\infty \le \bar{\gamma}^k \mu \tag{8}$$

Proof: See Appendix 1. □

Similarly to LTI systems, the superstability of LPV system implies the existence of an invariant set $\mathcal{Q} = \{\|x(k)\|_\infty \le \bar{\eta}\ \forall k,\ \forall w(k) \in l_\infty\}$ and a common Lyapunov function $\bar{V}(x) = \|x(k)\|_\infty \forall P_l \in \mathcal{P_L}$ with similar properties of $V(x)$: it is piecewise-linear and its time estimate is given by $\bar{V}(x(k)) \le \bar{\gamma}^k \bar{V}(x_0)$.

LPV models were introduced as an approximate description of non-linear dynamical systems for the benefit of system analysis and controller design. Under appropriate conditions, and through an analysis of the parametrisation it is possible to establish relevant system properties and formulate control design tools that aim at stability, optimality and robustness. For this class of models two possible cases for $S(k)$ can be considered:

- Systems in which $A(S(k))$ and $B_W(S(k))$ can attain any value inside $\mathcal{P_L}$

$$\mathcal{S} = \left\{S(k) : \sum_{l\in\mathcal{L}} s_l(k) = 1, s_l(k) \ge 0\right\}\ \forall k \tag{9}$$

known as polytopic linear systems (PLM).

- Systems in which $A(S(k))$ and $B_W(S(k))$ can attain only on the models of $\mathcal{P_L}$

$$\mathcal{S} = \left\{S(k) : s_j(k) = 1,\ s_i(k) = 0 \quad \forall i \ne j \quad i,j \in \mathcal{L}\right\}\ \forall k \tag{10}$$

known as 'switching systems'.

It is clear from Lemma 1 that switching or scheduling superstable LTI systems leads to superstable LPV systems. It is well known that this does not hold for the more general class of stable systems [22–24]. Furthermore, the analysis and design tools for these classes of LPV systems are dominated by Lyapunov stability theory and its generalisations (see [19, 25–29] and the references therein). In general, piecewise [26] or parameter-dependent [27] Lyapunov functions for all subsystems are required to guarantee system stability. Other approaches employ multiple storage functions to obtain passivity conditions [29], which are employed to design the passive controller. The construction of such Lyapunov functions and the derivation of passivity conditions involve the solution of linear matrix inequalities whose feasibility needs to be checked for each case.

Superstability is too rigid because a fixed Lyapunov function $V(x) = \|x(k)\|_\infty$ is specified and therefore is very difficult to attain, which is reflected in the difficulties to design the passive adaptive controller [18]. A more flexible approach based on a diagonal transformation to a superstable form can be used. In this way, the set of 'extended superstable matrices' $\mathcal{E}_S$ can be defined.

Definition 2: The system (1) belongs to the class of 'extended superstable systems' $\mathcal{E}_S$ if there is a diagonal matrix $D = \operatorname{diag} d_i > 0\ i \in \mathcal{I}_n$ such that $\|D^{-1}AD\|_1 < 1$ [30].

All the facts established for superstable systems can be extended to the class of extended superstable systems $\mathcal{E}_S$, and the Lyapunov function results from the compilation of piecewise-linear Lyapunov functions $V(x) = \max_{i\in\mathcal{I}_n} |x_i/d_i|$.

2.3 Superstability-based controller design

Stability is invariant to linear transformation of coordinates while superstability can be lost or acquired upon passing to other coordinates, since it is formulated only in terms of the entries of matrix A. Therefore a system can be superstabilised by means of state feedback. In this work, we consider two design problems: (i) the optimal rejection of bounded disturbances and (ii) the minimisation of performance index.

2.3.1 Rejection of bounded disturbances:

Let us consider an LTI system with control and bounded disturbances

$$x(k+1) = Ax(k) + B_U u(k) + B_W w(k) \tag{11}$$

where $u(k) \in \mathbb{R}^{n_u}$ and $w(k) \in l_\infty$. We are interested on determine a control law $u(k) = \mathcal{K}_c x(k)$ that superstabilises the closed-loop system and minimises the performance index

$$J_C = \sup_{w(k)\in l_\infty} \sup_{\forall k} \|x(k)\|_\infty \tag{12}$$

Theorem 1: The control law $\mathcal{K}_c$ that minimises (11) and guarantees $A + B_U\mathcal{K}_c \in \mathcal{E}_S$ is obtained by solving this optimisation problem

$$\min_{q\in[0,1)} \min_{\omega, Z_c, D} \frac{\omega}{1-q}\|B_W\|_1$$
$$\text{st.}$$
$$\sum_j \left|a_{ij}d_j + \sum_s b_{is}z^c_{sj}\right| \le qd_i \quad i,j,s \in \mathcal{I}_{n_x} \tag{13}$$
$$1 \le d_i \le \omega$$
$$\omega \ge 1$$

where $\mathcal{K}_c = Z_c D^{-1}$.

Proof: See Appendix 2. □

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2.3.2 Minimisation of performance index: Now, let us consider the optimal control problem with a linear performance functional

$$J_C = \sum_{k=0}^{\infty} \|x(k)\|_\infty + \alpha \|u(k)\|_\infty \qquad (14)$$

which is known as linear linear regulator problem. If we require $A + B_U\mathcal{K}_c \in \mathcal{E}_S$ and we use the estimate (3), then (see Appendix 3)

$$J_C \le \frac{1 + \alpha\|\mathcal{K}_c\|_1}{1 - \|\tilde{A}_c\|_1}\mu \qquad (15)$$

where $\tilde{A}_c = D^{-1}(A + B_U\mathcal{K}_c)D$. In turn, the minimisation of (13) becomes the parametric problem

$$\begin{gathered}\min_{q\in[0,1)} \min_{\omega,Z_c,D} \frac{1}{1-q}(1 + \alpha\|Z_c\|_1)\\ \text{st.}\\ \sum_j \left| a_{ij}d_j + \sum_s b_{is}z_{sj}^c \right| \le qd_i, \quad i,j,s \in \mathcal{I}_{n_x} \qquad (16)\\ 1 \le d_i \le \omega\\ \omega \ge 1\end{gathered}$$

2.3.3 Robust design: The effect of uncertainty on the controller design can be easily included into problems (12) and (15). The structure of the resulting problems will depend on the uncertainty description adopted. If the additive description is adopted

$$\bar{A} = A + \Delta, \quad |\Delta_{ij}| \le m_{ij}, \quad i,j \in \mathcal{I}_{n_x} \qquad (17)$$

the superstability constraints of problems (12) and (15) become

$$\sum_j \left| a_{ij}d_j + \sum_s b_{is}z_{sj}^c \right| + m_{ij}d_j \le qd_i, \quad i,j,s \in \mathcal{I}_{n_x} \qquad (18)$$

If PLM model (5) and (9) is employed to represent the uncertain system, the robust disturbance attenuation problem becomes a multi-objective optimisation problem

$$\begin{gathered}\min_{q\in[0,1)} \min_{\omega,Z_c,D} \frac{\omega}{1-q}\sum_{l\in\mathcal{L}} s_l(k)\|B_{W\,l}\|_1\\ \text{st.}\\ \sum_j \left| a_{ij}^l d_j + \sum_s \sum b_{is}^l z_{sj}^c \right| \le qd_i, \quad i,j,s \in \mathcal{I}_{n_x}, l \in \mathcal{L} \qquad (19)\\ 1 \le d_i \le \omega\\ \omega \ge 1\end{gathered}$$

This optimisation problem corresponds to a hybrid characterisation of the robust design problem where closed-loop performance is measured through a weighted-norm objective function

$$\frac{\omega}{1-q}\sum_{l\in\mathcal{L}} s_l(k)\|B_{W\,l}\|_1$$

that includes the closed-loop performances of each model $P_l \in \mathcal{P}_\mathcal{L}$, whereas the superstability is guaranteed through constraints that enforce the superstability of each model $P_l \in \mathcal{P}_\mathcal{L}$.

In the case of the robust optimal control problem, the optimisation becomes the simultaneous stabilisation problem

$$\begin{gathered}\min_{q\in[0,1)} \min_{\omega,Z_c,D} \frac{1}{1-q}(1 + \alpha\|Z_c\|_1)\\ \text{st.}\\ \sum_j \left| a_{ij}^l d_j + \sum_s b_{is}^l z_{sj}^c \right| \le qd_i, \quad i,j,s \in \mathcal{I}_{n_x}, l \in \mathcal{L}, \qquad (20)\\ 1 \le d_i \le \omega\\ \omega \ge 1\end{gathered}$$

Remark 2: The solutions of problems (19) and (20) guarantee the existence of a piecewise-linear Lyapunov function that is common to all models $\mathcal{P}_\mathcal{L}$, ensuring the positive realness of the PLM model and its dissipativity.

Finally, these approaches to robust design can be combined to tackle the uncertainty of individual models adding the extra terms $m_{ij}d_{ij}$ in the constraints of problems (19) and (20).

2.4 State estimation

Given the discrete-time LTI system

$$\begin{aligned}x(k+1) &= Ax(k) + B_W w(k), \quad x(0) = x_0, w(k), v(k) \in l_\infty\\ y(k) &= Cx(k) + D_V v(k) \qquad (21)\end{aligned}$$

we will consider the dual problem of rejection of bounded disturbances: estimate the state $x(k)$ using the output $y(k)$. In this case we are interested on determine the observer gain $\mathcal{K}_o$ such that the observer matrix $A_o = A + \mathcal{K}_oC \in \mathcal{E}_S$ and minimises the performance index

$$J_E = \max_{w(k)\in l_\infty} \max_{\forall k} \|x(k)\|_\infty \qquad (22)$$

This index provides the maximal reduction of the effect of bounded disturbances on the estimate $\hat{x}(k)$. Under these conditions, it is possible to estimate an invariant box $\mathcal{B}$ whose size is given by

$$\eta = \frac{d_{\max}(\|B_W + \mathcal{K}_oD_V\|_1)}{d_{\min}(1 - \|\tilde{A}_o\|_1)} \qquad (23)$$

where $\tilde{A}_o = D^{-1}(A + \mathcal{K}_oC)D$. In this context, best box is meant that having the minimum greatest side among the feasible ones. Again, the minimisation of (22) is transformed into the parametric linear optimisation problem

$$\begin{gathered}\min_{q\in[0,1)} \min_{\omega,Z_o,D} \frac{\omega}{1-q}(\|B_W\|_1 + \|Z_oD_V\|_1)\\ \text{st.}\\ \sum_j \left| a_{ij}d_j + \sum_s z_{is}^o c_{sj} \right| \le qd_i \quad i,j,s \in \mathcal{I}_{n_x} \qquad (24)\\ 1 \le d_i \le \omega\\ \omega \ge 1\end{gathered}$$

where $\mathcal{K}_o = Z_oD^{-1}$. The proof of these results follows the same lines of Theorem 1.

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Following the same lines like in Robust design subsection, it is possible to derive similar results on robust state estimation. The structure of the resulting optimisation problem for a polytopic description is the same of (19), but only differ in the cost function minimised: $\sum_{l\in\mathcal{L}} s_l(k)(\|B_{W\,l}\|_1 + \|Z_o D_{V\,l}\|_1)$ instead of $\sum_{l\in\mathcal{L}} s_l(k)\|B_{W\,l}\|_1$. Finally, the uncertainty of individual models can be considered by adding the extra terms $m_{ij}d_{ij}$ in the constraints of the problem, like in the controller design subsection.

3 Adaptive control using multiple models, switching and tuning

The objective of MMAC schemes is to control an unknown plant P that is subject to an exogenous unknown disturbance $w(k) \in l_\infty$. MMAC controllers use information obtained online to construct the compensator $\mathbb{C}(S(k))$ by switching or blending the candidates robust non-adaptive compensator $C_l \in \mathcal{C}_m$. The nature of signal $S(k)$ will depend on the type of MMAC considered:

- $S(k)$: $[0, \infty) \to \mathcal{L}$ is admissible if it is piecewise constant with a dwell-time $\tau > 0$ such that consecutive switching times $t_a < t_b$ satisfy $t_b - t_a \geq \tau$ for switching-based MMAC schemes,
- $S(k)$: $[0, \infty) \to s_l(k) \in [0, 1]$ $\forall l \in \mathcal{L}$ is admissible if it is piecewise constant and $\sum_l s_l(k) = 1$, $s_l(k) \geq 0$ for mixing-based MMAC schemes.

In this work, we adopt the same approach like in [18]: a soft-variable compensator $\mathbb{C}_{\mathcal{L}(k)}(\mathcal{K}(k))$ is designed using superstability concepts and a time-varying PLM $\mathcal{P}_{\mathcal{L}(k)}$. The structure of the adaptive controller is shown in Fig. 1*b*. At every sample, the monitoring block $\mathbb{M}$ selects, using information generated by estimation block $\mathbb{E}$. Only those models $P_l \in \mathcal{P}_{\mathcal{L}}$ that explain the input–output trajectory of the plant P such that $P \in \mathcal{P}_{\mathcal{L}(k)}$. Polytopes $\mathcal{P}_{\mathcal{L}}$ and $\mathcal{P}_{\mathcal{L}(k)}$ are characterised by the set of indexes $\mathcal{L}$ and $\mathcal{L}(k)$ respectively. Then, $\mathcal{L}(k)$ and the covariance matrix of each observer of $\mathbb{E}$ ($\mathbf{P}_l(k|k)$) are used by the supervisor block $\mathbb{S}$ to generate the switching signal $S(k)$

$$s_l(k) = \begin{cases} 1 & \forall l \in \mathcal{L}(k) \\ 0 & \forall l \notin \mathcal{L}(k) \end{cases} \tag{25}$$

and the optimisation weights $\Theta(k)$

$$\theta_l(k) = \begin{cases} 1 - \dfrac{\operatorname{tr}(\mathbf{P}_l(k|k))}{\sum_{l\in\mathcal{L}(k)} \operatorname{tr}(\mathbf{P}_l(k|k))} & \forall l \in \mathcal{L}(k) \\ 0 & \forall l \notin \mathcal{L}(k) \end{cases} \tag{26}$$

Remark 3: The switching signals $S(k)$ are used to built the polytope $\mathcal{P}_{\mathcal{L}(k)}$, selecting the corresponding models of $\mathcal{P}_{\mathcal{L}}$ and the optimisation weights $\Theta(k)$ are used to re parametrised the objective function used to measure the closed-loop performance in the optimisation problem.

3.1 Robust adaptive compensator

The plant to be considered is represented by

$$\begin{aligned} x(k+1) &= Ax(k) + B_U u(k) + B_W w(k) \\ y(k) &= Cx(k) + D_V v(k) \end{aligned} \tag{27}$$

which satisfies the standard assumptions ((A, B_U) is stabilisable and (A, C) is detectable) and $w(k), v(k) \in l_\infty$. A dynamic observer-based controller is chosen to achieve the desired control performance

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B_U u(k) + \chi(k) \\ \hat{y}(k) &= C\hat{x}(k) \\ \xi(k+1) &= A_o\xi(k) + B_o(y(k) - \hat{y}(k)) \\ \chi(k) &= C_o\xi(k) + \mathcal{K}_o(y(k) - \hat{y}(k)) \\ u(k) &= \mathcal{K}_c\hat{x}(k) \end{aligned} \tag{28}$$

where $A_o, B_o, C_o, \mathcal{K}_o$ and $\mathcal{K}_c$ are real matrices of appropriate dimensions to be designed, $\chi(k) \in \mathbb{R}^{n_x}$ is the correction signal and $\xi(k)$ is the auxiliary state. A dynamic observer can be regarded as an extension of the Luenberger observer that allows the inclusion of an additional dynamics (third and fourth equations of (28)) into the classical observer [first fourth equations of (28)] to achieve the desired closed-loop properties [31].

Remark 4: When $A_o = 0, B_o = 0$ and $C_o = 0$, the dynamic observer (28) is equivalent to the Luenberger observer.

Denoting $\varepsilon(k) = x(k) - \hat{x}(k)$, $\tilde{x}(k) = [x(k)^{\mathrm{T}}\varepsilon(k)^{\mathrm{T}}\xi(k)^{\mathrm{T}}]^{\mathrm{T}}$ and $d(k) = [w(k)^{\mathrm{T}}v(k)^{\mathrm{T}}]^{\mathrm{T}}$, the augmented system is given by

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}x(k) + \tilde{B}_D d(k) \\ y(k) &= \tilde{C}\tilde{x}(k) + D_V v(k) \end{aligned} \tag{29}$$

where

$$\tilde{A} = \begin{bmatrix} A + B_U\mathcal{K}_c & -B_U\mathcal{K}_c & 0 \\ 0 & A + \mathcal{K}_o C & -C_o \\ 0 & B_o C & A_o \end{bmatrix}$$
$$\tilde{B}_D = \begin{bmatrix} B_W & 0 \\ B_W & -\mathcal{K}_o D_V \\ 0 & B_o D_V \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \tag{30}$$

In the following we will assume, without loose of generality, a disturbance attenuation problem as a control and estimation objectives, which leads to the following combined performance index

$$\begin{aligned} J &= \alpha \sup_{\forall k} \|x(k)\|_\infty + (1-\alpha) \sup_{\forall k} \|\varepsilon(k)\|_\infty \quad \alpha \in [0, 1) \\ &= \alpha\|B_U Z_c\|_1 + (1-\alpha)\|Z_o D_V + C_o\|_1 + (1-\alpha)\|B_o D_V\|_1 \end{aligned} \tag{31}$$

where α is the parameter that allows to control the significance of each problem. The first term of (31) corresponds to control objective, which quantifies the effect of estimation error $\varepsilon(k)$ on the control signal. The effect of disturbance $w(k)$ was removed because it only offset J with a constant ($\|B_W\|_1$). The second term corresponds to the estimation objective measuring the effects of external disturbance $v(k)$ and observer states $\xi(k)$ on estimation error $\varepsilon(k)$. Finally, the third term corresponds to the effect of the residual $\varepsilon(k)$ and disturbance $v(k)$ on the states of the dynamic observer $\xi(k)$. These couplings can be clearly seen in the augmented system matrices (30).

The adaptive compensator $\mathbb{C}_{\mathcal{L}(k)}(\mathcal{K}(k))$ is obtained solving a modified version of optimisation problem (19) that includes the signals $S(k)$ and $\Theta(k)$, to select and weight

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the models $P_l \in \mathcal{P}_{\mathcal{L}(k)}$. The resulting optimisation problem is given by

$$\min_{q\in[0,1)} \min_{\substack{\omega, Z_c, Z_o, D,\\ A_o, B_o, C_o}} \frac{\omega}{1-q} \sum_{l\in\mathcal{L}} \theta_l(k) J_l$$

$$\text{st.}$$

$$s_l(k)\sum_j \left| a_{ij}^l d_j + \sum_s b_{is}^l z_{sj}^c \right| \le q d_i, \quad i,j,s \in \mathcal{I}_{n_x}, l \in \mathcal{L} \quad (32)$$

$$s_l(k)\sum_j \left| a_{ij}^l d_j + \sum_s z_{is}^o c_{sj}^l \right| \le q d_i$$

$$\sum_j \sum |a_{ij}^o| \le q d_i, \quad i,j \in \mathcal{I}_{n_o}$$

$$1 \le d_i \le \omega$$

$$\omega \ge 1$$

where J_l is the performance index for robust simultaneous design of the controller and observer (31).

If the desired closed-loop performance is defined through constraints on the closed-loop variables, the feasibility of optimisation problem (32) will depend on these constraints. One way of ensuring the feasibility of the optimisation problem is by softening performance constrains with slack variables and then penalised their deviation by including the slack variables in the objective function (32).

Remark 5: The use of an infinite number of controllers increase the flexibility in term of the control objectives, robustness and performance, since the values of the eigenvalues can continuously change and the regulation rate will thus be greater, than would be in the case of using a 'switching-based' controller.

Remark 6: The mechanism employed to built $\mathcal{P}_{\mathcal{L}(k)}$ allows a quick and robust adaptation to track system's changes, in comparison with 'mixing-based' controllers.

3.2 Supervisory algorithm

The role of the supervisory algorithm is to built $\mathcal{P}_{\mathcal{L}(k)}$ by excluding those models $P_l \in \mathcal{P}_{\mathcal{L}}$ that do not explain the time evolution of the input–output trajectories of plant P. $\mathcal{P}_{\mathcal{L}(k)}$ is build using the switching signal $S(k)$, which defines the models P_l that will be employed by the optimisation problem (32). The problem of 'disqualifying' models is addressed using set-valued observers [10]. This type of observers assume that the initial conditions of the plant is uncertain, there are bounded disturbances acting upon the plant and the measurements are corrupted with noise. Therefore the estimate is a set ('confidence set') instead of a single point. In this work, we use the robust set-valued observer proposed by El Ghaoui and Calafiore [33] that provides the minimal size ellipsoid of confidence $\mathcal{E}_l(\hat{x}_l(k), \mathbf{E}_l(k))$, defined by the central estimated $\hat{x}_l(k)$ and shape matrix $\mathbf{E}_l(k)$ and its size is measured by means of the sum of squared semi-axes lengths given by $\mathrm{tr}(\mathbf{E}_l(k|k)\mathbf{E}_l^{\mathrm{T}}(k|k)) = \mathrm{tr}(\mathbf{P}_l(k|k))$, computed recursively through two convex optimisation problems: (i) **Problem 1** computes the minimal size ellipsoid of confidence for **the estimate prediction** $\mathcal{E}_l(\hat{x}_l(k), \mathbf{E}_l(k|k-1))$ and (ii) **Problem 2** computes the minimal size ellipsoid of confidence for **the measurement update** $\mathcal{E}_l(\hat{x}_l(k), \mathbf{E}_l(k|k))$. If $\mathrm{tr}(\mathbf{P}_l(k|k)) \simeq 0$, then $\mathcal{E}_l(\hat{x}_l(k), \mathbf{E}_l(k|k-1)) \bigcap \mathcal{E}_l(\hat{x}_l(k|k), \mathbf{E}_l(k|k)) = \emptyset$ and the measurements are not compatible with model l, thus the model can be discarded. Based upon this fact, the algorithm to construct $\mathcal{P}_{(k)}$ is described in the Supervisory Algorithm showed in Fig. 2.

Data: $\mathcal{E}_l\left(\hat{x}_l(0), \mathbf{E}_l(0)\right), k = 1, \varepsilon > 0\ \forall l \in \mathcal{L}$

1 Measure the current system output $y(k)$
2 Initialize temporal variables $\mathcal{L}(k) = \emptyset$, $S(k) = \mathbf{0}$, $\Theta(k) = \mathbf{0}$
3 Select models compatible with the input–output trajectory
foreach $l \in \mathcal{L}$ **do**
  Compute $\mathcal{E}_l\left(\hat{x}_l(k), \mathbf{E}_l(k)\right)$
  Obtain $\mathbf{P}_l(k|k-1)$ and $\mathbf{P}_l(k|k)$
  $r_l = tr\left(\mathbf{P}_l(k|k)\right)$
  if $r_l > \varepsilon$ **then**
    $\mathcal{L}(k) = [\mathcal{L}(k)\ l]$, $s_l(k) = 1$
  else
    $s_l(k) = 0$, $\theta_l(k) = 0$
  end
end
4 Update the models' weights $\Theta(k)$
foreach $l \in \mathcal{L}(k)$ **do**
  $\theta_l(k) = 1 - \dfrac{r_l(k)}{\sum_{l\in\mathcal{L}(k)} r_l(k)}$
end
5 Select the model for the observer
$l_{\min}(k) = \arg\min_{l\in\mathcal{L}(k)} \{r_l\}$
$s_{l_{\min}}^o(k) = 1$, $s_{j\neq l_{\min}}^o(k) = 0 \quad \forall j \in \mathcal{L}$
6 Solve LP problem (32) to compute $\mathcal{K}_c, A_o, B_o, C_o, \mathcal{K}_o$
7 Update the compensator $\mathbb{C}_{\mathcal{L}(k)}\left(\mathcal{K}(k)\right)$ and compute $\hat{x}(k|k)$ and $u(k)$
$k \leftarrow k+1$

Fig. 2 *Supervisory algorithm*

4 Stability analysis

The proof of robust stability of the closed-loop is based upon the fact that the set-valued observers are non-conservative, that is, $\mathrm{tr}(\mathbf{P}_l(k|k)) > \varepsilon$ for some $l \in \mathcal{L}$, then the system trajectories can be explained by those models P_l that $\mathrm{tr}(\mathbf{P}_l(k|k)) > \varepsilon$ and then, the existence of the superstabilising compensator $\mathbb{C}_{\mathcal{L}(k)}(\mathcal{K}(k))$ can be guaranteed.

Assumption 1: The optimisation problems (32) are feasible for $\mathcal{L}(k) = \mathcal{L}$.

Assumption 2: The parameters of the set-valued observers are chosen such that the regions of the models of $\mathcal{P}_{\mathcal{L}}$ cover the entire uncertainty region.

Hereafter, the stability of the closed-loop system is discussed. The fact that the compensator $\mathbb{C}_{\mathcal{L}(k)}(\mathcal{K}(k))$ is computed such that all elements of $\mathcal{P}_{\mathcal{L}(k)}$ are superstable, leads to the first 'local' stability result.

Theorem 2: Supposed Assumptions 1 and 2 are satisfied and using Algorithm 2, then the resulting closed-loop system at time k is superstable.

Proof: To proof the closed-loop stability we need to guarantee the plant $P \in \mathcal{P}_{\mathcal{L}(k)}$ and the optimisation problem (32) is feasible at every sample. The set $\mathcal{P}_{\mathcal{L}(k)}$ is non-empty because of Assumption 2 and the current output $y(k) \in \mathcal{E}_l(x_l(k|k), \mathbf{E}_l(k|k))\ \forall l \in \mathcal{L}(k)$, therefore the true plant $P \in \mathcal{P}_{\mathcal{L}(k)}$ and the compensator $\mathbb{C}_{\mathcal{L}(k)}(\mathcal{K}(k))$ guarantees the superstability of the closed-loop system such that $\|x(k+1)\|_\infty \leq \gamma(k)\|x(k)\|_\infty\ \forall k$, where $\gamma(k) = \sum_{l\in\mathcal{L}(k)} |s_l(k)|\|A_l - B_l\mathcal{K}_c\|_1$. □

This result, valid for each individual polytope $\mathcal{P}_{\mathcal{L}(k)}\ \forall k$, can be also applied to the switching between different polytopes $\mathcal{P}_{\mathcal{L}(k)} \to \mathcal{P}_{\mathcal{L}(k+1)}$ since $\mathcal{P}_{\mathcal{L}(k)}\ \forall k$ is robustly superstable and Lemma 3. Furthermore, it is also clear from Lemma 3 that the trajectories of the closed-loop system decrease in norm along all trajectories of $\mathcal{P}_{\mathcal{L}(k)}$.

Finally, the stability results obtained for LTI systems also hold for non-linear systems

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \quad x(0) = x_0 \\ y(k) &= h(x(k)) \end{aligned} \tag{33}$$

in a compact and convex domain $\mathcal{D} = \mathcal{X} \times \mathcal{U}$, which contains the origins in its interior, if the distribution of the vertices's of $\mathcal{P}_\mathcal{L}$ covers $\mathcal{D}$ with m balls $B_l(x_{0l}, r_l)$ radius r_l centered at x_{0l} such that $\mathcal{X} \subseteq \left\{x | x \in \mathbb{R}^{n_x} \in \bigcup_{l\in\mathcal{L}} B_l(x_{0\,l}, r_l)\right\}$.

The accuracy ϵ of $\mathcal{P}_\mathcal{L}$ is related with the number of vertices's employed m, whose upper bound $m_{\max}$ for a uniform distribution over $\mathcal{D}$ is [34] [The operator $\lceil\cdot\rceil : R \to N_+$, that maps a real to the nearest integer towards infinity, and $\mathrm{eig}(A)$ denotes the eigenvalues of matrix A.]

$$m_{\max} = \left\lceil \frac{1}{\sqrt{2\epsilon}} \sqrt{\lambda(n_x + n_u)\sqrt{n_x}} \prod_{i=1}^{n_x} e_{x_i} \prod_{i=1}^{n_u} e_{u_i} \right\rceil \tag{34}$$

where

$$\lambda = \max_{i\in\mathcal{I}_{n_x}} \max_{\psi\in\mathcal{D}} \left| \mathrm{eig}\left(\frac{\partial^2 f_i(\psi)}{\partial^2 \psi}\right)\right| \tag{35}$$

and $e_{x_i} > 0$ and $e_{u_i} > 0$ are the upper bound of the error between linearisation points $(x_{0\,l}, u_{0\,l})$ and the operating point (x, u)

$$\begin{aligned} \|x - x_{0\,l}\|_2 &\leq e_x \qquad l \in \mathcal{L} \\ \|u - u_{0\,l}\|_2 &\leq e_u \end{aligned} \tag{36}$$

Assumption 3: The m LTI models employed to built $\mathcal{P}_\mathcal{L}$ are uniformly distributed on the operating domain $\mathcal{D}$ and $m \geq m_{\max}$.

Theorem 3: Supposed Assumptions 1–3 are satisfied and supervisory algorithm is used to control the non-linear system (33), then the stability of any switching sequences between $\mathcal{P}_{\mathcal{L}(k)}$ over $\mathcal{X}$ implies the stability of $f(x, u)$ over the same domain $\mathcal{D}$.

Proof: The proof of this theorem follows similar steps of Theorem 2 but it includes additional steps to show that error between the non-linear system and the model built by the supervisory algorithm is bounded, which allows us to extend to the non-linear system the properties of the model built by the supervisor.

In the first step, we need to guarantee the trajectories of the non-linear system $f(x, u)$ are also the trajectories of $\mathcal{P}_{\mathcal{L}(k)}$. This result is a direct consequence of Assumption 2, which guarantee the conservativeness of set–valued observers and the trajectory of $f(x(k), u(k))$ is also a trajectory of $\mathcal{P}_{\mathcal{L}(k)}\ \forall k$ [35].

Now, by Assumption 3 we can guarantee that the accuracy of the approximation model

$$\tilde{f}(x, u) = \sum_{l\in\mathcal{L}(k)} \theta_l(k)(A_l x + B_l u) \tag{37}$$

identified by the supervisory algorithm is bounded

$$\sup_{x\in\mathcal{X}} \|f(x, u) - \tilde{f}(x, u)\|_2 \leq \epsilon \quad \epsilon > 0 \tag{38}$$

Therefore the trajectories associated with the resulting non-linear closed-loop system $(\varsigma(k))$ and the PLMs closed-loop model $(\pi(k))$ verify

$$\begin{aligned} \|\varsigma(k)\|_2 - \|\pi(k)\|_2 &\leq \|\varsigma(k) - \pi(k)\|_2 \\ &= \|f(\varsigma(k-1), u(k-1)) \\ &\quad - \tilde{f}(\pi(k-1), u(k-1))\|_2 \leq \epsilon \quad \forall k \end{aligned} \tag{39}$$

For small ϵ the properties of $\tilde{f}(x, u)$ ensures the properties of $f(x, u)$ over the same domain [34]. Now, by Assumption 1 we can guarantee the existence of a superstabilising controller $\mathcal{K}_c(k)$ that guarantees the superstability of $f(x, u)$.

Finally, this result that is valid for each individual polytope $\mathcal{P}_{\mathcal{L}(k)}\ \forall k$, can be also applied to the switching between different polytopes $\mathcal{P}_{\mathcal{L}(k)} \to \mathcal{P}_{\mathcal{L}(k+1)}$. It is a direct consequence of the robust superstability of $\mathcal{P}_{\mathcal{L}(k)}\ \forall k$ and Lemma 3. Furthermore, it is also clear from Lemma 3 that the trajectories of the closed-loop system decrease in norm. □

Remark 7: The stability analysis holds equally for the general system $x(k+1) = f(x(k), u(k))$ with non-zero equilibrium condition (x_d, u_d), such that $f(x_d, u_d) = x_d$.

Indeed, the error dynamics satisfy

$$e(k+1) = x(k+1) - x_d = f(e(k) + x_d, u(k)) - x_d \tag{40}$$

and the control law in this case $u(k) = u_d + \varrho(k)$, which leads to the closed-loop error dynamics $e(k+1) = E(e(k), \varrho(k))$ such that $E(0, 0) = 0$ and $E(e(k), \varrho(k)) = f(e(k) + x_d, u_d + \varrho(k)) - x_d$. Furthermore, the linear models used to construct the polytopic linear model approximation from the error dynamics, are simply the linearised models of the original system.

5 Simulations and results

In this section, we provide an illustrative numerical example. Let us consider the problem of controlling a continuous stirred tank reactor (CSTR) in which an irreversible exothermic reaction $A \to B$ occurs in a constant volume reactor. This non-linear system was originally used by Morningred *et al.* [36] for testing discrete control algorithms. It is

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modelled by the following equations

$$\dot{Ca} = \frac{q}{V}[Ca_0 - Ca] - k_0 Ca \exp\left(\frac{-E}{R\,T}\right)$$
$$\dot{T} = \frac{q}{V}[T_0 - T] - \frac{k_0\,\Delta H}{\rho\,c_\rho} Ca\,\exp\left(\frac{-E}{R\,T}\right) + \frac{\rho_C\,c_{\rho C}}{\rho\,c_\rho V} q_C \left[1 - \exp\left(\frac{-hA}{q_C\,\rho_C\,c_{\rho C}}\right)\right][T_{CO} - T] \tag{41}$$

The nominal values of the variables and parameters can be found in Morningred's paper [36]. The objective is to control the output concentration $y(k) = Ca(k)$ using the coolant flow rate $u(k) = q_C(k)$. The reactor has a second output, the reactor temperature $T(t)$, and the state vector $x(k)$ includes both variables ($x(k) = [Ca(k)\ T(k)]$).

The non-linear nature of the system is shown in Fig. 3, where the T and Ca open-loop response are shown to changes in the manipulated variable $\Delta q_C = +10, -10, -10$ and $+10$ lt min^{-1} around its nominal value $q_{C0} = 100$ lt min^{-1}. The operation of the reactor is quite difficult because of the changes in the dynamics, the presence of complex poles near the imaginary axis, and because of it becomes uncontrollable when $q_C(k)$ goes to beyond of 113 lt min^{-1}. The operating space region $\mathcal{D}$ is defined by the cube

$$\begin{aligned} 0.05 \le x_1(k) \le 0.13\,\text{mol lt}^{-1} \\ 430 \le x_2(k) \le 445°\text{K} \\ 85 \le u(k) \le 110\,\text{lt min}^{-1} \end{aligned} \tag{42}$$

It is possible to approximate the non-linear model of the reactor within the specified working space using four LTI models, leading to an estimated error $\epsilon = 7 \times 10^{-4}$. The discrete LTI models shown in Table 1 with $\mathcal{L} = \{1, 2, 3, 4\}$ were determined from the responses shown in Fig. 3, using a subspace identification algorithm. They define the polytopic model $\mathcal{P}_\mathcal{L}$ associated with the non-linear behaviour in the considered operating region. The sampling time period was fixed at 0.1 min, which gives about four sampled-data points in the dominant time constant when the reactor is operating in the high concentration region.

Table 1 Vertices of $\mathcal{P}_\mathcal{L}$

P_1:	$A_1 = \begin{bmatrix} 0.222 & -4.455\,10^{-3} \\ 129.6 & 1.656 \end{bmatrix}$	$B_1 = \begin{bmatrix} 1.398\,10^{-4} \\ -8.272\,10^{-2} \end{bmatrix}$	$C_1 = [0\ 1]$
P_2:	$A_2 = \begin{bmatrix} 2.879\,10^{-2} & -3.789\,10^{-3} \\ 166.5 & 1.538 \end{bmatrix}$	$B_2 = \begin{bmatrix} 2.375\,10^{-4} \\ -0.1514 \end{bmatrix}$	$C_2 = [0\ 1]$
P_3:	$A_3 = \begin{bmatrix} -0.1687 & -2.616\,10^{-3} \\ 204.5 & 1.329 \end{bmatrix}$	$B_3 = \begin{bmatrix} 2.161\,10^{-4} \\ -0.1658 \end{bmatrix}$	$C_3 = [0\ 1]$
P_4:	$A_4 = \begin{bmatrix} 2.534\,10^{-2} & -3.774\,10^{-3} \\ 167.2 & 1.536 \end{bmatrix}$	$B_4 = \begin{bmatrix} 1.393\,10^{-4} \\ -8.898\,10^{-2} \end{bmatrix}$	$C_4 = [0\ 1]$

The structure of the compensator is given by (28), with the controller equation modified to include a reference signal $x_r(k)$ and an integral action

$$\begin{aligned} \tilde{u}(k) &= \mathcal{K}_I\,\tilde{u}(k-1) + \mathcal{K}_C\,(\hat{x}(k) - x_r(k)) \\ u(k) &= \tilde{u}(k) + q_{C0} \end{aligned} \tag{43}$$

The performance of the closed-loop system is defined through the minimisation of (13) and a settling time of 50 samples for an error of 5%

$$|e(k)| \le 0.05\,x_r(k) \quad \forall k \ge N_o + 50 \tag{44}$$

where N_o is the time instant when changes happen. To guarantee the system controllability over the whole operational region a hard constraint is set on q_C

$$q_C(k) \le 110\ \text{lt min}^{-1} \qquad \forall k \tag{45}$$

The proposed robust adaptive controller is compared with a supervisory adaptive controller with set-valued observers [10]. It employs robust non-adaptive controllers, designed using mixed-μ synthesis, and set-valued observers for each model of $\mathcal{P}_\mathcal{L}$. The simulation tests are similar to Morningred's work [36] and consists of a sequence of step

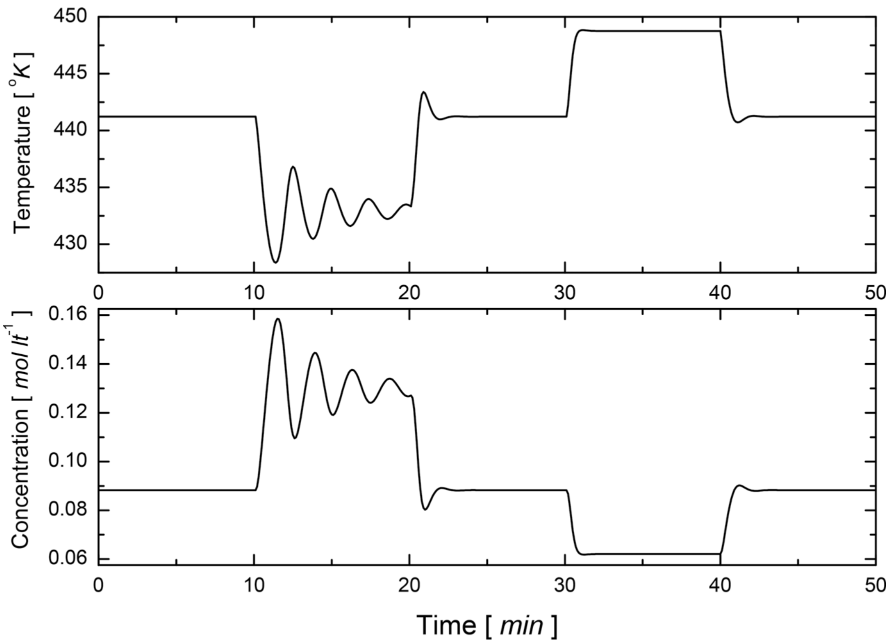

Fig. 3 *Open-loop responses of the CSTR to changes in q_C*

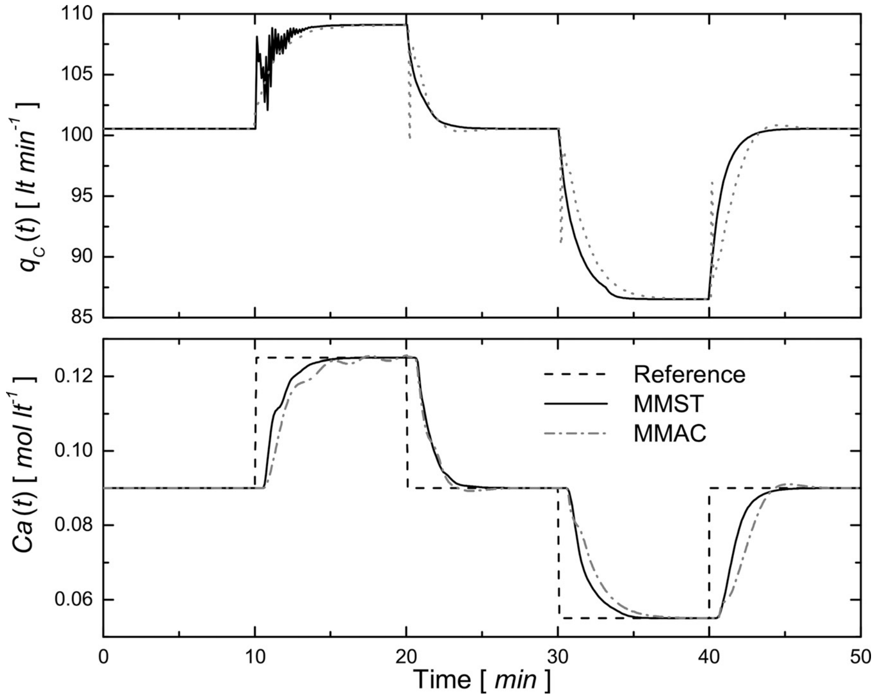

Fig. 4 *Simulation results for a multiple models switching and tuning controller (MMST) and a supervisory adaptive controller (MMAC)*

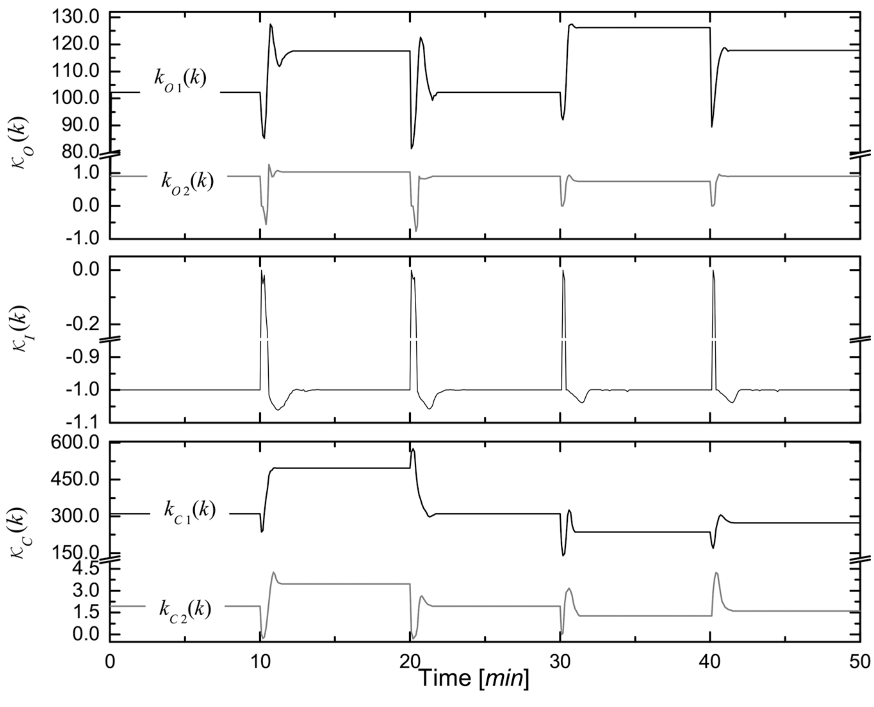

Fig. 5 *Time evolution of*
a Controller
b Observer gains

changes in the reference value. The set point was changed in intervals of 10 min from 0.09 to 0.125, returns to 0.09, then steps to 0.055 and finally returns to 0.09.

Figs. 4–6 show the closed-loop simulation results sustained by the controller described in the previous paragraphs. The good performance of the adaptive controller proposed is because of the combination of a switching scheme with the on-line design of the controller. The parameters of the adaptive controller are modified with respect to the reactor's operating region. An initial transient behaviour appears, after each change, before achieving their steady-state values. This fact can be appreciated in the behaviour of the switching variables $S(k)$, which show jitter during all reference changes. This controller 'hesitation' is due to the fact that the first and second models and the third and fourth models are computed around nearly the same state operating point,

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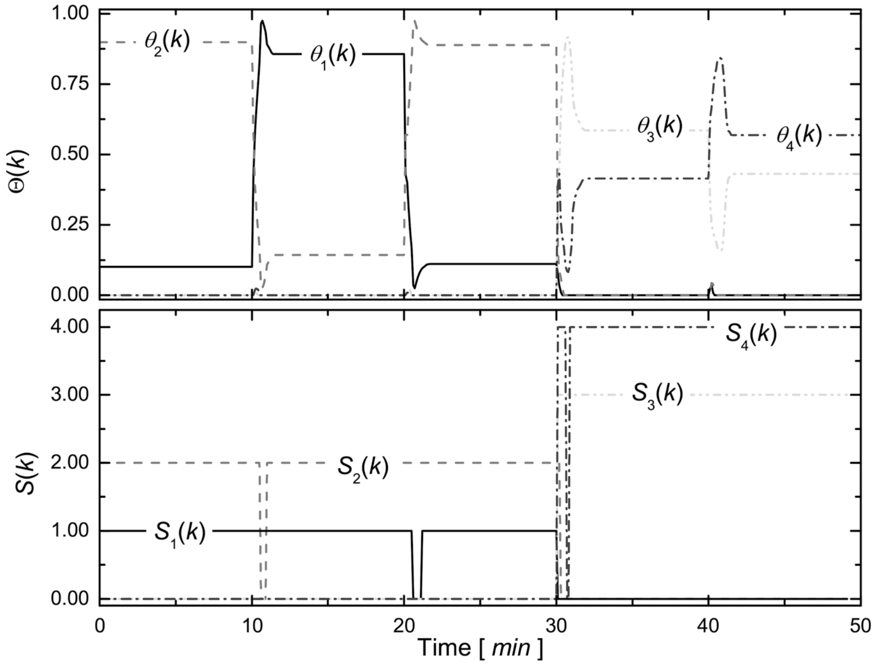

Fig. 6 *Time evolution*

a Switching variables ($s_l(k)$)

b Control weights ($\theta_l(k)$)

the only difference being the control operating point value. Hence, when the reference changes hold the controller hesitates between these models as long as the control value is not close to one of the control operating points.

The plant input ($u(k)$) and output ($y(k)$) resulting from the simulations with the controllers described above are shown in Fig. 4. The proposed adaptive scheme exhibits a fast regulation and easily satisfied the performance requirements [constraints (44) and (45)]. The closed-loop behaviour results from two facts: (a) all parameters of $\mathbb{C}_{\mathcal{L}(k)}(\mathcal{K}(k))$ are allowed to vary continuously due to its on-line design, and (b) the supervisor quickly identify the relevant models that explain the plant behaviour (see Fig. 6). At this point it is necessary to highlight the lack of oscillations and abrupt changes in the control signal, generated by controller switches, is because of the soft-variable nature of the controller.

The time evolution of the parameters of the proposed adaptive controller ($\mathcal{K}_{c\,i}(k)$, $\mathcal{K}_i(k)$ and $\mathcal{K}_{o\,i}(k)$) are shown in Fig. 5. An initial transient behaviour appears, after each change, before achieving their steady-state values. This fact can be appreciated in the behaviour of the switching variables ($S(k)$) and the models weights ($\Theta(k)$) in Fig. 6, that show oscillations during the initial samples after the reference changes.

6 Conclusions

The motivation for the development of multiple models switching and tuning adaptive control is to develop a deterministic approach capable of achieving high-performance by utilising robust LTI and switching-based adaptive tools, whereas avoiding issues of undesirable switching behaviours and uncertain disturbance models. In this way, in this work we extend the results obtained in [18] to MIMO and non-linear system, whereas the conservativeness of the resulting closed-loop system was reduced through the use of extended superstability. The properties of the resulting closed-loop system and design guidelines have been discussed. Comparative results with others supervisory adaptive and robust adaptive controllers, based on simulations of a non-linear system, have been presented to illustrate the effectiveness of the proposed controller.

The authors are currently working towards extending this line of research in two directions: (i) addressing the limtations imposed by the model selection scheme, and (ii) developing adaptive estimation and control algorithms for failure-robust design. For a practical implementation it is necessary to reduce the number of LTI models m required to represent the system in the operating space. This idea requires an efficient use of the information available in the supervisor $\mathbb{S}$ and find new ways of employing multiple models for robust system identification. The works of Narendra and Han [16, 17] show a promising line of research to explore. In fact, this idea has been used in this work to reduce the conservativeness of the closed-loop performance estimation through the reparametrisation of the optimisation cost function. In the context of LPV system, the framework proposed in this work is connected to some recent works in the area of switching (see [37, 38]) and piecewise ([39]) systems where the robust estimators and controllers are designed using passivity performance indexes in order to ensure closed-loop performance and stability.

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9 Appendix 1: Proof of Lemma 1

Given the LPV system (4), the norm of the states $x(k)$ is bounded by

$$\|x(k)\|_\infty \le \|A(S(k-1))\|_1 \|x(k-1)\|_\infty + \|B(S(k-1))\|_1 \|w(k-1)\|_\infty \quad (46)$$

Using the recursion backward in time we have

$$\|x(k)\|_\infty \le \prod_{i=0}^{k} \|A(S(k-i))\|_1 \|x(k-i)\|_\infty + \sum_{i=1}^{k} \prod_{l=0}^{i-1} \|A(S(k-l))\|_1 \|B(S(k-i))\|_1 \times \|w(k-i)\|_\infty \quad (47)$$

which is bounded by

$$\|x(k)\|_\infty \le \bar{\gamma}^k \mu + \sum_{i=0}^{k} \bar{\gamma}^i \bar{\beta} \quad (48)$$

where

$$\bar{\gamma} = \sup_{\forall k} \sup_{\forall S(k)\in\mathcal{S}} \|A(S(k))\|_1 \quad (49)$$

$$\bar{\beta} = \sup_{\forall k} \sup_{\forall S(k)\in\mathcal{S}} \|B(S(k))\|_1 \quad (50)$$

Then, using the formula for the sum of a geometric series for k terms we have

$$\|x(k)\|_\infty \le \bar{\gamma}^k \mu + \alpha \frac{1-\bar{\gamma}^k}{1-\bar{\gamma}} \le \bar{\eta} + q^k \max\{0, \mu - \bar{\eta}\} \quad (51)$$

where $\bar{\eta} = \bar{\beta}/(1-\bar{\gamma})$. This estimate shows that the PLM state trajectory is decreasing in norm until it reaches the

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invariant set $\mathcal{Q} = \{\|x(k)\|_\infty \le \bar{\eta}\}$, that is, the trajectories originating in this set stay in it for all admissible perturbations. If a disturbance drives the state out of the invariant set, the control law will drive it again to the invariant set.

10 Appendix 2: Proof of Theorem 1

After a change of variables $x(k) = D\nu(k)$, the closed-loop system becomes

$$\nu(k+1) = \tilde{A}_c \nu(k) + D^{-1} B_W w(k) \tag{52}$$

with $\tilde{A}_c = D^{-1}(A + B_U \mathcal{K}_c)D \in \mathcal{E}_S$. The estimate (2) for the closed-loop system is

$$\|\nu(k)\|_\infty \le \frac{\|D^{-1}\|_1 \|B_W\|_1}{1 - \|\tilde{A}_c\|_1} \quad \forall k \tag{53}$$

provided that $\|\nu(0)\|_\infty \le \|B_W\|_1/(1 - \|\tilde{A}_c\|_1)$. If this condition is not satisfied, the control law $\mathcal{K}_c$ will enforce this condition in few samples. For the original system the estimate (53) becomes

$$\|x(k)\|_\infty \le \frac{d_{\max}\|B_W\|_1}{d_{\min}(1 - \|\tilde{A}_c\|_1)} \tag{54}$$

where $d_{\min} = \min_{i \in \mathcal{I}_{n_x}} d_i$ and $d_{\max} = \max_{i \in \mathcal{I}_{n_x}} d_i$. The matrix D can be scaled such that $d_{\min} = 1$ and replace $d_{\max}$ by an upper bound $\omega : d_i \le \omega \quad \forall i \in \mathcal{I}_{n_x}$ such that

$$\|x(k)\|_\infty \le \eta \le \omega \frac{\|B_W\|_1}{1 - \|\tilde{A}_c\|_1} \tag{55}$$

The minimisation of this estimate over all possible $\mathcal{K}_c$, under the condition $A_c \in \mathcal{E}_S$, can be achieved solving

$$\begin{aligned} &\min_{q \in [0,1), \mathcal{K}_c, D} \frac{1}{1-q}\|B_W\|_1 \\ &\text{st.} \\ &\|(A + B_U \mathcal{K}_c)D\|_1 \le q\|D\|_1 \end{aligned} \tag{56}$$

Given the entries of the close–loop matrix $\tilde{A}_c$, the superstability constraint can be written as a set of n_x linear constraints

$$\sum_j \left| a_{ij} d_j + \sum_s b_{js} z^c_{sj} \right| < q d_i \quad i,j,s \in \mathcal{I}_{n_x} \tag{57}$$

leading to the following linear optimisation problem

$$\begin{aligned} &\min_{q \in [0,1)} \min_{\omega, Z_c, D} \frac{\omega}{1-q}\|B_W\|_1 \\ &\text{st.} \\ &\sum_j \left| a_{ij} d_j + \sum_s b_{is} z^c_{si} \right| \le q d_i \quad i,j,s \in \mathcal{I}_{n_x} \\ &1 \le d_i \le \omega \\ &\omega \ge 1 \end{aligned} \tag{58}$$

Using standard tools, problem (12) can be reformulated as a linear optimisation problem with respect to the parameters of $\mathcal{K}_c, \omega, Z_c, D$.

11 Appendix 3: upper bound of LLR

The linear performance functional (13) can be rewritten as follows

$$J_C = \sum_{k=0}^{\infty} (1 + \alpha\|\mathcal{K}_c\|_1)\|x(k)\|_\infty \tag{59}$$

If we require that the closed-loop system belongs to the class of extended superstable matrices and make use of the estimate (3), then

$$J_C \le \sum_{k=0}^{\infty} (1 + \alpha\|\mathcal{K}_c\|_1) q^k \mu \le \frac{1 + \alpha\|Z_c\|_1\|D^{-1}\|_1}{1 - \|\tilde{A}_c\|_1}\mu \tag{60}$$

Finally, since matrix D can be scaled such that $d_{\min} = 1$ the performance index becomes

$$J_C \le \frac{1 + \alpha\|Z_c\|_1}{1 - \|\tilde{A}_c\|_1}\mu \tag{61}$$

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