Extrapolation and weighted norm inequalities between Lebesgue and Lipschitz spaces in the variable exponent context

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Abstract

We give extrapolation results starting from weighted inequalities between Lebesgue and Lipschitz spaces, given by

$$\sup_B \frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |f(x) - m_B(f)|\, dx \leq C \|gw\|_s\,, \tag{0.1}$$

where $1 < \beta < \infty$, $0 \leq \delta < 1$, $\frac{\delta}{n} = \frac{1}{\beta} - \frac{1}{s}$, f and g are two measurable functions and w belongs to a suitable class of weights. From this hypothesis we obtain a large class of inequalities including weighted $L^p - L^q$ estimates and weighted L^p- Lipschitz integral spaces, generalizing well know results for certain sublinear operator.

From the same hypothesis (0.1) we obtain the corresponding results in the setting of variable exponent spaces. Particularly, we obtain estimates of the type $L^{p(\cdot)}$-variable versions of Lipschitz integral spaces. We also prove a surprising weighted inequalities of the type $L^{p(\cdot)}$-$L^{q(\cdot)}$.

An important tool in order to get the main results is an improvement of an estimate due to Calderon and Scott in [1], which allow us to relate different integral Lipschitz spaces.

Our results are new even in the classical context of constant exponents.

Keyword: Variable exponent spaces, Lipschitz spaces, weights, maximal operator, fractional integrals, Rubio de Francia extrapolation.

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1. Introduction

For $0 < \gamma < n$, the fractional integral operator of order γ, I_γ is usually defined by

$$I_\gamma f(x) := \int \frac{f(y)}{|x-y|^{n-\gamma}} dy, \tag{1.1}$$

whenever this integral is finite almost everywhere.

In 1974 Muckenhoupt and Wheeden ([17]), characterize the weights w for which the inequality

$$\|I_\gamma f\, w\|_q \leq C \|f\, w\|_p \tag{1.2}$$

holds for $1 < p < n/\gamma$ and $1/q = 1/p - \gamma/n$, as those weights belonging to the $\mathcal{A}(p,q)$ class, that is, the weights w such that the inequality

$$\left(\frac{1}{|B|}\int_B w^q\right)^{1/q} \left(\frac{1}{|B|}\int_B w^{-p'}\right)^{1/p'} \leq C$$

holds for every ball $B \subset \mathbb{R}^n$.

For the one dimensional case and $w \equiv 1$, the inequality (1.2) was proved by Hardy and Littlewood in [13]; and they also proved the result for $w(x) = |x|^\alpha$. This inequality is then extended to n dimensions with $w \equiv 1$ by Sobolev in [25] and with $w(x) = |x|^\alpha$ by Stein and Weiss in [26].

For the limiting case $p = n/\gamma$ and $q = \infty$, Muckenhoupt and Wheeden also characterized the weights w that satisfy the inequality

$$\frac{\|w\chi_B\|_\infty}{|B|} \int_B |I_\gamma f(x) - m_B(I_\gamma f)|\, dx \le C\, \|fw\|_{n/\gamma}$$

for every ball $B \subset \mathbb{R}^n$, as those belonging to the $\mathcal{A}(n/\gamma, \infty)$ class, which is known to be equivalent to $w^{-(n/\gamma)'} \in \mathcal{A}_1$. Here $m_B f$ denotes the average $|B|^{-1} \int_B f$. This result can be viewed as the boundedness of I_γ from $L_w^{n/\gamma}$ into BMO_w, one of the weighted versions of the space of functions with bounded mean oscillation BMO introduced in [17].

In 1997 Harboure, Salinas and Viviani ([12]) give two different versions of weighted BMO and Lipschitz integral spaces in order to obtain necessary and sufficient conditions on the weights that guarantee the boundedness of the fractional integral operator I_γ from L_w^p, for $n/\gamma \le p < n/(\gamma - 1)^+$, into these new weighted versions. They also characterize the weights for which I_γ can be extended to a bounded linear operator between weighted Lipschitz integral spaces. The corresponding results for the unweighted case have been established in different settings by several authors, see for instance [28], [27], [8] and [11].

On the other hand, in [20] Pradolini gives different versions from those given in [12] of weighted BMO and Lipschitz integral spaces. The author characterizes the pairs of weights (w, v) for which I_γ can be extended to a bounded linear operator from L_w^p into the new weighted versions defined with the weight v. Particularly, when $w = v$, $0 \le \delta < 1$ and p verify that $\delta = \gamma - n/p$, the weights w for which the inequality

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |I_\gamma f(x) - m_B(I_\gamma f)|\, dx \le C\|fw\|_p, \tag{1.3}$$

holds for every ball $B \subset \mathbb{R}^n$ are proved to be those such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta-\gamma}{n}}} \left(\int_B w^{-p'}\right)^{1/p'} \le C$$

for every ball $B \subset \mathbb{R}^n$. It is easy to check that the weights satysfying the inequality above are those in the class $\mathcal{A}(p, \infty) = \mathcal{A}(n/(\gamma - \delta), \infty)$.

Related with the extrapolation theory, in 1982, Rubio de Francia ([23]) proved that the $\mathcal{A}_p$ class enjoy a very interesting extrapolation property. More specifically, if for some $1 \le p_0 < \infty$, an operator preserves $L^{p_0}(w)$ for any $w \in \mathcal{A}_{p_0}$, then necessarily preserves the $L^p(w)$ space for every $1 < p < \infty$ and every $w \in \mathcal{A}_p$. Later, in 1988 Harboure, Macías and Segovia ([10]) proved that the $\mathcal{A}(p, q)$ classes have a similar extrapolation property, that is, if T is a sublinear operator such that the inequality

$$\|Tf\, w\|_q \le C\, \|fw\|_p \tag{1.4}$$

holds for some pair (p_0, q_0), $1 < p_0 \le q_0 < \infty$ and every $w \in \mathcal{A}(p_0, q_0)$, then (1.4) holds for every pair (p, q), $1 < p \le q < \infty$, which satisfy the condition

$$1/p - 1/q = 1/p_0 - 1/q_0$$

and every $w \in \mathcal{A}(p, q)$. Moreover, they also proved that this property is not only exclusive for the boundedness between weighted Lebesgue spaces, but also it is possible to extrapolate based on a continuity behavior of the type $L_w^\beta - BMO_w$ for some $1 < \beta < \infty$. In other words, if $1 < \beta < \infty$ and T is a sublinear operator satisfying

$$\frac{\|w\chi_B\|_\infty}{|B|} \int_B |Tf(x) - m_B(Tf)| \le C\, \|fw\|_\beta\,, \tag{1.5}$$

for every ball $B \subset \mathbb{R}^n$ and every weight $w \in \mathcal{A}(\beta, \infty)$; then, if $1 < p < \beta$, $1/p - 1/q = 1/\beta$ and $w \in \mathcal{A}(p, q)$, the inequality (1.4) holds for the pair (p, q) provided that the left hand side is finite.

The result above suggests the question whether it is possible to prove a version of extrapolation result for sublinear operators T satisfying inequalities in the spirit of (1.3), that is

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |Tf(x) - m_B(Tf)|\, dx \leq \|fw\|_p.$$

A considerable part of this paper is devoted to answering this question positively. In order to reach this objective we consider the weighted integral Lipschitz space given in [20] (see also [21]). More specifically, given a weight w and $0 \leq \delta < 1$, we say that a locally integrable function f belong to $\mathbb{L}_w(\delta)$ if there exists a positive constant C such that the inequality

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |f(x) - m_B f|\, dx \leq C \tag{1.6}$$

holds for every ball $B \subset \mathbb{R}^n$. The least constant C will be denoted by $|||f|||_{\mathbb{L}_w(\delta)}$. Let us observe that for $\delta = 0$, the space $\mathbb{L}_w(\delta)$ coincides with one of the versions of weighted bounded mean oscillation spaces, introduced in [18]. Moreover, for the case $w \equiv 1$, the space $\mathbb{L}_1(\delta)$ is the known Lipschitz integral space for $0 < \delta < 1$.

We first obtain extrapolation results that allow us to obtain continuity properties of certain operators of the type L^p_w - L^q_w or L^p_w - $\mathbb{L}_w(\tilde{\delta})$ starting with hypothesis of continuity of the type L^s_w - $\mathbb{L}_w(\delta)$ for some related parameters.

We are also interested in establish extrapolation results of the type described above in the variable exponent spaces context. In this direction we exhibit extrapolation results starting from hypothesis which involves inequalities of the type L^s_w- $\mathbb{L}_w(\delta)$, and obtaining unweighted estimates of the type $L^{p(\cdot)}$- $L^{q(\cdot)}$ or $L^{p(\cdot)}$- $\mathbb{L}(\tilde{\delta}(\cdot))$, where the last space is a variable version of the space defined in (1.6) and was introduced in [22]. Extrapolation in the scale of the variable Lebesgue spaces, to prove unweighted inequalities, was originally treated in [5]. In view of the results proved in [20] for I_γ, the extrapolation results in the variable context allow us to derive boundedness results for the same operator between $L^{p(\cdot)}$ and the variable version of the Lipschitz integral spaces we had mentioned. This result was previously proved in [22] with different techniques. Weighted versions of the results in the variable context are also obtained.

A useful tool used in obtaining our main results is an interesting estimate which allows us to generalize an inequality due to Calderón and Scott in [1]. This generalization gives us a way to relate the seminorms between weighed Lipschitz integral spaces associated to different orders.

Specifically, for a locally integrable function f and $0 \leq \delta < 1$ we consider the operator

$$f^\sharp_\delta(x) := \sup_{B \ni x} \frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |f - m_B f|\, dy.$$

In [1], the authors proved that

$$\|f^\sharp_0 w\|_r \leq C\|f^\sharp_\delta w\|_p$$

for $1 < p < n/\delta$ and $1/r = 1/p - \delta/n$. The fundamental key used in proving this result is the following pointwise estimate

$$f^\sharp_0 \leq C I_\delta f^\sharp_\delta,$$

where I_δ is the fractional integral operator of order δ. The improvement introduced in this paper is a pointwise estimate of the same type but replacing I_δ for M_δ, the fractional maximal operator or order δ. This estimate allows us to consider norm estimates for extreme values of the exponents of the type $(n/\delta,\infty)$ that can not be obtained with I_δ, since it is well known that this operator does not map $L^{n/\delta}$ into L^∞. Concretely, we prove that

$$f^\sharp_{\tilde{\delta}} \leq CM_{\delta-\tilde{\delta}} f^\sharp_\delta$$

and, as a consequence, we derive the following weighted norm estimate which is essential in the proof of our results,

$$\|f^\sharp_{\tilde{\delta}}w\|_r \leq C\|f^\sharp_\delta w\|_p$$

for suitable values of p and r, including extreme values, $0 \leq \tilde{\delta} \leq \delta < 1$ and w belonging to certain class of weights. In the variable exponent context some related estimates were obtained too. As far as we know, our results are new even when we consider constants parameters.

The remainder of this paper is organized as follows. In Section 2 we give some preliminaries and state our main theorems. In Section 3 we state and prove the auxiliary lemmas which are important tools in order to prove the theorems stated in Section 2. Finally, in Section 4, the proofs of the main results are given.

2. Preliminaries and main results

For a weight w we mean a locally integrable function such that $0 < w(x) < \infty$ a.e.

We say that the weight w belongs to the $\mathcal{A}_p$ class, for $1 < p < \infty$, if there exits a positive constant C such that the inequality

$$\left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}\right)^{p-1} \leq C$$

holds for every ball $B \subset \mathbb{R}^n$. The $\mathcal{A}_1$ class is defined as the set of weights w for which there exits a positive constant C such that the inequality

$$\frac{1}{|B|}\int_B w \leq C \inf_B w$$

holds for every ball $B \subset \mathbb{R}^n$.

By $L^p(\mathbb{R}^n)$ we mean the usual Lebesgue space on $\mathbb{R}^n$, that is, the set of the functions f such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p} < \infty$$

By L^p_w we mean the class of functions f such that $\|fw\|_p$ is finite, while $L^p(w)$ denotes the space of functions f such that

$$\int_{\mathbb{R}^n} |f(x)|^p w(x) dx$$

is finite.

A generalization of $\mathcal{A}_p$ classes defined above is given below. These classes were first introduced in [17].

A weight w is said to belong to the $\mathcal{A}(p,q)$ class, $w \in \mathcal{A}(p,q)$, if there exits a positive constant C such that the inequality

$$\left(\frac{1}{|B|}\int_B w^q dy\right)^{1/q}\left(\frac{1}{|B|}\int_B w^{-p'} dy\right)^{1/p'} \leq C,$$

holds for any ball $B \subset \mathbb{R}^n$.

For the limiting case $q = \infty$, we say that $w \in \mathcal{A}(p,\infty)$ if and only if there exits a positive constant C such that

$$\|w\chi_B\|_\infty \left(\frac{1}{|B|}\int_B w^{-p'} dy\right)^{1/p'} \leq C,$$

holds for any ball $B \subset \mathbb{R}^n$. It is easy to check that $w \in \mathcal{A}(p,\infty)$ is equivalent to $w^{-p'} \in \mathcal{A}_1$.

We are now in a position of stating one of our main theorems related to extrapolation results from Lipschitz spaces.

Theorem 2.1. *Let $1 < \beta < \infty$, $0 \le \delta < 1$ and $\frac{\delta}{n} = \frac{1}{\beta} - \frac{1}{s}$, f and g two positive measurable functions such that the inequality*

$$|||f|||_{\mathbb{L}_w(\delta)} \le C \left\|gw\right\|_s , \tag{2.2}$$

holds for every $w \in \mathcal{A}(s,\infty)$ and some positive constant $C = C(w)$. Then there exists a positive constant C such that the inequality

$$\left\|fw\right\|_q \le C \left\|gw\right\|_p , \tag{2.3}$$

holds for every p such that $1 < p < \beta$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{\beta}$ and for every weight $w \in \mathcal{A}(p,q)$, provided that the left hand side of (2.3) *is finite.*

If $\delta = 0$ a version of the theorem above for sublinear operators was proved by Harboure, Macías and Segovia in [10].

Remark 2.4. From the fact that $\frac{\delta}{n} = \frac{1}{\beta} - \frac{1}{s}$, and $0 \le \delta < 1$ it is easy to see that $1 < \beta \le s < \frac{\beta n}{(n-\beta)^+}$, where z^+ is defined by 0, if $z \le 0$ and z, if $z > 0$. Particularly, when $\beta = n/\alpha$, $0 < \alpha < n$ then $1 < n/\alpha \le s < n/(\alpha-1)^+$ and in this case, it was proved in [20] that the fractional integral operator I_α defined by

$$I_\alpha f(x) := \int \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

is bounded from $L^s(w)$ into $\mathbb{L}_w(\delta)$ when $w \in \mathcal{A}(s,\infty)$. Thus, a simple application of Theorem 2.1 leads to the boundedness of I_α from $L^p(w)$ into $L^q(w)$ for $w \in \mathcal{A}(p,q)$ from extrapolations results, that is, the result in [17].

As it can be seen the last theorem allows us to obtain boundedness result of $L^p_w - L^q_w$ type via extrapolation, from $L^s_w - \mathbb{L}_w(\delta)$ boundedness type result. It is natural to ask if it can be derived boundedness results of the $L^p_w - \mathbb{L}_w(\tilde{\delta})$ type from the same hypothesis as in Theorem 2.1. The answer to this question is given in the following theorem when $0 \le \tilde{\delta} \le \delta < 1$.

Theorem 2.5. *Let $1 < \beta < \infty$, $0 \le \delta < 1$, $\frac{\delta}{n} = \frac{1}{\beta} - \frac{1}{s}$, f and g be two measurable functions such that the inequality*

$$|||f|||_{\mathbb{L}_w(\delta)} \le C \left\|gw\right\|_s ,$$

holds for every $w \in \mathcal{A}(s,\infty)$ and some positive constant $C = C(w)$. Then, if $0 \le \tilde{\delta} \le \delta < 1$ and $\frac{\tilde{\delta}}{n} = \frac{1}{\beta} - \frac{1}{p}$, there exists a positive constant C such that the inequality

$$|||f|||_{\mathbb{L}_w(\tilde{\delta})} \le C \left\|gw\right\|_p , \tag{2.6}$$

holds for every $w \in \mathcal{A}(p,\infty)$.

Remark 2.7. When $\tilde{\delta} = 0$, then $p = \beta$ and Theorem 2.5 gives boundedness results from L^β_w into $\mathbb{L}_w(0)$, which, as we said previously, is a version of a weighted BMO spaces.

Particularly, when we consider $f = I_\alpha g$, in [20] it was proved that the hypothesis of Theorem 2.5 holds for $\beta = n/\alpha$. Thus, is $\tilde{\delta} = 0$ a simple application of this theorem lead us to the boundedness of I_α from $L^{n/\alpha}_w$ in to $\mathbb{L}_w(0)$ for weights $w \in \mathcal{A}(n/\alpha,\infty)$, which is the result in [17]. If $0 < \tilde{\delta} \le \delta$ we obtain the boundedness of I_α from L^p_w into $\mathbb{L}_w(\tilde{\delta})$ for $p > n/\alpha$.

We are now interested in obtaining extrapolation results of the type described above in the variable Lebesgue space context. Previous results related with this type of estimates were given in [5].

In order to establish the main theorem we give some definitions and notations.

Let $p : \mathbb{R}^n \to [1, \infty)$ be a mesurable function. For $A \subset \mathbb{R}^n$ we define

$$p_A^- := \operatorname{ess}\inf_{x\in A} p(x) \qquad p_A^+ := \operatorname{ess}\sup_{x\in A} p(x).$$

For simplicity we denote $p^+ = p^+_{\mathbb{R}^n}$ and $p^- = p^-_{\mathbb{R}^n}$. We shall also suppose that $1 < p^- \leq p(x) \leq p^+ < \infty$ for every $x \in \mathbb{R}^n$.

We say that $p \in \mathcal{P}(\mathbb{R}^n)$ if $1 < p^- \leq p(x) \leq p^+ < \infty$ and we say that $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ if $p \in \mathcal{P}(\mathbb{R}^n)$ and it satisfies the following inequalities

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x-y|)}, \qquad \text{for every } x, y \in \mathbb{R}^n. \tag{2.8}$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)}, \qquad \text{with } |y| \geq |x|. \tag{2.9}$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of the measurable functions f defined on $\mathbb{R}^n$ such that, for some positive λ, the convex functional modular

$$\varrho(f/\lambda) := \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)}\, dx$$

is finite. A Luxemburg norm can be defined in $L^{p(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \varrho(f/\lambda) \leq 1\}.$$

This spaces are special cases of Museliak-Orlicz spaces (see [19]), and generalize the classical Lebesgue spaces. For more information see, for example [15], [4], [6].

Let $p \in \mathcal{P}(\mathbb{R}^n)$ such that $\beta \leq p^- \leq p(x) \leq p^+ < \frac{n\beta}{(n-\beta)^+}$ and let $\frac{\delta(x)}{n} = \frac{1}{\beta} - \frac{1}{p(x)}$. The space $\mathbb{L}(\delta(\cdot))$ is defined by the set of measurable functions f such that

$$|||f|||_{\mathbb{L}(\delta(\cdot))} := \sup_B \frac{1}{|B|^{\frac{1}{\beta}} \|\chi_B\|_{p'(\cdot)}} \int_B |f - m_B f| < \infty$$

When $p(x)$ is equal to a constant p, this space coincide with the space $\mathbb{L}_1(\frac{n}{\beta} - \frac{n}{p})$.

The spaces $\mathbb{L}(\delta(\cdot))$ were $\frac{\delta(\cdot)}{n} = \frac{1}{\beta} - \frac{1}{p(\cdot)}$ was introduced in [22]. In this article, the author give conditions on the exponent $p(\cdot)$ that guarantee the boundedness of the fractional integral operator I_α from $L^{p(\cdot)}$ spaces into $\mathbb{L}(\delta(\cdot))$ spaces.

We shall give some unweighted boundedness results on variable Lebesgue spaces by extrapolating from the same hypothesis as in Theorems 2.1 and 2.5. The first one gives $L^{p(\cdot)} - L^{q(\cdot)}$ type estimates and the second result gives $L^{p(\cdot)} - \mathbb{L}(\delta(\cdot))$ estimates.

Theorem 2.10. *Let $1 < \beta < \infty$, $0 \leq \delta < 1$ and s such that $\frac{\delta}{n} = \frac{1}{\beta} - \frac{1}{s}$. Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$, $1 < p^- \leq p(x) \leq p^+ < \beta$ and $q(\cdot)$ such that $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{\beta}$. If f and g are two measurable functions such that the inequality*

$$|||f|||_{\mathbb{L}_w(\delta)} \leq C \|gw\|_s,$$

holds for every $w \in \mathcal{A}(s, \infty)$ and some positive constant $C = C(w)$. Then there exits a positive constant C such that the inequality

$$\|f\|_{q(\cdot)} \leq C \|g\|_{p(\cdot)} \tag{2.11}$$

holds, provided that the left hand side of (2.11) *is finite.*

Moreover

Theorem 2.12. *Let $1 < \beta < \infty$, $0 \le \delta < 1$, and let s be such that $\frac{\delta}{n} = \frac{1}{\beta} - \frac{1}{s}$. Suppose that $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $\frac{\tilde{\delta}(x)}{n} := \frac{1}{\beta} - \frac{1}{p(x)}$ with $0 \le \delta - \tilde{\delta}(x)$ for every $x \in \mathbb{R}^n$. If f and g are two measurable functions such that the inequality*

$$|||f|||_{\mathbb{L}_w(\delta)} \le C \|gw\|_s ,$$

holds for every weight w in $\mathcal{A}(s,\infty)$ and some positive constant $C = C(w)$. Then there exits a positive constant C such that the inequality holds

$$|||f|||_{\mathbb{L}(\tilde{\delta}(\cdot))} \le C \|g\|_{p(\cdot)} .$$

Remark 2.13. As we said in Remark 2.4 it was proved in [20] that the hypothesis in Theorem 2.10 and 2.12 hold with f replaced by $I_\alpha g$. Thus we can obtain the following Corollaries of Theorem 2.10 and 2.12 respectively for this operator.

Corollary 2.14. *Let $0 < \alpha < n$ and let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$. Let $1 < p^- \le p(x) \le p^+ < n/\alpha$ and $q(\cdot)$ be such that $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$. Then there exists a positive constant C such that*

$$\|I_\alpha f\|_{q(\cdot)} \le C \|f\|_{p(\cdot)} ,$$

provided that the left hand side is finite.

Corollary 2.15. *Let $0 < \alpha < n$ and let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $\frac{n}{\alpha} \le p^- \le p(x) \le p^+ < \frac{n}{(\alpha-1)^+}$ and $\frac{\tilde{\delta}(x)}{n} = \frac{\alpha}{n} - \frac{1}{p(x)}$ for every $x \in \mathbb{R}^n$. Then there exists a positive constant C such that*

$$|||I_\alpha f|||_{\mathbb{L}(\tilde{\delta}(\cdot))} \le C \|f\|_{p(\cdot)} .$$

Remark 2.16. When $p^- = n/\alpha$ the theorem above is a generalization to the variable Lebesgue context of the well known result in the classical setting, that gives the $L^{n/\alpha}$ - BMO boundedness of the operators I_α. See for example [27].

When $p^- > n/\alpha$, Corolary 2.15 is an extension to the variable contexts of the boundedness of I_α from L^r into $\mathbb{L}_1(\delta)$ for $0 < \delta < 1$ and $r > n/\alpha$.

While this paper was being written we found the paper of Cruz Uribe and D.Wang in [3]. Particularly the authors proved an interesting result which gives extrapolation results in variable Lebesgue spaces with weights. In order to establish it we need the following definition given in the same article. We say that $(p(\cdot), v)$ is an M-pair if and only if M is bounded on $L^{p(\cdot)}(v)$ and $L^{p'(\cdot)}(v^{-1})$, where $L^{p(\cdot)}(v)$ denotes the space of all measurable functions f such that $fv \in L^{p(\cdot)}$.

Theorem 2.17. *Suppose that for some p_0, q_0, $1 < p_0 \le q_0 < \infty$, and every $w_0 \in \mathcal{A}_{(p_0,q_0)}$, the inequality*

$$\|fw_0\|_{q_0} \le C \|gw_0\|_{p_0} ,$$

holds for some positive constant C.

Given $p(\cdot)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{\sigma'}.$$

If $w \in \mathcal{A}_{(p(\cdot),q(\cdot))}$ and $(q(\cdot)/\sigma,\ w^\sigma)$ is an M-pair, then

$$\|fw\|_{L^{q(\cdot)}} \le C \|gw\|_{L^{p(\cdot)}} .$$

The theorem holds for $p_0 = 1$ if we assume only that the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(w^{-q_0})$.

If $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $\frac{1}{p} - \frac{1}{q}$ is a constant, then $q \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Thus, if $w \in \mathcal{A}_{(p(\cdot),q(\cdot))}$ then it is easy to see that $(q(x)/\sigma, w^\sigma)$ is an M-pair. Then, as a consequence of Theorem 2.1 and [3] we obtain the following extrapolation result.

Theorem 2.18. *Let $1 < \beta < \infty$, $0 \le \delta < 1$ and s such that $\frac{\delta}{n} = \frac{1}{\beta} - \frac{1}{s}$. Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that $p^+ < \beta$ and let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{\beta}$. If f and g are two measurable functions such that $\|fw\|_{q_1} < \infty$ for some $1 < q_1$ and*

$$|||f|||_{\mathbb{L}_w(\delta)} \le \|gw\|_s ,$$

holds for every $w \in \mathcal{A}(s, \infty)$ then

$$\|fw\|_{q(\cdot)} \le C \|gw\|_{p(\cdot)} ,$$

holds for every $w \in \mathcal{A}_{(p(\cdot),q(\cdot))}$.

Remark 2.19. The result above provide us with a weighted version of Theorem 2.10 and a generalization to the variable context of Theorem 2.1.

3. Auxiliary Lemmas

Before proving the main results we give several technical lemmas. The first one is a version of the algorithm of Rubio de Francia in the general context of variable Lebesgue spaces and can be found in [4], [5], [6]. We include a short proof of this result.

Lemma 3.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Let h be a positive function such that $h \in L^{p(\cdot)}(\mathbb{R}^n)$, then there exists $H \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $H \in \mathcal{A}_1$ and*

$$h(x) \le H(x) \ a.e.\ x \in \mathbb{R}^n. \tag{3.2}$$

$$\|H\|_{L^{p(\cdot)}} \le 2\|h\|_{L^{p(\cdot)}}. \tag{3.3}$$

Proof. Following the algorithm of Rubio de Francia, it is sufficient to consider $H := \mathscr{R}h$ defined by

$$\mathscr{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{p(\cdot)}}},$$

where, for $k \ge 1$, M^k denotes k iterations of the Hardy-Littlewood maximal operator, and M^0 is the identity operator. The properties of H, follow immediately.

□

It is well known that a sufficient condition that guarantees the boundedness of the Hardy-Littewood maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$ is that the exponent function $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ (see, for example [4], [6]).

Definition 3.4. For a locally integrable function f and $0 \le \delta < 1$, the δ - sharp maximal operator, $f^\sharp_\delta$, is defined by

$$f^\sharp_\delta(x) := \sup_{B \ni x} \frac{1}{|B|^{1+\delta/n}} \int_B |f - m_B f|\, dy,$$

where $m_B f$ denotes the average of f over the ball $B \subset \mathbb{R}^n$, that is $m_B f = \frac{1}{|B|}\int_B |f|$.

Lemma 3.5. *Let $0 \le \delta < 1$ and v be a weight. Then there exits positive constants C_1 and C_2 such that*

$$C_1 \left\| v f^\sharp_\delta \right\|_\infty \le |||f|||_{\mathbb{L}_v(\delta)} \le C_2 \left\| v f^\sharp_\delta \right\|_\infty .$$

When $\delta = 0$ the lemma above is due to Harboure, Macías and Segovia ([10]), the proof of that lemma can be adapted to the general case, that is $0 < \delta < 1$. We omit it.

The following result will be used in the proof of Theorem 2.1. It was proved in [10].

Lemma 3.6. *Let $w^p \in \mathcal{A}_{p_0}$ and $f \in L^{p_0}(w^p)$ for some $1 < p_0 \leq p$. Then there exists a positive constant C, independent of p_0 and f, such that*

$$\|f\, w\|_p \leq C\|f_0^\sharp\, w\|_p.$$

The following lemma is an easy consequence of a result proved in [5] (see Corollary 2.3) and it will be useful in the proof of Theorem 2.10.

Lemma 3.7. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $f \in L^{p(\cdot)}$. Then there exists a positive constant C such that*

$$\|f\|_{p(\cdot)} \leq C \left\|f_0^\sharp\right\|_{p(\cdot)}.$$

In order to prove our main results we give the following generalization of a result given by Calderón and Scott in [1, Proposition 4.6].

Proposition 3.8. *Let $0 \leq \tilde{\delta} \leq \delta < 1$, $1 < p \leq \frac{n}{\delta-\tilde{\delta}}$ and $\frac{1}{r} = \frac{1}{p} - \frac{\delta-\tilde{\delta}}{n}$. If w is a weight such that $w \in \mathcal{A}(p, r)$, then there exists a positive constant C such that the following inequality*

$$\|f_{\tilde{\delta}}^\sharp\, w\|_r \leq C\|f_\delta^\sharp\, w\|_p$$

holds.

Remark 3.9. When $w \equiv 1$ and $\tilde{\delta} = 0$, the proposition above was proved in [1]. In proving that result the authors obtain the following pointwise estimate

$$f_0^\sharp(x) \leq C I_\delta f_\delta^\sharp(x),$$

and then the result follows from the boundedness properties of the fractional integral operator I_δ and the relation between p and r.

To state the next result we need the following definition: for $0 < \alpha < n$, the fractional maximal operator M_α is given by

$$M_\alpha f(x) := \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f|$$

were the supremum is taken over every ball $B \subset \mathbb{R}^n$ containing x.

The proof of Proposition 3.8 well be a consequence of the following improvement of the above inequality.

Lemma 3.10. *Let $0 \leq \tilde{\delta} \leq \delta < 1$, then*

$$f_{\tilde{\delta}}^\sharp(x) \leq M_{\delta-\tilde{\delta}} f_\delta^\sharp(x).$$

Proof. Let $B \subset \mathbb{R}^n$, if $x, z \in B$ then

$$\begin{aligned} \frac{1}{|B|^{1+\frac{\tilde{\delta}}{n}}} \int_B |f - m_B f| &= \frac{|B|^{\frac{\delta-\tilde{\delta}}{n}}}{|B|^{1+\frac{\delta}{n}}} \int_B |f - m_B f| \\ &\leq |B|^{\frac{\delta-\tilde{\delta}}{n}} f_\delta^\sharp(z). \end{aligned}$$

By integrating over the ball B we obtain that

$$\frac{|B|}{|B|^{1+\frac{\tilde{\delta}}{n}}}\int_B |f-m_Bf| \leq |B|^{\frac{\delta-\tilde{\delta}}{n}}\int_B f_\delta^\sharp(z)\,dz,$$

or equivalently

$$\begin{aligned}\frac{1}{|B|^{1+\frac{\tilde{\delta}}{n}}}\int_B |f-m_Bf| &\leq \frac{1}{|B|^{1-\frac{\delta-\tilde{\delta}}{n}}}\int_B f_\delta^\sharp(z)\,dz\\ &\leq M_{\delta-\tilde{\delta}}f_\delta^\sharp(x).\end{aligned}$$

Thus, the result follows by taking supremum over every ball containing x. □

Then the proof of Proposition 3.8 follows immediately from the continuity properties of the fractional maximal operator between weighted Lebesgue spaces ([17]) when $1<p<\frac{n}{\delta-\tilde{\delta}}$. For the case $p=\frac{n}{\delta-\tilde{\delta}}$ we use the following lemma.

Lemma 3.11. *Let $0\leq\alpha<n$ and $w\in\mathcal{A}(n/\alpha,\infty)$ if $\alpha>0$ or $w^{-1}\in\mathcal{A}_1$ if $\alpha=0$. Then there exists a positive constant C such that*

$$\|wM_\alpha f\|_\infty \leq C\,\|wf\|_{n/\alpha}$$

Proof. If $x\in B$, by applying Hölder's inequality and the hypothesis on the weight w we obtain that

$$\begin{aligned}w(x)\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|f| &\leq \frac{w(x)}{|B|^{1-\frac{\alpha}{n}}}\left(\int_B|fw|^{n/\alpha}\right)^{\alpha/n}\left(\int_B w^{-\frac{n}{n-\alpha}}\right)^{1-\frac{\alpha}{n}}\\ &\leq C\,\|w\chi_B\|_\infty\left(\fint_B w^{-\frac{n}{n-\alpha}}\right)^{1-\frac{\alpha}{n}}\|fw\|_{n/\alpha}\\ &\leq C\,\|fw\|_{n/\alpha}\,.\end{aligned}$$

□

As a direct consequence of Lemma 3.10 and the boundedness of the operator M_α in the variable context (see [5], Corollary 2.12) we have the following result.

Corollary 3.12. *Let $0\leq\tilde{\delta}\leq\delta<1$ and $p\in\mathcal{P}^{log}(\mathbb{R}^n)$. Let $p^+<n/(\delta-\tilde{\delta})$ and $r(\cdot)$ be such that $\frac{1}{p(\cdot)}-\frac{1}{r(\cdot)}=\frac{\delta-\tilde{\delta}}{n}$. Then there exists a positive constant C such that the following inequality*

$$\left\|f_{\tilde{\delta}}^\sharp\right\|_{r(\cdot)} \leq C\left\|f_\delta^\sharp\right\|_{p(\cdot)}$$

holds.

The next lemma is a result due to Rubio de Francia, (see [7] or [24])

Lemma 3.13. *Let w be an $\mathcal{A}_a$ weight, $1\leq a<\infty$. Then, for any $h\geq 0$ belonging to $L^{a'}(w)$, there exists H such that $h\leq H$, $Hw\in\mathcal{A}_1$ and $\|H\|_{L^{a'}(w)}\leq C\,\|h\|_{L^{a'}(w)}$. The constant C and the $\mathcal{A}_1$ constant corresponding to Hw do not depend on h.*

The following lemma is an unweighted version of Proposition 3.8 in the variable context. In order to state and prove it we give the following definition.

Let $1<\beta<\infty$ and $s(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ be such that $\frac{\delta(x)}{n}=\frac{1}{\beta}-\frac{1}{s(x)}\geq 0$, the $\delta(\cdot)$-sharp maximal operator $f_{\delta(\cdot)}^\sharp$ is defined by

$$f_{\delta(\cdot)}^\sharp(x)=\sup_{B\ni x}\frac{1}{|B|^{1/\beta}\,\|\chi_B\|_{s'(\cdot)}}\int_B|f-m_Bf|\,,$$

where the supremum is taking over every ball B contaning x.

Remark 3.14. From the definitions of $|||f|||_{\mathbb{L}(\delta(\cdot))}$ and $f^{\sharp}_{\delta(\cdot)}$ it is easy to see that

$$|||f|||_{\mathbb{L}(\delta(\cdot))} = \left\| f^{\sharp}_{\delta(\cdot)} \right\|_{\infty}.$$

Lemma 3.15. *Let $1 < \beta < \infty$, $s(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\delta(\cdot)$, $\tilde{\delta}(\cdot)$ such that*

$$\frac{\delta(x)}{n} = \frac{1}{\beta} - \frac{1}{s(x)}, \quad \frac{\tilde{\delta}(x)}{n} = \frac{1}{\beta} - \frac{1}{p(x)},$$

and $0 \leq \tilde{\delta}(x) \leq \delta(x) < n$ a.e. $x \in \mathbb{R}^n$. Then

$$\left\| f^{\sharp}_{\tilde{\delta}(\cdot)} \right\|_{\infty} \leq C \left\| f^{\sharp}_{\delta(\cdot)} \right\|_{\frac{n}{\delta(\cdot)-\tilde{\delta}(\cdot)}}.$$

Proof. Let $x, z \in B$, then

$$\frac{1}{|B|^{1/\beta}} \int_B |f - m_B f| = \frac{1}{|B|^{1/\beta}} \frac{\|\chi_B\|_{s'(\cdot)}}{\|\chi_B\|_{s'(\cdot)}} \int_B |f - m_B f| \leq \|\chi_B\|_{s'(\cdot)} f^{\sharp}_{\delta(\cdot)}(z),$$

or equivalently

$$\frac{1}{\|\chi_B\|_{s'(\cdot)} |B|^{1/\beta}} \int_B |f - m_B f| \leq f^{\sharp}_{\delta(\cdot)}(z).$$

Integrating over the ball B and by applying Hölder's inequality with exponents $n/(\delta - \tilde{\delta})$ and $n/[n - (\delta - \tilde{\delta})]$ we obtain that

$$\frac{|B|}{\|\chi_B\|_{s'(\cdot)} |B|^{1/\beta}} \int_B |f - m_B f| \leq \int_B f^{\sharp}_{\delta(\cdot)}(z)\, dz \leq C \left\| f^{\sharp}_{\delta} \right\|_{\frac{n}{\delta-\tilde{\delta}}} \|\chi_B\|_{\frac{n}{n-(\delta-\tilde{\delta})}}$$

since $\frac{n-(\delta-\tilde{\delta})}{n} = \frac{1}{p'(\cdot)} + \frac{1}{s(\cdot)}$, from the generalized version of Hölder's inequality in the variable Lebesgue spaces (see [4], [6] for details), we obtain that $\|\chi_B\|_{\frac{n}{n-(\delta-\tilde{\delta})}} \leq C \|\chi_B\|_{p'(\cdot)} \|\chi_B\|_{s(\cdot)}$, and hence

$$\frac{|B|}{\|\chi_B\|_{s'(\cdot)} \|\chi_B\|_{s(\cdot)}} \frac{1}{|B|^{1/\beta} \|\chi_B\|_{p'(\cdot)}} \int_B |f - m_B f| \leq C \left\| f^{\sharp}_{\delta} \right\|_{\frac{n}{\delta-\tilde{\delta}}}.$$

From the fact that $s \in \mathcal{P}^{\log}(\mathbb{R}^n)$, it follows that

$$\frac{|B|}{\|\chi_B\|_{s'(\cdot)} \|\chi_B\|_{s(\cdot)}} \geq C,$$

(see [2], [4], [6]).

Therefore the result follows by taking supremum over every ball B contaning x. □

4. Proof of main results

In this section we give the proofs of our main theorems. We begin with the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $w \in \mathcal{A}(p, q)$ and $f \in L^q(w^q)$ and suppose that $\|gw\|_p < \infty$, otherwise there is nothing to prove. Then there exists $h \in L^{(p/s)'}(w^{-p'})$ such that

$$\int h^{(p/s)'} w^{-p'}\, dx = 1. \tag{4.1}$$

and

$$\left(\int |g|^p w^p\,dx\right)^{1/p} = \left(\int (|gw^{p'}|^s) h\, w^{-p'}\,dx\right)^{1/s}, \tag{4.2}$$

in fact, it is sufficient to consider $h(x) := (g(x)w(x)^{p'}/\,\|gw\|_p)^{p-s}$.

Let $\tilde{h} = h^{-s'/s}$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{s}$. From (4.1) is follows that

$$\int \tilde{h}^{r/s'} w^{-p'}\,dx = 1.$$

Since $w \in \mathcal{A}(p,q)$ and $\frac{1}{r} - \frac{1}{q} = \frac{\delta}{n}$, it follows that $w^{-p'} \in \mathcal{A}_{1+p'/q} \subset \mathcal{A}_{1+p'/r}$, then $w \in \mathcal{A}(p,r)$. Therefore, by applying Lemma 3.13, with $a = 1+p'/r$, there exists a function $\tilde{H} \geq \tilde{h}$ satisfying

$$\|\tilde{H}^{1/s'} w^{-p'/r}\|_r^r = \int \tilde{H}^{r/s'} w^{-p'} dx \leq C. \tag{4.3}$$

Then

$$\begin{aligned}
\left(\int |g|^p w^p\,dx\right)^{1/p} &= \left(\int (|gw^{p'}|^s)(\tilde{h}^{-1/s'})^s w^{-p'}\,dx\right)^{1/s} \\
&\geq \left(\int (|gw^{p'}|^s)(\tilde{H}^{-1/s'})^s w^{-p'}\,dx\right)^{1/s} \\
&\geq \left(\int |g|^s \big[\tilde{H}^{-1/s'}\, w^{p'/s'}\big]^s\,dx\right)^{1/s}.
\end{aligned}$$

By Lemma 3.13 $\tilde{H}w^{-p'} \in \mathcal{A}_1$, hence $\tilde{H}^{-1/s'}w^{p'/s'} \in \mathcal{A}(s,\infty)$. Therefore, we use the hypothesis, Lemma 3.5 and (4.3) to obtain

$$\begin{aligned}
\left(\int |g|^p w^p\,dx\right)^{1/p} &\geq |||f|||_{\mathbb{L}_{\tilde{H}^{-1/s'}w^{p'/s'}}(\delta)} \\
&\geq C\|f_\delta^\sharp\, \tilde{H}^{-1/s'} w^{p'/s'}\|_\infty \\
&\geq C\|f_\delta^\sharp\, \tilde{H}^{-1/s'} w^{p'/s'}\|_\infty \|\tilde{H}^{1/s'} w^{-p'/r}\|_r \\
&\geq C\|f_\delta^\sharp\, w\|_r.
\end{aligned}$$

On the other hand, since $w \in \mathcal{A}(p,q)$ and $\frac{1}{r'} - \frac{1}{p'} > 0$, we obtain that $w^q \in \mathcal{A}_{1+q/p'} \subset \mathcal{A}_{1+q/r'}$ and hence $w \in \mathcal{A}(r,q)$.

Then by the Proposition 3.8 we have

$$\left(\int |g|^p w^p\,dx\right)^{1/p} \geq C\|f_0^\sharp\, w\|_q.$$

Finally, as we have assumed $f \in L^q(w^q)$, in accordance of Lemma 3.6 it follows

$$\left(\int |g|^p w^p\,dx\right)^{1/p} \geq C\|f\, w\|_q,$$

which concludes the proof.

□

Proof of Theorem 2.5. Let $w \in \mathcal{A}(p,\infty)$ and $\|gw\|_p < \infty$. Define $\frac{1}{r} := \frac{1}{p} - \frac{1}{s} = \frac{\delta-\tilde{\delta}}{n} \geq 0$, since $w^{-p'} \in \mathcal{A}_1 \subset \mathcal{A}_{1+p'/r}$, as in the proof of Theorem 2.1, we can deduce the existence of $\tilde{H} \in L^{r/s'}(w^{-p'})$ such that

$$\left\|\tilde{H}^{1/s'} w^{-p'/r}\right\|_r \leq C,$$

with $\tilde{H}^{-1/s'} w^{p'/r} \in \mathcal{A}(s, \infty)$ and

$$\left(\int |g|^p \, w^p \right)^{1/p} \geq C \left\| f_{\delta}^{\sharp} w \right\|_r .$$

Since $r' < p'$, by applying Hölder's inequality with exponents p'/r' and $p'/(p'-r')$, we obtain

$$\begin{aligned} \|w\chi_Q\|_\infty \left(\fint_Q w^{-r'} \right)^{1/r'} &\leq \|w\chi_Q\|_\infty \left(\int_Q w^{-p'} \right)^{1/p'} |Q|^{\frac{1}{r'} \frac{1}{(p'/r')'} - \frac{1}{r'}} \\ &= \|w\chi_Q\|_\infty \left(\fint_Q w^{-p'} \right)^{1/p'} \leq C, \end{aligned}$$

where in the last inequality we use that $w \in \mathcal{A}(p, \infty)$. Then $w \in \mathcal{A}(r, \infty)$, and by applying Proposition 3.8 we have that

$$\left(\int |g|^p \, w^p \right)^{1/p} \geq c \left\| f_{\tilde{\delta}}^{\sharp} w \right\|_\infty ,$$

which leads to the desired inequality after applying Lemma 3.5. □

Proof of Theorem 2.10. It is enough to consider $\|g\|_{p(\cdot)} < \infty$. Let $h(x) = \left(\frac{g(x)}{\|g\|_{p(\cdot)}} \right)^{p(x)-s}$, then $\|h\|_{\left(\frac{p(\cdot)}{s}\right)'} \leq 1$. In fact

$$\int |h(x)|^{(p(x)/s)'} \, dx = \int (|g(x)| \,/\, \|g\|_{p(\cdot)})^{p(x)} dx \leq 1.$$

Moreover

$$\int \frac{|g(x)|^s \, h(x)}{\|g\|_{p(\cdot)}^s} dx = \int (g(x)/\, \|g\|_{p(\cdot)})^{p(x)} dx \leq 1,$$

then

$$\|g\|_{p(\cdot)} \geq \left(\int |g|^s \, h \right)^{1/s} . \tag{4.4}$$

If $\tilde{h} := h^{-s'/s}$ and $\frac{1}{r(x)} := \frac{1}{p(x)} - \frac{1}{s}$ we have

$$\int \tilde{h}(x)^{r(x)/s'} dx = \int h^{-r(x)/s}(x) dx = \int \left(\frac{g(x)}{\|g\|_{p(\cdot)}} \right)^{p(x)} dx \leq 1,$$

Consequently

$$\|\tilde{h}\|_{r(\cdot)/s'} \leq 1.$$

By (4.4) and the definition of $\tilde{h}$ we obtain that

$$\|g\|_{p(\cdot)} \geq \left(\int |g|^s \, h \right)^{1/s} = \left(\int |g|^s \, (\tilde{h}^{-1/s'})^s \right)^{1/s} .$$

From the fact that $(r(\cdot)/s')^- > 1$ and $r(\cdot)/s' \in \mathcal{P}^{\log}(\mathbb{R}^n)$ since $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, from Lemma 3.1 there exists a function $\tilde{H} \geq \tilde{h}$ such that $\tilde{H} \in \mathcal{A}_1$ and

$$\|\tilde{H}\|_{r(\cdot)/s'} \leq 2\|\tilde{h}\|_{r(\cdot)/s'} \leq 2. \tag{4.5}$$

Thus, we have that

$$\|g\|_{p(\cdot)} \geq \left(\int |g|^s \, (\tilde{H}^{-1/s'})^s \right)^{1/s} .$$

Then, since $\tilde{H}^{-1/s'} \in \mathcal{A}(s,\infty)$, we can apply the hypothesis, Lemma 3.5 and (4.5) to obtain that

$$\begin{aligned}\|g\|_{p(\cdot)} &\geq C\||f|\|_{\mathbb{L}_{\tilde{H}^{-1/s'}}(\delta)} \\ &\geq C\left\|\tilde{H}^{-1/s'} f_\delta^\sharp\right\|_\infty \left\|\tilde{H}^{1/s'}\right\|_{r(\cdot)} \\ &\geq C\left\|f_\delta^\sharp\right\|_{r(\cdot)}.\end{aligned}$$

On the other hand, since $\frac{1}{r(\cdot)} - \frac{1}{q(\cdot)} = \frac{\delta}{n}$ then $r^+ < \frac{n}{\delta}$. Thus, by Corollary 3.12 we have that

$$\left\|f_0^\sharp\right\|_{q(\cdot)} \leq C\left\|f_\delta^\sharp\right\|_{r(\cdot)}.$$

Finally, since $f \in L^{q(\cdot)}$, and $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, from Lemma 3.7 we have that

$$\|f\|_{q(\cdot)} \leq C\left\|f_0^\sharp\right\|_{q(\cdot)},$$

which concludes the proof.

□

Proof of Theorem 2.12. By following the same argument as in proof of Theorem 2.10 we can obtain the existence of a function $\tilde{H}$ such that

$$\begin{aligned}\|g\|_{p(\cdot)} &\geq \left(\int |g|^s\, h\right)^{1/s} = \left(\int |g|^s\, (\tilde{h}^{-1/s'})^s\right)^{1/s} \\ &\geq \left(\int |g|^s\, (\tilde{H}^{-1/s'})^s\right)^{1/s} \geq c\left\|\tilde{H}^{-1/s'} f_\delta^\sharp\right\|_\infty \left\|\tilde{H}^{1/s'}\right\|_{r(\cdot)} \\ &\geq c\left\|f_\delta^\sharp\right\|_{r(\cdot)},\end{aligned}$$

where

$$\frac{1}{r(x)} := \frac{1}{p(x)} - \frac{1}{s} = \frac{\delta - \tilde{\delta}(x)}{n}.$$

Then, the result follows by applying Lemma 3.15 and Remark 3.14. □

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