Controllability of Schrödinger equation with a nonlocal term.

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Abstract

This paper is concerned with the internal distributed control problem for the 1D Schrödinger equation, $i\,u_t(x,t) = -u_{xx} + \alpha(x)\,u + m(u)\,u$, that arises in quantum semiconductor models. Here $m(u)$ is a non local Hartree–type nonlinearity stemming from the coupling with the 1D Poisson equation, and $\alpha(x)$ is a regular function with linear growth at infinity, including constant electric fields. By means of both the Hilbert Uniqueness Method and the contraction mapping theorem it is shown that for initial and target states belonging to a suitable small neighborhood of the origin, and for distributed controls supported outside of a fixed compact interval, the model equation is controllable. Moreover, it is shown that, for distributed controls with compact support, the exact controllability problem is not possible.

Keywords: Nonlinear Schrödinger–Poisson; Hartree potential; constant electric field; internal controllability.

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1 Introduction

We are mainly concerned with the internal distributed controllability for the following 1D Schrödinger equation

$$iu_t = -u_{xx} + \alpha(x)\,u + m(u)u, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$
$$u(x,0) = u_0(x), \tag{1.2}$$

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posed in the Sobolev space $\mathcal{H} = \{\phi \in H^1(\mathbb{R}) : \int \mu(x)|\phi|^2 < \infty\}$, where μ is a positive regular function that coincides with $|x|$ away from the origin. Here, the non linearity $m(u)$ is of non local nature:

$$m(\phi)(x) = \int \varrho(x,y)|\phi(y)|^2 dy, \tag{1.3}$$

where the kernel satisfies the estimate $|\varrho(x,y)| \leq \mu(y)$. This choice is motivated for the self–consistent 1D Schrödinger–Poisson equation used in quantum semiconductor theory

$$iu_t = -u_{xx} + u\left(|x| * \left(\mathcal{D} - |u|^2\right)\right) \tag{1.4}$$

where $\mathcal{D} \in C^\infty(\mathbb{R})$ denotes the fixed positively charged background or *impurities*, see [8] and references therein for semiconductor models. After a suitable rearrangement of terms the Hartree potential reads

$$\begin{aligned} |x| * \left(\mathcal{D} - |u|^2\right) &= \int (|x-y| - \mu(x))(\mathcal{D}(y) - |u(y,t)|^2)dy + \mu(x)\int (\mathcal{D}(y) - |u(y,t)|^2)dy \\ &= \int (|x-y| - \mu(x))\mathcal{D}(y)dy - \int (|x-y| - \mu(x))|u(y,t)|^2 dy + \\ &\quad \mu(x)\left(\|\mathcal{D}\|_{L^1(\mathbb{R})} - \|u(\cdot,t)\|^2_{L^2(\mathbb{R})}\right). \end{aligned}$$

Introducing $a := \|\mathcal{D}\|_{L^1(\mathbb{R})} - \|u(\cdot,t)\|^2_{L^2(\mathbb{R})}$, $F(x) := \int (|x-y| - \mu(x))\mathcal{D}(y)dy$, and $\varrho(x,y) = \mu(x) - |x-y|$, equation (1.4) thus becomes

$$iu_t(x,t) = -u_{xx}(x,t) + (a\mu(x) + F(x))u(x,t) + m(u(x,t))u(x,t),$$

taking $\alpha(x) := a\mu(x) + F(x)$ we show that the evolution equation (1.4) becomes (1.1).

We note that in the 1D case the kernel $\mu(x)$ is not bounded nor integrable so the classic theory developed in [1] does not apply and we refer to [3] for details on the well posedness. In this article we will consider a slightly extended version in which the term $a\mu(x)$ is replaced by a regular function $\alpha(x) \in C^\infty(\mathbb{R})$, with at most linear growth at infinity (i.e. with the asymptotics $\alpha(x) \sim C^\pm x$ for $x \sim \pm\infty$), in order to include constant electric fields $\alpha(x) = qx$. We note that due to the regularity requirements of the unique continuation technique displayed in Lemma 3.2, the regular function $\alpha(x)$ appears as a regularized approximation of a locally constant electric field, which is modelled with a continuous piecewise linear function. It is also worth to mention that since the impurities give rise to a bounded potential

$$V_d(x) = \int \left(|x-y| - |x|\right)\mathcal{D}(y)dy,$$

and hence enters in the model equation as a bounded multiplication operator, and since our results are still valid for bounded perturbations, there is no loss of generality in restricting ourselves to the case $\mathcal{D} \equiv 0$. Let us finally mention that results on controllability with local nonlinearities as $|u|^{2\sigma}u$ are widely developed, see [5, 11], and therefore local nonlinearities will not be taken into consideration.

The problem of exact internal controllability of equation (1.1)-(1.2) is usually described as the question of finding a control function $h \in L^2(0,T,\mathcal{H})$ and its associated state function

$u \in C(0,T,\mathcal{H})$ such that

$$iu_t = -u_{xx} + \alpha(x)\, u + m(u)\, u + \psi(x) h(x,t), \;\; x \in \mathbb{R}, \; t \in (0,T), \tag{1.5}$$
$$u(x,t_0) = u_0(x), \quad u(x,T) = u_T(x) \tag{1.6}$$

where $T > 0$ is a given target time and u_0 and u_T are the given initial and target states respectively, and $\psi : \mathbb{R} \to \mathbb{R}$ is a given C^1 function that localizes the control to $\mathrm{Supp}(\psi)$. The problem of distributed controllability for Schrödinger equations of nonlinear type appears often in nonlinear optics, see for instance [9, 4]. There are several results on controllability of the Schrödinger equation, for a review on this topic we refer to [13].

In this paper we discuss the internal distributed controllability for the problem

$$\begin{aligned} &iu_t = -u_{xx} + \alpha(x)\, u + m(u)\, u, \quad x \in \mathbb{R}, \; t > 0, \\ &u(x,t_0) = u_0(x) \end{aligned}$$

and present results concerning two different situations depending on the support of the control: on one hand controls that are supported outside a compact interval, in which case we shall give positive results, and on the other hand localized controls, in which case we shall give a non controllability result.

We start dealing with a distributed control given by $\psi(x)h(x,t)$ where $\psi \in C^1(\mathbb{R})$ satisfies:

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \geq R+1 \\ 0 & \text{for } |x| \leq R \end{cases} \tag{1.7}$$

We thus show that for a given $0 < T$ there exist a (small) constant δ such that for every $u_0,\; u_T \in \mathcal{H}$ with $\|u_0\|_{\mathcal{H}}, \|u_T\|_{\mathcal{H}} < \delta$ there exists a control $h(x,t) \in L^2(0,T,\mathcal{H})$ such that the nonlinear problem (1.5)-(1.6) has a unique solution $u \in C(0,T,\mathcal{H})$.

We then turn to the case in which $\psi \in C^1$ is compactly supported and show that for both $\alpha = \mu$ (linear operator with a discrete spectrum) and $\alpha(x) = x$ (constant electric field, which has a continuous spectrum), the linear system is not exactly controllable. More precisely we show that for any fixed finite time $T > 0$ and any fixed target state $u_T \in \mathcal{H}$ there exist an open bounded interval Ω and an initial state $u_0 \in \mathcal{H}$, such that for any ψ with $\mathrm{Supp}(\psi) \subset \Omega$, there is no control function $\psi(x)h(x,y)$, with $h \in L^2(0,T,\mathcal{H})$, and no constant $C = C(T,\Omega)$ such that

$$\begin{aligned} &iu_t = -u_{xx} + \alpha(x)\, u + \psi(x)h, \qquad x \in \mathbb{R}, \; t \in [0,T], \\ &u(x,0) = u_0(x), \qquad u(x,T) = u_T(x) \end{aligned}$$

with $\|h\|_{L^1(0,T,L^2(\Omega))} \leq C(T,\Omega)\left(\|u_0\|_{\mathcal{H}} + \|u_T\|_{\mathcal{H}}\right).$

The paper is organized as follows. We set the problem in section 1. In section 2, we deal with the existence of dynamics and establish useful estimates for the related evolution. Section 3 is devoted to the problem in which the control vanishes inside an open bounded interval. We start studying the linear system for which we prove global controllability in the space $\mathcal{H}$; we then prove the local controllability for the nonlinear system (1.5). In Section 4, we deal with the non controllability result for compactly supported controls.

2 Preliminaries

In this section we shall collect some results concerning spectral properties for the operator $-\partial_x^2 + \alpha(x)$. Since most of the estimates refer to different functional spaces we list them below:

- $H^1(\mathbb{R}) := \{\phi \in L^2(\mathbb{R}) : \phi_x \in L^2(\mathbb{R})\}$.
- $L^2_\mu(\mathbb{R}) := \{\phi : \mu^{1/2}\phi \in L^2(\mathbb{R})\}$ where μ is a regular even function satisfying $1 \leq \mu(x)$, and $\mu(x) \equiv |x|$ for $|x| \geq 2$.
- $\mathcal{H} := H^1(\mathbb{R}) \cap L^2_\mu(\mathbb{R})$ with $\|\phi\|^2_{\mathcal{H}} = \|\phi_x\|^2_{L^2} + \|\phi\|^2_{L^2_\mu}$

2.1 Existence of dynamics

To start with we consider the auxiliar operator L_+ defined by

$$\begin{aligned} L_+ : \mathcal{H} &\mapsto \mathcal{H}' \\ \phi &\mapsto L_+(\phi) := \left(-\partial_x^2 + |x|\right)\phi \end{aligned} \tag{2.1}$$

Although this operator does not enter directly in our model, because of the loss of regularity of $|x|$ in the origin, it provides the workspace $\mathcal{H}$ and also it possesses useful spectral properties, easily deduced from the ones of the Airy function, that are needed for the proof of the non controllability result of Theorem 4.1.

Lemma 2.1. *The operator L_+ satisfies the following properties:*

(a) It is self–adjoint in $L^2(\mathbb{R})$.

(b) It has a discrete spectrum $0 < \widetilde{\lambda}_1 < \cdots < \widetilde{\lambda}_N \nearrow +\infty$.

(c) It has a countable set of orthonormal (with respect to L^2) eigenfunctions $\{\phi_N : N \in \mathbb{N}\} \subseteq \mathcal{H}$ satisfying

$$\widetilde{\lambda}_N^{-1/4} \left(\int_\Omega |(\phi_N)_x|^2 \right)^{1/2} \leq C(\Omega), \tag{2.2}$$

where Ω is an arbitrary bounded interval.

Remark 2.2. *Self–adjointness of L_+ and the existence of both a discrete spectrum, $\{0 < \widetilde{\lambda}_1 < \widetilde{\lambda}_2 < \cdots\}$, and an orthonormal basis of eigenfunctions, $\{\phi_N\}_{N\in\mathbb{N}} \subseteq \mathcal{H}$, follows directly from [2] where by means of variational methods it is only shown that L_+^{-1} is a compact operator. However, the non–controllability result relies on some special feature of the eigenfunctions, given by claim (c), that are not considered there and we shall give an alternative proof.*

Proof. We first notice that the related quadratic form verifies $\langle \phi; L_+\phi\rangle = \|\phi_x\|^2_{L^2} + \||x|^{1/2}\phi\|^2_{L^2}$ and this is an equivalent norm for $\mathcal{H}$, from where we recover the self–adjointness of L_+. The operator L_+ has an explicit spectral decomposition expressed in terms of the Airy function Ai, defined as the solution of $-\mathrm{Ai}_{xx}(x) + x\mathrm{Ai}(x) = 0$ such that $\mathrm{Ai}(+\infty) = 0$, as follows. Let

$0 < z_0 < z_1 < \cdots \nearrow +\infty$, and $0 < w_0 < w_1 < \cdots \nearrow +\infty$ be the zeros of $\mathrm{Ai}'(-x)$ and $\mathrm{Ai}(-x)$ respectively, and take $\widetilde{\lambda}_{2N} = z_N$, $\widetilde{\lambda}_{2N+1} = w_N$, and $\phi_{2N}(x) = c_{2N}\mathrm{Ai}(|x| - \widetilde{\lambda}_{2N})$, $\phi_{2N+1}(x) = c_{2N+1}\,\mathrm{sgn}(x)\mathrm{Ai}(|x| - \widetilde{\lambda}_{2N+1})$, where c_N is a (bounded) sequence of normalization constants. A direct computation shows that $L_+(\phi_N) = \widetilde{\lambda}_N\phi_N$. This gives the spectral decomposition of L_+. Since for $|x| \sim +\infty$ it happens that $|x| - \widetilde{\lambda}_N > 0$, each eigenfunction ϕ_N inherits the decaying properties of the Airy function near $+\infty$ where it behaves as $e^{-r^{3/2}}$.

In order to get claim (c) we take profit of the integral expression for the Airy function and its derivative, with $x = -|x|$,

$$Ai(x) = (2\pi)^{-1/2}|x|^{1/2}\int e^{i|x|^{3/2}(k^3/3-k)}dk$$

$$Ai'(x) = (2\pi)^{-1/2}\int ike^{i|x|^{3/2}(k^3/3-k)}dk$$

from where, by means of the stationary phase method, we deduce the asymptotics

$$|Ai'(x)| \leq C(M)|x|^{1/4} \tag{2.3}$$

valid for $x \leq -M$, and also an estimate for the eigenvalues

$$\widetilde{\lambda}_N \sim N^{2/3}. \tag{2.4}$$

Let M be such that $\Omega \subseteq [-M, M]$, from estimate (2.4) there exists N_0 such that, for $N > N_0$, $\widetilde{\lambda}_{2N} - \widetilde{\lambda}_N > M$. Then, for $x \in \Omega$ one has $|x| - \widetilde{\lambda}_{2N} < M - \widetilde{\lambda}_{2N} < -\widetilde{\lambda}_N$. Using (2.3) we conclude $|\phi_{2N}(x)| < \widetilde{\lambda}_N^{1/4}$, and therefore $\|(\phi_N)_x\|_{L^2(\Omega)} \leq (2M)^{1/2}\widetilde{\lambda}_{N/2}^{1/4}$. This finishes the proof. □

As a direct consequence of previous result we get the existence of dynamics for the operator L_μ defined by

$$\begin{aligned} L_\mu : \mathcal{H} &\mapsto \mathcal{H}' \\ \phi &\mapsto L_\mu(\phi) := \left(-\partial_x^2 + \mu(x)\right)\phi \end{aligned} \tag{2.5}$$

where $\mu(x)$ is a regular even function satisfying $\mu(x) \equiv |x|$ for $|x| \geq 2$ and $\max\{1, |x|\} \leq \mu(x) \leq 1 + |x|$.

Lemma 2.3. *The operator L_μ is well defined and verifies*

(a) It is self adjoint.

(b) It has a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \lambda_N \nearrow +\infty$, and a countable set of orthonormal (with respect to L^2) eigenfunctions $\{\varphi_N : N \in \mathbb{N}\} \subseteq \mathcal{H}$.

Proof. It follows directly from the inequalities

$$\langle\phi; L_+\phi\rangle \leq \langle\phi; L_\mu\phi\rangle \leq \|\phi\|_{L^2}^2 + \langle\phi; L_+\phi\rangle,$$

and the compact embedding $\mathcal{H} \hookrightarrow L^2$.

Notice that previous estimates yield the asymptotics

$$\lambda_N \sim N^{2/3}.$$

□

In order to develop the observability inequality we need to build some appropriate Sobolev spaces, related to the operator L_μ defined by (2.5). This is done as follows. Let $\{\varphi_N\}_{N\in\mathbb{N}}$ be the orthonormal basis of L^2 given by Lemma 2.3 and, for $\phi \in L^2$, let $\widehat{\phi}$ be the Fourier coefficient: $\widehat{\phi}(N) := \langle \phi; \varphi_N \rangle$. We then set for $k = 0, 1, 2$ the Hilbert spaces $W^k := \{\phi \in L^2 : \sum_{N\geq 0} \lambda_N^k \widehat{\phi}(N)^2 < \infty\}$, with the inner product

$$\langle \psi; \phi \rangle_{W^k} := \sum_{N\geq 0} \lambda_N^k \widehat{\psi}(N)^* \widehat{\phi}(N). \tag{2.6}$$

Let $\mathcal{F} \subset W^0$ be the set of finite linear combinations of $\{\varphi_N\}_{N\in\mathbb{N}}$. Then for $k = -3, -2, -1$ the inner product (2.6) is well defined. We then define W^k as the Hilbert space obtained from the closure of $\mathcal{F}$ with the norm induced by $\langle \cdot; \cdot \rangle_{W^k}$. We have that $L_\mu : W^k \to W^{k-2}$ is an isometry: $\|L_\mu w\|_{W^{k-2}} = \|w\|_{W^k}$. Being L_μ positive, we have $L_\mu^{1/2} : W^k \to W^{k-1}$ which is also an isometry: $\|L_\mu^{1/2} w\|_{W^{k-1}} = \|w\|_{W^k}$.

We finally mention that $W^0 = L^2$, $W^1 = \mathcal{H}$, $W^2 = D(L_\mu)$, the domain of the operator $L_\mu : W^2 \mapsto L^2$, and $W^{-1} = \mathcal{H}'$, with compact embeddings

$$W^2 \subset W^1 \subset W^0 \subset W^{-1} \subset W^{-2} \tag{2.7}$$

Remark 2.4. *Since for any $\psi \in W^k$ and $\phi \in W^{-k}$ we have*

$$\begin{aligned}
\langle \psi; \phi \rangle_{L^2} &= \sum_{N\geq 0} \widehat{\psi}(N)^* \widehat{\phi}(N) \\
&= \sum_{N\geq 0} \lambda_N^{k/2} \widehat{\psi}(N)^* \lambda_N^{-k/2} \widehat{\phi}(N) \\
&\leq \|\psi\|_{W^k} \|\phi\|_{W^{-k}},
\end{aligned}$$

we also have for $k = -2, -1, 0, 1, 2$ that $\left(W^k\right)' = W^{-k}$.

We now turn to the general situation $L := -\partial_x^2 + \alpha(x)$, where $\alpha(x) \in C^\infty(\mathbb{R})$ is a regular function verifying $\alpha_x, \alpha_{xx} \in L^\infty$, and also the asymptotics

$$\lim_{x\to\pm\infty} \frac{\alpha(x)}{\mu(x)} = C^\pm \tag{2.8}$$

The following lemma states precisely the self–adjointness result.

Lemma 2.5. *Let $\alpha \in C^\infty(\mathbb{R})$ satisfying (2.8). Then $L : \mathcal{H} \mapsto \mathcal{H}'$ defined by $L := -\partial_x^2 + \alpha(x)$ is self–adjoint, and therefore $-iL$ generates a strongly continuous group of unitary operators in $L^2(\mathbb{R})$.*

Proof. To this purpose we first show that L is a closed operator. Let $\varphi \in C_0^\infty(\mathbb{R})$ and $(\phi_n; L(\phi_n)) \in \mathcal{H} \times \mathcal{H}'$ a sequence such that $(\phi_n; L(\phi_n)) \to (\phi; \psi)$ in $\mathcal{H} \times \mathcal{H}'$, since $\langle \varphi; \phi_{xx} - (\phi_n)_{xx} \rangle = \langle \varphi_{xx}; \phi - \phi_n \rangle \to 0$ and $\langle \varphi; \alpha(\phi - \phi_n) \rangle = \langle \mathrm{sgn}(\alpha)|\alpha|^{1/2}\varphi; |\alpha|^{1/2}(\phi - \phi_n) \rangle \to 0$ we thus have $\langle \varphi; L(\phi - \phi_n) \rangle \to 0$, and consequently we conclude $\langle \varphi; \psi - L\phi \rangle = \langle \varphi; \psi - L\phi_n \rangle + \langle \varphi; L(\phi_n - \phi) \rangle \to 0$. This shows that $L : \mathcal{H} \to \mathcal{H}'$ is a closed operator.

Since $L_\mu := -\partial_x^2 + \mu(x)$, with $\mu(x) \geq 1$ we deduce that $L_\mu \geq I$ (the identity operator). For $\varphi, \psi \in \mathcal{H}$ we introduce the (well defined) bilinear form $\mathcal{Q}(\phi, \psi) := \langle \phi_x; \psi_x \rangle + \langle \phi; \alpha(x)\psi \rangle$. We now establish two useful estimates

$$\begin{aligned} |\mathcal{Q}(\phi;\psi)| &\leq |\langle \phi_x; \psi_x \rangle| + |\langle \phi; \alpha\psi \rangle| \\ &\leq (1 + \|\alpha\mu^{-1}\|_{L^\infty}) \, |\langle \phi; L_\mu \psi \rangle| \\ &\leq (1 + \|\alpha\mu^{-1}\|_{L^\infty}) \|L_\mu^{1/2}\phi\|_{L^2} \, \|L_\mu^{1/2}\psi\|_{L^2} \end{aligned}$$

$$\begin{aligned} |\mathcal{Q}(L_\mu\phi;\psi) - \mathcal{Q}(\phi; L_\mu\psi)| &= |\langle \phi; [L_\mu : L]\psi \rangle| \\ &\leq |\langle \phi; (\mu-\alpha)_{xx}\psi \rangle| + 2|\langle (\mu-\alpha)_x \phi; \psi_x \rangle| \\ &\leq (\|(\mu-\alpha)_{xx}\|_{L^\infty} + 2\|(\mu-\alpha)_x\|_{L^\infty}) \, \|L_\mu^{1/2}\phi\|_{L^2} \, \|L_\mu^{1/2}\psi\|_{L^2} \end{aligned}$$

where we have used the identity $\|L_\mu^{1/2}\varphi\|_{L^2}^2 = \|\varphi_x\|_{L^2}^2 + \|\varphi\|_{L^2_\mu}^2$. Applying Theorem X.36' in [10] we obtain that L is a essentially self–adjoint operator in $\mathcal{H}$, since it is closed, it follows that L is self adjoint. □

2.2 Scattering properties for constant electric fields

The non controllability result, see Theorem 4.4, for a constant electric field $L_e := -\partial_x^2 - x$, follows from a well-known $L^1 - L^\infty$ estimate for the group $U_e(t)$ generated by $-iL_e$, which depends upon a result of Avron-Herbst, see [12] for details.

Lemma 2.6. *The operator L_e is essentially self–adjoint on $\mathcal{S}(\mathbb{R})$ and*

$$U_e(t) = e^{-it^3} e^{itx} e^{-i(p^2 t + t^2 p)} \tag{2.9}$$

where $p = -i\partial_x$ is the momentum operator.

Remark 2.7. *Identity (2.9) says that except for phase factors $U_e(t)\phi(x)$ is obtained by first translating by t^2 units to the right and then applying the free particle group $e^{it\partial_x^2}$*

Corollary 2.8. *For $\phi \in L^1(\mathbb{R})$ we have the following estimate:*

$$\|U_e(t)\phi\|_{L^\infty} \leq C|t|^{-1/2}\|\phi\|_{L^1}$$

2.3 Estimates for the evolution

Lemma 2.5 guarantees that $-iL$ generates a group $U(t)$. In the sequel we will exhibit useful bounds for the evolution related to both the homogeneous and inhomogeneous problem.

Lemma 2.9. *Let $U(t)$ be the group generated by $-iL$ where $L := -\partial_x^2 + \alpha$ in $\mathcal{H}$. Then*

- $\|(U(t)\phi)_x\|_{L^2} \leq \|\phi_x\|_{L^2} + |t|\|\alpha_x\|_{L^\infty}\|\phi\|_{L^2}$.
- $\|U(t)\phi\|_{L^2_\mu} \leq \|\phi\|_{L^2_\mu} + 2^{1/2}|t|^{1/2}\|\phi\|_{L^2}^{1/2}\|\phi_x\|_{L^2}^{1/2} + |t|\|\alpha_x\|_{L^\infty}\|\phi\|_{L^2}$.

- $\|U(t)\phi\|_{\mathcal{H}} \leq \|\phi\|_{\mathcal{H}}\big(1+|t|\cdot\|\mu_x-\alpha_x\|_{L^\infty}\big)$.

Proof. Let $u(t)=U(t)\phi$, since u verifies $iu_t=-u_{xx}+\alpha u$ and $\|u\|_{\mathcal{H}}^2=\|u\|_{L^2_\mu}^2+\|u_x\|_{L^2}^2$ we have

$$\begin{aligned}
\frac{d}{dt}\big\langle u_x;u_x\big\rangle_{L^2} &= 2\mathrm{Re}\big\langle u_{xt};u_x\big\rangle_{L^2}\\
&= 2\mathrm{Re}\big\langle u_x;-i\alpha_x u\big\rangle_{L^2}\\
\frac{d}{dt}\big\langle u;\mu u\big\rangle_{L^2} &= 2\mathrm{Re}\big\langle u_t;\mu u\big\rangle_{L^2}\\
&= 2\mathrm{Re}\big\langle u_x;i\mu_x u\big\rangle_{L^2}\\
\frac{d}{dt}\big\langle u;u\big\rangle_{\mathcal{H}} &= 2\mathrm{Re}\big\langle u_x;i(\mu_x-\alpha_x)u\big\rangle_{L^2}.
\end{aligned}$$

The inequalities are obtained by means of a standard ODE argument given by the following lemma. Details are given due to the lack of a suitable reference. □

Lemma 2.10. *Let $y:[0,T]\to[0,+\infty)$ be an L^1 function satisfying the inequality $y^2(t)\leq y^2(0)+C\int_0^t y(s)ds$ for some constant $C>0$. Then $y(t)\leq y(0)+Ct/2$.*

Proof. Let $w(t):=\int_0^t y(s)ds$ and $z(t):=\sqrt{y^2(0)+Cw(t)}$. Then $\dot z(t)\leq C/2$ and therefore $y(t)\leq z(t)\leq z(0)+Ct/2$. □

We now turn our attention to the non linear term in equation (1.5), and give the following estimates.

Lemma 2.11. *Let $m:\mathcal{H}\mapsto L^\infty(\mathbb{R})$ be given by*

$$m(\phi)(x)=\int \varrho(x,y)|\phi(y)|^2dy.$$

where $|\varrho(x,y)|\leq\mu(y)$ and $|\varrho_x(x,y)|\leq C$. Then for $\phi,\phi_1\in\mathcal{H}$ the following estimates hold.

- $\|m(\phi)\|_{L^\infty}\leq\|\phi\|_{L^2_\mu}^2$
- $\|m(\phi)\phi-m(\phi_1)\phi_1\|_{\mathcal{H}}\leq 3/2\left(\|\phi\|_{\mathcal{H}}^2+\|\phi\|_{\mathcal{H}}\|\phi_1\|_{\mathcal{H}}+\|\phi_1\|_{\mathcal{H}}^2\right)\|\phi-\phi_1\|_{\mathcal{H}}$

Proof. It is a straightforward computation and will be omitted.

□

We now turn to the non homogeneous problem (1.5) and give similar estimates in the lemma below, which in turn express the global well posedness of the problem.

Lemma 2.12. *Let $T>0$ be fixed, and let $u\in C(0,T,\mathcal{H})\cap C^1(0,T,\mathcal{H}')$ be a solution of (1.5) with fixed $h\in L^2(0,T,\mathcal{H})$ and $\psi\in C^1(\mathbb{R})$ such that ψ, and $\psi_x\in L^\infty(\mathbb{R})$. Then we have the following estimates:*

- $\|u\|_{L^\infty(0,T,L^2(\mathbb{R}))}\leq\|u_0\|_{L^2(\mathbb{R})}+T^{1/2}\|\psi\|_{L^\infty}\|h\|_{L^2(0,T,\mathcal{H})}$.

- $\|u_x\|_{L^\infty(0,T,L^2(\mathbb{R}))} \leq \|u_0\|_{H^1} + \|h\|_{L^2(0,T,\mathcal{H})}\, T^{3/2}\, C(u_0,\psi)$.
- $\|u\|_{L^\infty(0,T,L^2_\mu(\mathbb{R}))} \leq \|u_0\|_{L^2_\mu} + \|h\|_{L^2(0,T,\mathcal{H})}\, T^{3/2}\, C(u_0,\psi)$.
- $\|u\|_{L^\infty(0,T,\mathcal{H})} \leq \|u_0\|_{\mathcal{H}} + \|h\|_{L^2(0,T,\mathcal{H})}\, T^{3/2}\, C(u_0,\psi)$.

Proof. Using estimates for the linear and nonlinear term, given in Lemma 2.9 and Lemma 2.11 respectively, the proof relies on a procedure similar to the one displayed in Lemma 2.9 and will be omitted.

□

3 Controllability

3.1 Linear system

We start this section taking into consideration the controllability of the linear problem, which throughout this section means the existence of a control $h(x,t)$ such that the unique solution of the related non homogeneous linear equation

$$iu_t(x,t) = Lu(x,t) + \psi(x)h(x,t) \tag{3.1}$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R} \tag{3.2}$$

satisfies $u(x,T) = u_T(x)$, for given $T > 0$ and $u_0, u_T(x) \in \mathcal{H}$, where $L := -\partial_x^2 + \alpha(x)$ is the operator of Lemma 2.5, and ψ is defined in (1.7). The main result is given in the following theorem; its proof is based on the Hilbert Uniqueness Method (HUM), requires some technicalities, which we shall first develop, and will be delayed until the end of this subsection.

Theorem 3.1. *Global controllability: linear case.*

Let $T > 0$ be given. Then there exists a bounded linear operator $G : \mathcal{H}\times\mathcal{H} \to L^2(0,T,\mathcal{H})$ such that for any $u_0, u_T \in \mathcal{H}$ the system (3.1)-(3.2)*, with $h = G(u_0,u_T)$, admits a solution $u \in C(0,T,\mathcal{H})$ satisfying $u(x,T) = u_T$.*

As we stated before, we need first to present the ingredients to apply the HUM. To do this, we consider the corresponding adjoint problem in $\mathcal{H}'$:

$$iv_t(x,t) = Lv, \tag{3.3}$$

$$v(x,0) = v_0(x). \tag{3.4}$$

Let $\Lambda : \mathcal{H} \to \mathcal{H}'$ denote the usual isomorphism between the real spaces $\mathcal{H}$ and $\mathcal{H}'$ defined by $\Lambda(v) = \langle v, \cdot\rangle_{\mathcal{H}}$. Given $v_0 \in \mathcal{H}'$, let v be the solution of equation (3.3). Then, take $h(\cdot,t) = \Lambda^{-1}(\psi v(\cdot,t))$ and consider the problem

$$\begin{cases} iw_t(x,t) &= Lw + \psi(x)h(x,t), \\ w(x,T) &= u_1(x), \end{cases} \tag{3.5}$$

which we split into the two problems:

$$\begin{cases} iw_t^{(1)}(x,t) & = Lw^{(1)}, \\ w^{(1)}(x,T) & = u_1(x), \end{cases} \tag{3.6}$$

and

$$\begin{cases} iw_t^{(2)}(x,t) & = Lw^{(2)} + \psi(x)h(x,t), \\ w^{(2)}(x,T) & = 0. \end{cases} \tag{3.7}$$

Clearly, $w = w^{(1)} + w^{(2)}$. As usual with the HUM procedure, given $v_0 \in \mathcal{H}'$ the initial condition of equation (3.3), we define the linear operator $S : \mathcal{H}' \to \mathcal{H}$ by

$$S(v_0) = -iw^{(2)}(\cdot, 0) \tag{3.8}$$

where $w^{(2)}$ is the solution of (3.7).

If we can show that S is an isomorphism, then the inverse image by S of $-iu_0 + iw^{(1)}(\cdot, 0)$, is the initial condition for equation (3.3) that will provide the sought control $h = \Lambda^{-1}(\psi v(\cdot, t))$.

This is shown by establishing the observability inequality of system (3.3) in $\mathcal{H}'$ which we describe in the following lemma.

Lemma 3.2. *Let ψ be a C^1 function defined by* (1.7)*. There exists a constant $C > 0$ such that for all $v_0 \in \mathcal{H}'$, the solution v of* (3.3)-(3.4) *satisfies*

$$\int_0^T \|\psi v(.,t)\|^2_{\mathcal{H}'} dt \geq C\|v_0\|^2_{\mathcal{H}'}. \tag{3.9}$$

The proof of the observability inequality (3.9) is quite similar to the one given by L. Rosier and B. Zhang in [11]. We repeat most of the construction given in that paper for the sake of completeness.

In order to prove Lemma 3.2 we begin by proving the corresponding observability inequality in $\mathcal{H}$. We recall the isomorphism $L_\mu : \mathcal{H} \to \mathcal{H}'$, $L_\mu = -\partial_x^2 + \mu$. Consider the Schrödinger equation

$$iw_t(x,t) = Lw + P(w), \tag{3.10}$$

$$w(x,0) = w_0(x), \tag{3.11}$$

where $P(w) = L_\mu^{-1}[\nu, L_\mu](w)$, with $\nu := \alpha - \mu$.

Lemma 3.3. *Following assertions are true:*

(a) $\partial : W^k \mapsto W^{k-1}$ *is a bounded operator for* $k = 0, 1$.

(b) For $g \in L^\infty$ *such that* $g' \in L^\infty$ *the related multiplication operator* $g : W^k \mapsto W^k$ *is bounded for* $k = 0, 1, -1$.

(c) $P : W^k \mapsto W^k$ *is a bounded operator for* $k = 0, 1, -1$.

(d) $\|w\|_{L^\infty(0,T,W^k)} \leq C(T)\|w_0\|_{W^k}$ *for* $k = 0, 1, -1$.

Proof. For $k = 0$ claims (a) and (b) are evident. Claim (a) for $k = 1$ is obtained from $k = 0$ by duality $W^{-1} = (W^1)'$ (see Remark 2.4). Claim (b) for $k = 1$ follows from the estimate:

$$\|g\phi\|_{\mathcal{H}} \leq \|g'\|_{L^\infty}\|\phi\|_{L^2} + \|g\|_{L^\infty}\|\phi'\|_{L^2} + \|g\|_{L^\infty}\|\phi\|_{L^2_\mu}$$

By duality we also get claim (b) for $k = -1$. Claim (c) is a consequence of claims (a) and (b) applied to the identity

$$L_\mu^{-1}[\nu, L_\mu] = L_\mu^{-1}\left(2\nu_x\partial_x + \nu_{xx}\right)$$

where we have used that $\nu_x, \nu_{xx} \in L^\infty$.

Finally, claim (d) is a direct consequence of claim (c). □

Lemma 3.4. *Let ψ be a C^1 function defined by* (1.7)*. There exists a constant $C > 0$ such that for every $w_0 \in \mathcal{H}$, the solution w of* (3.10)-(3.11) *satisfies*

$$\int_0^T \|\psi\, w(\cdot, t)\ \|^2_{\mathcal{H}}\, dt \geq C\|w_0\|^2_{\mathcal{H}}. \tag{3.12}$$

Proof. By Duhamel, we know that there exists $C > 0$ such that for $w_0 \in \mathcal{H}$, the solution w of (3.10)-(3.11) satisfies

$$\|w_0\|^2_{\mathcal{H}} \leq C\int_0^T \|w(\cdot, t)\|^2_{\mathcal{H}}\, dt. \tag{3.13}$$

Therefore, (3.12) will follow if we prove the following inequality in $\mathcal{H}$:

$$\int_0^T \|w(\cdot, t)\|^2_{\mathcal{H}}\, dt \leq C\int_0^T \|\psi\, w(\cdot, t)\ \|^2_{\mathcal{H}}\, dt. \tag{3.14}$$

We use the multiplier technique. Define $q \in C_0^\infty(\mathbb{R})$

$$q(x) = \begin{cases} x & \text{for } |x| \leq R + 2 \\ 0 & \text{for } |x| \geq R + 3 \end{cases}. \tag{3.15}$$

We have that

$$\int_0^T \frac{d}{dt}\langle w, iqw_x\rangle dt = \langle w, iqw_x\rangle\big|_0^T. \tag{3.16}$$

Recall $L = -\partial_x^2 + \alpha$, then the l.h.s of the last equation reads:

$$\int_0^T \langle iw_{xx}, iqw_x\rangle - \langle i\alpha w + iP(w), iqw_x\rangle + \langle w, iqiw_{xxx}\rangle + \langle w, iq(-i)(\alpha w + P(w))_x\rangle dt \tag{3.17}$$

Integrating by parts we have that:

$$\langle w, iqw_x\rangle\big|_0^T = \int_0^T -2\langle w_x, q_x w_x\rangle - 2\langle\alpha w + P(w), qw_x\rangle - \langle q_{xx}w, w_x\rangle - \langle\alpha w + P(w), q_x w\rangle dt \tag{3.18}$$

and therefore, using that $\langle f, g\rangle = Re \int_{\mathbb{R}} fg^*$:

$$\frac{1}{2} Im \int_{\mathbb{R}} qw\bar{w}_x\Big|_0^T + Re\int_0^T \int_{\mathbb{R}} \left[q_x|w_x|^2 + \frac{1}{2}q_{xx}w\bar{w}_x + (\alpha w + P(w))(q\bar{w}_x + \frac{1}{2}q_x\bar{w})\right] dxdt = 0. \tag{3.19}$$

Then

$$\begin{aligned} \left|\int_0^T \int_{|x|\le R+2} |w_x|^2\right| &\le \frac{1}{2}\left|\int_{\{|x|\le R+3\}} qw\bar{w}_x\Big|_0^T\right| + \int_0^T \left[\left|\int_{\{R+2\le|x|\le R+3\}} q_x|w_x|^2\right|\right. \\ &+ \left.\frac{1}{2}\left|\int_{\{R+2\le|x|\le R+3\}} q_{xx}w\bar{w}_x\right| + \left|\int_{\{|x|\le R+3\}}(\alpha w + P(w))(q\bar{w}_x + \tfrac{1}{2}q_x\bar{w})\right|\right] \end{aligned} \tag{3.20}$$

and using Lemma 3.3 and

$$\|w(t_0,.)\|^2_{H^1(\mathbb{R})} \le C\int_0^T \|w(t,\cdot)\|^2_{H^1(\mathbb{R})}dt \ \forall t_0 \in [0,T] \tag{3.21}$$

$$\|\alpha w\|_{L^2(\{|x|\le R+3\})} \le C\|w\|_{L^2(\mathbb{R})} \tag{3.22}$$

we have that there exist $\varepsilon > 0$ and a constant C_ε such that

$$\int_0^T \int_{|x|\le R+2} |w_x|^2 dxdt \le \varepsilon\int_0^T \|w(t,\cdot)\|^2_{H^1}dt + C_\varepsilon\int_0^T \|w(t,\cdot)\|^2_{L^2}dt \tag{3.23}$$

$$+ C_2\int_0^T \int_{\{R+2\le|x|\le R+3\}} |w_x|^2 dxdt. \tag{3.24}$$

We have that

$$\|w\|_{\mathcal{H}} \le \|\psi w\|_{\mathcal{H}} + \|(1-\psi)w\|_{\mathcal{H}} \tag{3.25}$$

and since $1-\psi = 0$ for $|x| > R+1$

$$\|(1-\psi)w\|_{\mathcal{H}} \le C\|(1-\psi)w\|_{H^1}. \tag{3.26}$$

It is clear that

$$\|(1-\psi)w\|^2_{H^1} \le C\left(\int_{|x|\le R+1} |w_x|^2dx + \|w\|^2_{L^2(\mathbb{R})}\right), \tag{3.27}$$

and since $(\psi w)_x = w_x$ for $|x| \ge R+2$, we have that

$$\int_{|x|\ge R+2} |w_x|^2 dx \le \|\psi w\|^2_{\mathcal{H}}. \tag{3.28}$$

Therefore, if ε is chosen small enough, from (3.23) and (3.25)-(3.28), it follows the inequality

$$\int_0^T \|w(\cdot,t)\|^2_{\mathcal{H}}\, dt \le C\left(\int_0^T \|\psi\, w(\cdot,t)\ \|^2_{\mathcal{H}}\, dt + \int_0^T \|w(\cdot,t)\|^2_{L^2}\, dt\right). \tag{3.29}$$

It remains to prove that

$$\int_0^T \|w(\cdot,t)\|^2_{L^2}dt \le C\int_0^T \|\psi w(\cdot,t)\|^2_{\mathcal{H}}\, dt. \tag{3.30}$$

Assume inequality (3.30) is not true, then there exists a sequence $w_0^k \in \mathcal{H}$ such that the corresponding sequence w^k of solutions of (3.10) satisfies

$$1 = \int_0^T \|w^k(t)\|^2_{L^2(\mathbb{R})} dt \geq k \int_0^T \|\psi w^k(t)\|^2_{\mathcal{H}} dt, \quad k = 1, 2, \ldots \tag{3.31}$$

According to (3.29) and (3.31), the sequence $\{w^k\}$ is bounded in $L^2(0, T, \mathcal{H})$. Therefore by (3.13) the sequence $\{w_0^k\}$ is bounded in $\mathcal{H}$. Extracting a subsequence if needed, we may assume that

$$w_0^k \rightharpoonup w_0 \text{ weakly in } \mathcal{H} \text{ and } w^k \rightharpoonup w \text{ weakly in } L^2(0, T; \mathcal{H}) \tag{3.32}$$

where $w \in C([0, T]; \mathcal{H})$ solves equation (3.10)-(3.11) with initial data w_0. Indeed, we first have that $w_0^k \rightharpoonup w_0$ weakly in $\mathcal{H}$ and $w^k \rightharpoonup u$ weakly in $L^2(0, T, \mathcal{H})$. Being $\mathcal{H}$ compactly imbedded in $L^2(\mathbb{R})$, we may assume that $w_0^k \to w_0$ strongly in $L^2(\mathbb{R})$ and therefore

$$w^k \to w \text{ strongly in } L^2(0, T, L^2(\mathbb{R})) \tag{3.33}$$

where $w \in C(0, T, \mathcal{H})$ since it is the solution of equation (3.10)-(3.11) with initial data $w_0 \in \mathcal{H}$. From the uniqueness of weak limit in $L^2(0, T, L^2(\mathbb{R}))$ we obtain that $w = u$.

By (3.31), $\psi w^k \to 0$ strongly in $L^2(0, T, \mathcal{H})$ and since $\psi w^k \rightharpoonup \psi w$ weakly in $L^2(0, T, \mathcal{H})$, we conclude that $\psi w \equiv 0$ on $\mathbb{R} \times (0, T)$. Consequently,

$$w(x, t) = 0, \ |x| > R + 1, \ t \in (0, T). \tag{3.34}$$

Let $v = L_\mu w$, then v satisfies equation (3.3) and

$$v(x, t) = 0, \ |x| > R + 1, \ t \in (0, T). \tag{3.35}$$

We consider the new problem (similar to (3.3))

$$\begin{aligned} iv_t &= -v_{xx} + \alpha\tilde{\psi} v \\ v(x, 0) &= v_0. \end{aligned} \tag{3.36}$$

where $\tilde{\psi}$ is a $C_0^\infty(\mathbb{R})$ given by

$$\tilde{\psi}(x) = \begin{cases} 1 & \text{for } |x| \leq R + 1 \\ 0 & \text{for } |x| \geq R + 2 \end{cases}. \tag{3.37}$$

Then, problems (3.3) and (3.36) have the same solution which satisfy (3.35). Using Proposition 2.3 from [11] with $a = -\alpha\tilde{\psi}$ and $b = 0$ functions in $C_0^\infty(\mathbb{R})$ and being $v_0 \in \mathcal{H}'$ with compact support, we have that v is of class C^∞ on $\mathbb{R} \times (0, T)$.

By the unique continuation property for Schrödinger equation we conclude that $v \equiv 0$ on $\mathbb{R} \times (0, T)$. This implies $w \equiv 0$ on $\mathbb{R} \times (0, T)$. From (3.33) and (3.31) we have a contradiction.

Then observability inequality in $\mathcal{H}$ (3.12) is proved. □

We are now in position to prove the observability inequality (3.9) in $\mathcal{H}'$. We first prove a weaker inequality:

Lemma 3.5. *There exists a constant $C > 0$ such that for every $v_0 \in \mathcal{H}' = W^{-1}$ and v the solution of equation* (3.3)-(3.4)*, the following inequality is satisfied*

$$\|v_0\|_{W^{-1}}^2 \leq C \left(\int_0^T \|\psi v(t)\|_{W^{-1}}^2 dt + \|v_0\|_{W^{-2}}^2 \right). \tag{3.38}$$

Proof. Suppose that inequality (3.38) is false. Then there exist a sequence v_k of solutions of (3.3) in $C(0, T, \mathcal{H}')$ such that

$$1 = \|v_k(0)\|_{W^{-1}}^2 \geq k \left(\int_0^T \|\psi v_k(t)\|_{W^{-1}}^2 dt + \|v_k(0)\|_{W^{-2}}^2 \right). \tag{3.39}$$

Then we can extract a subsequence such that $v_k(0) \to v_0$ weak in $\mathcal{H}'$ for some $v_0 \in \mathcal{H}'$ and we can assume $v_k \to 0$ strongly en W^{-2} and therefore $v_0 = 0$. Moreover, we can assume $\psi v_k \to 0$ strongly in $L^2(0, T, \mathcal{H}')$.

Since $\mathcal{H} \subset H^1(\mathbb{R})$ continuosly, we have that

$$\|w_x\|_{W^0} \leq \|w\|_{W^1} \tag{3.40}$$

Now, let $v \in \mathcal{H}' = W^{-1}$, there exists $w \in \mathcal{H} = W^1$ such that $v = L_\mu w$, then

$$\|v_x\|_{W^{-2}} = \|L_\mu w_x + \mu_x w\|_{W^{-2}} = \|L_\mu^{-1}(L_\mu w_x + \mu_x w)\|_{W^0} \leq \|w_x\|_{W^0} + \|L_\mu^{-1}\mu_x w\|_{W^0} \tag{3.41}$$

using (3.40), we have $\|v_x\|_{W^{-2}} \leq C\|v\|_{W^{-1}}$. From Lemma 3.3 we also know that there exists a constant $C > 0$ such that for all $w \in L^2 = W^0$

$$\|w_x\|_{W^{-1}} \leq C\|w\|_{W^0}. \tag{3.42}$$

Next, we will prove that $v_k(0) \to 0$ strongly in W^{-1} arriving to a contradiction by (3.39).
Let $w_k = L_\mu^{-1}(v_k)$, then $w_k \in C([0, T], W^1)$ is a solution of equation (3.10) in $\mathcal{H}$ and

$$\psi w_k = \psi L_\mu^{-1} v_k = L_\mu^{-1}(\psi v_k) + [\psi, L_\mu^{-1}]v_k = L_\mu^{-1}(\psi v_k) + L_\mu^{-1}[L_\mu, \psi]w_k. \tag{3.43}$$

Since $\psi v_k \to 0$ strongly in $L^2(0, T, \mathcal{H}')$ and $\|L_\mu^{-1}(\psi v_k)\|_1 = \|\psi v_k\|_{-1}$, we deduce that $L_\mu^{-1}(\psi v_k) \to 0$ strongly in $L^2(0, T, \mathcal{H})$. On the other hand, using (3.42) and Lemma 3.3 we get

$$\begin{aligned}
\|L_\mu^{-1}[L_\mu, \psi](w_k)\|_{W^1} &= \|[L_\mu, \psi](w_k)\|_{W^{-1}} \\
&= \|\psi_{xx} w_k + 2\psi_x (w_k)_x\|_{W^{-1}} \\
&\leq C(\|v_k\|_{W^{-3}} + \|v_k\|_{W^{-2}}) \\
&\leq C\|v_k\|_{W^{-2}}
\end{aligned}$$

and this implies that $L_\mu^{-1}[L_\mu, \psi](w_k) \to 0$ strongly in $L^2(0, T, \mathcal{H})$, since $v_k(0) \to 0$ strongly in W^{-2}.

Therefore $\psi w_k \to 0$ strongly in $L^2(0, T, \mathcal{H})$. Since w_k is a solution of (3.10) we have from the observability inequality (3.12) that $w_k(0) \to 0$ strongly in $\mathcal{H}$. It follows that $v_k(0) = L_\mu w_k(0) \to 0$ strongly in $\mathcal{H}'$, which contradicts the fact that $\|v_k(0)\|_{\mathcal{H}'} = 1$. □

Proof of Lemma 3.2. Assume that inequality (3.9) is false, then there exists a sequence v_k of solutions of (3.3) in $C([0,T];\mathcal{H}')$ such that

$$1 = \|v_k(0)\|^2_{W^{-1}} \geq k \int_0^T \|\psi v_k(t)\|^2_{W^{-1}} dt \tag{3.44}$$

for all $k \geq 0$.

Extracting a subsequence, we may assume that

$$\begin{array}{l} v_k \to v \ \text{ in } \ L^\infty(0,T;\mathcal{H}') \ \text{ weak} - \star, \\ v_k(0) \to v(0) \ \text{ weakly in } \ \mathcal{H}' \end{array} \tag{3.45}$$

for some solution $v \in C(0,T;\mathcal{H}')$ of (3.3)-(3.4). From (3.44), $\psi v_k \to 0$ strongly in $L^2(0,T,\mathcal{H}')$ and since $\psi v_k \to \psi v$ in $L^\infty(0,T;\mathcal{H}')$ weak-$\star$, we have that $\psi v \equiv 0$. We deduce as before that $v \equiv 0$ in $\mathbb{R} \times (0,T)$.

$\{v_k(0)\}$ being a bounded sequence in W^{-1} and since W^{-1} is compactly imbedded in W^{-2}, see (2.7), there exists a subsequence such that $v_k(0)$ converges strongly in W^{-2} necessarily to 0.

We infer from (3.38) that $v_k(0)$ converges strongly to 0 in W^{-1} which is absurd from (3.44). This finishes the proof. □

Proof of Theorem 3.1. Let $v_0 \in \mathcal{H}'$ and $v(x,t)$ the solution of (3.3) such that $v(x,0) = v_0$. Let w be the solution of (3.7) with $u_1 = 0$ and $h = \Lambda^{-1}(\psi v(\cdot,t))$. Then

$$\int_0^T \langle v, iw_t - Lw \rangle_{\mathcal{H}',\mathcal{H}} dt = \int_0^T \langle v, \psi h \rangle_{\mathcal{H}',\mathcal{H}} dt. \tag{3.46}$$

Using that

$$\begin{array}{lcl} \langle v, iw_t \rangle_{\mathcal{H}',\mathcal{H}} & = & \frac{d}{dt}\langle v, iw \rangle_{\mathcal{H}',\mathcal{H}} + \langle iv_t, w \rangle_{\mathcal{H}',\mathcal{H}} \\ \langle v, \partial_x^2 w \rangle_{\mathcal{H}',\mathcal{H}} & = & \langle \partial_x^2 v, w \rangle_{\mathcal{H}',\mathcal{H}} \end{array} \tag{3.47}$$

we obtain

$$\int_0^T \frac{d}{dt}\langle v, iw \rangle_{\mathcal{H}',\mathcal{H}} dt = \int_0^T \langle -iv_t + Lv, w \rangle_{\mathcal{H}',\mathcal{H}} dt + \int_0^T \langle v, \psi h \rangle_{\mathcal{H}',\mathcal{H}} dt. \tag{3.48}$$

By (3.3), being $w(\cdot,T) = 0$ and $h(\cdot,t) = \Lambda^{-1}(\psi v(\cdot,t))$

$$\langle v_0, -iw(x,0) \rangle_{\mathcal{H}',\mathcal{H}} = \int_0^T \langle \psi v, \Lambda^{-1}(\psi v) \rangle_{\mathcal{H}',\mathcal{H}} dt, \tag{3.49}$$

and therefore

$$\langle v_0, S(v_0) \rangle_{\mathcal{H}',\mathcal{H}} = \int_0^T \|\psi v\|^2_{\mathcal{H}'} dt \geq C\|v_0\|^2_{\mathcal{H}'}. \tag{3.50}$$

It follows from Lax Milgram that S is an isomorphism. □

3.2 Non linear system

We are now in a position to present the local controllability of the non linear problem

$$iu_t(x,t) = Lu + m(u)u + \psi(x)h(x,t) \tag{3.51}$$
$$u(x,0) = u_0(x) \quad x \in \mathbb{R} \tag{3.52}$$

which, as in the linear case, means the existence of a *control* $h \in L^2(0,T,\mathcal{H})$ such that the related solution satisfies $u(x,T) = u_T(x)$.

Theorem 3.6. *Let $T > 0$ be fixed, then there exists $R > 0$ such that for every $u_0, u_T \in \mathcal{H}$ with $\max\{\|u_0\|_{\mathcal{H}}; \|u_T\|_{\mathcal{H}}\} < R$ there exists $h \in L^2(0,T;\mathcal{H})$ such that the unique solution of (3.51)–(3.52) satisfies $u(x,T) = u_T(x)$.*

Equation (3.51)-(3.52) can be written in its integral form

$$u(x,t) = e^{-iLt}u_0(x) - i\int_0^t e^{iL(s-t)}m(u(x,s))u(x,s)ds - i\int_0^t e^{iL(s-t)}\psi(x)h(x,s)ds.$$

We then set, for $v \in C(0,T,\mathcal{H})$, the mapping that defines the nonlinear term

$$\mathcal{N}(v,0,t) := -i\int_0^t e^{iL(s-t)}\big(m(v(s))v(s)\big)ds. \tag{3.53}$$

We next define $\Gamma : C(0,T,\mathcal{H}) \to C(0,T,\mathcal{H})$ as follows:

Given $v \in C(0,T,\mathcal{H})$, we compute $\mathcal{N}(v,0,t)$ as in (3.53). Given the initial state u_0 and the target state $u_T - \mathcal{N}(v,0,T)$, from Theorem 3.1 there exists a control $h^{lin} \in L^2(0,T,\mathcal{H})$ such that the solution $\tilde{w}$ of the linear equation (3.1)-(3.2) with $h = h^{lin}$

$$\tilde{w}(t) = e^{-iLt}u_0(x) - i\int_0^t e^{iL(s-t)}\psi(x)h^{lin}(x,s)ds. \tag{3.54}$$

satisfies $\tilde{w} \in C(0,T,\mathcal{H})$ and

$$\tilde{w}(T) = u_T - \mathcal{N}(v,0,T). \tag{3.55}$$

Observe that h^{lin} depends on v and therefore $\tilde{w}$ also depends on v.

Let

$$\Gamma(v)(t) := e^{-iLt}u_0 + \mathcal{N}(v,0,t) - i\int_0^t e^{iL(s-t)}\psi(x)h^{lin}(x,s)ds. \tag{3.56}$$

Since $\tilde{w} \in C(0,T,\mathcal{H})$ and is a solution of the linear equation (3.1), then $\Gamma(v)$ reads

$$\Gamma(v)(t) := \tilde{w}(t) + \mathcal{N}(v,0,t) \tag{3.57}$$

and therefore $\Gamma(v) \in C(0,T,\mathcal{H})$, $\Gamma(v)(0) = u_0$, and $\Gamma(v)(T) = u_T$. We shall remark that any fixed point of Γ yields the function needed to build the control $h \in L^2(0,T,\mathcal{H})$. Hence, it only remains to show that Γ has a fixed point. Let $\delta > 0$ and set $K_\delta := \{v \in C(0,T,\mathcal{H}) : v(0) = u_0,\ v(T) = u_T,\ \|v\|_{L^\infty(t_0,T,\mathcal{H})} \le \delta\}$. As usual, we must show that K_δ is left invariant by Γ, and also that this is a contractive mapping. With this in mind we list below some useful estimates.

Lemma 3.7. *Let* $R > 0$ *and let* $u_0, u_T \in \mathcal{H}$ *be such that* $\max\{\|u_0\|_{\mathcal{H}}; \|u_T\|_{\mathcal{H}}\} < R$, *let also* $\delta > 0$ *and take* $v, u \in K_\delta$. *Thus the following estimates hold,*

- $\|\Gamma(v)\|_{L^\infty(0,T,\mathcal{H})} \leq AR + B\delta^3$
- $\|\Gamma(v) - \Gamma(u)\|_{L^\infty(0,T,\mathcal{H})} \leq C\delta^2 \|u - v\|_{L^\infty(0,T,\mathcal{H})}$.

where A, B, C *are positive constants.*

Proof. These estimates follow from identities (3.53), (3.54), (3.57) and Lemmas (2.9) and (2.11):

$$\begin{aligned}\|\mathcal{N}(v,0,t)\|_{\mathcal{H}} &\leq \int_0^t \|e^{iL(s-t)}\big(m(v(s))v(s)\big)\|_{\mathcal{H}}\, ds \\ &\leq \int_0^t \|m(v(s))v(s)\|_{\mathcal{H}}\, \big(1 + (t-s)\|\mu_x - \alpha_x\|_{L^\infty}\big) ds \\ &\leq \frac{3}{2}\left(1 + T\|\mu_x - \alpha_x\|_{L^\infty}\right) \int_0^t \|v(s)\|_{\mathcal{H}}^3\, ds \\ &\leq B(T, \|\mu_x - \alpha_x\|_{L^\infty}) \|v\|^3_{L^\infty(0,T,\mathcal{H})} \\ &\leq B(T, \|\mu_x - \alpha_x\|_{L^\infty})\delta^3,\end{aligned}$$

and

$$\begin{aligned}\left\|\int_0^t e^{iL(s-t)}\psi(x)h^{lin}(x,s)ds\right\|_{\mathcal{H}} &\leq \int_0^t \|\psi h^{lin}(\cdot,s)\|_{\mathcal{H}}\big(1 + (t-s)\|\mu_x - \alpha_x\|_{L^\infty}\big)ds \\ &\leq C_\psi \|h^{lin}\|_{L^2(0,T,\mathcal{H})}\, \|1 + (t-s)\|\mu_x - \alpha_x\|_{L^\infty}\|_{L^2(0,t)} \\ &\leq C(\psi, T, \|\mu_x - \alpha_x\|_{L^\infty}) \left(\|u_0\|_{\mathcal{H}} + \|u_T - \mathcal{N}(v,0,T)\|_{\mathcal{H}}\right) \\ &\leq C(\psi, T, \|\mu_x - \alpha_x\|_{L^\infty}) \left(\|u_0\|_{\mathcal{H}} + \|u_T\|_{\mathcal{H}} + \|v\|^3_{L^\infty(0,T,\mathcal{H})}\right) \\ &\leq A_1(\psi, T, \|\mu_x - \alpha_x\|_{L^\infty})R + B(T, \|\mu_x - \alpha_x\|_{L^\infty})\delta^3\end{aligned}$$

For the second assertion note that

$$\Gamma(v)(t) - \Gamma(u)(t) = -i\int_0^t e^{iL(s-t)}\big(m(v)(s)v(s) - m(u)(s)u(s)\big)ds. \tag{3.58}$$

A similar reasoning leads us to the inequality

$$\begin{aligned}\|\Gamma(v)(t) - \Gamma(u)(t)\|_{\mathcal{H}} &\leq \int_0^t \|e^{iL(s-t)}\big(m(v(s))v(s) - m(u(s))u(s)\big)\|_{\mathcal{H}}\, ds \\ &\leq B(T, \|\mu_x - \alpha_x\|_{L^\infty}) \int_0^t \|m(v(s))v(s) - m(u(s))u(s)\|_{\mathcal{H}}\, ds \\ &\leq B(T, \|\mu_x - \alpha_x\|_{L^\infty}) \int_0^t \left(\|v\|^2_{\mathcal{H}} + \|v\|_{\mathcal{H}}\|u\|_{\mathcal{H}} + \|u\|^2_{\mathcal{H}}\right) \|v - u\|_{\mathcal{H}}\, ds \\ &\leq B(T, \|\mu_x - \alpha_x\|_{L^\infty})\delta^2 \|v - u\|_{L^\infty(0,T,\mathcal{H})},\end{aligned}$$

from where second estimate follows easily. This finishes the proof.

□

Proof of Theorem 3.6. As we state above, it relies on a fixed point argument. Set $K_\delta := \{v \in C(0,T,\mathcal{H}) : v(0) = u_0,\ v(T) = u_T,\ \|v\|_{L^\infty(0,T,\mathcal{H})} \le \delta\}$. Using the estimates given by Lemma 3.7, we get the following sufficient conditions

$$\begin{aligned} AR + B\delta^3 &\le \delta \\ C\delta^2 &< 1 \end{aligned}$$

which are easily satisfied taking $\delta = 2RA$ and $R < \min\{\frac{1}{2\sqrt{C}A}, \frac{1}{2\sqrt{2B}A}\}$. □

4 Non controllability for compactly supported controls

Throughout this section we shall focus our attention to controls $\psi(x)h(x,t)$ with $\mathrm{Supp}(\psi)$ compact, and consider two different situations, depending on the linear term: $L_\mu = -\partial_x^2 + \mu$, which has a discrete spectrum, and $L_e = -\partial_x^2 - x$ with a continuous spectrum. The negative result concerning the related exact controllability for the linear problem is similar to the one given in [6], however our problem is posed in $\mathcal{H}$ which is not L^2 but a suitable Sobolev space. For this reason we shall adapt both the result and its proof, and this heavily relies upon the spectral properties reported in section 2. Actually, since the proof relies on a special feature of the eigenstates of the linear operator, we shall use the unitary group U_+, and the eigenfunctions $\{\phi_N\}_{N\in\mathbb{N}}$ of the auxiliar operator $L_+ := -\partial_x^2 + |x|$ yielded by Lemma 2.1.

4.1 Discrete spectrum

We first consider the non–controllability result for the model equation,

$$iu_t(x,t) = L_\mu u(x,t) + \psi(x)h(x,t), \quad x \in \mathbb{R}, \tag{4.1}$$

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x), \tag{4.2}$$

with $\mathrm{Supp}(\psi)$ compact. The main result reads as follows.

Theorem 4.1. *The exact internal distributed control is not possible, i.e. for a given target state $u_T \in \mathcal{H}$ there exist a bounded open set $\Omega \subseteq \mathbb{R}$ and an initial function u_0 such that there is no control function h and no constant $C = C(\Omega,T) > 0$ such that the equation* (4.1) *holds with $u(0) = u_0$, $u(T) = u_T$, and $\|h\|_{L^1(0,T,\mathcal{H})} \le C\left(\|u_0\|_{\mathcal{H}} + \|u_T\|_{\mathcal{H}}\right)$*

Proof. As in [6] we argue by contradiction. Let Ω be a fixed finite interval and take ϕ_N, the N–th eigenfunction of L_+, as a target state, and assume that there exist a time $T > 0$, a control function $h_N \in L^2(0,T,\mathcal{H})$, a constant $C(\Omega,T)$, with $\|h\|_{L^2(0,T,\mathcal{H})} \le C(\Omega,T)(\|u_0\|_{\mathcal{H}} + \|\phi_N\|_{\mathcal{H}})$ an initial state u_0 and a solution u_N of (4.1). Let $U_+(t)$ be the unitary group generated by $-iL_+$ in $\mathcal{H}$, since $L_\mu = L_+ + b$ where $b(x) = \mu(x) - |x|$ has compact support, from Duhamel identity we have:

$$\phi_N(x) = U_+(T)u_0(x) - i\int_0^T U_+(T-s)(\psi h_N + bu_N)ds.$$

Since $U_+(t)\psi = \sum e^{-it\widetilde{\lambda}_N}\widehat{\psi}(N)\phi_N(x)$, where $\widehat{\psi}(N) = \int \psi(x)\phi_N(x)dx$ are the related Fourier coefficients, after taking the L^2-inner product with ϕ_N we get

$$1 = e^{-iT\widetilde{\lambda}_N}\langle u_0;\phi_N\rangle - i\int_0^T e^{-i(T-s)\widetilde{\lambda}_N}\langle \psi h_N + bu_N;\phi_N\rangle ds. \tag{4.3}$$

Since $u_0 \in \mathcal{H}$ the first term goes to zero. The second term verifies

$$\begin{aligned}\langle \psi h_N + bu_N;\phi_N\rangle &= \widetilde{\lambda}_N^{-1}\langle \psi h_N + bu_N; L_+\phi_N\rangle \\ &= \widetilde{\lambda}_N^{-1}\langle \partial(\psi h_N + bu_N);(\phi_N)_x\rangle + \widetilde{\lambda}_N^{-1}\langle |x|\psi h_N + |x|bu_N;\phi_N\rangle \\ &= \widetilde{\lambda}_N^{-1}\langle \psi_x h_N + \psi(h_N)_x + b_x u_N + b(u_N)_x;(\phi_N)_x\rangle + \widetilde{\lambda}_N^{-1}\langle |x|\psi h_N + |x|bu_N;\phi_N\rangle\end{aligned}$$

From Lemma 2.1 we see that the eigenfunctions $\{\phi_N\}_{N\in\mathbb{N}}$ satisfy $\|\phi_N\|_{L^2} = 1$, $\|\phi_N\|_{\mathcal{H}} = \widetilde{\lambda}_N^{1/2}$, and $\|(\phi_N)_x\|_{L^2(\Omega)} \sim \widetilde{\lambda}_N^{1/4}$, we also recall that both ψ and b have compact support, and verifies b_x, $\psi_x \in L^\infty$. With this in mind we get:

$$\begin{aligned}\left| i\int_0^T e^{-i(T-s)\widetilde{\lambda}_N}\langle \psi h_N + bu_N;\phi_N\rangle\right| &\le \int_0^T |\langle \psi h_N + bu_N;\phi_N\rangle| \\ &\le C(\psi,b)\widetilde{\lambda}_N^{-1}\|\phi_N\|_{L^2}\int_0^T (\|u_N\|_{\mathcal{H}} + \|h_N\|_{\mathcal{H}}) \\ &\quad + C(\psi,b)\widetilde{\lambda}_N^{-1}\|(\phi_N)_x\|_{L^2(\Omega)}\int_0^T (\|u_N\|_{\mathcal{H}} + \|h_N\|_{\mathcal{H}}) \\ &\le \widetilde{\lambda}_N^{-1}C(\psi,\Omega,T)(\|u_0\|_{\mathcal{H}} + \widetilde{\lambda}_N^{1/2})(1 + \widetilde{\lambda}_N^{1/4})\end{aligned}$$

which goes to zero as N goes to infinity. This contradicts identity (4.3), and finishes the proof. □

4.2 Continuous spectrum

We now consider the non–controllability result for the linear model equation, with $L_e = -\partial_x^2 - x$,

$$iu_t(x,t) = L_e u(x,t) + \psi(x)h(x,t), \quad x \in \mathbb{R}, \tag{4.4}$$
$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x), \tag{4.5}$$

with Supp(ψ) compact. The main result reads as follows.

Theorem 4.2. *The exact internal distributed control is not possible, i.e. for a given target state $u_T \in \mathcal{H}$ there exist a bounded open set $\Omega \subseteq \mathbb{R}$ and an initial function u_0 such that there is no control function h and no constant $C = C(\Omega,T) > 0$ such that the equation (4.4) holds with $u(0) = u_0$, $u(T) = u_T$, and $\|h\|_{L^1(0,T,\mathcal{H})} \le C\left(\|u_0\|_{\mathcal{H}} + \|u_T\|_{\mathcal{H}}\right)$*

Remark 4.3. *As for the result of the previous subsection we follow the ideas of Theorem 3 of [6], but in order to accomplish the task we need an extra ingredient given by the following Lemma.*

Lemma 4.4. *Let $U_e(t)$ be the group generated by $-iL_e$ where $L_e := -\partial_x^2 + x$. Then $[\partial_x : U_e(t)] = -itU_e(t)$.*

Proof. We start noting that $[\partial_x : L_e] = [\partial_x : x] = 1$ and $[\partial_x : L_e^{M+1}] = [\partial_x : L_e^M]L_e + L_e^M[\partial_x : L_e]$. An inductive argument shows the identity $[\partial_x : L_e^{M+1}] = (M+1)L_e^M$. For ϕ in the Schwarz space we have

$$\begin{aligned}[\partial_x : U_e(t)]\phi &= \sum_{M\geq 0} \frac{(-it)^M}{M!}[\partial_x : L_e^M]\phi \\ &= \sum_{M\geq 0} \frac{(-it)^{M+1}}{(M+1)!}[\partial_x : L_e^{M+1}]\phi \\ &= -itU_e(t)\phi\end{aligned}$$

A density argument allows us to extend the result for $\phi \in \mathcal{H}$. □

Proof of Theorem 4.2. We first set $\Psi(x) \in C_0^\infty(\mathbb{R})$ such that $0 \leq \Psi(x)$, $\mathrm{Supp}(\Psi) = [-1; 1]$, and $1 = \int \Psi(x)$, and take $\Psi_\varepsilon = \varepsilon^{-1}\Psi(\varepsilon^{-1}x)$. We below collect the behavior of the different norms involved in the proof, their validity is evident and will not be reported.

$$\|\Psi_\varepsilon\|_{L^1} = \|\Psi\|_{L^1} = 1 \tag{4.6}$$

$$\|\Psi_\varepsilon\|_{L^2} = \varepsilon^{-1/2}\|\Psi\|_{L^2} \tag{4.7}$$

$$\|\Psi_\varepsilon\|_{L^2_\mu} \leq \varepsilon^{-1/2}(1+\varepsilon)^{1/2}\|\Psi\|_{L^2_\mu} \tag{4.8}$$

$$\|(\Psi_\varepsilon)_x\|_{L^1} = \varepsilon^{-1}\|\Psi_x\|_{L^1} \tag{4.9}$$

$$\|(\Psi_\varepsilon)_x\|_{L^2} = \varepsilon^{-3/2}\|\Psi_x\|_{L^2} \tag{4.10}$$

We also add, for a fixed $T > 0$, the function $\phi_\varepsilon := U_e(2T)\Psi_\varepsilon$, where U_e is the related unitary group, and notice that $\|\phi_\varepsilon\|_{L^2}^2 = \|\Psi_\varepsilon\|_{L^2}^2 = \varepsilon^{-1}\|\Psi\|_{L^2}^2$. We now argue by contradiction. Assume the exact controllability of (4.4), then there exist $h_\varepsilon \in L^2(0, T, \mathcal{H})$ such that

$$\|h_\varepsilon\|_{L^2(0,T,\mathcal{H})} \leq C(\|u_0\|_{\mathcal{H}} + \|\phi_\varepsilon\|_{\mathcal{H}}),$$

and a solution $u_\varepsilon(x,t)$ of (4.4) with $u_\varepsilon(\cdot, T) = \phi_\varepsilon$, and $u_0 \in \mathcal{H}$ arbitrary.

From Duhamel identity we have

$$\phi_\varepsilon = U_e(T)u_0 - i\int_0^T U_e(T-s)(\psi h_\varepsilon)ds$$

and taking the L^2 inner–product with $L_\mu\phi_\varepsilon$ we get

$$\langle \phi_\varepsilon; L_\mu\phi_\varepsilon\rangle = \langle U_e(T)u_0 - i\int_0^T U_e(T-s)(\psi h_\varepsilon)ds; L_\mu\phi_\varepsilon\rangle, \tag{4.11}$$

left hand side reads:

$$\langle \phi_\varepsilon; L_\mu\phi_\varepsilon\rangle = \langle \Psi_\varepsilon; U_e(-2T)(-\partial_x^2)U_e(2T)\Psi_\varepsilon\rangle + \langle \phi_\varepsilon; \mu\phi_\varepsilon\rangle.$$

Before going further we develop an useful identity, based on the commutator relation given by Lemma 4.4:

$$
\begin{aligned}
U_e(r)(-\partial_x^2)U_e(s) &= -[U_e(r):\partial_x]\partial_x U_e(s) - \partial_x U_e(r)\partial_x U_e(s) \\
&= -irU_e(r)\partial_x U_e(s) + is\partial_x U_e(r)U_e(s) - \partial_x U_e(r)U_e(s)\partial_x \\
&= r^2 U_e(r+s) - i(r-s)\partial_x U_e(r+s) - \partial_x U_e(r+s)\partial_x. \qquad (4.12)
\end{aligned}
$$

With this result, the left hand side of (4.11) reads

$$
\begin{aligned}
\langle \phi_\varepsilon; L_\mu \phi_\varepsilon\rangle &= 4T^2\|\Psi_\varepsilon\|_{L^2}^2 + 4iT\langle \Psi_\varepsilon;\partial_x\Psi_\varepsilon\rangle - \langle \Psi_\varepsilon;\partial_x^2\Psi_\varepsilon\rangle + \langle \phi_\varepsilon;\mu\phi_\varepsilon\rangle \\
&= 4T^2\varepsilon^{-1}\|\Psi\|_{L^2}^2 + \varepsilon^{-3}\|\Psi_x\|_{L^2}^2 + \|\phi_\varepsilon\|_{L^2_\mu}^2.
\end{aligned}
$$

The last term is bounded with the help of Lemma 2.9

$$
\|\phi_\varepsilon\|_{L^2_\mu}^2 \le \|\Psi_\varepsilon\|_{L^2_\mu}^2 + 4T\|\Psi_\varepsilon\|_{L^2}\|(\Psi_\varepsilon)_x\|_{L^2} + 4T^2\|\Psi_\varepsilon\|_{L^2}^2.
$$

Previous estimates altogether yield:

$$
\varepsilon^3\langle \phi_\varepsilon; L_\mu\phi_\varepsilon\rangle = \|\Psi_x\|_{L^2}^2 + O(\varepsilon). \qquad (4.13)
$$

After multiplying by ε^3, the right hand side of (4.11) reads:

$$
\varepsilon^3\langle U_e(T)u_0; L_\mu U_e(2T)\Psi_\varepsilon\rangle - i\varepsilon^3\int_0^T \langle U_e(T-s)(\psi h); L_\mu U_e(2T)\Psi_\varepsilon\rangle ds\,.
$$

The first term goes to zero as easily follows from Lemma 2.9 and estimates (4.6):

$$
\begin{aligned}
\varepsilon^3\,|\langle U_e(T)u_0; L_\mu U_e(2T)\Psi_\varepsilon\rangle| &\le \varepsilon^3\|\phi_\varepsilon\|_{\mathcal{H}}\|U_e(T)u_0\|_{\mathcal{H}} \\
&\le \varepsilon^3 C(T)\|\Psi_\varepsilon\|_{\mathcal{H}}\|u_0\|_{\mathcal{H}} \\
&\le C(T,u_0,\Psi)\varepsilon^{3/2}.
\end{aligned}
$$

The second term is splitted as

$$
-i\varepsilon^3\int_0^T \langle U_e(T-s)(\psi h); (-\partial_x^2)U_e(2T)\Psi_\varepsilon\rangle\, ds - i\varepsilon^3\int_0^T \langle U_e(T-s)(\psi h); \mu U_e(2T)\Psi_\varepsilon\rangle\, ds.
$$

and each term is treated separately. For the later we apply a similar procedure as for the initial datum:

$$
\begin{aligned}
\varepsilon^3\,|\langle U_e(T-s)(\psi h); \mu U_e(2T)\Psi_\varepsilon\rangle| &\le \|\phi_\varepsilon\|_{L^2_\mu}\|U_e(T-s)\psi h_\varepsilon\|_{L^2_\mu} \\
&\le C(T,\Psi)\varepsilon^2\|\psi h_\varepsilon\|_{L^2}^{1/2}\|(\psi h_\varepsilon)_x\|_{L^2}^{1/2} \\
&\le C(T,\Psi,\Omega)\varepsilon^2\|h_\varepsilon\|_{H^1} \\
&\le C(T,\Psi,\Omega)\varepsilon^{1/2},
\end{aligned}
$$

and the former is handled using the L^1-L^∞ estimate displayed in Corollary 2.8. To see this we first apply the identity (4.12) and get:

$$
U_e(-2T)(-\partial_x^2)U_e(T-s) = 4T^2U_e(-T-s) - i(s-3T)\partial_x U_e(-T-s) - \partial_x U_e(-T-s)\partial_x.
$$

This leads to:

$$
\begin{aligned}
\left|\langle U_e(T-s)(\psi h); (-\partial_x^2)U_e(2T)\Psi_\varepsilon\rangle\right| &\leq 4T^2\|\Psi_\varepsilon\|_{L^1}\|U_e(-T-s)(\psi h_\varepsilon)\|_{L^\infty} \\
&\quad + 3T\|(\Psi_\varepsilon)_x\|_{L^1}\|U_e(-T-s)(\psi h_\varepsilon)\|_{L^\infty} \\
&\quad + \|(\Psi_\varepsilon)_x\|_{L^1}\|U_e(-T-s)(\psi h_\varepsilon)_x\|_{L^\infty} \\
&\leq C(\Omega,T)\|h_\varepsilon\|_{L^2} + C(\Omega,T,\Psi)\varepsilon^{-1}\|h_\varepsilon\|_{H^1} \\
&\leq C(\Omega,T,\Psi,u_0)\varepsilon^{-5/2}
\end{aligned}
$$

where we have used the estimates $\|\psi h_\varepsilon\|_{L^1} \leq C(\Omega,T)\|h_\varepsilon\|_{L^2}$, $\|(\psi h_\varepsilon)_x\|_{L^1} \leq C(\Omega,T)\|h_\varepsilon\|_{H^1}$, and the fact that $|-T-s|^{-1/2} \leq T^{-1/2}$. Integrating in $[0,T]$ and multiplying by ε^3 we see that the right hand side of (4.11) tends to zero, contradicting the estimate (4.13). This finishes the proof.

□

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