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Symplectic structures on nilmanifolds: an obstruction for its existence

Viviana del Barco*

Abstract. In this work we introduce an obstruction for the existence of symplectic structures on nilpotent Lie algebras. Indeed, a necessary condition is presented in terms of the cohomology of the Lie algebra. Using this obstruction we obtain both positive and negative results on the existence of symplectic structures on a large family of nilpotent Lie algebras. Namely the family of nilradicals of minimal parabolic subalgebras associated to the real split Lie algebra of classical complex simple Lie algebras.
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1. Introduction

A nilmanifold is an homogeneous manifold $M = \Gamma\backslash N$ where N is a simply connected nilpotent Lie group and Γ is co-compact discrete subgroup of N. For these compact manifolds the natural map from $H^i_{dR}(\mathfrak{n})$, $\mathfrak{n}$ the Lie algebra of N, to the de Rham cohomology group $H^i(M,\mathbb{R})$ is an isomorphism for all $0 \leq i \leq 2n$, as showed by Nomizu in [13].

In particular this implies that any symplectic structure on a nilmanifold is cohomologous to an invariant one. Thus to solve the problem of existence of symplectic structures on the nilmanifold $\Gamma\backslash N$ reduces to find a non-degenerate closed 2-form ω on the Lie algebra $\mathfrak{n}$; if it exists $\mathfrak{n}$ is called a symplectic Lie algebra. Here we work from this Lie algebra point of view.

The goal of this work is to prove that every symplectic nilpotent Lie algebra has a certain non-zero component on its cohomology. Actually, the intermediate cohomology of a Lie algebra $\mathfrak{n}$ (concept presented by the author in [3]) is used in Theorem 3.1 to give a necessary condition for $\mathfrak{n}$ to admit a symplectic structure. As an application, we study the validity of this property on a particular subfamily of nilpotent Lie algebras.

Benson and Gordon in [1] proved that the Hard Lefschetz Theorem fails for any symplectic non-abelian nilpotent Lie algebra. In order to show this, they deduce some general structure results of symplectic nilpotent Lie algebras.

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Nevertheless there are not many general conditions to determine whether a given nilpotent Lie algebra is symplectic or not.

Until now there are known all the symplectic nilpotent Lie algebras up to dimension 6 (see [6, 15] for instance) and this list is mostly build-up by studying case by case. But the lack of a full classification of real nilpotent Lie algebras of dimension ≥ 8 makes this method non-feasible in greater dimensions.

Moreover, several authors studied the problem on different subfamilies of nilpotent Lie algebras. For example, the classification of symplectic filiform Lie algebras, which are Lie algebras $\mathfrak{n}$ of nilpotency index $k = \dim \mathfrak{n} - 1$, is given in [11]. Moreover, in [5] the authors work with Heisenberg type nilpotent Lie algebras. Among nilpotent Lie algebras associated with graphs, a complete description of the symplectic ones can be made in terms of the corresponding graph [14]. The full classification of the symplectic free nilpotent Lie algebras is done in [2].

In this context, the aim of this work is to contribute with a better understanding of the structure of symplectic nilpotent Lie algebras. Its organization is as follows. Section 2. is devoted to an introduction to the intermediate cohomology of nilpotent Lie algebras and the development of the properties that will be used later on the presentation. In Section 3. we study the relationship between symplectic structures and intermediate cohomology. This leads us to a necessary condition for a nilpotent Lie algebra to admit a symplectic structure. We notice that this condition is not sufficient in general.

In Section 4 we restrict ourselves to the study of the existence of symplectic structures in the family of nilradicals of minimal parabolic subalgebras associated to the real split Lie algebras corresponding to complex classical simple Lie algebras. We prove that the obstruction in Theorem 3.1 is also sufficient for that family. This allows us to obtain both positive and negative results about the existence of symplectic structures in this case.

Recall that for the nilpotent complex case Kostant in [10] describes the Lie algebra cohomology groups of the nilradicals of Borel subalgebras for any irreducible representation as a direct sum of one dimensional modules of multiplicity one. The real version of his description was recently given in [16]. Here we also use a decomposition of the cohomology groups but the summands are not in one to one correspondence with those of neither Kostant (in the complex version) nor Šilhan in the real case.

2. Intermediate Cohomology of nilpotent Lie algebras

The concept of intermediate cohomology of nilpotent Lie algebras and a deep study of its properties were analyzed by the author in [3]. For completeness of this work we give here a brief introduction to this cohomology by quickly reviewing its definition and the properties that will be used later.

Let $\mathfrak{g}$ denote a real Lie algebra. The central descending series of $\mathfrak{g}$, $\{\mathfrak{g}^i\}$ for all $i \geq 0$, is given by

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}], \; i \geq 1.$$

A Lie algebra $\mathfrak{g}$ is k-step nilpotent if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$; this number k is called the nilpotency index of $\mathfrak{g}$. Nilpotent Lie algebras will be denoted by $\mathfrak{n}$. Abelian Lie algebras are 1-step nilpotent. Moreover, 2-step nilpotent Lie algebras verify $\mathfrak{n}^1 \subseteq \mathfrak{z}(\mathfrak{n})$, where $\mathfrak{z}(\mathfrak{n})$ denotes the center of $\mathfrak{n}$.

The Chevalley-Eilenberg complex of a Lie algebra $\mathfrak{g}$ of dimension m is

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g}^* \xrightarrow{d_1} \Lambda^2\mathfrak{g}^* \xrightarrow{d_2} \ldots\ldots \xrightarrow{d_{m-1}} \Lambda^m\mathfrak{g}^* \longrightarrow 0 \ . \tag{1}$$

We identify the exterior product $\Lambda^p\mathfrak{g}^*$ with the space of skew-symmetric p-linear forms on $\mathfrak{g}$, thus each differential $d_p : \Lambda^p\mathfrak{g}^* \longrightarrow \Lambda^{p+1}\mathfrak{g}^*$ is defined by:

$$d_p c\,(x_1, \ldots, x_{p+1}) = \sum_{1\le i<j\le p+1} (-1)^{i+j-1} c([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{p+1}).$$

The first differential d_1 coincides with the dual mapping of the Lie bracket $[\,,\,] : \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}$ and the collection of d_p is a derivation of the exterior algebra $\Lambda^*(\mathfrak{g}^*)$. We will denote d instead of d_p independently of p.

The cohomology of $(\Lambda^*\mathfrak{g}^*, d)$ is called the Lie algebra cohomology of $\mathfrak{g}$ (with real coefficients) and it is denoted by $H^*(\mathfrak{g}, \mathbb{R})$ and more often as $H^*(\mathfrak{g})$ if there is no place to confusion. For nilpotent Lie algebras $H^1(\mathfrak{n}) \cong \mathfrak{n}/\mathfrak{n}^1$ and $\dim H^2(\mathfrak{n}) \geq 2$ [4].

When the Lie algebra is nilpotent, a filtration of the cochain complex in Eq. (1) arises in the following manner. Consider the subspaces of $\mathfrak{n}^*$ defined by Salamon in [15]

$$V_0 = 0 \qquad\qquad V_i = \{\alpha \in \mathfrak{n}^* : d\alpha \in \Lambda^2 V_{i-1}\} \qquad i \geq 1. \tag{2}$$

Then $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_i \subseteq \cdots \subseteq \mathfrak{n}^*$ and V_i is the annihilator of $\mathfrak{n}^i$, the ith-ideal in the central descending series; that is $V_i = (\mathfrak{n}^i)^\circ$. In particular, $\mathfrak{n}$ is a k-step nilpotent Lie algebra if and only if $V_k = \mathfrak{n}^*$ and $V_{k-1} \neq \mathfrak{n}^*$.

Suppose $\mathfrak{n}$ is a k-step nilpotent Lie algebra of dimension m, then for any $q = 0, \ldots, m$, the space of skew symmetric q-forms $\Lambda^q\mathfrak{n}^*$ is filtered since

$$0 = \Lambda^q V_0 \subsetneq \Lambda^q V_1 \subsetneq \ldots \subsetneq \Lambda^q V_{k-1} \subsetneq \Lambda^q V_k = \Lambda^q \mathfrak{n}^*. \tag{3}$$

In addition each of these subspaces is invariant under the differential, therefore

$$F^pC^* : 0 \longrightarrow \mathbb{R} \longrightarrow V_{k-p} \longrightarrow \Lambda^2 V_{k-p} \longrightarrow \cdots \longrightarrow \Lambda^m V_{k-p} \longrightarrow 0 \tag{4}$$

is a subcomplex of the Chevalley-Eilenberg complex for each fixed p and $\{F^pC^*\}_{p\geq 0}$ constitutes a filtration of the complex in Eq. (1).

As any filtration of a cochain complex, $\{F^pC^*\}_{p\geq 0}$ gives rise to a spectral sequence $\{E_r^{p,q}(\mathfrak{n})\}_{r\geq 0}^{p,q\in\mathbb{Z}}$. In this case, this spectral sequence always converges to the Lie algebra cohomology of $\mathfrak{n}$ (see [3] and references therein). In particular this implies that each cohomology group $H^i(\mathfrak{n})$ can be written as a direct sum of the limit terms of the spectral sequence. Namely

$$H^i(\mathfrak{n}) \cong \bigoplus_{p+q=i} E_\infty^{p,q}(\mathfrak{n}) \quad \text{ for all } i = 0, \ldots, m. \tag{5}$$

This way of describing the cohomology groups as a sum of smaller spaces suggests us the following definition.

Definition 2.1. Let $\mathfrak{n}$ be a nilpotent Lie algebra of dimension m. Then, for each $i = 0, \dots, m$, the intermediate cohomology groups of degree i of $\mathfrak{n}$ are the vector spaces $E_\infty^{p,q}(\mathfrak{n})$ with $p + q = i$.

Notice that for each $i = 0, \dots, m$ there is a finite amount of non-zero intermediate cohomology groups of degree i.

Each intermediate cohomology group can be described using the Lie algebra differential restricted to the subspaces in the filtration:

$$E_\infty^{p,q}(\mathfrak{n}) \cong \frac{\{x \in \Lambda^{p+q} V_{k-p} : dx = 0\}}{d(\{x \in \Lambda^{p+q-1}\mathfrak{n}^* : dx \in \Lambda^{p+q} V_{k-p}\}) + \{x \in \Lambda^{p+q} V_{k-p-1} : dx = 0\}}. \tag{6}$$

If a nilpotent Lie algebra $\mathfrak{n}$ can be decomposed as a direct sum of a one dimensional ideal $\mathbb{R}$ and a nilpotent Lie algebra of dimension one less than $\mathfrak{n}$, a similar formula to the Künneth formula can be stated for the intermediate cohomology.

Theorem 2.2 ([3]). *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra which can be decomposed as a direct sum of ideals $\mathfrak{n} = \mathbb{R} \oplus \mathfrak{h}$. Then $\mathfrak{h}$ is k-step nilpotent and for all $0 \leq r \leq \infty$ it holds*

1. $E_r^{p,-p}(\mathfrak{n}) = 0$ *for all* $p = 0, \dots, k-2$ *and* $E_r^{k-1,1-k}(\mathfrak{n}) \cong \mathbb{R}$.
2. $E_r^{k-1,2-k}(\mathfrak{n}) \cong E_r^{k-1,2-k}(\mathfrak{h}) \oplus \mathbb{R}$,
3. $E_r^{p,1-p}(\mathfrak{n}) \cong E_r^{p,1-p}(\mathfrak{h}) \quad$ *if* $p \leq k-2$,
4. $E_r^{p,q}(\mathfrak{n}) \cong E_r^{p,q}(\mathfrak{h}) \oplus E_r^{p,q-1}(\mathfrak{h}) \quad$ *if* $p + q \geq 2$.

Throughout an inductive procedure the next result follows.

Corollary 2.3. *Suppose $\mathfrak{n}$ is a non-abelian nilpotent Lie algebra. Then $E_\infty^{0,2}(\mathbb{R}^s \oplus \mathfrak{n}) = E_\infty^{0,2}(\mathfrak{n})$ for any $s \geq 0$.*

3. Symplectic structures and the $E_\infty^{0,2r}$ intermediate cohomology groups

A symplectic structure on a differentiable manifold M is a differentiable closed 2-form Ω that is non-singular at every point of M. Not every manifold admits such a structure. It is well known that for a compact symplectic manifold, its even de Rham cohomology groups are non-zero. When M is a nilmanifold this criteria is useless to determine the non-existence of symplectic strucutres since $H_{dR}^{2p}(M) \cong H^{2p}(\mathfrak{n})$ and always non-zero for nilpotent Lie algebras [4]. And yet there exists non-symplectic nilmanifolds. We present here an adapted version of this criteria that can be used to determine non-existence of symplectic structures on nilmanifolds.

Recall that a Lie algebra is symplectic if it admits a skew-symmetric bilinear form ω which is both closed and non-degenerate; in that case it is necessarily even dimensional. In Theorem 3.1 we prove that there is a close relationship between the existence of symplectic structures on a nilpotent Lie algebra $\mathfrak{n}$ and its even dimensional intermediate cohomology groups $E_\infty^{0,2r}(\mathfrak{n})$. Thus a new general obstruction for the existence of these structures on nilpotent Lie algebras is introduced.

Theorem 3.1. *Let $\mathfrak{n}$ be a symplectic nilpotent Lie algebra. Then $E_\infty^{0,2r}(\mathfrak{n}) \neq 0$ for all $r = 1, \dots, \dim \mathfrak{n}/2$.*

Proof. Given a symplectic form ω on a k-step nilpotent Lie algebra $\mathfrak{n}$, the fact that is non-degenerate implies $\omega \notin \Lambda^2 V_{k-1}$. Formula (6) for $p = 0$ and $q = 2$ gives

$$E_\infty^{0,2}(\mathfrak{n}) = \frac{\{x \in \Lambda^2\mathfrak{n}^* : dx = 0\}}{\{x \in \Lambda^2 V_{k-1} : dx = 0\}}.$$

So ω defines a non-zero element in $E_\infty^{0,2}(\mathfrak{n})$. The wedge product $\omega^r = \omega \wedge \overset{r}{\dots} \wedge \omega$ defines an even order non-exact closed form which also defines a non-zero element in

$$E_\infty^{0,2r}(\mathfrak{n}) = \frac{\{x \in \Lambda^{2r}\mathfrak{n}^* : dx = 0\}}{d(\Lambda^{2r-1}\mathfrak{n}^*) + \{x \in \Lambda^{2r} V_{k-1} : dx = 0\}}.$$

Thus the theorem follows. ■

Remark 3.2. Notice that if $\mathfrak{n}$ is nilpotent, the fact $E_\infty^{0,2}(\mathfrak{n}) = 0$ states not only that $\mathfrak{n}$ is not symplectic but also $\mathbb{R}^s \oplus \mathfrak{n}$ is not symplectic for all $s \geq 0$ as a consequence of Corollary 2.3.

The converse of this result is not valid in general as the next example shows.

Example 3.3. Let $\mathfrak{n}_{m,2}$ be the free 2-step nilpotent Lie algebra on m generators. Recall that $\mathfrak{n}_{m,2} = \mathfrak{f}_m/(\mathfrak{f}_m)^2$ where $\mathfrak{f}_m$ is the free Lie algebra on m generators. On the one hand, when $m \geq 4$ the Lie algebra $\mathfrak{n}_{m,2}$ does not admit symplectic structures (see [2, 5, 14]).

On the other hand, $E_\infty^{0,2r}(\mathfrak{n}_{m,2}) \neq 0$ for all m. Indeed, consider the Hall basis $\mathcal{B}$ of $\mathfrak{n}_{m,2}$ for a set of generators $\{e_1, \dots, e_m\}$, explicitely

$$\mathcal{B} = \{e_i,\ [e_j, e_k]\ :\ i = 1, \dots, m,\ 1 \leq k < j \leq m\}.$$

These basis were introduced by Hall in [7] and they are the usual ones to work with when dealing with free Lie algebras. Notice that $\dim \mathfrak{n}_{m,2} = m(m+1)/2$. The dual basis of $\mathcal{B}$ consists of 1-forms α^i, α^{jk} and their differentials, by Maurer-Cartan formulas, are

$$\begin{cases} d\alpha^i = 0, & \text{for} \quad i = 1, \dots, m, \\ d\alpha^{ij} = -\alpha^i \wedge \alpha^j, & \text{for} \quad 1 \leq j < i \leq m. \end{cases}$$

The filtration in Eq. (2) of $\mathfrak{n}_{m,2}^*$ is

$$\begin{aligned} V_1 &= \operatorname{span}\{\alpha^i,\ i = 1 \dots, m\}, \\ V_2 &= \operatorname{span}\{\alpha^i,\ \alpha^{jk}, i = 1, \dots, m, 1 \leq k < j \leq m\} \ = \ \mathfrak{n}_{m,2}^*. \end{aligned}$$

Assume $\dim \mathfrak{n}_{m,2}$ is even. It is possible to construct a closed $2r$-form defining a non-zero element in $E_\infty^{0,2r}(\mathfrak{n}_{m,2})$ for each $r = 1, \dots, \dim \mathfrak{n}_{m,2}/2$.

In fact, for $r = 1$, the 2-form $\alpha^1 \wedge \alpha^{12}$ is closed and fits into $\Lambda^2 \mathfrak{n}_{m,2}^*$ but not into $\Lambda^2 V_1$, so $\alpha^1 \wedge \alpha^{12}$ represents a non-zero class in $E_\infty^{0,2}(\mathfrak{n}_{m,2})$. To give a 4-form we concatenate $\alpha^1 \wedge \alpha^{12}$ with the closed 2-form $\alpha^{13} \wedge \alpha^{14}$ thus obtaining $\alpha^1 \wedge \alpha^{12} \wedge \alpha^{13} \wedge \alpha^{14}$. Using formulas above it is possible to see that this 4-form is non-exact, therefore, it represents a non-zero element in $E_\infty^{0,4}(\mathfrak{n}_{m,2})$ (see (6)). If $m = 3$ then the 4-form $\alpha^1 \wedge \alpha^{12} \wedge \alpha^{13} \wedge \alpha^2$ can be used.

To obtain a closed $2r$-form the idea is to concatenate 1-forms in the following way

$$\alpha^1 \wedge \alpha^{12} \wedge \alpha^{13} \wedge \dots \wedge \alpha^{1m} \wedge \alpha^2 \wedge \alpha^{23} \wedge \dots \wedge \alpha^{2m} \wedge \alpha^3 \wedge \dots \wedge \alpha^{3m} \wedge \alpha^4 \dots$$

up to obtain the form of the desired degree. One verifies that it defines a non-zero element in $E_\infty^{0,2r}(\mathfrak{n}_{m,2})$. Therefore each 2-step free nilpotent Lie algebra has $E_\infty^{0,2r}(\mathfrak{n}_{m,2}) \neq 0$ for all $1 \leq r \leq \dim \mathfrak{n}/2$ even though they are not symplectic for $m \geq 4$. Thus the converse of Theorem 3.1 does not hold.

A similar procedure applies on free nilpotent Lie algebras of greater nilpotency index which are also non-symplectic as proved in [2].

3.1. $\mathrm{Aut}(\mathfrak{n})$ action on $E_\infty^{0,2}(\mathfrak{n})$.

Once it is known that a certain Lie algebra is symplectic, it is interesting to classify its symplectic forms up to equivalence. In the case of symplectic nilpotent Lie algebras the subspace $E_\infty^{0,2}$ is non-zero. What we study here is how this subspace helps to this classification problem.

The automorphism group of a Lie algebra $\mathfrak{g}$ is

$$\mathrm{Aut}(\mathfrak{g}) = \{A \in \mathrm{GL}(\mathfrak{g}) : [Ax, Ay] = [x, y] \quad \text{for all } x, y \in \mathfrak{g}\}.$$

This group acts on $H^2(\mathfrak{g})$ in the following way: $A \cdot [\omega] = [(A^{-1})^* \omega]$, for all $A \in \mathrm{Aut}(\mathfrak{g})$ and $[\omega] \in H^2(\mathfrak{g})$. Here $(A^{-1})^*$ denotes the automorphism of the exterior algebra $\Lambda^* \mathfrak{n}^*$ induced by $(A^{-1})^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$.

When the Lie algebra is nilpotent its group of automorphisms $\mathrm{Aut}(\mathfrak{n})$ acts similarly in $E_\infty^{0,2}(\mathfrak{n})$. Given a closed 2-form ω in $\Lambda^2 \mathfrak{n}^*$ denote with $[\omega]^{0,2}$ its class as an element of the quotient space

$$\frac{\{x \in \Lambda^2 \mathfrak{n}^* : dx = 0\}}{\{x \in \Lambda^2 V_{k-1} : dx = 0\}} \cong E_\infty^{0,2}(\mathfrak{n}).$$

Any element in $\mathrm{Aut}(\mathfrak{n})$ preserves the filtration in (3) of $\mathfrak{n}^*$ and in particular $(A^{-1})^* V_{k-1} = V_{k-1}$. This implies that if ω_1, ω_2 are closed 2-forms on $\Lambda^2 \mathfrak{n}^*$ with $[\omega_1]^{0,2} = [\omega_2]^{0,2}$ then $[(A^{-1})^* \omega_1]^{0,2} = [(A^{-1})^* \omega_2]^{0,2}$. Therefore, the following action is well defined:

$$\begin{array}{rcl} \mathrm{Aut}(\mathfrak{n}) \times E_\infty^{0,2}(\mathfrak{n}) & \longrightarrow & E_\infty^{0,2}(\mathfrak{n}) \\ (A, [\omega]^{0,2}) & \mapsto & A \cdot [\omega]^{0,2} = [(A^{-1})^* \omega]^{0,2}. \end{array} \tag{7}$$

Proposition 3.4. *For any nilpotent Lie algebra* $\mathfrak{n}$ *the map*

$$p: H^2(\mathfrak{n}) \longrightarrow E_\infty^{0,2}(\mathfrak{n}), \quad [\omega] \mapsto [\omega]^{0,2}$$

is an $\operatorname{Aut}(\mathfrak{n})$ *equivariant map. Moreover, the orbit map* $\tilde{p}: H^2(\mathfrak{n})/\operatorname{Aut}(\mathfrak{n}) \longrightarrow E_\infty^{0,2}(\mathfrak{n})/\operatorname{Aut}(\mathfrak{n})$ *is surjective.*

Proof. The fact that $d(\mathfrak{n}^*) \subseteq \Lambda^2 V_{k-1}$ implies that the map $p: H^2(\mathfrak{n}) \longrightarrow E_\infty^{0,2}$, $p([\omega]) = [\omega]^{0,2}$ is well defined and surjective. Hence so is $\tilde{p}$.

Notice that p is injective if and only if $\dim H^2(\mathfrak{n}) = \dim E_\infty^{0,2}(\mathfrak{n})$ and this situation occurs only when $\mathfrak{n}$ is a 2-step free nilpotent Lie algebra. ■

In the next example we show that the quotient map $\tilde{p}$ is not always injective, even when $E_\infty^{0,2} \neq 0$.

Example 3.5. Let $\mathfrak{n}$ be the six dimensional nilpotent Lie algebra having non-zero Lie brackets

$$[e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_1, e_4] = e_6.$$

Denote by $\{e^1, \ldots, e^6\}$ the dual basis of $\mathfrak{n}^*$. The followings are symplectic forms on $\mathfrak{n}$:

$$\omega_1 = e^1 \wedge e^6 - e^2 \wedge e^4 + e^3 \wedge e^5, \quad \omega_2 = e^1 \wedge e^6 + e^2 \wedge e^5 + e^3 \wedge e^4.$$

They verify $0 \neq [\omega_1]^{0,2} = [\omega_2]^{0,2} = [e^1 \wedge e^6]^{0,2}$. However through direct computations one can prove that the de Rham cohomology classes of ω_1 and ω_2 do not belong to the same $\operatorname{Aut}(\mathfrak{n})$-orbit.

In spite of the previous example, it is possible to find nilpotent Lie algebras for which the cardinal of the quotient space $E_\infty^{0,2}/\operatorname{Aut}(\mathfrak{n})$ coincides with the amount of $\operatorname{Aut}(\mathfrak{n})$ equivalence classes of symplectic structures on $\mathfrak{n}$.

4. Classification of symplectic nilradicals.

In this section we study the intermediate cohomology of the real nilpotent Lie algebras $\mathfrak{n}$ arising as nilradicals of minimal parabolic subalgebras of the real split forms of semisimple complex Lie algebras $\mathfrak{g}$. In particular we show that when considering $\mathfrak{g}$ to be a classical simple complex Lie algebra, it holds $E_\infty^{0,2}(\mathfrak{n}) = 0$ in almost every case. According to the results in the previous section those nilpotent Lie algebras do not admit symplectic structures. Even more, we prove that if $E_\infty^{0,2}(\mathfrak{n}) \neq 0$ then $\mathfrak{n}$ admits such structures.

To determine the intermediate cohomology group $E_\infty^{0,2}$ of those nilpotent Lie algebras, our main tool is the root decomposition of semisimple Lie algebras. For this subject we give the book of Helgason [8] as a reference. The understanding of those systems allows the description of the filtration in Eq. (2) in terms of the root spaces.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and let $\triangle$ be a root system of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\triangle-\{0\}} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote as $\triangle^+$ the set of positive roots. The Lie algebra

$$\mathfrak{n} = \bigoplus_{\alpha\in\triangle^+} \mathfrak{g}_\alpha \tag{8}$$

is complex nilpotent.

Remark 4.1. Given a complex nilpotent Lie algebra $\mathfrak{n}$, the filtration described in Eq. (2) and the induced spectral sequence are also canonically determined by $\mathfrak{n}$. Therefore each Lie algebra cohomology group with complex coefficients $H^i(\mathfrak{n}, \mathbb{C})$ decomposes as in Eq. (5).

In the particular case that $\mathfrak{n}$ is the nilradical in (8) of a Borel subalgebra of a complex semisimple Lie algebra, Kostant proved that $H^i(\mathfrak{n}, \mathbb{C})$ is a direct sum of T-modules of dimension one (see [9, Theorem 6.1]) where T is the diagonal subgroup of the semisimple Lie group. The action of T on $\mathfrak{n}$ can be induced to $\Lambda^k\mathfrak{n}^*$ and commutes with the Lie algebra differential. As a consequence, the canonical filtration of $\mathfrak{n}^*$ is preserved by the T-action and the intermediate cohomology groups $E_\infty^{p,q}$ are T-modules. But $E_\infty^{p,q}$ is not irreducible in general. In particular, each (complex) intermediate cohomology group of $\mathfrak{n}$ is a sum of Kostant's one dimensional modules.

The Lie algebra $\mathfrak{n}$ in Eq. (8) admits a basis $\{X_\alpha\}_{\alpha\in\triangle^+}$ such that $\mathfrak{g}_\alpha = \mathbb{C}\, X_\alpha$ and the structure constants of $\mathfrak{n}$ in this basis are in $\mathbb{R}$. The object of study in this section is the real nilpotent Lie algebra having those real structure coefficients; we will also denote it as $\mathfrak{n}$.

This real nilpotent Lie algebra $\mathfrak{n}$ is the nilradical of the minimal parabolic subalgebra of the split form corresponding to the semisimple Lie algebra $\mathfrak{g}$.

We pursue the computation of the filtration of the Chevalley-Eilenberg complex of $\mathfrak{n}$ and the intermediate cohomology group $E_\infty^{0,2}(\mathfrak{n})$.

Denote as $\triangle_0 = \{\alpha_1, \dots, \alpha_r\}$ the subset of positive simple roots of $\mathfrak{g}$. Then for any positive root α there are non-negative integers n_i, $i = 1, \dots, r$ such that

$$\alpha = \sum_{i=1}^{r} n_i\alpha_i.$$

In this case we say that the *level* of the root is $\ell(\alpha) = \sum_{i=1}^r n_i$. Clearly the roots of level 1 are only the simple roots. There exists a unique positive root $\alpha_{\max}$ of maximal level, that is, such that $\ell(\alpha) \le \ell(\alpha_{\max})$ for all $\alpha \in \triangle^+$.

For each $i \in \mathbb{N}$ define $L_i = \bigoplus_{\alpha:\ell(\alpha)=i} \mathbb{R}X_\alpha$ where X_α is as before. Then

$$\mathfrak{n} = \bigoplus_{j\ge 1} L_j \quad \text{and} \quad [L_j, L_i] \subseteq L_{i+j}$$

and $\mathfrak{n}$ is an $\mathbb{N}$-graded Lie algebra.

Let α be a positive root of level m and let $\gamma_\alpha \in \mathfrak{n}^*$ be the dual element of X_α. Since $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_m$ for all $i+j=m$, it holds

$$d\gamma_\alpha \in \bigoplus_{i+j=m} L_i^* \wedge L_j^*. \tag{9}$$

In particular $d\gamma_\alpha = 0$ if and only if α is a simple root. This accounts into a description of the subspaces in (2) of $\mathfrak{n}^*$ as follows

$$V_0 = 0, \qquad V_j = \operatorname{span}\{\gamma_\beta : \ell(\beta) \leq j\} = L_1^* \oplus \cdots \oplus L_j^*, \quad j = 1, \ldots, k.$$

Notice that the nilpotency index of $\mathfrak{n}$ is $k = \ell(\alpha_{\max})$.

We proceed by making an insight into the space of closed 2-forms which we denote as Z^2. Such a form is an element of $\Lambda^2\mathfrak{n}^* = \bigoplus_{1\leq i<j\leq k} L_i^* \wedge L_j^*$. In this context, the result of Benson and Gordon [1, Lemma 2.8] assures that Z^2 is contained in a strictly smaller subspace, namely

$$Z^2 \subseteq L_k^* \wedge L_1^* \oplus \bigoplus_{1\leq i<j\leq k-1} L_i^* \wedge L_j^*.$$

Therefore any $\omega \in Z^2$ can be written as a sum $\omega = \sigma + \tilde{\omega}$ where

$$\sigma \in L_k^* \wedge L_1^*, \qquad \tilde{\omega} \in \bigoplus_{1\leq i<j\leq k-1} L_i^* \wedge L_j^* \quad \text{and} \quad d\sigma = -d\tilde{\omega}. \tag{10}$$

The vector space L_k^* has dimension one and is spanned by $\gamma_{\alpha_{\max}}$. Moreover $L_1^* = V_1$ is spanned by the 1-forms γ_{α_i}, $i = 1, \ldots, r$ where $\alpha_1, \ldots, \alpha_r$ are the simple roots. Hence the 2-form σ in (10) is in fact $\sigma = \gamma_{\alpha_{\max}} \wedge \eta$, with $\eta = \sum_{i=1}^r r_i \gamma_{\alpha_i} \in V_1$, $r_i \in \mathbb{R}$.

Since $d\eta = 0$, $d\sigma = d\gamma_{\alpha_{\max}} \wedge \eta$ and by Eq. (9),

$$d\sigma \;\in\; L_{k-1}^* \wedge L_1^* \wedge L_1^* \;\oplus \bigoplus_{\substack{i+j=k \\ 1\leq i<j\leq k-2}} L_i^* \wedge L_j^* \;\wedge L_1^*. \tag{11}$$

In addition,

$$d\tilde{\omega} \in \Lambda^3(L_{k-1}^* \oplus L_{k-2}^* \oplus \cdots \oplus L_1^*). \tag{12}$$

The key here is to compare components of $d\sigma$ and $-d\omega$ in particular subspaces of those in Eqns. (11) and (12). For classical complex simple Lie algebras this idea allows us to prove that $Z^2 \subset \Lambda^2 V_{k-1}$ which by (6) implies $E_\infty^{0,2}(\mathfrak{n}) = 0$.

Remark 4.2. The structure of the semisimple Lie algebra $\mathfrak{g}$ is independent of the root system. Therefore, the real nilpotent Lie algebra associated to a certain root system is isomorphic to the nilpotent Lie algebra arising from a different one. For this reason we can choose the root system of $\mathfrak{g}$ that is more convenient for us to make calculations easier.

The result we obtain is the following.

Lemma 4.3. *Let $\mathfrak{n}$ be the nilradical of a minimal parabolic subalgebra of the real split Lie algebra corresponding to the complex classical simple Lie algebra $\mathfrak{g}$.*

1. *If $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ for some $n \geq 3$ then $E_\infty^{0,2}(\mathfrak{n}) = 0$.*
2. *If $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$ for some $n \geq 3$ then $E_\infty^{0,2}(\mathfrak{n}) = 0$.*
3. *If $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ for some $n \geq 3$ then $E_\infty^{0,2}(\mathfrak{n}) = 0$.*
4. *If $\mathfrak{g} = \mathfrak{so}(2n,\mathbb{C})$ for some $n \geq 4$ then $E_\infty^{0,2}(\mathfrak{n}) = 0$.*

The full classification of the nilradicals admitting symplectic structures follows.

Theorem 4.4. *Suppose $\mathfrak{n}$ is a nilradical of a minimal parabolic subalgebra of the real split Lie algebra corresponding to the complex classical simple Lie algebra $\mathfrak{g}$. The Lie algebra of even dimension $\mathbb{R}^s \oplus \mathfrak{n}$, $s \geq 0$ admits symplectic structures if and only if $\mathfrak{g}$ is one of the followings:*

$$\mathfrak{sl}(2,\mathbb{C}), \quad \mathfrak{sl}(3,\mathbb{C}), \quad \mathfrak{so}(5,\mathbb{C}).$$

Proof. The nilradical $\mathfrak{n}$ of $\mathfrak{sl}(2,\mathbb{C})$ is the abelian Lie algebra of dimension one and clearly $\mathbb{R} \oplus \mathfrak{n}$ admits symplectic structures. When $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, then $\mathfrak{n}$ is the Heisenberg Lie algebra of dimension 3. It is well known that $\mathbb{R} \oplus \mathfrak{n}$ admits symplectic structures in this case too.

The nilradical corresponding to $\mathfrak{so}(5,\mathbb{C})$ is the 4-dimensional 3-step nilpotent Lie algebra which can be endowed with a symplectic structure.

When $\mathfrak{g}$ is not one of the Lie algebras above, Lemma 4.3 implies that its nilradical $\mathfrak{n}$ has $E_\infty^{0,2}(\mathfrak{n}) = 0$. By Corollary 2.3, given $s \geq 0$ the Lie algebra $\mathbb{R}^s \oplus \mathfrak{n}$ also has zero intermediate cohomology group $E_\infty^{0,2}$. After Theorem 3.1, those nilpotent Lie algebras are not symplectic. ■

We proceed with the proof of Lemma 4.3. This is made using the canonical root systems known for classical simple Lie algebras. Moreover it is performed separately by cases because of the differences between those root systems. The order in which the cases are exposed is from the easiest to the most difficult one.

Proof. Part (1) of Lemma 4.3. Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$, $n \geq 3$. If $n = 3$, the Lie algebra $\mathfrak{n}$ is isomorphic to the Lie algebra of strictly upper triangular matrices 4×4 for which can be easily shown that $E_\infty^{0,2}(\mathfrak{n}) = 0$.

Suppose $n \geq 4$ and consider the Cartan subalgebra $\mathfrak{h}$ for which the positive roots are $e_i \pm e_j$, $1 \leq i < j \leq n+1$. The set of simple roots is $\triangle_0 = \{\alpha_i = e_i - e_{i+1} : i = 1, \ldots, n\}$. Moreover the maximal root is $\alpha_{\max} = \sum_{i=1}^n \alpha_i$, hence the nilpotency index k of $\mathfrak{n}$ is $k = n$. There are two different roots of level $n-1$, namely $\delta_1 = \sum_{i=1}^{n-1} \alpha_i$ and $\delta_2 = \sum_{i=2}^n \alpha_i$; in particular $\dim L_{k-1} = 2$.

Let ω be a closed 2-form in $\mathfrak{n}^*$, then $\omega = \sigma + \tilde{\omega}$ where σ and $\tilde{\omega}$ satisfy the conditions in (10). The fact $d\sigma = -d\tilde{\omega}$ implies that the components of $d\sigma$ and $-d\tilde{\omega}$ in the subspace $L_{k-1}^* \wedge L_1^* \wedge L_1^*$ are equal. So we compute both components.

As before, $\sigma = \gamma_{\alpha_{\max}} \wedge \eta$ where $\eta = \sum_{i=1}^{n} r_i \gamma_{\alpha_i}$, $r_i \in \mathbb{R}$ $i = 1, \ldots, n$ and $d\sigma = d\gamma_{\alpha_{\max}} \wedge \eta$. Using Equation (9) and the fact that $\alpha_{\max} = \delta_1 + \alpha_n = \delta_2 + \alpha_1$ we obtain $d\gamma_{\alpha_{\max}} = a_1 \gamma_{\delta_1} \wedge \gamma_{\alpha_n} + a_2 \gamma_{\delta_2} \wedge \gamma_{\alpha_1} + \tau$ with $\tau \in \Lambda^2 V_{k-2} = \Lambda^2(L_{k-2}^* \oplus \cdots \oplus L_1^*)$ and $a_1, a_2 \in \mathbb{R}$ are both non-zero. This implies that the component of $d\sigma = d\gamma_{\alpha_{\max}} \wedge \eta$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^*$ is

$$\sum_{i=1}^{n} a_1 r_i \gamma_{\delta_1} \wedge \gamma_{\alpha_n} \wedge \gamma_{\alpha_i} + \sum_{i=1}^{n} a_2 r_i \gamma_{\delta_2} \wedge \gamma_{\alpha_1} \wedge \gamma_{\alpha_i}. \tag{13}$$

To find the component of $d\tilde{\omega}$ in the same subspace write $\tilde{\omega} = \omega_1 + \tilde{\omega}_1$ with

$$\omega_1 \in L_{k-1}^* \wedge L_2^*, \qquad \tilde{\omega}_1 \in L_{k-1}^* \wedge \Big(\bigoplus_{\substack{j \leq k-1 \\ j \neq 2}} L_j^*\Big) \oplus \Lambda^2(L_{k-2}^* \oplus \cdots \oplus L_1^*). \tag{14}$$

Hence by Eq. (9),

$$d\omega_1 \in L_{k-1}^* \wedge L_1^* \wedge L_1^* \oplus \bigoplus_{i+j=k-1} L_i^* \wedge L_j^* \wedge L_2^*. \tag{15}$$

By the same equation $d\tilde{\omega}_1$ has no component in that subspace. Therefore the component of $d\sigma$ equals the component of $-d\omega_1$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^*$.

Since $n \geq 4$ it is $L_{k-1}^* \neq L_2^*$ and the roots of level 2 are $\{\alpha_i + \alpha_{i+1} : i = 1, \ldots, n-1\}$. Hence $L_{k-1}^* \wedge L_2^*$ has a basis of the form $\{\gamma_{\delta_1} \wedge \gamma_{\alpha_i+\alpha_{i+1}}, \gamma_{\delta_2} \wedge \gamma_{\alpha_i+\alpha_{i+1}} : i = 1, \ldots, n-1\}$. Then

$$\omega_1 = \sum_{i=1}^{n-1} b_i \gamma_{\delta_1} \wedge \gamma_{\alpha_i+\alpha_{i+1}} + \sum_{i=1}^{n-1} c_i \gamma_{\delta_2} \wedge \gamma_{\alpha_i+\alpha_{i+1}}, \qquad \text{where } b_i,\, c_i \in \mathbb{R} \text{ for all } i.$$

Equation (9) implies that $d\gamma_{\alpha_i+\alpha_{i+1}} = \xi_i \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}}$ where $\xi_i \neq 0$, for all $i = 1, \ldots, n-1$. Then

$$\begin{aligned} -d\omega_1 \;=\; & \sum_{i=1}^{n-1} b_i \xi_i \gamma_{\delta_1} \wedge \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}} + \sum_{i=1}^{n-1} c_i \xi_i \gamma_{\delta_2} \wedge \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}} \\ & - \sum_{i=1}^{n} b_i d\gamma_{\delta_1} \wedge \gamma_{\alpha_i+\alpha_{i+1}} - \sum_{i=1}^{n} c_i d\gamma_{\delta_2} \wedge \gamma_{\alpha_i+\alpha_{i+1}}. \end{aligned}$$

The component of $-d\omega_1$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^*$ is

$$\sum_{i=1}^{n-1} b_i \xi_i \gamma_{\delta_1} \wedge \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}} + \sum_{i=1}^{n-1} c_i \xi_i \gamma_{\delta_2} \wedge \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}}. \tag{16}$$

Indeed, the element $d\gamma_{\delta_i}$ belongs to $\Lambda^2 V_{k-2}$ for $i = 1, 2$ because $\gamma_{\delta_i} \in L_{k-1}^* \subseteq V_{k-1}$.

Formulas (13) and (16) give the components in $L_{k-1}^* \wedge L_1^* \wedge L_1^*$ of $d\sigma$ and $-d\omega_1$ which must be equal, that is,

$$\begin{aligned} \sum_{i=1}^{n-1} b_i \xi_i \gamma_{\delta_1} \wedge \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}} + \sum_{i=1}^{n-1} c_i \xi_i \gamma_{\delta_2} \wedge \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}} \;=\; & \\ \sum_{i=1}^{n-1} a_1 r_i \gamma_{\delta_1} \wedge \gamma_{\alpha_n} \wedge \gamma_{\alpha_i} + \sum_{i=2}^{n} a_2 r_i \gamma_{\delta_2} \wedge \gamma_{\alpha_1} \wedge \gamma_{\alpha_i}. & \end{aligned}$$

This expression implies that $r_i = 0$ for all $i = 1, \dots, n$ which means $\eta = 0$ yielding to $\sigma = 0$. The conclusion here is that if $\omega = \sigma + \tilde{\omega}$ is closed then $\sigma = 0$, that is, $\omega \in \Lambda^2 V_{k-1}$ and therefore $E_\infty^{0,2}(\mathfrak{n}) = 0$ as we wanted to prove. ■

Proof. Part (3) of Lemma 4.3. Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ for some $n \geq 4$.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ associated to the set of positive roots $\triangle^+ = \{e_i \pm e_j : 1 \leq i < j \leq n, 2e_i, i = 1, \dots, n\}$; the subset of simple roots is $\triangle_0 = \{\alpha_i := e_i - e_{i+1} : i = 1, \dots, n-1, \alpha_n := 2e_n\}$. The maximal root is $\alpha_{\max} = \sum_{i=1}^{n-1} 2\alpha_i + \alpha_n$ from which we deduce that $\mathfrak{n}$ is $2n-1$-step nilpotent; set $k = 2n-1$. Unlike the previous case, $\dim L_{k-1} = 1$; the root of level $k-1$ is $\delta = \alpha_1 + \sum_{i=2}^{n-1} 2\alpha_i + \alpha_n$.

Consider a closed 2-form $\omega \in \Lambda^2 \mathfrak{n}^*$ with $\omega = \sigma + \tilde{\omega}$ as in Eq. (10). We are interested in computing the components of $d\sigma$ and $d\omega$ in the subspace $L_{k-1}^* \wedge L_1^* \wedge L_1^* \;\oplus\; L_{k-2}^* \wedge L_2^* \wedge L_1^*$ which must be opposite. Below we compute them both.

As before $\sigma = \gamma_{\alpha_{\max}} \wedge \eta$ where η is a linear combination of the 1-forms γ_{α_i}, $i = 1, \dots, n$. Moreover $\alpha_{\max} = \delta + \alpha_1$. The roots $\rho = \alpha_1 + \alpha_2 + \sum_{i=3}^{n-1} 2\alpha_i + \alpha_n$ and $\rho' = \sum_{i=2}^{n-1} 2\alpha_i + \alpha_n$ are the only ones of level $k-2$ so

$$d\gamma_{\alpha_{\max}} = a_1 \gamma_{\alpha_1} \wedge \gamma_\delta + a_2 \gamma_{\alpha_1+\alpha_2} \wedge \gamma_\rho + \tau, \qquad a_1 \neq 0, a_2 \neq 0, \quad \text{with } \tau \in \Lambda^2 V_{k-3}. \tag{17}$$

In fact, $\alpha_{\max} = \delta + \alpha_1$ and there do not exists any positive root β such that $\beta + \rho' = \alpha_{\max}$.

From Eq. (17),

$$d\sigma = d\gamma_{\alpha_{max}} \wedge \eta = a_1 \gamma_{\alpha_1} \wedge \gamma_\delta \wedge \eta + a_2 \gamma_{\alpha_1+\alpha_2} \wedge \gamma_\rho \wedge \eta + \tau \wedge \eta,$$

being $\tau \wedge \eta$ an element in $\Lambda^3 V_{k-3}$. The component of $d\sigma$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^* \oplus L_{k-2}^* \wedge L_2^* \wedge L_1^*$ is

$$a_1 \gamma_{\alpha_1} \wedge \gamma_\delta \wedge \eta + a_2 \gamma_{\alpha_1+\alpha_2} \wedge \gamma_\rho \wedge \eta. \tag{18}$$

To find the component of $d\tilde{\omega}$ let ω_1 and $\tilde{\omega}_1$ be 2-forms such that $\tilde{\omega} = \omega_1 + \tilde{\omega}_1$ where

$$\begin{aligned} \omega_1 &\in L_{k-1}^* \wedge L_2^* \oplus L_3^* \wedge L_{k-2}^*, \\ \tilde{\omega}_1 &\in L_{k-1}^* \wedge \Big(\bigoplus_{\substack{j \leq k-1 \\ j \neq 2}} L_j^* \Big) \oplus L_{k-2}^* \wedge \Big(\bigoplus_{\substack{j \leq k-1 \\ j \neq 3}} L_j^* \Big) \oplus \Lambda^2 (L_{k-3}^* \oplus \cdots \oplus L_1^*). \end{aligned}$$

Using Eq. (9) for the differential of basic 1-forms one obtains

$$d\omega_1 \in L_{k-1}^* \wedge L_1^* \wedge L_1^* \;\oplus\; L_{k-2}^* \wedge L_2^* \wedge L_1^*.$$

and $d\tilde{\omega}_1$ has zero component in the same subspace. Hence the component of $d\tilde{\omega}$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^* \;\oplus\; L_{k-2}^* \wedge L_2^* \wedge L_1^*$ is the component of $d\omega_1$ in that subspace.

In this case, $\ell(\beta) = 2$ if and only if $\beta = \alpha_i + \alpha_{i+1}$ for some $i = 1, \dots, n-1$. Thus $L_{k-1}^* \wedge L_2^*$ admits the set $\{\gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_\delta,\ i = 1, \dots, n-1\}$ as a basis. The roots of level three are $\alpha_i + \alpha_{i+1} + \alpha_{i+2}$, $i = 1, \dots, n-2$ and $2\alpha_{n-1} + \alpha_n$.

Since $n \geq 4$, $3 \neq k-2$ and $L_3^* \wedge L_{k-2}^*$ is spanned by $\{\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_\rho, \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_{\rho'} : i = 1, \ldots, n-2, \ \gamma_{2\alpha_{n-1}+\alpha_n} \wedge \gamma_\rho, \gamma_{2\alpha_{n-1}+\alpha_n} \wedge \gamma_{\rho'}\}$. The 2-form ω_1 can be written as

$$\begin{aligned}\omega_1 &= \sum_{i=1}^{n-1} c_i \gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_\delta + \sum_{i=1}^{n-2} d_i \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_\rho + d_{n-1}\gamma_{2\alpha_{n-1}+\alpha_n} \wedge \gamma_\rho \\ &\quad + \sum_{i=1}^{n-2} e_i \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_{\rho'} + e_{n-1}\gamma_{2\alpha_{n-1}+\alpha_n} \wedge \gamma_{\rho'}, \quad c_i, d_i, e_i \in \mathbb{R}.\end{aligned}$$

From (10),

$$\begin{aligned} d\gamma_{\alpha_i+\alpha_{i+1}} &= \xi_i \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}}, \\ d\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} &= s_i^1 \gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_{\alpha_{i+2}} + s_i^2 \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}+\alpha_{i+2}}; \\ d\gamma_{2\alpha_{n-1}+\alpha_n} &= s_{n-1}\gamma_{\alpha_n-1} \wedge \gamma_{\alpha_{n-1}+\alpha_n}, \end{aligned}$$

with ξ_i, s_i^1, s_i^2, s_{n-1} all non-zero.

Putting this all together we reach:

$$\begin{aligned} d\omega_1 &= \sum_{i=1}^{n-1} c_i \xi_i \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}} \wedge \gamma_\delta - \sum_{i=1}^{n-1} c_i \gamma_{\alpha_i+\alpha_{i+1}} \wedge d\gamma_\delta \\ &+ \sum_{i=1}^{n-2} d_i (s_i^1 \gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_{\alpha_{i+2}} + s_i^2 \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}+\alpha_{i+2}}) \wedge \gamma_\rho \\ &- \sum_{i=1}^{n-2} d_i \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge d\gamma_\rho + d_{n-1}s_{n-1}\gamma_{\alpha_{n-1}} \wedge \gamma_{\alpha_{n-1}+\alpha_n} \wedge \gamma_\rho \\ &- d_{n-1}\gamma_{2\alpha_{n-1}+\alpha_n} \wedge d\gamma_\rho - \sum_{i=1}^{n-2} e_i \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge d\gamma_{\rho'} \\ &+ \sum_{i=1}^{n-2} e_i (s_i^1 \gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_{\alpha_{i+2}} + s_i^2 \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}+\alpha_{i+2}}) \wedge \gamma_{\rho'} \\ &- e_{n-1}s_{n-1}\gamma_{\alpha_{n-1}} \wedge \gamma_{\alpha_{n-1}+\alpha_n} \wedge \gamma_{\rho'} - e_{n-1}\gamma_{2\alpha_{n-1}+\alpha_n} \wedge d\gamma_{\rho'}. \end{aligned}$$

Since $\ell(\rho) = \ell(\rho') = k-2$, $d\gamma_\rho, d\gamma_{\rho'} \in \Lambda^2 V_{k-3}$. In addition, $\delta = \alpha_2 + \rho = \alpha_1 + \rho'$ implies

$$d\gamma_\delta = b_1 \gamma_{\alpha_2} \wedge \gamma_\rho + b_2 \gamma_{\alpha_1} \wedge \gamma_{\rho'} + \tau', \qquad b_1 \neq 0, b_2 \neq 0, \ \text{ with } \tau' \in \Lambda^2 V_{k-3}.$$

Therefore, the component of $-d\omega_1$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^* \oplus L_{k-2}^* \wedge L_2^* \wedge L_1^*$ is

$$\begin{aligned} &-\sum_{i=1}^{n-1} c_i \xi_i \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}} \wedge \gamma_\delta + \sum_{i=1}^{n-1} c_i \gamma_{\alpha_i+\alpha_{i+1}} \wedge (b_1 \gamma_{\alpha_2} \wedge \gamma_\rho + b_2 \gamma_{\alpha_1} \wedge \gamma_{\rho'}) \\ &- \sum_{i=1}^{n-2} d_i (s_i^1 \gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_{\alpha_{i+2}} + s_i^2 \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}+\alpha_{i+2}}) \wedge \gamma_\rho \\ &- \sum_{i=1}^{n-2} e_i (s_i^1 \gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_{\alpha_{i+2}} + s_i^2 \gamma_{\alpha_i} \wedge \gamma_{\alpha_{i+1}+\alpha_{i+2}}) \wedge \gamma_{\rho'} \\ &- d_{n-1}s_{n-1}\gamma_{\alpha_{n-1}} \wedge \gamma_{\alpha_{n-1}+\alpha_n} \wedge \gamma_\rho + e_{n-1}s_{n-1}\gamma_{\alpha_{n-1}} \wedge \gamma_{\alpha_{n-1}+\alpha_n} \wedge \gamma_{\rho'}. \end{aligned} \tag{19}$$

The components of $d\sigma$ and $-d\tilde{\omega}_1$ expressed in Eqns. (18) and (19) respectively, coincide. Following usual computations we get that c_i, e_j are zero for all $i = 1, \dots, n-1$ and $j = 1, \dots, n-2$ which simplify Eq. (19). After some other simplifications we obtain once again that $\sigma = 0$ and then any closed 2-form in $\mathfrak{n}$ belongs to $\Lambda^2 V_{k-1}$ which is equivalent to $E_\infty^{0,2}(\mathfrak{n}) = 0$.

When $\mathfrak{g} = \mathfrak{sp}(6, \mathbb{C})$ the Lie algebra $\mathfrak{n}$ has dimension 9 and the proof is made by direct computations using a mathematical software. ∎

Proof. Part (2) of Lemma 4.3. Suppose $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ for some $n \geq 5$.

Consider the Cartan Lie subalgebra such that the set of positive roots is $\triangle^+ = \{e_i \pm e_j : 1 \leq i < j \leq n, \, e_i, \, i = 1, \dots, n\}$. Then $\dim \mathfrak{n} = n^2$. The simple roots are $\{\alpha_i := e_i - e_{i+1} : i = 1, \dots, n-1, \alpha_n := e_n\}$ and the maximal root $\alpha_{\max}$ is $e_1 + e_2 = \alpha_1 + 2\sum_{i=2}^n \alpha_i$. As a consequence $\mathfrak{n}$ is $2n-1$-step nilpotent; set $k = 2n-1$. Here we also have $\dim L_{k-1} = 1$; the root of level $k-1$ is now $\delta = \alpha_1 + \alpha_2 + \sum_{i=2}^{n-1} 2\alpha_i$.

Let $\omega \in \Lambda^2 \mathfrak{n}^*$ be a closed form and let σ and $\tilde{\omega}$ be as in (10). In this case we study the components of $d\sigma$ and $d\tilde{\omega}$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^* \, \oplus \, L_{k-2}^* \wedge L_2^* \wedge L_1^* \oplus L_{k-3}^* \wedge L_3^* \wedge L_1^*$ which must be opposite since $0 = d\omega = d\sigma + d\tilde{\omega}$.

Recall that $\sigma = \gamma_{\alpha_{\max}} \wedge \eta$ with η a linear combination of the 1-forms γ_{α_i}, $i = 1, \dots, n$. There are two roots of level $k-2$, namely $\rho = \alpha_1 + \alpha_2 + \alpha_3 + \sum_{i=4}^n 2\alpha_i$ and $\rho' = \alpha_2 + \sum_{i=3}^n 2\alpha_i$. The roots of level $k-3$ are $\theta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\sum_{i=5}^n \alpha_i$ and $\theta' = \alpha_2 + \alpha_3 + 2\sum_{i=4}^n \alpha_i$.

Notice that

$$\alpha_{\max} = \delta + \alpha_2, \quad \alpha_{\max} = \rho + (\alpha_2 + \alpha_3) = \rho' + (\alpha_1 + \alpha_2) \quad \text{and}$$
$$\alpha_{\max} = \theta + (\alpha_2 + \alpha_3 + \alpha_4) = \theta' + (\alpha_1 + \alpha_2 + \alpha_3).$$

This implies

$$\begin{aligned} d\gamma_{\alpha_{\max}} &= a_1 \gamma_{\alpha_2} \wedge \gamma_\delta + a_2 \gamma_{\alpha_2+\alpha_3} \wedge \gamma_\rho + a_3 \gamma_{\alpha_1+\alpha_2} \wedge \gamma_{\rho'} \\ &\quad + a_4 \gamma_{\alpha_2+\alpha_3+\alpha_4} \wedge \gamma_\theta + a_5 \gamma_{\alpha_1+\alpha_2+\alpha_3} \wedge \gamma_{\theta'} + \tau, \end{aligned} \tag{20}$$

where $a_i \neq 0$, for all i and $\tau \in \Lambda^2 V_{k-4}$. Then $d\sigma = d\gamma_{\alpha_{\max}} \wedge \eta$ is

$$\begin{aligned} d\sigma &= a_1 \gamma_{\alpha_2} \wedge \gamma_\delta \wedge \eta + a_2 \gamma_{\alpha_2+\alpha_3} \wedge \gamma_\rho \wedge \eta + a_3 \gamma_{\alpha_1+\alpha_2} \wedge \gamma_{\rho'} \wedge \eta \\ &\quad + a_4 \gamma_{\alpha_2+\alpha_3+\alpha_4} \wedge \gamma_\theta \wedge \eta + a_5 \gamma_{\alpha_1+\alpha_2+\alpha_3} \wedge \gamma_{\theta'} \wedge \eta + \tilde{\tau}, \end{aligned} \tag{21}$$

with $\tilde{\tau} \in \Lambda^2 V_{k-4}$. Thus the element

$$\begin{aligned} &a_1 \gamma_{\alpha_2} \wedge \gamma_\delta \wedge \eta + a_2 \gamma_{\alpha_2+\alpha_3} \wedge \gamma_\rho \wedge \eta + a_3 \gamma_{\alpha_1+\alpha_2} \wedge \gamma_{\rho'} \wedge \eta \\ &\quad + a_4 \gamma_{\alpha_2+\alpha_3+\alpha_4} \wedge \gamma_\theta \wedge \eta + a_5 \gamma_{\alpha_1+\alpha_2+\alpha_3} \wedge \gamma_{\theta'} \wedge \eta \end{aligned} \tag{22}$$

is the component of $d\sigma$ in $L_{k-1}^* \wedge L_1^* \wedge L_1^* \, \oplus \, L_{k-2}^* \wedge L_2^* \wedge L_1^* \oplus L_{k-3}^* \wedge L_3^* \wedge L_1^*$.

To compute the component of $d\omega$ in the same subspace, take ω_1 and $\tilde{\omega}_1$ such that $\tilde{\omega} = \omega_1 + \tilde{\omega}_1$ where

$$\omega_1 \in L_2^* \wedge L_{k-1}^* \, \oplus \, L_3^* \wedge L_{k-2}^* \, \oplus \, L_4^* \wedge L_{k-3} \qquad \text{and}$$

$$\tilde{\omega}_1 \in L_{k-1}^* \wedge \Big(\bigoplus_{\substack{j\leq k-1\\ j\neq 2}} L_j^*\Big) \oplus L_{k-2}^* \wedge \Big(\bigoplus_{\substack{j\leq k-1\\ j\neq 3}} L_j^*\Big) \oplus L_{k-3}^* \wedge \Big(\bigoplus_{\substack{j\leq k-3\\ j\neq 4}} L_k^*\Big) \oplus \Lambda^2(L_{k-4}^* \oplus \cdots \oplus L_1^*).$$

The 3-form $d\tilde{\omega}_1$ has zero component in $L_{k-1}^* \wedge L_1^* \wedge L_1^* \oplus L_{k-2}^* \wedge L_2^* \wedge L_1^* \oplus L_{k-3}^* \wedge L_3^* \wedge L_1^*$. Therefore the component of $d\tilde{\omega}$ in that subspace is that one of $d\omega_1$ and to compute it we will make use of the following formulas obtained from Eq. (9):

$$\begin{aligned} d\gamma_\delta &= b_1\, \gamma_{\alpha_3} \wedge \gamma_\rho + b_2\, \gamma_{\alpha_1} \wedge \gamma_{\rho'} + b_3\, \gamma_\theta \wedge \gamma_{\alpha_3+\alpha_4} + \tau', \\ d\gamma_\rho &= \nu_1\, \gamma_{\alpha_1} \wedge \gamma_{\theta'} + \nu_2\, \gamma_{\alpha_4} \wedge \gamma_\theta + \tau'', \\ d\gamma_{\rho'} &= \mu_1\, \gamma_{\alpha_3} \wedge \gamma_{\theta'} + \tau''', \qquad \text{with } \tau', \tau'', \tau'' \in \Lambda^2 V_{k-4} \end{aligned} \tag{23}$$

The difference between $d\gamma_\rho$ and $d\gamma_{\rho'}$ is due to the lack of simple roots β verifying $\theta + \beta = \rho'$; in opposite to ρ which verifies $\rho = \theta + \alpha_4$.

Notice that $\ell(\theta) = \ell(\theta') = k-3$, hence γ_θ and $\gamma_{\theta'}$ are in V_{k-3}; $d\gamma_\theta$ and $d\gamma_{\theta'}$ are elements in $\Lambda^2 V_{k-4} = \Lambda^2(L_{k-4}^* \oplus \cdots \oplus L_1^*)$.

The roots of level two are $\alpha_i + \alpha_{i+1}$, $i = 1, \ldots, n-1$, which gives the following basis of $L_{k-1}^* \wedge L_2^*$: $\{\gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_\delta,\ i = 1, \ldots, n-1\}$. In addition, the roots of level three are $\alpha_i + \alpha_{i+1} + \alpha_{i+2}$, $i = 1, \ldots, n-2$ and $\alpha_{n-1} + 2\alpha_n$. Therefore $L_3^* \wedge L_{k-2}^*$ is spanned by $\{\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_\rho, \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_{\rho'} : i = 1, \ldots, n-2,\ \gamma_{\alpha_{n-1}+2\alpha_n} \wedge \gamma_\rho, \gamma_{\alpha_{n-1}+2\alpha_n} \wedge \gamma_{\rho'}\}$. Finally, the roots of level four are $\alpha_i + \alpha_{i+1} + \alpha_{i+2} + \alpha_{i+3}$, $i = 1, \ldots, n-3$, and $\alpha_{n-2} + \alpha_{n-1} + 2\alpha_n$. Since $n \geq 5$, $k-3 \neq 4$ and $\{\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}} \wedge \gamma_\theta, \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}} \wedge \gamma_{\theta'}, i = 1, \ldots, n-3, \gamma_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} \wedge \gamma_\theta, \gamma_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} \wedge \gamma_{\theta'}\}$ is a basis of $L_{k-3}^* \wedge L_4^*$.

To make computations easier, write $\omega_1 \in L_2^* \wedge L_{k-1}^* \oplus L_3^* \wedge L_{k-2}^* \oplus L_4^* \wedge L_{k-3}^*$ as $\omega_1 = \omega_1^a + \omega_1^b + \omega_1^c$ where for some coefficients $c_i, d_i, e_i, f_i, g_i \in \mathbb{R}$, it holds

$$\begin{aligned} \omega_1^a &= \sum_{i=1}^{n-1} c_i\, \gamma_{\alpha_i+\alpha_{i+1}} \wedge \gamma_\delta, \\ \omega_1^b &= \sum_{i=1}^{n-2} d_i\, \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_\rho + d_{n-1}\, \gamma_{\alpha_{n-1}+2\alpha_n} \wedge \gamma_\rho + \\ &\quad + \sum_{i=1}^{n-2} e_i\, \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} \wedge \gamma_{\rho'} + e_{n-1}\, \gamma_{\alpha_{n-1}+2\alpha_n} \wedge \gamma_{\rho'}, \\ \omega_1^c &= \sum_{i=1}^{n-3} f_i\, \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}} \wedge \gamma_\theta + f_{n-2}\, \gamma_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} \wedge \gamma_\theta + \\ &\quad + \sum_{i=1}^{n-3} g_i\, \gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}} \wedge \gamma_{\theta'} + g_{n-2}\, \gamma_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} \wedge \gamma_{\theta'}. \end{aligned}$$

Notice that $\omega_1^a \in L_{k-1}^* \wedge L_2^*$, $\omega_1^b \in L_3^* \wedge L_{k-2}^*$ and $\omega_1^c \in L_4^* \wedge L_{k-3}^*$. The differential of the basic elements of L_2^*, L_3^* y L_4^* are

$$\begin{array}{lcl}
d\gamma_{\alpha_i+\alpha_{i+1}} & = & \xi_i\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}},\\
d\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} & = & s_i^1\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_{\alpha_{i+2}}+s_i^2\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}},\\
d\gamma_{\alpha_{n-1}+2\alpha_n} & = & s_{n-1}\gamma_{\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n},\\
d\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}} & = & t_i^1\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}}+t_i^2\,\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_{\alpha_{i+2}+\alpha_{i+3}}\\
 & & \quad +t_i^3\,\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_{i+3}},\\
d\gamma_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} & = & t_{n-2}^1\,\gamma_{\alpha_{n-2}}\wedge\gamma_{\alpha_{n-1}+2\alpha_n}+t_{n-2}^2\,\gamma_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n},
\end{array}$$

where ξ_i, s_i^j, s_{n-1}, t_i^j, are all non-zero.

The differential of ω_1^a is by (23)

$$d\omega_1^a=\sum_{i=1}^{n-1}c_i\,d\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_\delta-\sum_{i=1}^{n-1}c_i\,\gamma_{\alpha_i+\alpha_{i+1}}\wedge d\gamma_\delta=\sum_{i=1}^{n-1}c_i\,\xi_i\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}}\wedge\gamma_\delta-$$

$$\sum_{i=1}^{n-1}c_i\,\gamma_{\alpha_i+\alpha_{i+1}}\wedge\left(b_1\,\gamma_{\alpha_3}\wedge\gamma_\rho+b_2\,\gamma_{\alpha_1}\wedge\gamma_{\rho'}+b_3\,\gamma_\theta\wedge\gamma_{\alpha_3+\alpha_4}+\tau'\right).$$

Then the component of $d\omega_1^a$ in $L_{k-1}^*\wedge L_1^*\wedge L_1^*\oplus L_{k-2}^*\wedge L_2^*\wedge L_1^*\oplus L_{k-3}^*\wedge L_3^*\wedge L_1^*$ is

$$\sum_{i=1}^{n-1}c_i\,\xi_i\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}}\wedge\gamma_\delta-\sum_{i=1}^{n-1}b_1\,c_i\,\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_{\alpha_3}\wedge\gamma_\rho+b_2\,c_i\,\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_{\alpha_1}\wedge\gamma_{\rho'},\quad(24)$$

since $\tau'\in\Lambda^2V_{k-4}=\Lambda^2(L_{k-4}^*\oplus\cdots\oplus L_1^*)$ and $\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_\theta\wedge\gamma_{\alpha_3+\alpha_4}\in L_2^*\wedge L_2^*\wedge L_{k-3}^*$.

In a similar way we compute the components of $d\omega_1^b$ and $d\omega_1^c$ in $L_{k-1}^*\wedge L_1^*\wedge L_1^*\oplus L_{k-2}^*\wedge L_2^*\wedge L_1^*\oplus L_{k-3}^*\wedge L_3^*\wedge L_1^*$ which are:

- component of $d\omega_1^b$:

$$\begin{aligned}
&\sum_{i=1}^{n-2}\left(d_is_i^1\,\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_{\alpha_{i+2}}\wedge\gamma_\rho+s_i^2d_i\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_\rho\right)\\
&\quad+d_{n-1}s_{n-1}\,\gamma_{\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n}\wedge\gamma_\rho-d_{n-1}\nu_1\,\gamma_{\alpha_{n-1}+2\alpha_n}\wedge\gamma_{\alpha_1}\wedge\gamma_{\theta'} \qquad (25)\\
&\quad-\sum_{i=1}^{n-2}\left(d_i\nu_1\,\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_1}\wedge\gamma_{\theta'}+d_i\nu_2\,\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_4}\wedge\gamma_\theta\right)\\
&\quad-\nu_2d_{n-1}\gamma_{\alpha_{n-1}+2\alpha_n}\wedge\gamma_{\alpha_4}\wedge\gamma_\theta+\sum_{i=1}^{n-2}d_is_i^1\,\gamma_{\alpha_i+\alpha_{i+1}}\wedge\gamma_{\alpha_{i+2}}\wedge\gamma_{\rho'}\\
&\quad+\sum_{i=1}^{n-2}d_is_i^2\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\rho'}-\sum_{i=1}^{n-2}d_i\mu_1\,\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_3}\wedge\gamma_{\theta'}\\
&\quad+d_{n-1}s_{n-1}\,\gamma_{\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n}\wedge\gamma_{\rho'}-d_{n-1}\mu_1\,\gamma_{\alpha_{n-1}+2\alpha_n}\wedge\gamma_{\alpha_3}\wedge\gamma_{\theta'};
\end{aligned}$$

notice that, in fact, the elements of (25) belong to $L_{k-2}^*\wedge L_2^*\wedge L_1^*\oplus L_{k-3}^*\wedge L_3^*\wedge L_1^*$.

- component of $d\omega_1^c$:

$$\sum_{i=1}^{n-3} f_i\,\left(t_i^1\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}}+t_i^3\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_{i+3}}\right)\wedge\gamma_\theta$$
$$+f_{n-2}(t_{n-2}^1\gamma_{\alpha_{n-2}}\wedge\gamma_{\alpha_{n-1}+2\alpha_n}+t_{n-2}^2\gamma_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n})\wedge\gamma_\theta$$
$$+\sum_{i=1}^{n-3} g_i\,\left(t_i^1\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}}+t_i^3\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_{i+3}}\right)\wedge\gamma_{\theta'}$$
$$+g_{n-2}(t_{n-2}^1\gamma_{\alpha_{n-2}}\wedge\gamma_{\alpha_{n-1}+2\alpha_n}+t_{n-2}^2\gamma_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n})\wedge\gamma_{\theta'}. \qquad (26)$$

In this case, the elements of (26) are in $L_{k-3}^*\wedge L_3^*\wedge L_1^*$.

The component of $-d\omega_1$ in $L_{k-1}^*\wedge L_1^*\wedge L_1^*\,\oplus\,L_{k-2}^*\wedge L_2^*\wedge L_1^*\oplus L_{k-3}^*\wedge L_3^*\wedge L_1^*$ is obtained from Eqns. (24), (25) and (26) and, at the same time, it coincides with the 3-form in (22).

The part in $L_{k-1}^*\wedge L_1^*\wedge L_1^*$ of (22) and that of $-d\omega_1$ are equal, that is,

$$\sum_{i=1}^{n} a_1\,r_i\,\gamma_{\alpha_2}\wedge\gamma_\delta\wedge\gamma_{\alpha_i}=\sum_{i=1}^{n-1} c_i\,\xi_i\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}}\wedge\gamma_\delta.$$

This imply $r_i=0$ for all $4\le i\le n$. Putting this in (22) and looking at the part in $L_{k-3}^*\wedge L_3^*\wedge L_1^*$ of $-d\omega_1$ we obtain

$$\sum_{i=1}^{3}\eta_i\,(a_4\,\gamma_{\alpha_2+\alpha_3+\alpha_4}\wedge\gamma_\theta+a_5\,\gamma_{\alpha_1+\alpha_2+\alpha_3}\wedge\gamma_{\theta'})\wedge\gamma_{\alpha_i}=$$
$$\sum_{i=1}^{n-3} f_i\,\left(t_i^1\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}}+t_i^3\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_{i+3}}\right)\wedge\gamma_\theta$$
$$+f_{n-2}(t_{n-2}^1\gamma_{\alpha_{n-2}}\wedge\gamma_{\alpha_{n-1}+2\alpha_n}+t_{n-2}^2\gamma_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n})\wedge\gamma_\theta$$
$$+\sum_{i=1}^{n-3} g_i\,\left(t_i^1\,\gamma_{\alpha_i}\wedge\gamma_{\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+3}}+t_i^3\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_{i+3}}\right)\wedge\gamma_{\theta'}$$
$$+g_{n-2}(t_{n-2}^1\gamma_{\alpha_{n-2}}\wedge\gamma_{\alpha_{n-1}+2\alpha_n}+t_{n-2}^2\gamma_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}\wedge\gamma_{\alpha_n})\wedge\gamma_{\theta'} \qquad (27)$$
$$-\sum_{i=1}^{n-2}\left(d_i\nu_1\,\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_1}\wedge\gamma_{\theta'}+d_i\nu_2\,\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_4}\wedge\gamma_\theta\right)$$
$$-d_{n-1}\nu_1\,\gamma_{\alpha_{n-1}+2\alpha_n}\wedge\gamma_{\alpha_1}\wedge\gamma_{\theta'}-\nu_2 d_{n-1}\gamma_{\alpha_{n-1}+2\alpha_n}\wedge\gamma_{\alpha_4}\wedge\gamma_\theta$$
$$+\sum_{i=1}^{n-2} d_i\mu_1\,\gamma_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}\wedge\gamma_{\alpha_3}\wedge\gamma_{\theta'}-d_{n-1}\mu_1\,\gamma_{\alpha_{n-1}+2\alpha_n}\wedge\gamma_{\alpha_3}\wedge\gamma_{\theta'}.$$

Being careful and comparing term by term we deduce that $r_2=r_3=0$ and $f_i=d_i=0$ for all $i\ge 2$. Comparing one more time we reach $r_1=0$ and therefore $\sigma=0$. As before, this implies $E_\infty^{0,2}(\mathfrak{n})=0$.

If $\mathfrak{g}=\mathfrak{so}(7,\mathbb{C})$ or $\mathfrak{g}=\mathfrak{so}(9,\mathbb{C})$ then the intermediate cohomology groups $E_\infty^{0,2}(\mathfrak{n})$ were proved to be zero throughout a computational program. ■

Proof. Part (4) of Lemma 4.3. The family of simple Lie algebras $\mathfrak{so}(2n,\mathbb{C})$ is defined for $n \geq 4$.

Let $\mathfrak{h}$ be the Cartan subalgebra for which corresponds the following set of positive roots $\triangle^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\}$. Hence $\dim \mathfrak{n} = n(n-1)$. The set of simple roots is $\triangle_0 = \{\alpha_i := e_i - e_{i+1} : i = 1, \dots, n-1, \alpha_n := e_{n-1} + e_n\}$ and the maximal root $\alpha_{\max}$ is $e_1 + e_2$ and can be obtained as $\alpha_{\max} = \alpha_1 + 2\sum_{i=2}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n$. Then $\mathfrak{n}$ is $2n-3$-step nilpotent. Define $k = 2n-3$. As in the previous case $\dim L_{k-1} = 1$ and the root of level $k-1$ is $\delta = \alpha_1 + \alpha_2 + \sum_{i=2}^{n-2} 2\alpha_i + \alpha_{n-1} + \alpha_n$.

If $n \geq 6$, then the proof of the previous case applies. Actually, the root system corresponding to $\mathfrak{so}(2n,\mathbb{C})$ has two different roots of level $k-2$ which are $\rho = \alpha_1 + \alpha_2 + \alpha_3 + 2\sum_{i=4}^{n} \alpha_i + \alpha_{n-1} + \alpha_n$ and $\rho' = \alpha_2 + \sum_{i=3}^{n} 2\alpha_i + \alpha_{n-1} + \alpha_n$.

There are two roots of level three if $n \geq 6$ and in there is only one in other case; this is why we add the hypothesis $n \geq 6$ to repeat the proof made for the family $\mathfrak{so}(2n+1,\mathbb{C})$. In this case the roots of level $k-3$ are $\theta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\sum_{i=5}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n$ and $\theta' = \alpha_2 + \alpha_3 + 2\sum_{i=4}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n$.

For these roots the same relations as for the last case hold:

$$\alpha_{\max} = \delta + \alpha_2, \qquad \alpha_{\max} = \rho + (\alpha_2 + \alpha_3) \ = \ \rho' + (\alpha_1 + \alpha_2)$$

$$\alpha_{\max} = \theta + (\alpha_2 + \alpha_3 + \alpha_4) \ = \ \theta' + (\alpha_1 + \alpha_2 + \alpha_3).$$

Then Eq. (20) is valid, and so are Eqns. (22) and (23). The roots of level two, three and four are

$\ell = 2:$ $\alpha_i + \alpha_{i+1}$, $i = 1, \dots, n-2$ and $\alpha_{n-2} + \alpha_n$.
$\ell = 3:$ $\alpha_i + \alpha_{i+1} + \alpha_{i+2}$, $i = 1, \dots, n-2$ and $\alpha_{n-3} + \alpha_{n-2} + \alpha_n$.
$\ell = 4:$ $\alpha_i + \alpha_{i+1} + \alpha_{i+2} + \alpha_{i+3}$, $i = 1, \dots, n-3$ and $\alpha_{n-4} + \alpha_{n-3} + \alpha_{n-2} + \alpha_n$.

Notice that the differentials of the elements of the basis of L_2^*, L_3^* and L_4^* do not coincide with those in the previous case. Nevertheless, they have the same behavior. Proceeding in an analogous manner we obtain also in this case that $E_\infty^{0,2}(\mathfrak{n}) = 0$.

For $n = 4$ and $n = 5$ we used a computational program to verify that $E_\infty^{0,2}(\mathfrak{n}) = 0$ in both cases. ■

From the proof of the classification Theorem we can state the following:

Corollary 4.5. *For a nilpotent Lie algebra $\mathfrak{n}$ as in Theorem 4.4 of dimension greater or equal than 2 the followings conditions are equivalent:*

1. *any even dimensional trivial extension $\mathbb{R}^s \oplus \mathfrak{n}$ is symplectic,*
2. $E_\infty^{0,2}(\mathfrak{n}) \neq 0$,
3. $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ *or* $\mathfrak{g} = \mathfrak{so}(5,\mathbb{C})$.

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Viviana del Barco
ECEN-FCEIA
Universidad Nacional de Rosario
Pellegrini 250
S2000BTP Rosario
Argentina
delbarc@fceia.unr.edu.ar