SCHRÖDINGER TYPE SINGULAR INTEGRALS: WEIGHTED ESTIMATES FOR $p = 1$

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Abstract. A critical radius function ρ assigns to each $x \in \mathbb{R}^d$ a positive number in a way that its variation at different points is somehow controlled by a power of the distance between them. This kind of function appears naturally in the harmonic analysis related to a Schrödinger operator $-\Delta + V$ with V a non-negative potential satisfying some specific reverse Hölder condition. For a family of singular integrals associated to such critical radius function, we prove boundedness results in the extreme case $p = 1$. On one side we obtain weighted weak $(1, 1)$ results for a class of weights larger than Muckenhoupt class A_1. On the other side, for the same weights, we prove continuity from appropriate weighted Hardy spaces into weighted L^1. To achieve the latter result we define weighted Hardy spaces by means of a ρ-localized maximal heat operator. We obtain a suitable atomic decomposition and a characterization via ρ-localized Riesz Transforms for these spaces. For the case of ρ derived from a Schrödinger operator, we obtain new estimates for of many of the operators appearing in [16].

1. Introduction

Let us consider the Schrödinger operator $\mathcal{L} = -\Delta + V$ in $\mathbb{R}^d$, $d \geq 3$, where V is a non-negative locally integrable function satisfying a reverse Hölder condition for $q > d/2$,

$$\left(\frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy, \tag{1}$$

for every ball $B \subset \mathbb{R}^d$. It is well known that if in addition V is not identically zero, the function

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V \leq 1 \right\}, \quad x \in \mathbb{R}^d, \tag{2}$$

is non-zero and finite and for some constants $c_\rho, k_0 \geq 1$, the following property holds

$$c_\rho^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq c_\rho \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{\frac{k_0}{k_0+1}} \tag{3}$$

for all $x, y \in \mathbb{R}^d$, (see Lemma 1.4 in [16]).

Since the pioneering work of Shen (see [16]) many authors deal with the harmonic analysis associated to Schrödinger operators under the above assumptions (see for example [3], [8], [9], [10], [12], [17]).

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From now on, we shall call a critical radius function any continuous function $\rho : \mathbb{R}^d \mapsto (0, \infty)$ satisfying (3). Let us point out that even the left inequality may be deduced from the other, we choose to write it in this way for future references.

Associated to such function ρ, we introduce in Section 2 two classes of operators, $\mathcal{S}_0^\rho$ and $\mathcal{S}^\rho$, different singular integrals appearing in the Schrödinger setting (see [16]).

The first class $\mathcal{S}_0^\rho$ consists of operators associated to kernels satisfying some special size and smoothness point-wise conditions. In particular, they are classical Calderón-Zygmund kernels with an extra decay at infinity related to the function ρ.

Examples of this kind of operators are Schrödinger Riesz transforms $\nabla(-\Delta + V)^{-1/2}$, provided V satisfies a reverse-Hölder condition (1) for some $q \geq d$. However, when (1) is satisfied only for q such that $d/2 \leq q < d$, those Riesz transforms fail to be Calderón-Zygmund. In fact, they are bounded on L^p only for p in a finite interval. Anyway, their kernels still satisfy some integral size and smoothness conditions.

To cover such case and that of other operators associated to the Schrödinger semi-group, we also deal with the family $\mathcal{S}^\rho$ (see Section 2 for the definition).

Let us mention that weighted L^p estimates for both kind of operators and $p > 1$ can be derived from the results obtained in [2] (see Proposition 7 below)

In this work we concentrate our attention on weighted inequalities for both classes of operators at the extreme case $p = 1$. In this sense, in Section 2, we analyze weak type $(1, 1)$ weighted inequalities for classes of weights larger than Muckenhoupt class A_1. Then we turn into the study of boundedness on appropriate weighted Hardy spaces. To this end we introduce in Section 3 weighted Hardy spaces associated to ρ defined in terms of ρ-localized maximal operator associated to the usual heat kernel. For the Schrödinger case we identify such spaces with those given in terms of the maximal operator of the semi-group as defined in [9] in the unweighted case and in [17] for A_1 Muckenhoupt weights.

As in those papers, we obtain in Section 4 an atomic decomposition of the spaces and we also give a characterization in terms of ρ-localized classical Riesz transforms. With these tools at hand we obtain in Section 5 boundedness results for the ρ-singular integrals over these weighted Hardy spaces.

Finally, we apply all our results to the Schrödinger setting, obtaining continuity properties for $p = 1$ of many of the operators appearing in [16]. We also derived the corresponding characterization of the Schrödinger-Hardy spaces in terms of Schrödinger Riesz transforms, extending the results in [9] and [17] to a wider class of weights.

Let us introduce the classes of weights we are going to deal with. We recall that a weight is in the A_p class of Muckenhoupt, $1 < p < \infty$, if the inequality

$$\left(\int_B w\right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \leq C|B| \tag{4}$$

holds for any ball $B \subset \mathbb{R}^d$.

For case $p = 1$, a weight w is in the Muckenhoupt A_1 class, if the inequality

$$\int_B w \leq C|B| \inf_{x \in B} w \tag{5}$$

holds for any ball $B \subset \mathbb{R}^d$.

Given a critical radius function ρ we shall consider two families of weights. We introduce the $A_p^{\rho,\text{loc}}$ class of weights as those w satisfying (5) or (4) for ρ-local balls, that is, $B = B(x,r)$ with $r \le \rho(x)$. Also, given $\theta > 0$ and $1 < p < \infty$ we will say that $w \in A_p^{\rho,\theta}$ if the inequality

$$\left(\int_B w\right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \le C|B| \left(1 + \frac{r}{\rho(x)}\right)^{\theta} \tag{6}$$

holds for all balls $B = B(x,r)$.

For $p = 1$ we define $A_1^{\rho,\theta}$ as the set of weights w such that

$$\int_B w \le C|B| \left(1 + \frac{r}{\rho(x)}\right)^{\theta} \inf_{x\in B} w. \tag{7}$$

holds for all balls $B = B(x,r)$.

Remark 1. It is not difficult to see that in (7) it is equivalent to consider cubes instead of balls, due to (3).

We denote $A_p^{\rho} = \cup_{\theta>0} A_p^{\rho,\theta}$, for $1 \le p < \infty$. It is clear that $A_p \subset A_p^{\rho} \subset A_p^{\rho,\text{loc}}$ and it is easy to check that for $\rho \equiv 1$, the inclusions are strict. In fact the weight $w_\gamma(x) = (1+|x|)^\gamma$ belong to A_p^{ρ} for any $\gamma > d(p-1)$ but they do not lie in A_p. On the other hand, $w(x) = e^{x_k}$, for any $k = 1, \dots, d$, are weights in $A_p^{\rho,\text{loc}}$ but not in A_p^{ρ}. Also, observe that in the limiting case $V \equiv 0$, the above classes coincide.

Classes A_p^{ρ} are intimately connected with the family of maximal operators M_ρ^θ defined by

$$M_\rho^\theta f(x) = \sup_{r>0} \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|, \tag{8}$$

for any $\theta > 0$. In fact, for $1 < p < \infty$ they are bounded on $L^p(w)$, provided $w \in A_p^{\rho}$ (see Proposition 3 in [1]).

There are some useful facts about weights in these classes that we shall use in the sequel.

Proposition 1. *(see* [3]*) If $w \in A_1^{\rho,\text{loc}}$ then for any $\beta > 1$, $w \in A_1^{\beta\rho,\text{loc}}$.*

With this in mind, it is easy to get the following property for these weights.

Proposition 2. *Let $w \in A_1^{\rho,loc}$ and $B_0 = B(x_0, \rho(x_0))$ for a fixed $x_0 \in \mathbb{R}^d$. Then there exists a weight $v \in A_1$ such that $v|_{B_0} = w|_{B_0}$ and moreover the constant of v in A_1 depends only on the constant of w in $A_1^{\rho,loc}$.*

Proof. We use the fact that if a weight satisfies inequality (5) for all balls contained in B_0, it is possible to find such extension of w (see [3]). To check that $w|_{B_0}$ satisfies such property, let $B = B(x,r) \subset B_0$. By (3) it holds that $c_1\rho(x_0) \le \rho(x) \le c_2\rho(x_0)$ and therefore if $r \le \rho(x_0)$ we have $r \le \rho(x)/c_1$ and by the above property $w \in A_1^{\rho/c_1,\text{loc}}$, then inequality (5) is true for the ball B. $\square$

Remark 2. Since by Proposition 1, $w \in A_1^{\rho,\text{loc}}$ implies $w \in A_1^{\beta\rho,\text{loc}}$, we may also find an A_1-extension of $w|_{\beta B_0}$ for any fixed $\beta > 1$ according to Proposition 1.

The next proposition is a consequence of Lemma 5 in [3] which resembles a reverse Hölder property.

Proposition 3. *If $w \in A_p^\rho$, $1 \le p < \infty$, then there exist positive constants δ, η and C such that*

$$\left(\frac{1}{|B|}\int_B w^{1+\delta}\right)^{\frac{1}{1+\delta}} \le C\left(\frac{1}{|B|}\int_B w\right)\left(1+\frac{r}{\rho(x)}\right)^{\eta},$$

for every ball $B = B(x,r)$. Moreover, if $w \in A_1^{\rho,\theta}$ then η depends only on θ and the constants appearing in (3).

Also, as in the Muckenhoupt case we have the following (see [4]).

Proposition 4. *If $w \in A_p^\rho$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}^\rho$.*

Another property of weights in A_1^ρ that resembles A_1 is presented in the following result (see [4]).

Proposition 5. *If $w \in A_1^\rho$, then there exists $\nu > 1$ such that $w^\nu \in A_1^\rho$.*

Finally, let us recall a very useful tool to work with critical radius functions. For a proof we refer to [9]. Even though the proof is given for ρ in the context of Schrödinger operators, the only property used is inequality (3).

Proposition 6. *There exists a sequence of points x_j, $j \ge 1$, in $\mathbb{R}^d$, so that the family $Q_j = B(x_j, \rho(x_j))$, $j \ge 1$, satisfies*

i) $\cup_j Q_j = \mathbb{R}^d$.

ii) For every $\kappa \ge 1$ there exist constants C and N_1 such that, $\sum_j \chi_{\kappa Q_j} \le C\kappa^{N_1}$.

2. Weak type $(1,1)$ of singular integrals of ρ-type

In [2] some special kind of singular integrals have been considered in a way that when ρ comes from a Schrödinger operator, several Riesz transforms associated to $\mathcal{L}$ fall under that scope. Moreover weighted strong inequalities can be derived from the results therein.

In this section we introduce appropriate classes of singular integrals related to critical radius function ρ and prove weighted weak type $(1,1)$ inequalities for a wide class of weights.

First we deal with the class $\mathcal{S}_0^\rho$ consisting of singular integrals T given by a kernel $K(x,y)$ satisfying

(a) T is bounded on L^p for any $p > p_0$, for some $p_0 \ge 1$.

(b) For each $N > 0$, there exists $C_N > 0$ such that

$$|K(x,y)| \le C_N \frac{1}{|x-y|^d}\left(1+\frac{|x-y|}{\rho(y)}\right)^{-N}.$$

(c) There exists $0 < \lambda < 1$ such that for each $M > 0$

$$|K(x,y)-K(x,y_0)| \le C_M \frac{|y-y_0|^\lambda}{|x-y_0|^{d+\lambda}}\left(1+\frac{|x-y_0|}{\rho(y_0)}\right)^{-M},$$

for every $x, y \in \mathbb{R}^d$, provided $|y-y_0| \le \frac{|x-y_0|}{2}$.

Remark 3. Let us observe that from inequality (3) it is easy to get that

$$1+\frac{|x-y|}{\rho(x)} \le \left(1+\frac{|x-y|}{\rho(y)}\right)^{k_0+1}, \tag{9}$$

for any $x, y \in \mathbb{R}^d$. In particular, condition (b) is equivalent to ask the same inequality but with $\rho(y)$ replaced by $\rho(x)$.

Operators in the class $\mathcal{S}_0^\rho$ appear in the Schrödinger context when the potential V satisfies a reverse Hölder inequality of order $q \geq d$. Nevertheless, when $d/2 < q < d$, even Riesz transforms are not in $\mathcal{S}_0^\rho$. For example, they are bounded on L^p only for a limited rage of p.

To deal with those cases we introduce the class $\mathcal{S}^\rho$ as the operators T associated to a kernel K satisfying for some $s > 1$, the following properties.

(a') T is bounded on L^p, $1 < p < s$.

(b') For each $N > 0$ there exists $C_N > 0$ such that

$$\left(\int_{R<|x-x_0|\leq 2R} |K(x,y)|^s dx\right)^{1/s} \leq C_N R^{-d/s'}\left(1 + \frac{R}{\rho(x_0)}\right)^{-N},$$

for every $y \in B(x_0, r)$ and $R \geq 2r$.

(c') For every $\theta > 0$, there exists C_θ such that

$$\sum_{k\geq 1} (2^k r)^{d/s'}\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{\theta}\left(\int_{2^{k+1}B\setminus 2^k B} |K(x,y) - K(x,x_0)|^s dx\right)^{1/s} \leq C_\theta,$$

for every ball $B = B(x_0, r)$ with $r \leq \rho(x_0)$ and every $y \in B$.

Using results from [1] and [2] we have the following proposition concerning strong weighted inequalities for operators in $\mathcal{S}_0^\rho$ and $\mathcal{S}^\rho$.

Proposition 7. *(i) If $T \in \mathcal{S}_0^\rho$ then T is bounded on $L^p(w)$ with $1 < p < \infty$ and any $w \in A_p^\rho$.*

(ii) If $T \in \mathcal{S}^\rho$, then T is bounded on $L^p(w)$ with $1 < p < s$ and any weight w, $w^{-\frac{1}{p-1}} \in A^\rho_{p'/s'}$.

Proof. We start with the proof of (ii) since it is more delicate. Let us fix p, with $1 < p < s$ and $w^{-\frac{1}{p-1}} \in A^\rho_{p'/s'}$. From Proposition 4 there exists s_1 with $p < s_1 < s$ such that $w^{-\frac{1}{p-1}} \in A^\rho_{p'/s_1'}$. Besides, assumptions (b)' and (c)' also hold for s_1 instead of s. Therefore, the adjoint operator T^* satisfies the hypothesis of Theorem 5 of [2] with $s = s_1$. Consequently,

$$\int_{\mathbb{R}^d} |T^* f(x)|^q v(x)dx \leq C \int_{\mathbb{R}^d} |M^\theta_{\rho,s_1'} f(x)|^q v(x)dx, \tag{10}$$

for any $\theta > 0$ and every $v \in A_\infty^{\rho,\text{loc}} = \cup_{q\geq 1} A_q^{\rho,\text{loc}}$.

By Proposition 3 in [1] the operator $M^\theta_{\rho,s_1'}$ is bounded in $L^q(v)$ for $v \in A^\rho_{q/s_1'}$ with $q > s_1'$. Inequality (10) and a duality argument imply that T is bounded on $L^q(v)$, $1 < q < s_1$, for $v^{-\frac{1}{q-1}} \in A^\rho_{q'/s_1'}$, particularly on $L^p(w)$.

The proof of (i) is easier using Theorem 6 in [2] instead of Theorem 5.

□

In order to get weighted weak type $(1,1)$ inequalities for operators in $\mathcal{S}_0^\rho$ and $\mathcal{S}^\rho$ we use an adapted version of Calderón-Zygmund decomposition lemma.

Lemma 1. *(See [4]) For any $\theta \geq 0$ there exists an at most countable family of cubes $\{P_j\}$, $P_j = P(x_j, r_j)$ such that for all $\lambda > 0$*

$$\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta}\lambda \;\leq\; \frac{1}{|P_j|}\int_{P_j}|f| \;\leq\; C\,\lambda\left(1+\frac{r_j}{\rho(x_j)}\right)^{k_0\theta}, \tag{11}$$

where k_0 is the constant appearing in (3). Moreover,

$$|f(x)| \leq \lambda, \quad a.e. \quad x \notin \cup_j P_j. \tag{12}$$

Now we are in position to state and prove the main result of this section.

Theorem 1. *(i) If $T \in \mathcal{S}_0^{\rho}$ and $w \in A_1^{\rho}$, then T is of weak type $(1,1)$ with respect to w.*

(ii) If $T \in \mathcal{S}^{\rho}$ and $w^{s'} \in A_1^{\rho}$, then T is of weak type $(1,1)$ with respect to w.

Proof. We use an argument close to the proof of Theorem 3 in [4]. We start showing statement (ii). If w satisfies that $w^{s'} \in A_1^{\rho}$, then $w^{s'} \in A_1^{\rho,\beta}$, for some $\beta \geq 0$. In this case it is also true that $w \in A_1^{\rho,\theta}$ with $\theta = \beta/s'$.

Given $f \in L^1(w)$, let us consider $P_j = P(x_j, r_j)$ the Calderón-Zygmund decomposition given in Lemma 1 associated to θ. We define the set of indexes

$$J_1 = \{j : \; r_j \leq \rho(x_j)\}, \quad J_2 = \{j : \; r_j > \rho(x_j)\},$$

and

$$\Omega_1 = \cup_{j\in J_1} P_j, \quad \Omega_2 = \cup_{j \in J_2} P_j.$$

Now we split $f = g + h + h'$, as

$$g(x) = \begin{cases} \frac{1}{|P_j|}\int_{P_j} f, & \text{if } x \in P_j,\; j \in J_1, \\ 0, & \text{if } x \in P_j,\; j \in J_2, \\ f(x), & \text{if } x \notin \Omega, \end{cases}$$

with $\Omega = \Omega_1 \cup \Omega_2$,

$$h(x) = \begin{cases} f(x) - \frac{1}{|P_j|}\int_{P_j} f, & \text{if } x \in P_j,\; j \in J_1, \\ 0, & \text{otherwise}, \end{cases}$$

and therefore $h'(x) = \chi_{\Omega_2} f$.

Let $\tilde{P}_j = P_j(x_j, 2\sqrt{n}\, r_j)$ and $\tilde{\Omega} = \cup_j \tilde{P}_j$. Now,

$$w(\{x : \; |Tf|(x) > \lambda\} \;\leq\; w(\tilde{\Omega}) + w(\{x \notin \tilde{\Omega} : \; |Tf|(x) > \lambda\}. \tag{13}$$

The first term of the last expression, can be controlled using (11) and the fact that $w \in A_1^{\rho,\theta}$ (see Remark 1), as

$$\begin{aligned} w(\tilde{\Omega}) \;&\leq\; \sum_j w(\tilde{P}_j) \;\lesssim\; \frac{1}{\lambda}\sum_j \frac{w(\tilde{P}_j)}{|\tilde{P}_j|}\left(1+\frac{r_j}{\rho(x_j)}\right)^{-\theta}\int_{P_j}|f| \\ &\lesssim\; \frac{1}{\lambda}\sum_j \inf_{P_j} w \int_{P_j}|f| \;\lesssim\; \frac{1}{\lambda}\int_{\mathbb{R}^d}|f|\,w. \end{aligned} \tag{14}$$

For the second term of (13), we estimate $I = w(\{x : \; |Tg(x)| > \lambda\})$, $II = w(\{x \notin \tilde{\Omega} : \; |Th(x)| > \lambda\})$ and $III = w(\{x \notin \tilde{\Omega} : \; |Th'(x)| > \lambda\})$.

To deal with I first notice that, from Lemma 1 it follows that $|g| \lesssim \lambda$. Now, from $w^{s'} \in A_1^{\rho}$ we get $w^{s'\nu} \in A_1^{\rho}$ for some $\nu > 1$ (see Proposition 5) and taking p

such that $p(1-s')+s'=\frac{1}{\nu}$ it is easy to check that $w^{-\frac{1}{p-1}}\in A^{\rho}_{p'/s'}$ with $1<p<s$. Then from Proposition 7 (item (ii)) it follows that T is bounded on $L^p(w)$.

Therefore,

$$w(\{x:\ |Tg(x)|>\lambda\})\ \lesssim\ \frac{1}{\lambda^p}\int_{\mathbb{R}^d}|g|^p\,w\ \lesssim\frac{1}{\lambda}\left(\sum_{j\in J_1}\frac{w(P_j)}{|P_j|}\int_{P_j}|f|+\int_{\Omega^c}|f|\,w\right). \tag{15}$$

Since $w\in A_1^{\rho}$ and for $j\in J_1$, $r_j\le\rho(x_j)$ we have $\frac{w(P_j)}{P_j}\lesssim\inf_{P_j}w$, and hence the last expression in (15) can be easily bounded by $\frac{1}{\lambda}\int_{\mathbb{R}^d}|f|\,w$.

In order to deal with II we apply Tchebysheff's inequality to get

$$\begin{aligned} II &\le\frac{1}{\lambda}\int_{(\tilde\Omega)^c}|Th(x)|w(x)\ dx\le\frac{1}{\lambda}\int_{(\tilde\Omega)^c}\left|\sum_{j\in J_1}\int_{P_j}K(x,y)h(y)\ dy\right|w(x)\ dx\\ &\le\frac{1}{\lambda}\int_{(\tilde\Omega)^c}\left(\sum_{j\in J_1}\int_{P_j}|K(x,y)-K(x,x_j)||h(y)|dy\right)w(x)\ dx\\ &\le\frac{1}{\lambda}\sum_{j\in J_1}\int_{P_j}|h(y)|\left(\int_{(\tilde P_j)^c}|K(x,y)-K(x,x_j)|w(x)\ dx\right)dy, \end{aligned}\tag{16}$$

where we used that h has zero average on P_j, $j\in J_1$.

The inner integrals may be estimated splitting into square annuli and applying Hölder's inequality with exponent s. In this way, setting $P_j^k=P(x_j,2^k\sqrt{n}\,r_j)$ we have

$$\begin{aligned} &\int_{(\tilde P_j)^c}|K(x,y)-K(x,x_j)|w(x)\ dx\\ &\quad\le\sum_{k\ge1}\left(\int_{P_j^{k+1}\setminus P_j^k}|K(x,y)-K(x,x_j)|^s dx\right)^{1/s}\left(\int_{P_j^{k+1}}w(x)^{s'}dx\right)^{1/s'}\\ &\quad\lesssim\inf_{P_j}w\ \sum_{k\ge1}(2^kr_j)^{d/s'}\left(1+\frac{2^kr_j}{\rho(x_j)}\right)^{\theta}\left(\int_{P_j^{k+1}\setminus P_j^k}|K(x,y)-K(x,x_j)|^s dx\right)^{1/s}, \end{aligned}\tag{17}$$

since $w^{s'}\in A_1^{\rho,\beta}$ and $\theta=\beta/s'$. In the last expression we may apply Hörmander's type condition (c'), since $P_j^{k+1}\setminus P_j^k\subset B_j^{k+2}\setminus B_j^k$, where $B_j^k=B(x_j,2^k\sqrt{n}\,r_j)$, to obtain

$$II\lesssim\frac{1}{\lambda}\sum_{j\in J_1}\inf_{P_j}w\int_{P_j}|h|\ \lesssim\ \frac{1}{\lambda}\sum_{j\in J_1}\int_{P_j}|f|w\ \lesssim\ \frac{1}{\lambda}\int_{\mathbb{R}^d}|f|w.$$

Finally, we take care of III which involves $h'=f\chi_{\Omega_2}$. Proceeding as for II,

$$III\le\frac{1}{\lambda}\sum_{j\in J_2}\int_{P_j}|h'(y)|\left(\int_{(\tilde P_j)^c}|K(x,y)|w(x)\ dx\right)dy. \tag{18}$$

Now, for each $j\in J_2$ we bound the inner integral splitting into square annuli and applying Hölder's inequality as in (17). Then from condition (b') and the fact that $P_j^{k+1} \setminus P_j^k \subset B_j^{k+2} \setminus B_j^k$ we obtain

(19)
$$\begin{aligned}
\int_{(\tilde{P}_j)^c} |K(x,y)| w(x)\, dx &\le \sum_{k\ge 1} \left(\int_{P_j^{k+1}\setminus P_j^k} |K(x,y)|^s dx \right)^{1/s} \left(\int_{P_j^{k+1}} w(x)^{s'} dx \right)^{1/s'} \\
&\lesssim \sum_{k\ge 1} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{-N} \left(\frac{1}{|P_j^{k+1}|} \int_{P_j^{k+1}} w(x)^{s'} dx \right)^{1/s'} \\
&\lesssim \inf_{P_j} w \sum_{k\ge 1} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{-N+\frac{\beta}{s'}} \\
&\lesssim \inf_{P_j} w,
\end{aligned}$$

with a choice of N large enough.

Therefore, the right hand side of (18) can be bounded by a constant times

$$\frac{1}{\lambda} \sum_{j\in J_2} \inf_{P_j} w \int_{P_j} |h'| \;\lesssim\; \frac{1}{\lambda} \sum_{j\in J_2} \int_{P_j} |f| w \;\lesssim\; \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| w.$$

Now, let us see the statement (i). Let $w \in A_1^{\rho}$, more precisely $w \in A_1^{\rho,\theta}$, for some $\theta \ge 0$. Proceeding as for (ii), we obtain from (13) and (14) the inequality

$$\begin{aligned}
w(\{x : \; |Tf|(x) > \lambda\} \;&\le\; w(\tilde{\Omega}) \;+\; w(\{x \notin \tilde{\Omega} : \; |Tf|(x) > \lambda\} \\
&\lesssim\; \frac{1}{\lambda} \int_{\mathbb{R}^d} |f|\, w \;+\; w(\{x \notin \tilde{\Omega} : \; |Tf|(x) > \lambda\}.
\end{aligned}$$

As before it is enough to estimate $IV = w(\{x : \; |Tg(x)| > \lambda\})$, $V = w(\{x \notin \tilde{\Omega} : \; |Th(x)| > \lambda\})$ and $VI = w(\{x \notin \tilde{\Omega} : \; |Th'(x)| > \lambda\})$.

To deal with IV we observe that (15) holds since $A_1^{\rho} \subset A_p^{\rho}$.

To estimate V as for (16) we get

$$V \le \frac{1}{\lambda} \sum_{j\in J_1} \int_{P_j} |h(y)| \left(\sum_{k\ge 1} \int_{P_j^{k+1}\setminus P_j^k} |K(x,y) - K(x,x_j)| w(x) dx \right) dy. \tag{20}$$

Since for each $k \ge 1$, $x \in P_j^{k+1} \setminus P_j^k$ and $y \in P_j$ it follows $|y - x_j| \le \frac{|x-x_j|}{2}$ and then, according to condition (c) we obtain for some $0 < \delta < 1$ and $M > 0$,

$$\begin{aligned}
|K(x,y) - K(x,x_j)| &\le C_M \frac{|y-x_j|^{\delta}}{|x-x_j|^{d+\delta}} \left(1 + \frac{|x-x_j|}{\rho(x_0)}\right)^{-M} \\
&\le C_M \frac{2^{-k\delta}}{|P_j^{k+1}|} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{-M}.
\end{aligned} \tag{21}$$

From (21), and the fact that $w \in A_1^{\rho,\theta}$,

$$\begin{aligned}\sum_{k\geq 1}\int_{P_j^{k+1}\setminus P_j^k}&|K(x,y)-K(x,x_j)|w(x)dx\\ &\lesssim \sum_{k\geq 1} 2^{-k\delta}\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{-M}\frac{1}{|P^{k+1}|}\int_{P^{k+1}} w(x)dx\\ &\lesssim \sum_{k\geq 1} 2^{-k\delta}\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{-M+\theta}\inf_{P_j^k} w\\ &\lesssim \inf_{P_j} w\sum_{k\geq 1} 2^{-k\delta}\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{-M+\theta}.\end{aligned}$$

If we choose $M > \theta$ the last sum is finite.

Therefore, we can estimate the inner integral of (20) to get

$$V \lesssim \frac{1}{\lambda}\sum_{j\in J_1}\inf_{P_j} w\int_{P_j}|h(y)|dy \lesssim \frac{1}{\lambda}\sum_{j\in J_1}\int_{P_j}|f|w \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^d}|f|w.$$

Finally, VI is estimated in the same way as III, observing that for each $y \in P_j$ we have $\rho(y) \simeq \rho(x_j)$. Hence, from condition (b) we obtain for $j \in J_2$

$$\begin{aligned}\int_{(\tilde{P}_j)^c}|K(x,y)|w(x)\,dx &\leq C_N\sum_{k\geq 1}\left(\int_{P_j^{k+1}\setminus P_j^k}\frac{1}{|x-y|^d}\left(1+\frac{|x-y|}{\rho(y)}\right)^{-N} w(x)\,dx\right)\\ &\lesssim \sum_{k\geq 1}\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{-N}\left(\frac{1}{|P_j^{k+1}|}\int_{P_j^{k+1}} w(x)\,dx\right).\\ &\lesssim \inf_{P_j} w\sum_{k\geq 1}\left(1+\frac{2^k r_j}{\rho(x_j)}\right)^{-N+\theta}\\ &\lesssim \inf_{P_j} w,\end{aligned}$$

choosing $N > \theta$.

□

3. Weighted Hardy spaces associated to ρ

Given a function ρ we introduce the following two maximal operators

$$W_\rho^* f(x) = \sup_{0<t<\rho^2(x)}|W_t f(x)|$$

and

$$W_\rho^{*,0} f(x) = \sup_{0<t<\rho^2(x)}|W_t^{\mathrm{loc}} f(x)|$$

where W_t is the classical heat kernel

$$W_t(x,y) = \frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x-y|^2}{4t}}$$

and W_t^{loc} is the integral operator associated to the kernel

$$W_t^{\mathrm{loc}}(x,y) = W_t(x,y)\chi_{B(x,\rho(x))}(y).$$

When ρ comes from a Schrödinger operator $\mathcal{L} = -\Delta + V$ with $V \in RH_q$ for $q > d/2$, we shall also consider two other maximal operators, namely

$$T^* f(x) = \sup_{t>0} |T_t f(x)|$$

and

$$T^{*,0} f(x) = \sup_{t>0} |T_t^{\mathrm{loc}} f(x)|$$

where T_t is the semi-group operator with infinitesimal generator $\mathcal{L}$. It is known that T_t is an integral operator with a kernel that, by abuse of notation, we denote $T_t(x,y)$. Moreover, since V is non-negative, we have $0 \leq T_t(x,y) \leq W_t(x,y)$ (see [9]). Analogously, T_t^{loc} is the integral operator against the kernel

$$T_t^{\mathrm{loc}}(x,y) = T_t(x,y)\chi_{B(x,\rho(x))}(y).$$

In what follows our aim is to prove that any of these maximal operators applied to a certain function f are bounded below by $|f|$. To this end we want to check that, for appropriate weights w, they are the maximal operators associated to some approximations of the identity in $L^1(w)$, in the a.e. sense.

First we shall see that W_ρ^* as well as T^* are controlled by any maximal function of the type

$$M_\rho^\theta f(x) = \sup_{r>0} \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|,$$

with $\theta \geq 0$, while the local operators $W_\rho^{*,0}$ and $T^{*,0}$ are bounded by

$$M_\rho^{\mathrm{loc}} f(x) = \sup_{0<r<\rho(x)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|.$$

In fact the following proposition holds

Proposition 8. *Let $f \in L^1_{\mathrm{loc}}$. Then, there exist constants C_θ and C such that*

a) $W_\rho^*(|f|) \leq C_\theta M_\rho^\theta f$ and $T^*(|f|) \leq C_\theta M_\rho^\theta f$, for any $\theta > 0$.
b) $W_\rho^{*,0}(|f|) \leq C M_\rho^{\mathrm{loc}} f$ and $T^{*,0}(|f|) \leq C M_\rho^{\mathrm{loc}} f$.

Proof. For the first inequality in *a*) we write for some $N > d+1$, and $B_k = B(x, 2^k\sqrt{t})$

$$\begin{aligned} W_t(|f|)(x) &= \int_{\mathbb{R}^d} W_t(x,y)|f(y)|\,dy \\ &\leq C\frac{1}{t^{d/2}} \int_{\mathbb{R}^d} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-N} |f(y)|\,dy \\ &\leq C\frac{1}{t^{d/2}} \sum_{k\geq 0} \int_{B_{k+1}\setminus B_k} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-N} |f(y)|\,dy \\ &\qquad + C\frac{1}{t^{d/2}} \int_{B(x,\sqrt{t})} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-N} |f(y)|\,dy. \end{aligned}$$

The first term is bounded by a constant times

$$\begin{aligned}\sum_{k\geq 0}\frac{1}{t^{d/2}}\frac{1}{(1+2^k)^N}\int_{B_{k+1}}|f(y)|\,dy\\ \leq C\sum_{k\geq 0}2^{-k}\,\frac{(1+2^k)^{-N+d+1}}{|B_{k+1}|}\int_{B_{k+1}}|f(y)|\,dy.\end{aligned}$$

Now if $t\leq\rho(x)^2$ we have $2(1+2^k)\geq 1+\frac{2^{k+1}\sqrt{t}}{\rho(x)}$ and hence the last expression is bounded by $M_\rho^{N-d-1}f(x)$. Taking N large enough we can reach any $\theta>0$.

To deal with second term observe that if $t\leq\rho(x)^2$ we have $2^{-\theta}\leq(1+\frac{\sqrt{t}}{\rho(x)})^{-\theta}$ for any $\theta>0$. Therefore

$$\begin{aligned}\frac{1}{t^{d/2}}\int_{B(x,\sqrt{t})}\left(1+\frac{|x-y|}{\sqrt{t}}\right)^{-N}|f(y)|\,dy\\ &\leq\frac{1}{t^{d/2}}\int_{B(x,\sqrt{t})}|f(y)|\,dy\\ &\leq C\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-\theta}\frac{1}{|B(x,\sqrt{t})|}\int_{B(x,\sqrt{t})}|f(y)|\,dy\\ &\leq CM_\rho^\theta f(x).\end{aligned}$$

Next observe that the first inequality in *b*) follows with the same steps but now the sum is extended up to k_0-1 with $k_0=\max\{k:2^k\sqrt{t}<\rho(x)\}$ and also appears the term

$$C\frac{1}{t^{d/2}}\int_{B(x,\rho(x))\setminus B(x,2^{k_0}\sqrt{t})}\left(1+\frac{|x-y|}{\sqrt{t}}\right)^{-N}|f(y)|\,dy$$

Thus, having in mind that $(1+2^k)^{-N+d+1}<1$ and $2^{k_0}\sqrt{t}\simeq\rho(x)$ we obtain

$$W_t^{\mathrm{loc}}|f|(x)\lesssim\sum_{k=0}^{k_0}2^{-k}\fint_{B(x,2^k\sqrt{t})}|f(y)|\,dy\;+\;2^{-k_0}\fint_{B(x,\rho(x))}|f(y)|\,dy,$$

and the inequality follows since the radius of the balls are at most $\rho(x)$. Notice that we do not use $t\leq\rho(x)^2$ here.

As for the second inequality in *b*) the fact that $0\leq T_t(x,y)\leq W_t(x,y)$ and the remark above give

$$T^{*,0}|f|(x)=\sup_{t>0}T_t^{\mathrm{loc}}|f|(x)\leq\sup_{t>0}W_t^{\mathrm{loc}}|f|(x)\leq M_\rho^{\mathrm{loc}}f(x). \tag{22}$$

Finally the remaining inequality follows from (20) in [3] which gives for any $N>0$,

$$\begin{aligned}(T_t-T_t^{\mathrm{loc}})|f|(x)&\leq\frac{1}{\rho(x)^d}\sum_{k\geq 0}2^{-kN}\int_{B(x,2^k\rho(x))}|f|\\ &\leq C\sum_{k\geq 0}2^{-k}\left(\frac{1}{1+2^k}\right)^{N-d-1}\frac{1}{|B(x,2^k\rho(x))|}\int_{B(x,2^k\rho(x))}|f|\\ &\leq CM_\rho^{N-d-1}f(x).\end{aligned}$$

This, together with (22), gives the desired estimate since $M_\rho^{\mathrm{loc}}\leq 2^\theta M_\rho^\theta$ for any $\theta\geq 0$.

□

It is known that $M_\rho^{\rm loc}$ is of weak type $(1,1)$ with respect to the weight w if and only if, $w \in A_1^{\rho,\rm loc}$ (see [3], Theorem 1).

As we mentioned before, in [1], L^p-weighted inequalities were studied for the operator M_ρ^θ whenever $1 < p < \infty$. Following the ideas developed there, it is not hard to see the behavior of M_ρ^θ for $p = 1$.

Proposition 9. *Given a weight $w \in A_1^\rho$, the maximal function M_ρ^θ is of weak type $(1,1)$ with respect to w, for θ large enough.*

Proof. Assume $w \in A_1^\sigma$ for some $\sigma > 0$. Let $Q_k = B(x_k, \rho(x_k))$, $k \geq 1$, be a covering of $\mathbb{R}^d$ as in Proposition 6. Following the proof of Proposition 3 in [1], we can write

$$M_\rho^\theta f(x) \leq M_\rho^{\rm loc} f(x) \ + \ M_\rho^{\theta,2} f(x),$$

where

$$M_\rho^{\theta,2} f(x) = \sup_{r>\rho(x)} \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|.$$

Since $M_\rho^{\rm loc}$ is of weak type $(1,1)$ for $w \in A_1^{\rho,\rm loc}$ and $A_1^\rho \subset A_1^{\rho,\rm loc}$, it is enough to bound $M_\rho^{\theta,2} f$.

As in [1], for $x \in Q_k$, setting $Q_k^j = 2^j Q_k$, we have

$$\begin{aligned} M_\rho^{\theta,2} f(x) &\lesssim \sup_{j\geq 1} 2^{-j\theta} \frac{1}{|Q_k^j|} \int_{Q_k^j} |f| \lesssim \sum_{j\geq 1} \frac{2^{-j(\theta-\sigma)}}{w(Q_k^j)} \int_{Q_k^j} |f| w \\ &\lesssim \frac{1}{w(Q_k)} \sum_{j\geq 1} 2^{-j(\theta-\sigma)} \int_{Q_k^j} |f| w = \frac{A_k}{w(Q_k)} \end{aligned}$$

where in the second inequality we use $w \in A_1^\sigma$. Then

$$\begin{aligned} w(\{x \in \mathbb{R}^d : \ M_\rho^{\theta,2} f(x) > \lambda\}) &\leq \sum_{k\geq 1} w(\{x \in Q_k : \ A_k / w(Q_k) > \lambda\}) \\ &\leq \frac{1}{\lambda} \sum_{j\geq 1} 2^{-j(\theta-\sigma)} \sum_{k\geq 1} \int_{Q_k^j} |f| w \\ &\leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| w \left(\sum_{j\geq 1} 2^{-j(\theta-\sigma-N_1)} \right). \end{aligned}$$

and the proposition follows taking $\theta > \sigma + N_1$.

□

Remark 4. The operators W_ρ^* and T^* are of weak type $(1,1)$ with respect to A_1^ρ-weights. The same holds for $W_\rho^{*,0}$ and $T^{*,0}$ with respect to $A_1^{\rho,\rm loc}$-weights. The truth of these statements can be seen as a consequence of Proposition 8 and Proposition 9 and the weak type $(1,1)$ of $M_\rho^{\rm loc}$ for weights in $A_1^{\rho,\rm loc}$ (see [3], Theorem 1). In particular, the first two operators applied to functions in $L^1(w)$, $w \in A_1^\rho$, are finite almost everywhere and the same is valid for the other two for functions in $L^1(w)$, $w \in A_1^{\rho,\rm loc}$.

Lemma 2. *If $f \in C_0$, the continuous functions of compact support, then $W_t f$, $T_t f$, $W_t^{\rm loc} f$ and $T_t^{\rm loc} f$ converge point-wisely to f when t goes to zero.*

Proof. The statement is well known for the heat semi-group. For the remaining cases we shall prove that their differences with W_t go to zero. In fact

$$
\begin{aligned}
|(W_t - W_t^{\mathrm{loc}})f(x)| &\leq \int_{|x-y|>\rho(x)} W_t(x,y)|f(y)|\,dy \\
&\leq C\|f\|_\infty \int_{|z|>\rho(x)/2\sqrt{t}} e^{-z^2}\,dz.
\end{aligned} \tag{23}
$$

Since $\rho(x)>0$ for all x, the last integral goes to zero with t.

Also from the estimate given in [10] (see Proposition 2.16) for $t \leq \rho(x)^2$

$$
|T_t(x,y) - W_t(x,y)| \leq \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\varepsilon} \frac{1}{t^{d/2}}\, g\left(\frac{x-y}{\sqrt{t}}\right)
$$

where g is a positive Schwartz function and $\varepsilon>0$, we obtain the result for $T_t f$.

Finally we observe that

$$
|(T_t - T_t^{\mathrm{loc}})f(x)| \leq (W_t - W_t^{\mathrm{loc}})|f|(x)
$$

which follows from the point-wise estimate $T_t(x,y) \leq W_t(x,y)$ between the kernels. Now proceeding as above in (23) the proof is finished.

□

With these observations we may prove the next result.

Proposition 10. *The following inequalities hold a.e.*

i) $|f| \leq W_\rho^* f$ *and* $|f| \leq T^* f$ *for* $f \in L^1(w)$ *and* $w \in A_1^\rho$.

ii) $|f| \leq W_\rho^{*,0} f$ *and* $|f| \leq T^{*,0} f$ *for* $f \in L^1(w)$ *and* $w \in A_1^{\rho,\mathrm{loc}}$.

Proof. As a consequence of Remark 4, the previous lemma and the density of C_0 in $L^1(w)$, by standard arguments, we obtain the a.e. convergence to f of $W_t f$, $T_t f$, W_t^{loc} and T_t^{loc} for f as stated. Using that the limit is bounded by the supremum and that $\rho(x)>0$ we get *i)* and *ii)*.

□

For any given ρ satisfying (3) and a weight w, we may define the Hardy spaces

$$
H_\rho^1(w) = \left\{ f \in L^1(w) : \|W_\rho^* f\|_{L^1(w)} < \infty \right\},
$$

and

$$
H_{\rho,0}^1(w) = \left\{ f \in L^1(w) : \|W_\rho^{*,0} f\|_{L^1(w)} < \infty \right\},
$$

If $w \in A_1^\rho$, by Proposition 10 the quantity $\|W_\rho^* f\|_{L^1(w)}$ becomes a norm and the same occurs when $w \in A_1^{\rho,\mathrm{loc}}$ for $\|W_\rho^{*,0} f\|_{L^1(w)}$. Therefore, in such cases we set

$$
\|f\|_{H_\rho^1(w)} \doteq \|W_\rho^* f\|_{L^1(w)} \quad \text{and} \quad \|f\|_{H_{\rho,0}^1(w)} \doteq \|W_\rho^{*,0} f\|_{L^1(w)}.
$$

Moreover we have $\|f\|_{L^1(w)} \leq \|f\|_{H_\rho^1(w)}$ and $\|f\|_{L^1(w)} \leq \|f\|_{H_{\rho,0}^1(w)}$ for $w \in A_1^\rho$ and $w \in A_1^{\rho,\mathrm{loc}}$ respectively.

When ρ is associated to some $\mathcal{L} = -\Delta + V$ (through (2)) we may also define

$$
H_{\mathcal{L}}^1(w) = \left\{ f \in L^1(w) : \|T^* f\|_{L^1(w)} < \infty \right\},
$$

and

$$
H_{\mathcal{L},0}^1(w) = \left\{ f \in L^1(w) : \|T^{*,0} f\|_{L^1(w)} < \infty \right\},
$$

with norms given by

$$
\|f\|_{H_{\mathcal{L}}^1(w)} \doteq \|T^* f\|_{L^1(w)} \quad \text{and} \quad \|f\|_{H_{\mathcal{L},0}^1(w)} \doteq \|T^{*,0} f\|_{L^1(w)}.
$$

Analogous considerations hold for these spaces

$$\|f\|_{L^1(w)} \leq \|f\|_{H^1_{\mathcal{L}}(w)}, \quad \text{for} \quad w \in A_1^{\rho},$$

and

$$\|f\|_{L^1(w)} \leq \|f\|_{H^1_{\mathcal{L},0}(w)}, \quad \text{for} \quad w \in A_1^{\rho,\mathrm{loc}}.$$

Let us remark that the spaces $H^1_{\mathcal{L}}(w)$ with $w = 1$ was introduced by Dziubański and Zienkiewicz in [9] as the natural Hardy spaces in the Schrödinger context and also this definition appears in [17] for $w \in A_1$.

Our first result deals with some relationships among these spaces. We point out that the first statement of item b) of next theorem was obtained in [10] for the unweighted case, although with a different proof.

Theorem 2. *Given ρ and a weight w, we have*

a) $H^1_{\rho}(w) = H^1_{\rho,0}(w)$ for $w \in A_1^{\rho}$ with equivalent norms.
b) When ρ comes from a Schrödinger operator, $H^1_{\rho}(w) = H^1_{\mathcal{L}}(w)$ for $w \in A_1^{\rho}$ and $H^1_{\rho,0}(w) = H^1_{\mathcal{L},0}(w)$ for $w \in A_1^{\rho,\mathrm{loc}}$ with equivalent norms in both cases.

In particular, the four spaces coincide when ρ comes from $\mathcal{L}$ and $w \in A_1^{\rho}$.

Proof. In view of Proposition 10 it will be enough to show

i) $|W^*_{\rho} - W^{*,0}_{\rho}| \lesssim S_1$,
ii) $|T^* - T^{*,0}| \lesssim S_1$,
iii) $|W^{*,0}_{\rho} - T^{*,0}| \lesssim S_2$,

with S_1 bounded on $L^1(w)$ for weights $w \in A_1^{\rho}$, and S_2 bounded on $L^1(w)$ for $w \in A_1^{\rho,\mathrm{loc}}$.

Let $Q_k = B(x_k, \rho(x_k))$, $k \geq 1$, be a covering of $\mathbb{R}^d$ as in Proposition 6 and $w \in A_1^{\rho,\sigma}$.

For i) we observe that given $N \geq d$,

$$\begin{aligned}
|(W^*_{\rho} - W^{*,0}_{\rho})f(x)| &\leq \sup_{t \leq \rho^2(x)} \int_{|x-y|>\rho(x)} \frac{e^{-\frac{|x-y|^2}{4t}}}{t^{d/2}} |f(y)|\, dy \\
&\lesssim \sup_{t \leq \rho^2(x)} t^{(N-d)/2} \int_{|x-y|>\rho(x)} \frac{|f(y)|}{|x-y|^N}\, dy \\
&\lesssim \rho(x)^{(N-d)} \sum_{j \geq 0} \int_{|x-y| \simeq 2^j \rho(x)} \frac{|f(y)|}{|x-y|^N}\, dy \\
&\lesssim \sum_{j \geq 0} \frac{2^{-j(N-d)}}{|B(x, 2^j \rho(x))|} \int_{B(x, 2^j \rho(x))} |f(y)|\, dy \\
&= S_1 f(x).
\end{aligned}$$

From (3) it is easy to check that there exists a fixed dilation such that $B(x, \rho(x)) \subset \tilde{Q}_k = \mathcal{C} Q_k$, for any $x \in Q_k$, just taking $\mathcal{C} = c_{\rho} 2^{k_0} + 1$. Since $\rho_k = \rho(x_k) \simeq \rho(x)$, using that $w \in A_1^{\rho,\sigma}$,

$$\begin{aligned}\int_{\mathbb{R}^d} S_1 f\ w &\le \sum_{k\ge 0} w(Q_k) \sum_{j\ge 0} \frac{2^{-j(N-d)}}{|2^j \tilde{Q}_k|} \int_{2^j\tilde{Q}_k} |f| \\ &\lesssim \sum_{j\ge 0} 2^{-j(N-d)} \sum_{k\ge 0} \frac{w(2^j\tilde{Q}_k)}{|2^j\tilde{Q}_k|} \int_{2^j\tilde{Q}_k} |f| \\ &\lesssim \sum_{j\ge 0} 2^{-j(N-d-\sigma)} \sum_{k\ge 0} \int_{2^j\tilde{Q}_k} |f| w \\ &\lesssim \int_{\mathbb{R}^d} |f| w \left(\sum_{j\ge 0} 2^{-j(N-d-\sigma-N_1)} \right).\end{aligned}$$

The estimate is achieved taking N large enough.

As for ii), we use estimate (20) in [3] obtaining, for any $N \ge d$,

$$|(T^* - T^{*,0}) f(x)| \lesssim \rho(x)^{-d} \sum_{j\ge 1} 2^{-jN} \int_{2^j B(x,\rho(x))} |f(y)|\, dy \lesssim S_1 f(x).$$

Finally to check iii), let $w \in A_1^{\rho,\mathrm{loc}}$ and observe that

$$\begin{aligned}|(W_\rho^{*,0} - T^{*,0}) f(x)| &\le \sup_{t>0} |(W_t^{\mathrm{loc}} - T_t^{\mathrm{loc}}) f(x)| \ + \sup_{t>\rho(x)^2} |(W_t^{\mathrm{loc}} f(x)| \\ &= S_{2,1} f(x) \ + \ S_{2,2} f(x).\end{aligned} \tag{24}$$

For the first term we use the estimate (see (3.2) in [9])

$$|W_t(x,y) - T_t(x,y)| \le C \left(\frac{|x-y|}{\rho(x)} \right)^{\epsilon} \frac{1}{|x-y|^d},$$

when $|x-y| < \rho(x)$.

As before if $x \in Q_k$, then $B(x,\rho(x)) \subset \tilde{Q}_k = \mathcal{C} Q_k$, and $\rho_k \simeq \rho(x)$. Therefore,

$$S_{2,1} f(x) \le C \rho_k^{-\epsilon} \int_{\tilde{Q}_k} \frac{|f(y)|}{|x-y|^{d-\epsilon}}\, dy$$

and thus

$$\int_{Q_k} S_{2,1} f(x\ ) w(x)\, dx \le C \rho_k^{-\epsilon} \int_{\tilde{Q}_k} |f(y)| \left(\int_{Q_k} \frac{w(x)}{|x-y|^{d-\epsilon}}\, dx \right) dy$$

Since $y \in \tilde{Q}_k$, the ball $Q_k \subset \tilde{B} = B(y, M\rho(y))$ with $M = c_\rho (\mathcal{C}+1)^{k_0+1}$, and $\rho(y) \simeq \rho_k$, the inner integral is estimated by

$$\begin{aligned}\int_{\tilde{B}} \frac{w(x)}{|x-y|^{d-\varepsilon}}\, dx &\le C \rho_k^{\varepsilon} \sum_{j\ge 0} \frac{2^{-j\epsilon}}{|2^{-j}\tilde{B}|} \int_{2^{-j}\tilde{B}} w(x)\ dx \\ &\le C \rho_k^{\varepsilon} \sum_{j\ge 0} 2^{-j\varepsilon} \inf_{2^{-j}\tilde{B}} w \\ &\le C \rho_k^{\varepsilon} w(y),\end{aligned}$$

where in the second inequality we use that $A_1^{\rho,\mathrm{loc}} = A_1^{M\rho,\mathrm{loc}}$.

Therefore,

$$\int_{Q_k} S_{2,1} f\ w \le C \int_{\tilde{Q}_k} |f|\ w,$$

for some constant C that depends only on the constants of (3). By the finite overlapping property of the covering, the sum in k gives

$$\int_{\mathbb{R}^d} S_{2,1}f\ w \leq C \int_{\mathbb{R}^d} |f|\ w.$$

For the second term in (24) we have

$$\begin{aligned}\sup_{t>\rho(x)^2} |W_t^{\text{loc}} f(x)| &\leq \sup_{t>\rho(x)^2} \frac{1}{t^{d/2}} \int_{B(x,\rho(x))} |f(y)|\ dy \\ &\leq \frac{1}{\rho(x)^d} \int_{B(x,\rho(x))} |f(y)|\ dy,\end{aligned}$$

thus, with the same notation as above,

$$\begin{aligned}\int_{\mathbb{R}^d} S_{2,2}f(x)w(x)\ dx &\leq \sum_{k\geq 1} \int_{Q_k} \frac{1}{\rho(x)^d} \left(\int_{B(x,\rho(x))} |f(y)|\ dy \right) w(x)\ dx \\ &\leq C \sum_{k\geq 0} \frac{w(\tilde{Q}_k)}{|\tilde{Q}_k|} \int_{\tilde{Q}_k} |f(y)| dy \\ &\leq C \sum_{k\geq 0} \inf_{x\in\tilde{Q}_k} w(x) \int_{\tilde{Q}_k} |f(y)| dy \\ &\leq C \sum_{k\geq 0} \int_{\tilde{Q}_k} |f(y)| w(y)\ dy \\ &\leq C \int_{\mathbb{R}^d} |f(y)| w(y)\ dy.\end{aligned}$$

□

4. Atomic decomposition

In this section we are going to obtain two characterizations for the weighted Hardy spaces just introduced: one in terms of (ρ, w)-atoms and the other by means of ρ-localized Riesz transforms. Both results will be carried out in the context of the space $H^1_{\rho,0}(w)$, with $w \in A_1^{\rho,\text{loc}}$ defined in the previous section in terms of the classical ρ-localized heat semi-group. In view of Theorem 2 we shall obtain the corresponding characterizations for the other spaces. In particular when ρ comes from a Schrödinger operator we get a generalization of the atomic decomposition given in [9] and [17]. As for the Riesz Transforms characterization we do not recover that given in [9] and [17] since it involves Schrödinger Riesz Transforms which can not be expressed in terms of ρ. Nevertheless, we shall go back to this issue later on (see Theorem 5 below).

In order to obtain the atomic decomposition of $H^1_{\rho,0}(w)$ we shall proceed as in [9] and [17], starting from the one presented in [6] by Bui for local Hardy spaces with weights in A_1. Recall that those spaces were defined as

$$h^1(w) = \left\{ f \in L^1(w) : \sup_{0<t<1} |W_t f|(x) \in L^1(w) \right\},$$

a particular case of $H^1_\rho(w)$ with $\rho \equiv 1$. Moreover, defining for $R > 0$,

$$h^1_R(w) = \left\{ f \in L^1(w) : \sup_{0<t<R^2} |W_t f|(x) \in L^1(w) \right\},$$

and applying Theorem 2, we get

$$\|f\|_{h^1_R(w)} = \|W^*_\rho f\|_{L^1(w)} \simeq \|W^{*,0}_\rho f\|_{L^1(w)},$$

for $\rho \equiv R$ and $w \in A_1$.

Given $R > 0$ and a weight w, following [6], we define an h^1_R-atom as a function a satisfying that there exists x_0 and r such that $\operatorname{supp} a \subset B(x_0, r)$, and

i) $\|a\|_\infty \leq \frac{1}{w(B)}$;
ii) $\int_B a = 0$, if $r < \frac{R}{2}$.

Theorem 3 (Theorem 5.2 in [6])**.** *Let $w \in A_1$. A function f belongs to $h^1_R(w)$ if and only if there exists a sequence of h^1_R–atoms $\{a_i\}$ and numbers $\{\lambda_i\}$ such that*

$$f = \sum_i \lambda_i a_i,$$

in the sense of $L^1(w)$ and $\sum_i |\lambda_i| < \infty$. Further,

$$\|f\|_{h^1_R(w)} \simeq \inf \left\{ \sum_i |\lambda_i| : \ f = \sum_i \lambda_i a_i, \ \textit{for } a_i \ h^1_R\textit{-atom} \right\}.$$

Remark 5. In [9] and [17] it also appears the following property of that decomposition: if $f \in h^1_R(w)$ is such that $\operatorname{supp} f \subset B(x, r)$ with $r \geq R$, then the atoms may be chosen with supports contained in $B(x, C_0 r)$ for a constant C_0 independent of f and r.

The proof of Theorem 3 appears in [6] for $R = 1$. With respect to Remark 5, although it was cited in [9] as proved in [6] we were not able to found that result there. In fact Bui obtains the atomic decomposition as a consequence of an analog result for $H^1(w)$, the classical weighted Hardy space and the following equivalence: $f \in h^1_1(w)$ if and only if $f - \psi * f \in H^1(w)$, where ψ belongs to the Schwartz class, $\int \psi = 1$ and with null moments of higher order.

It is worth noting that an arbitrary atomic decomposition of a compactly supported $f \in h^1_1(w)$ may not share the property in Remark 5. Also, the decomposition constructed in [6] for functions in $H^1(w)$ with compact support does not seem to enjoy that property either. To our believe a suitable atomic decomposition should be provided. To this end a crucial step is building the atoms with supports in the level sets of a smaller grand maximal function that involves convolutions only with compactly supported functions ψ.

That such grand maximal function characterizes the space $h^1(w)$ follows from the next observation.

Proposition 11. *Let $w \in A_1$. Then $f \in h^1_1(w)$, if and only if, $\mathcal{M}_{\mathfrak{A}} f(x) = \sup_{t<1, \psi \in \mathfrak{A}} |\psi_t * f(x)| \in L^1(w)$, where $\mathfrak{A}$ is a subclass of B_1.*

Here, B_1 is the class given in [6] and the proof of the proposition is a direct consequence of Corollary 1 there.

Then for a fixed constant c, we may choose $\mathfrak{A} = \{\psi \in B_1 : \ \operatorname{supp} \psi \subset B(0, c)\}$. It turns out that a construction of atoms can be carried out with a similar pattern to that given in [6] for $H^1(w)$ but with some variants adapted to the local nature of the spaces, and as long as c is taken large enough.

Nevertheless, c is a constant that depends only on the dimension. In this construction atoms are built supported on cubes contained in $\Omega_0 = \{x : \ \mathcal{M}_{\mathfrak{A}} f(x) >$

$0\}$. The advantage is that when f is supported in $B(x_0, r)$, with $r \geq 1$, then $\Omega_0 \subset B(x_0, r + c) \subset B(x_0, (1 + c)r)$. Calling $C_0 = 1 + c$, Remark 5 follows for $R = 1$.

Let us point out here that in [18] (in a more general setting) the author introduces weighted local Hardy spaces by means of a grand maximal function involving also convolutions with functions ψ of compact support. He also obtains an atomic decomposition proceeding very much in the spirit we just outlined.

For $R \neq 1$, the problem can be reduced to $R = 1$ by means of the following observation.

Remark 6. A function $f \in h^1_R(w)$ if and only if $f(R\,\cdot)$ belongs to $h^1_1(w_{1/R})$, where, as usual, $w_{1/R}(x) = w(Rx)/R^d$. Moreover, the mapping $f \mapsto f(R\,\cdot)$ is an isometry. Also, we point out that $w_{1/R}$ belongs to A_1 if and only if $w \in A_1$ and with the same constant. This allows to show that the equivalence between the atomic and maximal norms given in Theorem 3 can be written with constants independent of R.

In what follows we will denote by γ the constant

$$\gamma = \gamma(\rho, d) = 2c_\rho C_0(1 + 2C_0)^{k_0}$$

where C_0 is the constant of Remark 5, c_ρ and k_0 are the constants of ρ given in (3). With this notation we may introduce the notion of (ρ, w)-atoms.

Definition 1. *An integrable function a is said to be a (ρ, w)-atom if it satisfies:*

(i) There exists a ball $B(x_0, r)$ with $r \leq \gamma\rho(x_0)$ such that $\operatorname{supp} a \subset B(x_0, r)$.
(ii) $\|a\|_\infty \leq \frac{1}{w(B(x_0,r))}$.
(iii) $\int_{\mathbb{R}^d} a = 0$ whenever $r < \gamma^{-1}\rho(x_0)$.

Now we are ready to state and prove the following characterization of the space $H^1_{\rho,0}(w)$.

Theorem 4. *Let ρ be a function satisfying* (3) *and $w \in A_1^{\rho,\mathrm{loc}}$. Then a function $f \in H^1_{\rho,0}(w)$ if and only if there exist a sequence of (ρ, w)-atoms $\{a_i\}$ and scalars $\{\lambda_i\}$ such that*

$$f = \sum_i \lambda_i a_i,$$

in the sense of $L^1(w)$. Further,

$$\|f\|_{H^1_{\rho,0}(w)} \simeq \inf\left\{\sum_i |\lambda_i| : f = \sum_i \lambda_i a_i,\ a_i\ (\rho, w)\text{-atom}\right\}$$

Proof. First we show that for any decomposition of f we have $\|f\|_{H^1_{\rho,0}(w)} \leq \sum_i |\lambda_i|$. By standard arguments it is enough to show that for any (ρ, w)-atom a we have $a \in H^1_{\rho,0}(w)$ and moreover there is a fixed constant C such that

$$\|a\|_{H^1_{\rho,0}(w)} \leq C.$$

So let a and $B_0 = B(x_0, r)$ as in Definition 1, that is, $\operatorname{supp} a \subset B(x_0, r)$, $r \leq \gamma\rho(x_0)$. We shall evaluate the norms $\|W^{*,0}_\rho a\|_{L^1(\lambda B_0, w)}$ and $\|W^{*,0}_\rho a\|_{L^1((\lambda B_0)^c, w)}$, for some $\lambda > 1$ to be chosen later.

Clearly, since $W_t \geq 0$ and $\|W_t(x, \cdot)\|_{L^1} = 1$, $W^{*,0}_\rho a \leq \|a\|_\infty$ and then

$$\|W^{*,0}_\rho a\|_{L^1(\lambda B_0, w)} \leq \frac{w(\lambda B_0)}{w(B_0)} \leq C,$$

where C depends on λ, γ and the $A_1^{\rho,\mathrm{loc}}$ constant of w (see Proposition 1).

To estimate $\|W_\rho^{*,0}a\|_{L^1((\lambda B_0)^c,w)}$, let us remind that

$$W_\rho^{*,0}a(x) = \sup_{t<\rho^2(x)} \left| \int_{B(x,\rho(x))\cap B_0} W_t(x,y)a(y)dy \right|.$$

Therefore, $W_\rho^{*,0}a(x) > 0$ implies $|x-x_0| < r+\rho(x) \le (\gamma+(2+\gamma)^{k_0}c_\rho)\rho(x_0) = \tilde{\gamma}\rho(x_0)$. Hence,

$$\|W_\rho^{*,0}a\|_{L^1((\lambda B_0)^c,w)} = \int_{\lambda r<|x-x_0|<\tilde{\gamma}\rho(x_0)} W_\rho^{*,0}a(y)dy.$$

Choosing $\lambda = \gamma\tilde{\gamma} \ge 2$, it follows that we have to consider only the case $r \le \rho(x_0)/\gamma$ and therefore we have $\int_{\mathbb{R}^d} a = 0$. Using that property and that $|x-x_0| \ge \lambda r$ and $|y-x_0| < r$ imply $|x-x_0| \simeq |x-y|$, the mean value theorem gives,

$$\begin{aligned} W_\rho^{*,0}a(x) &\le \sup_{t>0} \int_{|y-x_0|<r} |W_t(x,y) - W_t(x,x_0)|\,|a(y)|dy \\ &\lesssim \|a\|_\infty \left(\frac{r}{|x-x_0|}\right)^{d+1}, \end{aligned}$$

where we have used that $|\nabla W|(z) \lesssim \frac{1}{|z|^{d+1}}$. Therefore, choosing j_0 such that $2^{j_0-1}r < \tilde{\gamma}\rho(x_0) < 2^{j_0}r$ it follows that

$$\begin{aligned} \|W_\rho^{*,0}a\|_{L^1((\lambda B_0)^c,w)} &\le \|a\|_\infty r^{d+1} \int_{2r<|x-x_0|<\tilde{\gamma}\rho(x_0)} \frac{w(x)}{|x-x_0|^{d+1}}dx \\ &\lesssim \|a\|_\infty \sum_{j=2}^{j_0} \frac{1}{2^{j(1+d)}} \int_{|x-x_0|<2^j r} w(x)dx \qquad (25)\\ &\lesssim \frac{|B_0|}{w(B_0)} \inf_{B_0} w \sum_{j=2}^{j_0} 2^{-j} \lesssim 1, \end{aligned}$$

where we have used that $w \in A_1^{\beta\rho,\mathrm{loc}}$ for $\beta = 2\tilde{\gamma}$.

In order to prove the converse let us consider a covering $\{Q_k\}_k$ by balls of critical radius $Q_k = B(x_k,\rho(x_k))$ as in Proposition 6. Related to this covering, there is a partition of unity $\{\psi_k\}_k$ that may be chosen to satisfy (see [8])

i) $0 \le \psi_k \le 1$, $\operatorname{supp}\psi_k \subset 2Q_k$.
ii) $\psi_k \in C^1(\mathbb{R}^d)$ with $|\nabla\psi_k| \le \frac{C}{\rho_k}$, where $\rho_k = \rho(x_k)$.
iii) $\sum_k \psi_k = 1$.

Associated with such covering, there exists also a sequence $\{w_k\}_k$ of weights in the class A_1 such that $w_k|_{2C_0Q_k} = w|_{2C_0Q_k}$, according to Remark 2 (where C_0 is the constant of Remark 5). In particular, all the weights w_k have a A_1 constant independent of k. Clearly, $f = \sum_k f\psi_k$ since by the finite overlapping property of $\{2Q_k\}$ the sum has a finite number of non-zero terms for each x. Moreover, we shall prove the following claim:

If $f \in H^1_{\rho,0}(w)$, then $f\psi_k \in h^1_{\rho_k}(w_k)$ and for some constant C, we have

$$\sum_k \|f\psi_k\|_{h^1_{\rho_k}(w_k)} \le C\|f\|_{H^1_{\rho,0}(w)}.$$

To prove the claim first observe that since $\operatorname{supp} f\psi_k \subset 2Q_k$ the function $W^{*,0}_{\rho_k}(f\psi_k)$ is supported in $3Q_k$ and for $x \in 3Q_k$ we have

$$\begin{aligned}
W^{*,0}_{\rho_k}(f\psi_k)(x) &= \sup_{t<\rho_k^2}\left|\int_{|x-y|<\rho_k} W_t(x,y)f(y)\psi_k(y)dy\right| \\
&\lesssim \sup_{t<\rho_k^2}\int_{|x-y|<\rho_k} W_t(x,y)|f(y)|\,|\psi_k(y)-\psi_k(x)|dy \\
&\quad + \sup_{t<\rho_k^2}\left|\int_{|x-y|<\rho_k} W_t(x,y)f(y)dy\right|\psi_k(x) \\
&= A_k^1(x) + A_k^2(x).
\end{aligned}$$

Using that $|\psi_k(x)-\psi_k(y)| \lesssim \frac{|x-y|}{\rho_k}$ and that $W_t(x,y) \lesssim \frac{1}{|x-y|^d}$ we have

$$A_k^1(x) \lesssim \frac{1}{\rho_k}\int_{|x-y|<\rho_k} \frac{|f(y)|}{|x-y|^{d-1}}dy.$$

Since $C_0 \geq 2$ we have $w_k = w$ on $3Q_k$ and hence

$$\int_{3Q_k} A_k^1(x)w(x)\,dx \lesssim \frac{1}{\rho_k}\int_{4Q_k}|f(y)|\int_{|x-y|<\rho_k}\frac{w(x)}{|x-y|^{d-1}}dx\,dy \tag{26}$$

We decompose dyadically the inner integral and we bound it by

$$\begin{aligned}
\rho_k\sum_{j\geq 0}\frac{2^{-j}}{(2^{-j}r)^d}\int_{B(y,2^{-j}\rho_k)} w &\lesssim \rho_k\sum_j 2^{-j}\inf_{B(y,2^{-j}\rho_k)} w \\
&\lesssim \rho_k w(y)
\end{aligned} \tag{27}$$

for every $y \in 4Q_k$, where we have taken into account that $w \in A_1^{\rho,\mathrm{loc}}$. Therefore, from the finite overlapping property of $\{4Q_k\}$ given in Proposition 6 we obtain

$$\sum_k\int_{3Q_k} A_k^1\,w \lesssim \sum_k\int_{4Q_k}|f|w \lesssim \int_{\mathbb{R}^d}|f|w \lesssim \|f\|_{H^1_{\rho,0}(w)}.$$

Coming back to A_k^2 and having in mind that is supported in $2Q_k$, and for some constants c_1 and c_2 depending only on ρ, we have $c_1\rho_k \leq \rho(x) \leq c_2\rho_k$ for any $x \in 2Q_k$,

$$\begin{aligned}
A_k^2(x) &\lesssim \psi_k(x)\sup_{t>0}\int_{c_1\rho_k<|x-y|<c_2\rho_k} W_t(x,y)|f(y)|dy \\
&\quad + \psi_k(x)\sup_{t<\rho_k^2}\left|\int_{|x-y|<\rho(x)} W_t(x,y)f(y)dy\right| \\
&= A_k^{2,1}(x) + A_k^{2,2}(x).
\end{aligned}$$

For $x \in 2Q_k$, calling $\tilde{c}_2 = c_2 + 2$ we have

$$A_k^{2,1}(x) \lesssim \frac{1}{\rho_k^d}\int_{\tilde{c}_2 Q_k}|f(y)|dy$$

and hence,

$$\int_{2Q_k} A_k^{2,1}(x)w(x)dx \lesssim \frac{w(2Q_k)}{\rho_k^d}\int_{\tilde{c}_2 Q_k}|f(y)|dy \lesssim \int_{\tilde{c}_2 Q_k}|f(y)|w(y)dy, \tag{28}$$

where we used that $w \in A_1^{\beta\rho,\mathrm{loc}}$ for $\beta = \tilde{c}_2$. Again, the sum over k is bounded by $\|f\|_{L^1(w)}$.

Finally for $x \in 2Q_k$ we also have

$$A_k^{2,2}(x) \lesssim \psi_k(x) \sup_{t<\rho(x)^2} \left| \int_{|x-y|<\rho(x)} W_t(x,y) f(y) dy \right|$$
$$+ \psi_k(x) \sup_{c_1\rho_k^2<t<c_2\rho_k^2} \int_{|x-y|<\rho(x)} W_t(x,y)|f(y)|dy$$

Clearly the sum over k of the first terms gives $W_\rho^{*,0} f(x)$ and then $\|W_\rho^{*,0} f\|_{L^1(w)} = \|f\|_{H^1_{\rho,0}(w)}$.

On the other hand, the second term is bounded by

$$\frac{1}{\rho_k^d} \int_{\tilde{c}_2 Q_k} |f(y)|dy$$

for each $x \in 2Q_k$, and the estimate follows as in (28).

The proof of the claim is now complete. Next, since each w_k belongs to A_1 and $f\psi_k \in h^1_{\rho_k}(w_k)$ we may apply Theorem 3 to obtain for each k sequences of $h^1_{\rho_k}(w_k)$-atoms $\{a_j^k\}$ and scalars $\{\lambda_j^k\}$ such that

$$f\psi_k = \sum_j \lambda_j^k a_j^k,$$

in the sense of $L^1(w_k)$ and

$$\sum_j |\lambda_j^k| \lesssim \|f\psi_k\|_{h^1_{\rho_k}(w_k)},$$

where in the last inequality the constant is independent of k.

Therefore, $f = \sum_{j,k} \lambda_j^k a_j^k$ in the sense of $L^1(w)$. Moreover, by the claim,

$$\sum_{j,k} |\lambda_j^k| \lesssim \sum_k \|f\psi_k\|_{h^1_{\rho_k}(w_k)} \lesssim \|f\|_{H^1_{\rho,0}(w)}.$$

It only remains to prove that each a_j^k is a (ρ, w)-atom.

Assume k if fixed. Since a_j^k are $h^1_{\rho_k}(w_k)$-atoms they satisfy

(a) $\operatorname{supp} a_j^k \subset B_j^k = B(x_j^k, r_j^k)$,

(b) $\|a_j^k\|_\infty \leq \frac{1}{w_k(B_j^k)}$,

(c) if $r_j^k \leq \frac{1}{2}\rho_k$ then $\int_{\mathbb{R}^d} a_j^k = 0$.

Moreover, by Remark 5, since $f\psi_k$ is supported in $B(x_k, 2\rho_k)$ we have $B(x_j^k, r_j^k) \subset B(x_k, 2C_0\rho_k)$. Therefore

$$|x_j^k - x_k| < 2C_0\rho_k \quad \text{and} \quad r_j^k \leq 2C_0\rho_k \tag{29}$$

From inequality (3) it follows that

$$\rho_k = \rho(x_k) \leq c_\rho (1 + 2C_0)^{k_0} \rho(x_j^k)$$

and hence

$$r_j^k \leq 2c_\rho C_0 (1 + 2C_0))^{k_0} \rho(x_j^k) = \gamma \rho(x_j^k).$$

Then condition (i) in the Definition 1 is satisfied.

Next, since $B(x_k, 2C_0\rho_k) = 2C_0 Q_k$ and w and w_k coincide there, we have $w_k(B(x_j^k, r_j^k)) = w(B(x_j^k, r_j^k))$ and thus (ii) follows from (b).

Finally, let us assume that $r_j^k < \rho(x_j^k)/\gamma$. From (29) and inequality (3) we deduce

$$\rho(x_j^k) \leq c_\rho(1+2C_0)^{k_0/(k_0+1)}\rho(x_k) \leq \frac{1}{2}\gamma\rho_k.$$

Therefore, $r_j^k < \frac{1}{2}\rho_k$ and by (c) we arrive to $\int_{\mathbb{R}^d} a_j^k = 0$ as we wanted.

□

Remark 7. Let us remark that when $\rho(x) = R$, Theorem 4 gives an extension of Bui's result, stated as Theorem 3 above.

In fact, when $w \in A_1^R$ we showed that $h_R^1(w)$ coincides with $h_{R,0}^1(w)$ and since $A_1^R \subset A_1^{R,\mathrm{loc}}$ we get an atomic decomposition for the space $h_R^1(w)$ when $w \in A_1^R$, that is, for some $\theta \geq 0$ it satisfies

$$\frac{w(B(x,r))}{|B(x,r)|} \lesssim \left(1+\frac{r}{R}\right)^\theta \inf_{B(x,r)} w.$$

It is not hard to see that weights of the kind $w(x) = 1 + |x|^\beta$ satisfy the above inequality but they may not be A_1 weights.

Remark 8. From the proof of the above theorem it follows that

$$\|f\|_{H^1_{\rho,0}(w)} \simeq \sum_{k\geq 1} \|f\psi_k\|_{h^1_{\rho_k}(w_k)}.$$

One of the inequalities is contained in the claim. For the other, taking the atomic decomposition given in the proof, it holds

$$\|f\|_{H^1_{\rho,0}(w)} \lesssim \sum_{k,j} |\lambda_j^k| \leq \sum_k \sum_j |\lambda_j^k| \lesssim \sum_k \|f\psi_k\|_{h^1_{\rho_k}(w_k)}.$$

An advantage of having an atomic decomposition of some space X is the possibility of reducing the proof of boundedness of a linear operator T from X into some Banach space Y to check that for any atom and a fixed constant A,

$$\|T(a)\|_Y \leq A. \tag{30}$$

It is worth mentioning that such reduction is not always possible for every operator T when, for instance, $X = H^1(\mathbb{R}^d)$, Y a Banach space and ∞-atoms are used in the decomposition (see [5], [13], [14], [15]). Nevertheless, for the case $X = H^1(\mathbb{R}^d)$, $Y = L^1(\mathbb{R}^d)$ and T a Calderón-Zygmund operator, condition (30) suffices to extend T to a bounded operator from $H^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. A proof of this fact can be found, for instance, in [11].

In the next proposition we adapt Grafakos' argument to our situation.

Proposition 12. *Let T be a linear operator of weak type $(1,1)$ with respect to a weight $w \in A_1^{\rho,\mathrm{loc}}$. If there exists a constant C such that*

$$\|Ta\|_{L^1(w)} \leq C, \tag{31}$$

for every (ρ,w)-atom a, then T is bounded from $H^1_{\rho,0}(w)$ into $L^1(w)$.

Proof. Let $f \in H^1_{\rho,0}(w)$, by Theorem 4 there exist sequences $\{a_j\}$ and $\{\lambda_j\}$, such that

$$f = \sum_j \lambda_j a_j, \tag{32}$$

in the sense of $L^1(w)$ and a constant C independent of f satisfying

$$\sum_j |\lambda_j| \leq C\|f\|_{H^1_{\rho,0}(w)}. \tag{33}$$

Let us observe that since $f \in L^1(w)$ then Tf belongs to $L^{1,\infty}(w)$ and hence is finite almost everywhere. Also, by (31) and (33), the series $\sum_j \lambda_j Ta_j$ converges in $L^1(w)$ and hence it is finite almost everywhere.

Thus, given $\delta > 0$ and $N \in \mathbb{N}$, using the weak type $(1,1)$ and Tchebysheff's inequality, we have

$$\begin{aligned} w(\{|Tf - \sum_j \lambda_j Ta_j| > \delta\}) &\leq w(\{|Tf - \sum_{j=1}^N \lambda_j Ta_j| > \delta/2\}) \\ &\quad + w(\{|\sum_{j=N+1}^\infty \lambda_j Ta_j| > \delta/2\}) \\ &\lesssim \frac{1}{\delta}\left\| f - \sum_{j=1}^N \lambda_j a_j \right\|_{L^1(w)} + \frac{1}{\delta}\left\| \sum_{j=N+1}^\infty \lambda_j Ta_j \right\|_{L^1(w)}. \end{aligned}$$

The first term goes to zero since the sum converges in $L^1(w)$. The second term also goes to zero since the series converges in $L^1(w)$.

Therefore $Tf = \sum_j \lambda_j Ta_j$ a.e. and thus

$$\|Tf\|_{L^1(w)} \leq \sum_j |\lambda_j| \|Ta_j\|_{L^1(w)} \lesssim \sum_j |\lambda_j| \lesssim \|f\|_{L^1(w)}.$$

$\square$

Remark 9. For a weight $w \in A_1^\rho$, the above proposition holds true substituting $H^{1,0}_\rho(w)$ by $H^1_\rho(w)$, according to Theorem 2.

To finish this section we give a characterization of the Hardy space $H^1_{\rho,0}(w)$ in terms of $\rho-$local Riesz transforms.

Let η be a smooth function, $\eta \in C_0^\infty(\mathbb{R}^d)$ radial, $0 \leq \eta \leq 1$, $\operatorname{supp} \eta \subset B(0,2)$ and $\eta \equiv 1$ in $B(0,1)$. For the classical Riesz transforms R_j with kernel $k_j(z) = z_j/|z|^{d+1}$ we consider the $\rho-$localized operators

$$\begin{aligned} R_j^\rho f(x) &= p.v. \int k_j(x-y)\, \eta\left(\frac{|x-y|}{\rho(x)}\right) f(y)\, dy \\ &= p.v. \int k_j^\rho(x-y) f(y)\, dy. \end{aligned}$$

These operators are known to be bounded on $L^p(w)$, $1 < p < \infty$, $w \in A_p^{\rho,\mathrm{loc}}$ (see [3]), even if we consider a non-smooth cut function like $\eta = \chi_{B(0,1)}$.

In [6] a characterization of the local Hardy space $h_1^R(w)$, with $w \in A_1$ in terms of R_j^ρ with $\rho = R$ was given. In fact, it is done for $R = 1$ but is easily extended to any $R > 0$ (see Remark 6). We point out that the constants in the equivalence of the norms are independent of R and that they depend on the weight only through the A_p constant. To obtain a Riesz transforms characterization of $H^1_{\rho,0}(w)$, $w \in A_1^{\rho,\mathrm{loc}}$ we use a procedure similar to Theorem 4 in order to reduce to the case of $h_1^R(w)$.

Theorem 5. *Let ρ be a critical radius function and $w \in A_1^{\rho,\mathrm{loc}}$. Then $f \in H^1_{\rho,0}(w)$ if and only if f and $R_j^\rho f$, $j = 1, \ldots, d$ belong to $L^1(w)$ and moreover*

$$\|f\|_{H^1_{\rho,0}(w)} \simeq \|f\|_{L^1(w)} + \sum_{j=1}^{d} \|R_j^\rho f\|_{L^1(w)}.$$

Proof. First observe that the operator R_j^ρ is of weak type $(1,1)$ with respect to $w \in A_1^{\rho,\mathrm{loc}}$ (see [3]). Then, in order to show that R_j^ρ is bounded from $H^1_{\rho,0}(w)$ into $L^1(w)$, according to Proposition 12, it is enough to check

$$\|R_j^\rho a\|_{L^1(w)} \leq C,$$

where a is a (ρ, w)-atom supported in $B = B(x_0, r)$, $r \leq \gamma\rho(x_0)$. As before, for a choice of λ similar to the proof of Theorem 4, we estimate $\|R_j^\rho a\|_{L^1(\lambda B,w)}$ and $\|R_j^\rho a\|_{L^1((\lambda B)^c,w)}$. For the first one, using the boundedness of R_j^ρ on $L^p(w)$, $1 < p < \infty$, and $w \in A_p^{\rho,\mathrm{loc}}$, we get

$$\begin{aligned}\|R_j^\rho a\|_{L^1(\lambda B,w)} &\leq \|R_j^\rho a\|_{L^p(w)} w(\lambda B)^{1/p'} \lesssim \|a\|_{L^p(w)} w(\lambda B)^{1/p'} \\ &\lesssim \left(\frac{w(\lambda B)}{w(B)}\right)^{1/p'} \lesssim 1,\end{aligned}$$

where we used that $w \in A_1^{\lambda\gamma\rho,\mathrm{loc}}$ according to Proposition 1.

By the choice of λ, we have as above that $R_j^\rho a(x) \equiv 0$ for $x \notin \lambda B$ unless $r \leq \gamma^{-1}\rho(x_0)$ and hence $\int a = 0$. Then

$$\begin{aligned}|R_j^\rho a(x)| &\leq \int |k_j^\rho(x-y) - k_j^\rho(x-x_0)||a(y)|\, dy \\ &\leq C\|a\|_\infty \left(\frac{r}{|x-x_0|}\right)^{d+1}\end{aligned}$$

where we used $|\nabla k_j^\rho(z)| \leq C/|z|^{d+1}$ and that $|x-y| \simeq |x-x_0|$ for $y \in B$ and $x \notin \lambda B$.

Hence, following as in (25), we obtain the desired estimate.

For the converse inequality let us assume that f and $R_j^\rho f$ are in $L^1(w)$. We shall prove a similar result to the claim stated in the proof of Theorem 4, namely for each j

$$\sum_{k\geq 1} \|R_j^{\rho_k}(f\psi_k)\|_{L^1(w_k)} \lesssim \|R_j^\rho f\|_{L^1(w)} + \|f\|_{L^1(w)} \tag{34}$$

with the same notation in there.

In fact we set

$$\begin{aligned}|R_j^{\rho_k}(f\psi_k)(x)| &\leq |\psi_k(x) R_j^{\rho_k} f(x)| + |R_j^{\rho_k}(f\psi_k)(x) - \psi_k(x) R_j^{\rho_k} f(x)| \\ &= I(x) + II(x).\end{aligned}$$

For the second term, by the properties of ψ_k we obtain

$$II(x) \leq \int_{B(x,2\rho_k)} \frac{|\psi_k(y) - \psi_k(x)|}{|x-y|^d} |f(y)|\, dy \leq \frac{C}{\rho_k} \int_{B(x,2\rho_k)} \frac{|f(y)|}{|x-y|^{d-1}} dy$$

and the estimate follows as in (26) and (27), since the function $R_j^{\rho_k}(f\psi_k)$ is supported in $4Q_k$ and $w_k \equiv w$ there.

Also

$$I(x) \leq |\psi_k(x)(R_j^{\rho_k} - R_j^{\rho})f(x)| \ + \ |\psi_k(x)R_j^{\rho}f(x)| = I_1(x) \ + \ I_2(x).$$

But for $x \in 2Q_k$, since $c_1\rho_k \leq \rho(x) \leq c_2\rho_k$, we have

$$I_1(x) \leq C\psi_k(x) \int_{c_1\rho_k<|x-y|<2c_2\rho_k} |k_j(x-y)||f(y)| \, dy \leq C\frac{\psi_k(x)}{\rho_k^d} \int_{cQ_k} |f(y)| \, dy,$$

where $c = 2c_2 + 3$. Continuing as in (28) and having in mind that $\operatorname{supp} I_1 \subset 2Q_k$ we arrive to $\|\psi_k f\|_{L^1(w)}$.

Therefore, summing over k we get (34).

Next we use the Riesz characterization of $h_1^{\rho_k}(w_k)$, being w_k an A_1 weight given in [6]. Therefore for each k,

$$\|f\psi_k\|_{h_1^{\rho_k}(w_k)} \simeq \|f\psi_k\|_{L^1(w_k)} \ + \ \sum_{j=1}^{d} \|R_j^{\rho_k}(f\psi_k)\|_{L^1(w_k)},$$

where the constants are independent of k. Therefore, summing over k and using Remark 8 and (34) we get the remaining inequality, finishing the proof of the theorem.

□

Remark 10. Arguing as in Remark 7, we may obtain from the above result the characterization of $h_1^R(w)$ in terms of local Riesz transforms for w in A_1^R, extending Bui's result.

5. Boundedness on ρ-Hardy spaces of singular integrals of ρ-type

Theorem 6. *Let $T \in \mathcal{S}_0^{\rho}$ and $w \in A_1^{\rho}$. Then T maps $H_{\rho}^1(w)$ into $L^1(w)$ continuously.*

Proof. According to Theorem 1 and Proposition 12, it is enough to check that for any (ρ, w)-atom a, $\|Ta\|_{L^1(w)} \leq C$.

Assume that $\operatorname{supp} a \subset B = B(x_0, r)$ with $r \leq \gamma\rho(x_0)$. Then if we denote $\tilde{B} = 2B$

$$\int_{\tilde{B}} |Ta|w \leq \left(\int_{\mathbb{R}^d} |Ta|^p \right)^{1/p} \left(\int_{\tilde{B}} w^{p'} \right)^{1/p'}.$$

If we choose p large enough such that w satisfies a p' reverse Hölder inequality (see Proposition 3) and $p \geq p_0$ we have

$$\int_{\tilde{B}} |Ta|w \leq C\left(\int_{\tilde{B}} |a|^p \right)^{1/p} |\tilde{B}|^{-1/p} w(\tilde{B}) \leq C\|a\|_{\infty} w(\tilde{B}) \leq C,$$

where in the last inequality we use Proposition 1.

Next we observe that for $x \notin \tilde{B}$ and $y \in B$ we have $|x - x_0| \simeq |x - y|$.

Setting $B_0 = B(x_0, \gamma^{-1}\rho(x_0))$ we have $\tilde{B}^c = (\tilde{B}^c \cap B_0^c) \cup (B_0 \setminus \tilde{B})$. Then, using $\rho(y) \simeq \rho(x_0)$ for $y \in B$ and condition (b) above

$$\begin{aligned}
\int_{\tilde{B}^c \cap B_0^c} |Ta(x)|w(x)\, dx &\lesssim \int_{B_0^c} w(x) \int_B \frac{\|a\|_\infty}{|x-x_0|^d}\left(\frac{|x-x_0|}{\rho(x_0)}\right)^{-N} dy\, dx \\
&\lesssim \|a\|_\infty \frac{r^d}{\rho(x_0)^{-N}} \sum_{j\geq 1} \int_{2^{j+1}B_0 \setminus 2^j B_0} \frac{w(x)}{|x-x_0|^{N+d}} dx \\
&\lesssim \frac{r^d}{w(B)} \sum_{j\geq 1} 2^{-jN} \frac{w(2^{j+1}B_0)}{|2^j B_0|} \\
&\lesssim \frac{r^d}{w(B)} \inf_B w \sum_{j\geq 1} 2^{-j(N-\theta)} \\
&\lesssim 1,
\end{aligned}$$

where θ is such that $w \in A_1^{\rho,\theta}$ and N is taken large enough.

Now we notice that $B_0 \setminus \tilde{B} \neq \emptyset$ if and only if $2r < \gamma^{-1}\rho(x_0)$ and hence we may assume $\int a = 0$. Therefore for $x \in B_0 \setminus \tilde{B}$, using condition (c) above, we get

$$|Ta(x)| \leq \int_B |K(x,y) - K(x,x_0)||a(y)|\, dy \leq C\|a\|_\infty \left(\frac{r}{|x-x_0|}\right)^{d+\lambda}.$$

Then for j_0 such that $2^{j_0} r \simeq \gamma^{-1}\rho(x_0)$,

$$\begin{aligned}
\int_{B_0 \setminus \tilde{B}} |Ta(x)|w(x)\, dx &\leq C\|a\|_\infty r^{d+\lambda} \sum_{j=1}^{j_0} \int_{|x-x_0|\simeq 2^j r} \frac{w(x)}{|x-x_0|^{d+\lambda}} dx \\
&\leq C \frac{r^d}{w(B)} \sum_{j=1}^{j_0} 2^{-j\lambda} \frac{w(2^j B)}{|2^j B|} \\
&\leq C \frac{r^d}{w(B)} \inf_B w \\
&\leq C,
\end{aligned}$$

since for any $1 \leq j \leq j_0$, $2^j r \leq \gamma^{-1}\rho(x_0)$. □

Theorem 7. *Let $T \in \mathcal{S}^\rho$. Then T is bounded from H_1^ρ into $L^1(w)$ for w such that $w^{s'} \in A_1^\rho$.*

Proof. From to Theorem 1 and Proposition 12 it is enough to check boundedness on atoms.

Let a be an atom and $B = B(x_0, r)$ its support. In this case we proceed as in the proof of Theorem 6 cutting the domain of integration. With the same notation there, we have

$$\int_{\tilde{B}} |Ta|w \leq \left(\int |Ta|^p\right)^{1/p} \left(\int_{\tilde{B}} w^{p'}\right)^{1/p'}.$$

Choosing $p < s$ but close enough, $w^{s'}$ would satisfy the reverse Hölder with exponent p'/s' and being $r \leq \gamma\rho(x_0)$ the quantity above is bounded by a constant times

$$\frac{|B|}{w(B)} \left(\frac{1}{|\tilde{B}|}\int_{\tilde{B}} w^{s'}\right)^{1/s'}.$$

Using the A_1^ρ condition for $w^{s'}$ the estimate follows.

Next, as before, we split $\tilde{B}^c = (\tilde{B}^c \cap B_0^c) \cup (B_0 \setminus \tilde{B})$. Again we observe that $B_0 \setminus \tilde{B} \neq \emptyset$ if and only if $2r < \gamma^{-1}\rho(x_0)$ and hence we may assume $\int a = 0$. Using that, Hölder inequality and setting j_0 so that $2^{j_0} r \simeq \gamma^{-1}\rho(x_0)$ we have

$$\begin{aligned}
&\int_{B_0\setminus\tilde{B}} |Ta(x)|w(x)\ dx \\
&\qquad \lesssim \|a\|_\infty \sum_{j\geq 0}^{j_0} [w^{s'}(2^{j+1}B_0)]^{1/s'} \int_B \left(\int_{2^{j+1}B\setminus 2^j B} |K(x,y)-K(x,x_0)|^s dx \right)^{1/s} dy \\
&\qquad \lesssim \frac{|B|}{w(B)} \inf_B w,
\end{aligned}$$

where we have used that $2^j r \leq \gamma^{-1}\rho(x_0)$ for $j \leq j_0$.

For the remaining integral, we decompose the integral dyadically to get

$$\begin{aligned}
&\int_{\tilde{B}^c\cap B_0^c} |Ta(x)|w(x)\ dx \\
&\qquad \leq C\|a\|_\infty \int_B \left(\int_{B_0^c} |K(x,y)|w(x)\ dx \right) dy \\
&\qquad \leq C\|a\|_\infty \sum_{j\geq 0} [w^{s'}(2^{j+1}B_0)]^{1/s'} \ \int_B \left(\int_{2^{j+1}B_0\setminus 2^j B_0} |K(x,y)|^s dx \right)^{1/s} dy.
\end{aligned} \tag{35}$$

Now, since $2r < \gamma^{-1}\rho(x_0)$ we apply condition (b') with $R = 2^j\gamma^{-1}\rho(x_0)$ and using that $w^{s'} \in A_1^{\rho,\theta}$, we get

$$\begin{aligned}
\int_{\tilde{B}^c\cap B_0^c} |Ta(x)|w(x)\ dx &\leq C\|a\|_\infty |B| \sum_{j\geq 1} 2^{-j(N-\frac{\theta}{s'})} \inf_{2^j B_0} w \\
&\leq C\frac{|B|}{w(B)} \inf_{B_0} w \\
&\leq C\frac{|B|}{w(B)} \inf_{B} w \\
&\leq C,
\end{aligned} \tag{36}$$

by taking $N > \theta/s'$.

On the the other hand, in the case $2r \geq \gamma^{-1}\rho(x_0)$, that is $r \simeq \rho(x_0)$ and $B_0 \setminus \tilde{B} = \emptyset$, we proceed as in (35) and (36) but with $R = 2^j r$, to finally obtain

$$\begin{aligned}
&\int_{\tilde{B}^c \cap B_0^c} |Ta(x)| w(x)\, dx \\
&\qquad \leq C\|a\|_\infty \int_B \left(\int_{\tilde{B}^c} |K(x,y)| w(x)\, dx \right) dy \\
&\qquad \leq C\|a\|_\infty \sum_{j\geq 1} [w^{s'}(2^j \tilde{B})]^{1/s'} \int_B \left(\int_{2^{j+1}\tilde{B}\setminus 2^j \tilde{B}} |K(x,y)|^s dx \right)^{1/s} dy \\
&\qquad \leq C\|a\|_\infty |B| \sum_{j\geq 1} \left(1 + \frac{2^{j+1} r}{\rho(x_0)} \right)^{-(N - \frac{\theta}{s'})} \inf_{2^j \tilde{B}} w \\
&\qquad \leq C \frac{|B|}{w(B)} \inf_{\tilde{B}} w \\
&\qquad \leq C,
\end{aligned}$$

taking $N > \theta/s'$.

□

6. The Schrödinger case

In this section we are going to deal with singular integrals arising from the Schrödinger differential operator $\mathcal{L} = -\Delta + V$, V a non-negative potential and satisfying a reverse Hölder inequality of order q, $q > d/2$ and $d \geq 3$. So, from now on, ρ is the function defined by (2) that, as we said, satisfies inequality (3).

As we proved in section 3, when $w \in A_1^\rho$ the Hardy space $H^1_{\mathcal{L}}(w)$ coincides with $H^1_\rho(w) = H^1_{\rho,0}(w)$ so we may apply all the results from previous sections. Therefore we already have an atomic decomposition for $H^1_{\mathcal{L}}(w)$, its characterization by ρ-local Riesz transforms, as well as boundedness of the associated ρ-singular integrals from $H^1_\rho(w)$ into $L^1(w)$. More precisely, we conclude from Theorems 6 and 7 above the following result.

Theorem 8. *If an operator T belongs to $\mathcal{S}_0^\rho$ (respectively to $\mathcal{S}^\rho$) then T maps $H^1_{\mathcal{L}}(w)$ into $L^1(w)$ for $w \in A_1^\rho$ (respectively for $w^{s'} \in A_1^\rho$).*

Now we apply these results and those of Section 2 to some operators arising from $\mathcal{L}$ that satisfy the above assumptions.

Theorem 9. *Assume $V \in RH_q$, $q > d$. Then the operators $\nabla(-\Delta + V)^{-1/2}$, $(-\Delta + V)^{-1/2}\nabla$ and $\nabla(-\Delta + V)^{-1}\nabla$ are of weak type $(1,1)$ and bounded from $H^1_{\mathcal{L}}(w)$ into $L^1(w)$ for $w \in A_1^\rho$. Further, if $V \in RH_q$, $q > d/2$ the same holds for the operator $(-\Delta + V)^{i\zeta}$, $\zeta \in \mathbb{R}$.*

Proof. It is enough to check that these operators satisfy conditions (a), (b) and (c) in the definition of the class $\mathcal{S}_0^\rho$.

As it was shown in [16] the above operators are Calderón-Zygmund operators, so (a) holds. Further their kernels satisfy (b) and (c) (see Theorem 9 and Theorem 10 in [2] and references therein). Then, the result follows from Theorem 1 and Theorem 8. □

Theorem 10. *Assume $V \in RH_q$, $q > d/2$ and let $\mathcal{T}_1 = V^{1/2}(-\Delta+V)^{-1/2}$, $\mathcal{T}_2 = V(-\Delta+V)^{-1}$ and $\mathcal{T}_3 = \nabla(-\Delta+V)^{-1/2}$. Then the operators $\mathcal{T}_j$, $j=1,2,3$ are of weak type $(1,1)$ and bounded from $H^1_{\mathcal{L}}(w)$ into $L^1(w)$ for weights such that $w^{s_j'} \in A_1^\rho$ where $s_1 = 2q$, $s_2 = q$ and s_3 satisfies $1/s_3 = 1/q - 1/d$ provided $q < d$.*

Proof. We need to check conditions (a'), (b') and (c') in the definition of the class $\mathcal{S}^\rho$.

From [16] (see Theorem 5.10, Theorem 3.1 and Theorem 0.5), each operator $\mathcal{T}_j$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \le s_j$, with $j = 1,2,3$.

We denote by K_j the kernel of the operator $\mathcal{T}_j$, $j=1,2,3$. The proof that K_3 satisfies condition (c') is given in Lemma 6 of [4]. As for K_1 and K_2 it follows from the proof of Theorem 11 in [2] (notice that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the adjoint operators of those appearing there).

Let us check that K_1 and K_2 satisfy condition (b'). It is known (see [12]), that if $V \in RH_q$ with $q > d/2$, for any $N > 0$, there exists $C_N > 0$ such that, for $j = 1,2$,

$$|K_j(x,y)| \le C_N \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N} \frac{1}{|x-y|^{d-j}} V(x)^{j/2},$$

and in view of Remark 3 the above inequality also holds with $\rho(x)$ in place of $\rho(y)$. Therefore, for $y \in B(x_0,r)$ and $R \ge 2r$, using that $|x-x_0| \simeq |x-y|$ as long as $R \le |x-x_0| < 2R$, we have

$$\begin{aligned}
&\left(\int_{R\le|x-x_0|<2R} |K_j(x,y)|^{2q/j} dx\right)^{j/(2q)} \\
&\qquad \lesssim \left(\int_{R\le|x-x_0|<2R} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-2qN/j} \frac{V(x)^q}{|x-y|^{(d-j)2q/j}} dx\right)^{j/(2q)} \\
&\qquad \lesssim \left(1+\frac{R}{\rho(x_0)}\right)^{-N/k_0+1} \frac{1}{R^{d-j}} \left(\int_{B(x_0,2R)} V(x)^q dx\right)^{j/(2q)} \\
&\qquad \lesssim \left(1+\frac{R}{\rho(x_0)}\right)^{-N/k_0+1} \frac{R^{-jd/2q'}}{R^{d-j}} \left(\int_{B(x_0,2R)} V(x)\, dx\right)^{j/2},
\end{aligned}$$

where we have also used inequalities (9) and (1).

Finally, according to Lemma 1 in [12],

$$\int_{B(x_0,cR)} V(y)\, dy \le C R^{d-2} \left(1 + \frac{R}{\rho(x_0)}\right)^{\mu} \tag{37}$$

for some $\mu > 1$ and $c > 0$. Therefore we obtain

$$\left(\int_{R\le|x-x_0|<2R} |K_j(x,y)|^{2q/j} dx\right)^{j/(2q)} \lesssim R^{-d/s_j'} \left(1+\frac{R}{\rho(x_0)}\right)^{-(N/(k_0+1)-\mu/2)}. \tag{38}$$

Since N is arbitrary, the desired estimate follows.

Next, we check that K_3 satisfies condition (b'). According to [16] (see page 538) for every $N > 0$, there exists $C_N > 0$ such that

$$|K_3(x,y)| \le C_N \left(1+\frac{|x-y|}{\rho(y)}\right)^{-N} \frac{1}{|x-y|^{d-1}} \left(\frac{1}{|x-y|} + \int_{B(y,2|x-y|)} \frac{V(z)}{|z-x|^{d-1}} dz\right).$$

Proceeding as before for $y \in B(x_0, r)$ and $R \geq 2r$, and $R \leq |x - x_0| < 2R$, we have

$$|K_3(x,y)| \lesssim \left(1 + \frac{R}{\rho(x_0)}\right)^{-N/k_0+1} \frac{1}{R^{d-1}} \left(\frac{1}{R} + I_1(\chi_{B(x_0,3R)}V)(y)\right),$$

where I_1 is the classical fractional integral of order 1. Therefore,

$$\left(\int_{R\leq|x-x_0|<2R} |K(x,y)|^{s_3} dx\right)^{1/s_3} \lesssim R^{1-d} \left(1 + \frac{R}{\rho(x_0)}\right)^{-N/k_0+1} \left(\frac{1}{R} + \|I_1(\chi_{B(x_0,3R)}V)\|_{s_3}\right).$$

From the boundedness of I_1, the fact that $V \in RH_q$, and inequality (37), we get

$$\begin{aligned}\left(\int_{\mathbb{R}^d} I_1(\chi_{B(x_0,3R)}V)^{s_3} dx\right)^{1/s_3} &\leq C\left(\int_{B(x_0,3R)} |V(x)|^q dx\right)^{1/q} \\ &\leq CR^{d/q'} \int_{B(x_0,3R)} |V(x)| dx \\ &\leq CR^{d/q-2} \left(1 + \frac{R}{\rho(x_0)}\right)^{\mu}.\end{aligned}$$

Then, since $d/s - 1 = d/q - 2$ and $d/s' = d - d/q + 1$, we obtain

$$\begin{aligned}\left(\int_{R\leq|x-x_0|<2R} |K(x,y)|^s dx\right)^{1/s} &\lesssim R^{1-d} \left(1 + \frac{R}{\rho(x_0)}\right)^{-(N/(k_0+1)-\mu)} \left(R^{d/s-1} + R^{d/q-2}\right) \\ &\lesssim R^{-d/s'} \left(1 + \frac{R}{\rho(x_0)}\right)^{-(N/(k_0+1)-\mu)},\end{aligned}$$

and the proof is finished since N is arbitrary.

Then we have proved that the operators $\mathcal{T}_i$, $i = 1, 2, 3$, belong to the class $\mathcal{S}^\rho$ and therefore the result follows from Theorem 1 and Theorem 8.

□

Finally we are going to show that, as in the unweighted case, the space $H^1_{\mathcal{L}}(w)$ can also be characterized in terms of Schrödinger-Riesz transforms, namely by the components $\mathcal{R}_j$ of the vector operator $\nabla(-\Delta + V)^{-1/2}$. It is worth mentioning that this kind of result was obtained in [17] only for weights in the classical class A_1.

Theorem 11. *Let $\mathcal{L} = -\Delta + V$ with $V \in RH_q$ and w a weight. Then the equivalence $f \in H^1_{\mathcal{L}}(w)$ if and only if $f, \mathcal{R}_j f \in L^1(w)$ holds provided either*
a) $q \geq d$ and $w \in A^\rho_1$.
b) $d/2 < q < d$ and $w^{s'} \in A^\rho_1$ with $1/s = 1/q - 1/d$.
Moreover, in both cases,

$$\|f\|_{H^1_{\mathcal{L}}(w)} \simeq \|f\|_{L^1(w)} + \sum_{j=1}^{d} \|\mathcal{R}_j f\|_{L^1(w)}.$$

Proof. From the previous theorems we only need to prove that f and $\mathcal{R}_j f$ in $L^1(w)$ imply $f \in H^1_{\mathcal{L}}(w)$.

Our goal is to use the characterization given in Theorem 5. Following the proof of Theorem 3 in [3] we write

$$\begin{aligned} R^\rho_j f(x) = (R^\rho_j f(x) - R_j(f\chi_{E_x})(x)) \ + \ (R_j - \mathcal{R}_j)(f\chi_{E_x})(x) \ - \ \mathcal{R}_j(f\chi_{E^c_x})(x) \\ + \ \mathcal{R}_j f(x) = A^1_j f(x) \ + \ A^2_j f(x) \ + \ A^3_j f(x) \ + \ \mathcal{R}_j f(x), \end{aligned}$$

where R_j denotes the corresponding classical Riesz transform and $E_x = \{y \in \mathbb{R}^d : |x-y| < \rho(y)\}$.

Following the argument in [3], page 13, the $L^1(w)$ norm of A^2_j and A^3_j are bounded by $\|f\|_{L^1(w)}$ for $w \in A^\rho_1$ if $q \geq d$ or $w^{s'} \in A^\rho_1$ if $d/2 < q < d$.

Now, for A^1_j, we observe that $B(x, c^{-1}\rho(x)) \subset E_x \subset B(x, c\rho(x))$, with $c = c_\rho 2^{k_0}$. Also, since $\eta \equiv 1$ in $B(0,1)$ and $\eta \equiv 0$ in $B(0,2)^c$ we obtain,

$$\begin{aligned} |A^1_j f(x)| &\lesssim \int_{\mathbb{R}^d} \left| \eta\left(\frac{|x-y|}{\rho(x)}\right) - \chi_{E_x}(y) \right| \frac{|f(y)|}{|x-y|^d} \, dy \\ &\lesssim \frac{1}{\rho(x^d)} \int_{B(x,(c+2)\rho(x))} |f(y)| \, dy. \end{aligned}$$

Finally since for $x \in Q_k$, $\rho(x) \simeq \rho(x_k)$, the norm $\|A^1_j f\|_{L^1(w)}$ is bounded by $\|f\|_{L^1(w)}$ proceeding as in (28) and for any $w \in A^{\rho,\text{loc}}_1$.

Therefore using Theorem 2 and Theorem 5 we get

$$\begin{aligned} \|f\|_{H^1_{\mathcal{L}}(w)} &\lesssim \|f\|_{H^1_{\rho,0}(w)} \\ &\lesssim \|f\|_{L^1(w)} + \sum_{j=1}^d \|R^\rho_j f\|_{L^1(w)} \\ &\lesssim \|f\|_{L^1(w)} + \sum_{j=1}^d \|\mathcal{R}_j f\|_{L^1(w)}, \end{aligned}$$

and the result follows. □

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