RESEARCH PAPER

DYADIC NONLOCAL DIFFUSIONS IN METRIC MEASURE SPACES

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Abstract

In this paper we solve the initial value problem for the nonlocal diffusion generated by the space fractional derivative induced by the dyadic tilings of M. Christ on a space of homogeneous type. We consider the problems of pointwise and norm convergence to the initial data. The main tool is the use of the Haar system induced by a dyadic tiling, which is actually the set of eigenfunctions for the fractional derivative operator.

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1. Introduction

Since the beginning of the use of real analysis techniques in harmonic analysis and partial differential equations in the middle fifties of the past century, it was appreciated that the robustness of the real methods allows its use in much more general settings than Euclidean spaces (see for example [23]). Since then, the subject of analysis on metric measure spaces has overcome remarkable developments. The corner-stones of these results are contained in the book by Coifman and Weiss [8] and the in two papers by Macías and Segovia [17] and [16].

In the last decade, motivated by problems in PDE's and in fractal geometry, some of the relevant work is related to the functional spaces

associated to elliptic equations. In particular, extensions of Sobolev spaces to these settings are consider in [15], [10], [6], [21] and [18]. On the other hand for fractals, the work by Kigami in [14] and Strichartz in [24] using scaling invariances deal with extensions of the idea of harmonic functions starting from energy methods.

On the other hand, a more adapted differentiation theory in these settings seems to be provided by fractional analysis. In the Euclidean space $\mathbb{R}^n$, for $0 < s < 2$, the fractional derivative of order s of f is given by

$$D^s f(x) = p.v. \int \frac{f(x) - f(y)}{|x - y|^{n+s}} dy, \qquad (1.1)$$

see [20]. The above formula is a kernel representation of the generalized Dirichlet to Neumman operator (see [5]).

The solution of the diffusion problem

$$\begin{cases} \frac{\partial u}{\partial t} & = -D^s u, \quad \text{in } \mathbb{R}_+^{n+1}, \\ u(x,0) & = f(x), \quad \text{in } \mathbb{R}^n, \end{cases} \qquad (1.2)$$

for adequate initial data f is provided by the Fourier transform with respect to the space variable

$$\widehat{u}(\xi, t) = e^{-|\xi|^s t} \widehat{f}(\xi).$$

In the classical context, the theory of parabolic equations with pseudo-differential operators of fractional type can be seen in [9].

The kernel approach for D^s contained in (1.1) can be generalized to metric measure spaces. Nevertheless the classical Fourier method is not available in such a general setting. In [4] the authors make use of the Haar system in $\mathbb{R}$ do consider a particular fractional derivative. Since the seminal work of M. Christ (see [7]) on the existence of dyadic families on spaces of homogeneous type, the basic tool to provide a dyadic distance associated to a Haar-Fourier analysis is also available.

As we shall describe with some detail in §2, on a space of homogeneous type (quasi-metric space with a doubling measure, such as manifolds and typical fractals) dyadic tilings and Haar type bases can be constructed. These two ingredients give the basic tools to have natural fractional diffusions on each quadrant of X with respect to the dyadic distance. The eigenfunctions of such fractional derivatives are precisely the Haar functions and the eigenvalues have the expected size related to scale.

In order to introduce our results we will sketch here briefly the basic ingredients of our theory, although in Section 2 we shall be more precise.

Let $\mathscr{D}$ be the dyadic family in the space of homogeneous type (X, d, μ) (see [7]). Let $\delta(x,y) = \inf\{\mu(Q) : x, y \in Q \text{ and } Q \in \mathscr{D}\}$. Then δ is a distance on each quadrant of X with respect to $\mathscr{D}$. Associated to the

system $\mathscr{D}$ we also have Haar bases $\mathcal{H}$ for $L^2(X,\mu)$ (see [3]). For $0 < \sigma < 1$, set

$$D^\sigma f(x) = \int_X \frac{f(x) - f(y)}{\delta(x,y)^{1+\sigma}} d\mu(y)$$

which is well defined if f belongs to the linear span of $\mathcal{H}$. The next statement summarizes the more relevant results contained in this paper.

THEOREM. Let $0 < \sigma < 1$ be given. Then,

(1) **(spectral analysis of D^σ)** for each $h \in \mathcal{H}$ we have

$$D^\sigma h(x) = m_h \mu(Q(h))^{-\sigma} h(x), \tag{1.3}$$

with m_h bounded above and below by positive constants and $Q(h)$ denotes the dyadic cube on which h is based;

(2) **(diffusions induced by D^σ)** for $f \in \mathbb{L}^p(X,\mu)$, $1 < p < \infty$, the function u defined in $X \times \mathbb{R}^+$ by

$$u(x,t) = \sum_{h \in \mathcal{H}} e^{-m_h \mu(Q(h))^{-\sigma} t} \langle f, h \rangle h(x)$$

solves the problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) &= -D^\sigma u(x,t), \quad x \in X, t > 0, \\ u(x,0) &= f(x), \qquad x \in X \end{cases} \tag{1.4}$$

in the L^p-sense;

(3) **(pointwise convergence to the initial data)** with f and u as before we have, for some constant C, that

$$\sup_{t>0} |u(x,t)| \leq C M_{dy} f(x),$$

where M_{dy} is the dyadic Hardy-Littlewood maximal operator associated to the dyadic family $\mathscr{D}$. Hence $u(x,t)$ tends to $f(x)$ for almost every $x \in X$ as $t \to 0^+$.

The paper is organized as follows. In Section 2 we introduce the geometric and analytic settings. Section 3 is devoted to statement (1) in the above theorem, Section 4 to statement (2) and Section 5 to statement (3). In the last section we provide a particular non classical example; the Sierpinski triangle.

2. **Dyadic and Haar systems on spaces of homogeneous type**

A quasi-metric on the set X is a nonnegative symmetric function $\rho(x,y)$ defined on $X \times X$ which vanishes if and only if $x = y$ and satisfies the

following weak form of the triangle inequality

$$\rho(x, y) \le K \left(\rho(x, z) + \rho(z, y) \right)$$

for some constant $K \ge 1$ and every $x, y, z \in X$.

In the non-metric case, when $K > 1$, it is still possible to give a topology in X through the neighborhood system induced by ρ.

It is worthy noticing that with this topology the ρ balls

$$B_\rho(x, r) = \{ y \in X : \rho(x, y) < r \}$$

could be excluded from the topology. Even more, the balls may not be Borel sets (see [1], also [12]). R. Macías and C. Segovia in [17] show that it is always possible to change the quasi-metric ρ by another equivalent for which the balls are open sets. From the analytical point of view, the change from one quasi-metric to another equivalent is generally possible when we are dealing with positive operators.

With the above remarks we proceed to introduce our definition of space of homogeneous type. A space of homogeneous type is a quasi-metric space (X, d) equipped with a measure μ defined on a σ-algebra containing the ρ-balls such that for some constant $A \ge 1$ the inequalities

$$0 < \mu(B(x, 2r)) \le A\mu(B(x, r)) < \infty$$

hold for every $x \in X$ and every $r > 0$.

Based on the construction in [7], more recently, in [2], [3] and [13] dyadic families on spaces of homogeneous type are built. We shall proceed to state only the properties of those families which will be relevant in our further work.

Given a space of homogeneous type (X, d, μ), a dyadic system in (X, d, μ) is a family $\mathscr{D}$ of measurable subsets of X that can be organized according to scales as

$$\mathscr{D} = \bigcup_{j \in \mathbb{Z}} \mathscr{D}^j.$$

For fixed $j \in \mathbb{Z}$ the family $\mathscr{D}^j$ gathers all the *cubes* $Q \in \mathscr{D}$ of j-th level. We assume, as usual, that the resolution increases when $j \to \infty$. Nevertheless some caveat is needed regarding the intuition of the families $\mathscr{D}^j$ since our setting contains bounded or purely atomic spaces. In the euclidean setting the families $\mathscr{D}^i$ and $\mathscr{D}^j$ are disjoint when $i \ne j$. This is no longer true in general. Let us list the basic properties of the cubes in $\mathscr{D}$:

(1) For Q and Q' in $\mathscr{D}^j$ we have that $Q = Q'$ or $Q \cap Q' = \emptyset$;

(2) for each $j \in \mathbb{Z}$, $X = \bigcup_{Q \in \mathscr{D}^j} Q$;

(3) there exists $\theta \in (0, 1)$ such that for each $Q \in \mathscr{D}^j$ there exists $x_Q \in X$ with

$$B(x_Q, c_1 \theta^j) \subseteq Q \subseteq B(x_Q, c_2 \theta^j)$$

for some constants c_1 and c_2;

(4) for a given $Q \in \mathscr{D}^j$ the number of elements of the set

$$\mathcal{L}_Q = \{Q' \in \mathscr{D}^{j+1} : Q' \subseteq Q\}$$

of the *children* of Q is bounded above by a geometric constant $N \in \mathbb{N}$, i.e. $1 \le \#\mathcal{L}_Q \le N$;

(5) X can be written as the disjoint union of a finite number $\mathcal{K}_1, \mathcal{K}_2, \ldots,$ $\mathcal{K}_n$ of quadrants

$$\mathcal{K}(Q) = \bigcup \{Q' \in \mathscr{D} : Q \subseteq Q'\}.$$

Moreover, each $(\mathcal{K}_i, d, \mu)$ is a space of homogeneous type.

Once such a $\mathscr{D}$ is given, an orthonormal Haar type basis for $L^2(X, \mu)$ supported by $\mathscr{D}$ can be constructed (see [3]). Let us briefly sketch the construction of $\mathcal{H}$.

Set V^j to denote the closed subspace of $L^2(X, \mu)$ of those functions which are constant on each $Q \in \mathscr{D}^j$. The sequence $\{V_j : j \in \mathbb{Z}\}$ provides the multiresolution analysis for $L^2(X, \mu)$ which in the metric sense satisfies a weak self-similarity property. The projector P_j from $L^2(X, \mu)$ onto V_j provides a good approximation of the identity for $j \to +\infty$ and $P_j f \to 0$ as $j \to -\infty$ pointwise and in $L^p(X, \mu)$ when $\mu(X) = +\infty$ and $P_j f = \mu(X)^{-1} \int_X f \, d\mu$ for every $j \le j_0$, when $\mu(X) < +\infty$.

Let W^j be the orthogonal complement of V^j in V^{j+1}. In other words $V^{j+1} = V^j \oplus W^j$. The system

$$\left\{ \frac{\chi_Q}{\sqrt{\mu(Q)}} : Q \in \mathscr{D}^j \right\}$$

is an orthonormal basis for V^j. This basis can be completed to an orthonormal basis of V^{j+1} by adding a particular basis $\mathcal{H}^j$ of W^j. This basis $\mathcal{H}^j$ has the following properties:

(1) for each $h \in \mathcal{H}^j$ there exist a cube $Q = Q(h) \in \mathscr{D}^j$ such that h vanishes outside $Q(h)$ and is minimal with this property, occasionally we will say that h is based on Q;

(2) h is orthogonal to $\chi_{Q(h)}$ in $L^2(X, \mu)$, i.e. $\int_X h \, d\mu = 0$.

(3) each $h \in \mathcal{H}^j$ belongs to V^{j+1}, i.e. h is constant on each cube of $\mathscr{D}^{j+1}$;

(4) for each $Q \in \mathscr{D}^j$ there exist $\#\mathcal{L}_Q - 1$ functions $h \in \mathcal{H}^j$ such that $Q(h) = Q$.

The Haar system $\mathcal{H} = \bigcup_{j \in \mathbb{Z}} \mathcal{H}^j$, is an orthonormal basis for $L^2(X, \mu)$ if $\mu(X) = +\infty$. When the measure of X is finite, equivalently, when X is bounded, $\mathcal{H} \cup \{\mu(X)^{-1/2}\}$ is an orthonormal basis of $L^2(X, \mu)$. To avoid the frequent distinction of the cases $\mu(X) = +\infty$ and $\mu(X) < +\infty$ we

shall write $\mathbb{L}^p$ to denote the space $L^p(X, \mu)$ when $\mu(X) = +\infty$ and, when $\mu(X) < +\infty$,

$$\mathbb{L}^p = \{f \in L^p(X, \mu) : \int_X f d\mu = 0\}.$$

For a given $Q \in \mathscr{D}^j$ we shall write $h_Q^\ell : \ell = 1, \ldots, \#\mathcal{L}_Q - 1$ to identify each one of the wavelets $h \in \mathcal{H}$ for which $Q(h) = Q$.

It is known that $\mathcal{H}$ is an unconditional basis for $\mathbb{L}^p$, for $1 < p < \infty$, in particular $f = \sum_{h \in \mathcal{H}} \langle f, h \rangle h$ in the $\mathbb{L}^p$ sense and

$$C_1 \|f\|_p \leq \left\| \left(\sum_{h \in \mathcal{H}} |\langle f, h \rangle|^2 |h|^2 \right)^{1/2} \right\|_p \leq C_2 \|f\|_p \tag{2.1}$$

for some constants C_1 and C_2 and every $f \in \mathbb{L}^p$ (see [3]).

It is worthy at this point noticing that the expression space of homogeneous type include still quite inhomogeneous situation. For our purposes in Section 4 we shall consider the question of whether or not cubes of the same scale have comparable measures. It turns out that this is not true even in some elementary situation on the real numbers. Consider for example the space of homogeneous type $X = \mathbb{R}$, $d(x, y) = |x - y|$ and $\mu(E) = \int_E |x|^{-1/2} dx$. The standard dyadic intervals in $\mathbb{R}$ constitute a dyadic family satisfying the desired properties. Nevertheless $\mu((n, n+1)) \simeq n^{-1/2}$ for $n \in \mathbb{N}$. In the proof of Theorem 4.2 we shall need a classification of the Haar system $\mathcal{H}$ in terms of the measure of cubes in which the functions are based. We shall write

$$\mathcal{H}_\mu^j := \{h \in \mathcal{H} : 2^{-j} \leq \mu(Q(h)) \leq 2^{-j+1}\}.$$

Observe that the families $\mathcal{H}_\mu^j$ are pairwise disjoint and also satisfy that $\mathcal{H} = \cup_{j \in \mathbb{Z}} \mathcal{H}_\mu^j$.

3. Spectral analysis of the dyadic fractional differential operator

The first result in this section is an elementary lemma which reflects the one dimensional character of X equipped with the distance δ. For the sake of notational simplicity we shall write $A \simeq B$ when the quotient A/B is bounded above and below by positive and finite constants. In a similar way we write $A \lesssim B$ when A/B is bounded above.

LEMMA 3.1. *Let $0 < \epsilon < 1$, and let Q be a given dyadic cube in X. Then, for $x \in Q$, we have*

$$\int_{X \setminus Q} \frac{d\mu(y)}{\delta(x, y)^{1+\epsilon}} \simeq \mu(Q)^{-\epsilon}$$

and

$$\int_Q \frac{d\mu(y)}{\delta(x,y)^{1-\epsilon}} \simeq \mu(Q)^\epsilon.$$

P r o o f. In order to prove the estimate over $X \backslash Q$, suppose that Q belongs to $\mathscr{D}^j$. For every $k \in \mathbb{N}$ let Q^k denote the only dyadic cube of $\mathscr{D}^{j-k}$ that contains Q.

Hence

$$
\begin{aligned}
\int_{X \backslash Q} \delta(x,y)^{-1-\epsilon} d\mu(y) &= \sum_{k=1}^\infty \int_{Q^k \backslash Q^{k-1}} \delta(x,y)^{-1-\epsilon} d\mu(y) \\
&= \sum_{k=1}^\infty \left[\mu\left(Q^k\right) - \mu\left(Q^{k-1}\right) \right] \mu\left(Q^k\right)^{-1-\epsilon} \\
&= \sum_{k=1}^\infty \left[1 - \frac{\mu\left(Q^{k-1}\right)}{\mu\left(Q^k\right)} \right] \mu\left(Q^k\right)^{-\epsilon}. \quad (3.1)
\end{aligned}
$$

Let us observe that for the non-vanishing terms in the above series we necessarily have that the measure of a any cube and its "father" are proportional independently of the scale, then

$$0 < C_1 \le \frac{\mu\left(Q^{k-1}\right)}{\mu\left(Q^k\right)} \le C_2 < 1,$$

where C_1 and C_2 are constant independent of k and Q. Generalizing the above expression by induction we have that

$$C_1^k \le \frac{\mu\left(Q\right)}{\mu\left(Q^k\right)} \le C_2^k. \quad (3.2)$$

Therefore

$$
\begin{aligned}
\sum_{k=1}^\infty \left[1 - \frac{\mu\left(Q^{k-1}\right)}{\mu\left(Q^k\right)} \right] \mu\left(Q^k\right)^{-\epsilon} &\le (1 - C_1) \sum_{k=1}^\infty \left(C_2^{-k} \mu(Q) \right)^{-\epsilon} \\
&= \mu(Q)^{-\epsilon} (1 - C_1) \sum_{k=1}^\infty \left(C_2^\epsilon \right)^k \\
&= \mu(Q)^{-\epsilon} \frac{1 - C_1}{C_2^{-\epsilon} - 1}. \quad (3.3)
\end{aligned}
$$

In an analogous way we can get the following estimate

$$\sum_{k=1}^\infty \left[1 - \frac{\mu\left(Q^{k-1}\right)}{\mu\left(Q^k\right)} \right] \mu\left(Q^k\right)^{-\epsilon} \ge \mu(Q)^{-\epsilon} \frac{1 - C_2}{C_1^{-\epsilon} - 1}. \quad (3.4)$$

So, from (3.1), (3.3) and (3.4) we have that

$$\frac{1-C_2}{C_1^{-\epsilon}-1}\mu(Q)^{-\epsilon} \leq \int_{X\setminus Q}\delta(x,y)^{-1-\epsilon}d\mu(y) \leq \frac{1-C_1}{C_2^{-\epsilon}-1}\mu(Q)^{-\epsilon}.$$

The proof of the second identity follows the same lines.   □

The dyadic fractional differential operator of order σ, with $0 < \sigma < 1$, is given by

$$D^\sigma f(x) = \int_X \frac{f(x)-f(y)}{\delta(x,y)^{1+\sigma}}d\mu(y), \tag{3.5}$$

provided that the above expression is finite. Lipschitz spaces are suitable domains for these operators. Let $\Lambda^r(X,\delta,\mu)$ be the space of bounded Lipschitz continuous function of order r, i.e. $f \in \Lambda^r(X,\delta,\mu)$ when $f \in L^\infty(X,\mu)$ and there exists a positive constant C such that for every $x,y \in X$ we have that

$$|f(x)-f(y)| \leq C\delta(x,y)^r. \tag{3.6}$$

This space turns out to be a Banach space once it is equipped with the norm

$$\|f\|_{\Lambda_r} := \|f\|_{L^\infty} + [f]_{\Lambda_r},$$

where $[f]_{\Lambda_r}$ denotes the seminorm induced by the infimum of the constant that satisfies (3.6), i.e.

$$[f]_{\Lambda_r} := \sup_{x\in X}\sup_{y\in X\setminus\{x\}}\frac{|f(x)-f(y)|}{d(x,y)^r}.$$

LEMMA 3.2.   *Let* $0 < \sigma < 1$. *For every function* $f \in \Lambda^r(X,\delta,\mu)$, *with* $\sigma < r < 1$, $D^\sigma f$ *is well-defined and moreover*

$$\|D^\sigma f\|_{L^\infty} \leq C\|f\|_{\Lambda^r}. \tag{3.7}$$

P r o o f. Let Q be any cube containing x, then

$$\int_X \frac{|f(x)-f(y)|}{\delta(x,y)^{1+\sigma}}d\mu(y) \leq \int_Q \frac{|f(x)-f(y)|}{\delta(x,y)^{1+\sigma}}d\mu(y) + \int_{X\setminus Q}\frac{|f(x)-f(y)|}{\delta(x,y)^{1+\sigma}}d\mu(y)$$

$$\leq [f]_{\Lambda^r}\int_Q \frac{d\mu(y)}{\delta(x,y)^{1-(r-\sigma)}} + 2\|f\|_{L^\infty}\int_{X\setminus Q}\frac{d\mu(y)}{\delta(x,y)^{1+\sigma}}.$$

Hence, by Lemma 3.1 we obtain that

$$|D^\sigma f(x)| \leq \int_X \frac{|f(x)-f(y)|}{\delta(x,y)^{1+\sigma}}d\mu(y) \leq C\|f\|_{\Lambda^r},$$

and the proof is complete.   □

As a consequence of the above lemma we can conclude that the operator D^σ is well-defined on the linear span $S(\mathcal{H})$ of the Haar system $\mathcal{H}$, since the Haar functions are Lipschitz functions of order 1 with respect to δ.

In [4] the authors prove that the Haar functions are eigenfunctions of D^σ in $\mathbb{R}^+$. The following theorem shows that we can extend the result to spaces of homogeneous type and the corresponding eigenvalues have the expected size.

THEOREM 3.1.   *Let $0 < \sigma < 1$. For each $h \in \mathcal{H}$ we have*

$$D^\sigma h(x) = m_h \mu(Q(h))^{-\sigma} h(x), \tag{3.8}$$

where m_h is a constant that could depends of $Q(h)$ but there exists two finite and positive constants m_1 and m_2 such that

$$m_1 < m_h < m_2, \quad \text{for all } h \in \mathcal{H}. \tag{3.9}$$

Before starting the proof of the above theorem we would like to comment on the integrability properties the eigenfunctions of D^σ. In the classical Euclidean case the eigenfunctions of the fractional Laplacian are exponential functions which do not belong to L^2. Our setting is quite different. Each eigenfunction belongs to L^∞ and has bounded support independently of the boundedness of the whole space itself. This fact seems to be in contradiction with the nonlocal character of D^σ, since the support of D^σ applied to some function f is expected to be larger than the support of f itself, eventually the whole space X. However, as becomes evident in the proof of Theorem 3.1, the geometric fact that $\delta(x, y)$ is constant for $y \in Q \in \mathscr{D}$ whenever $x \notin Q$, implies that the operator D^σ applied to a vanishing mean function supported on a dyadic cube has bounded support. This is the case of the Haar functions system.

P r o o f. Notice that for $Q, Q' \in \mathscr{D}$, with $Q \cap Q' = \emptyset$, we have that $\delta(x, y)$ is constant for all $x \in Q$ and all $y \in Q'$. Moreover

$$\delta(x, y) = \mu(\widetilde{Q}), \quad \text{for all } x \in Q \text{ and all } y \in Q'. \tag{3.10}$$

where $\widetilde{Q}$ is the first common ancestor of Q and Q'.

Let h be a fixed Haar function, $Q = Q(h)$ and $x \in X \backslash Q$. Since h has its support contained in Q then $h(x) = 0$, then in order to prove (3.8) we must see that $D^\sigma h(x) = 0$. Observe that

$$D^\sigma h(x) = \int_{X \backslash Q} \frac{-h(y)}{\delta(x, y)^{1+\sigma}} d\mu(y) + \int_Q \frac{-h(y)}{\delta(x, y)^{1+\sigma}} d\mu(y). \tag{3.11}$$

The first integral of the right hand side is zero since $h(y) \equiv 0$ for all $y \in X \backslash Q$. For the second integral, since $x \notin Q$ and $y \in Q$, we apply

(3.10) to obtain

$$\int_Q \frac{-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y) = -\mu(\widetilde{Q})^{-1-\sigma}\int_Q h(y)d\mu(y) = 0.$$

Therefore, we have proved (3.8) for $x \in X\backslash Q$.

Suppose now that $x \in Q$. Let us denote with Q^* the child of Q which contanis x. Hence

$$\int_Q \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y) = \int_{Q^*} \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y)$$
$$+ \int_{Q\backslash Q^*} \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y).$$

Since h is constant on each child of Q, then the integral over Q^* vanishes. Note that in the integral over $Q\backslash Q^*$ we have that $\delta(x,y) = \mu(Q)$, then

$$\int_Q \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y) = \int_{Q\backslash Q^*} \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y)$$
$$= \mu(Q)^{-1-\sigma}\int_{Q\backslash Q^*} h(x)-h(y)d\mu(y)$$
$$= \mu(Q)^{-1-\sigma}\int_Q h(x)-h(y)d\mu(y)$$
$$= \mu(Q)^{-1-\sigma}\left[\int_Q h(x)d\mu(y) - \int_Q h(y)d\mu(y)\right]$$
$$= \mu(Q)^{-1-\sigma}[h(x)\mu(Q) - 0]$$
$$= \mu(Q)^{-\sigma}h(x). \tag{3.12}$$

Notice also that

$$\int_{X\backslash Q} \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y) = h(x)\int_{X\backslash Q}\delta(x,y)^{-1-\sigma}d\mu(y). \tag{3.13}$$

Therefore, from (3.12) and (3.13) it holds that

$$D^\sigma h(x) = \int_Q \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y) + \int_{X\backslash Q} \frac{h(x)-h(y)}{\delta(x,y)^{1+\sigma}}d\mu(y)$$
$$= \mu(Q)^{-\sigma}h(x) + h(x)\int_{X\backslash Q}\delta(x,y)^{-1-\sigma}d\mu(y)$$
$$= \left[1 + \mu(Q)^\sigma\int_{X\backslash Q}\delta(x,y)^{-1-\sigma}d\mu(y)\right]\mu(Q)^{-\sigma}h(x)$$
$$=: m_h\mu(Q)^{-\sigma}h(x). \tag{3.14}$$

Finally, applying Lemma 3.1 we get that

$$0 < 1 + \frac{1 - C_2}{C_1^{-\sigma} - 1} \le m_h \le 1 + \frac{1 - C_1}{C_2^{-\sigma} - 1} < \infty,$$

and the proof is complete. □

4. Diffusions induced by D^σ

The first part of this section is devoted to find a large class of functions f defined on X for which D^σ is well defined. It turns out that, if we think in terms of weak derivatives of distributions, the class is given by a natural extension of the weighted L^1 space introduced by L. Silvestre in [22]. After that we give a wavelet (Haar) characterization of the corresponding Sobolev spaces. Then we precisely state and prove the statement 2 of our main result contained in the introduction. The next lemma provides a bound for the fractional derivative of a regular function.

LEMMA 4.1. Let $0 < \sigma < r < 1$ and $f \in \Lambda^r(X, \delta, \mu)$ with $K := \text{supp}(f)$ bounded. Then for any $x_0 \in K$ and $x \in X$ we have that

$$|D^\sigma f(x)| \le C \frac{\|f\|_{\Lambda^r}}{1 + \delta(x, x_0)^{1+\sigma}}, \tag{4.1}$$

where C is a constant that depends on σ, r and K.

P r o o f. Given $x \in B(x_0, 2\,\text{diam}(K))$ where $\text{diam}(K) := \sup\{\delta(x, y) : x, y \in K\}$, by Lemma 4.1 we have that

$$|D^\sigma f(x)| \le C(1 + (2\,\text{diam}(K))^{1+\sigma}) \frac{\|f\|_{\Lambda^r}}{1 + \delta(x, x_0)^{1+\sigma}}. \tag{4.2}$$

On the other hand, if $x \notin B(x_0, 2\,\text{diam}(K))$ it is easy to see that $\delta(x, x_0) \le 2\delta(x, y)$ for all $y \in K$. Hence, since $x \notin K$ we get that

$$|D^\sigma f(x)| \le \int_K \frac{|f(y)|}{\delta(x, y)^{1+\sigma}} d\mu(y) \le \frac{\|f\|_{L^\infty} \mu(K)}{\delta(x, x_0)^{1+\sigma}}. \tag{4.3}$$

Therefore from (4.2) and (4.3) we obtain (4.1) and the proof is complete.
□

The above estimate allows us to extend D^σ, by duality, to a greater class of functions. The duality will be interpreted in terms of the theory of distributions on spaces of homogeneous type introduced by R. Macías and C. Segovia in [16]. Let us briefly skecht it. For a fixed positive γ, following [16] we define the test space $\mathcal{D}$ as the space of those functions φ with bounded support such that φ belongs to every Lipschitz space Λ^r with respect to δ for $r < \gamma$. For $x_0 \in X$ fixed and $n \in \mathbb{N}$ define $\mathcal{D}_n$ as the class of those functions in $\mathcal{D}$ with support in $B(x_0, n)$. The space $\mathcal{D}_n$

has the natural metrizable topology induced by the family of seminorms $\{[\cdot]_{\Lambda^{\gamma-1/k}} : k \in \mathbb{N}\}$. As in the classical euclidean case $\mathcal{D}$ can be equipped with the inductive limit topology τ. The space of distributions $\mathcal{D}'$ is, then, the dual of $(\mathcal{D}, \tau)$. The central fact regarding this topology on $\mathscr{D}$ is provided be the next lemma.

LEMMA 4.2. *Let* $T : \mathcal{D} \to \mathbb{C}$ *be a lineal functional. Then* T *is continuous if and only if for every sequence* $\{\phi_k\}$ *in* $\mathcal{D}$ *such that the supports of* ϕ_k *are uniformly bounded and* $\|\phi_k\|_{\Lambda^r} \to 0$ *when* $k \to \infty$ *for all* $r < \gamma$, *we have that*

$$\langle T, \phi_k \rangle := T(\phi_k) \to 0,$$

when $k \to \infty$.

The next lemma shows that the natural extension of Silvestre's class (see [22]) works in our setting.

LEMMA 4.3. *Let* $f \in L^1_{loc}(X, \mu)$ *such that*

$$\int_X \frac{|f(x)|}{(1 + \delta(x, x_0))^{\alpha+\sigma}} \, d\mu(x) < \infty.$$

Then the functional $D^\sigma f$ *defined on* $\mathcal{D}$ *by*

$$\langle D^\sigma f, \varphi \rangle := \langle f, D^\sigma \varphi \rangle,$$

belongs to $\mathcal{D}'$.

P r o o f. Given $\varphi \in \mathcal{D}(X, \delta, \mu)$, by Lemma 4.1 we have that

$$\begin{aligned}
|\langle D^\sigma f, \varphi \rangle| &\leq \int_X |f(x)||D^\sigma \varphi(x)| \, d\mu(x) \\
&\leq C\|\varphi\|_{\Lambda^r} \int_X \frac{|f(x)|}{(1 + \delta(x, x_0))^{\alpha+\sigma}} \, d\mu(x) \\
&\leq C\|\varphi\|_{\Lambda^r}.
\end{aligned} \tag{4.4}$$

Hence $D^\sigma f$ is well-defined. Moreover, since the constant C in (4.4) only depends on K, by Lemma 4.2, $D^\sigma f \in \mathcal{D}'$. □

Notice that if $f \in \mathbb{L}^p$, with $1 < p < \infty$, then

$$\int_X \frac{|f(x)|}{(1 + \delta(x, x_0))^{1+\sigma}} \, d\mu(x) \leq \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}$$
$$\times \left(\int_X \frac{1}{(1 + \delta(x, x_0))^{(1+\sigma)p'}} \, d\mu(x) \right)^{\frac{1}{p'}} \leq C\|f\|_{L^p},$$

and therefore $D^\sigma f \in \mathcal{D}'(X, d, \mu)$. We say that a function $f \in \mathbb{L}^p$, with $1 < p < \infty$, is in the Sobolev space $\mathbb{L}_\sigma^p(X, \delta, \mu)$ if the weak fractional derivative $D^\sigma f$ is a function of $\mathbb{L}^p$, i.e., if there exist $g \in \mathbb{L}^p$ such that for all $\phi \in \mathcal{D}(X, \delta, \mu)$ we have that

$$\langle D^\sigma f, \phi \rangle = \int_X g\phi \, d\mu.$$

Even more, we can endow this space with the norm given by

$$\|f\|_{L_\sigma^p} := \|f\|_{L^p} + \|D^\sigma f\|_{L^p}.$$

Let us point out here that there are other ways of defining Sobolev spaces in metric measure spaces (see [15], [10], [6], [11], [21] and [18]). The preceding gives to the fractional differential operator the role that the gradient plays in the classical cases and, of course, brings us back a trivial but basic property: $D^\sigma : \mathbb{L}_\sigma^p(X, \delta, \mu) \to \mathbb{L}^p$ continuously.

We want to point out that Theorem 3.1 allows us to give an alternative definition of D^σ. In fact, given $f \in S(\mathcal{H})$ we that the series defining f, i.e.

$$f(x) = \sum_{h \in \mathcal{H}} \langle f, h \rangle h(x),$$

has only finite non vanishing terms. Then

$$D^\sigma f(x) = \sum_{h \in \mathcal{H}} m_h \mu(Q(h))^{-\sigma} \langle f, h \rangle h(x).$$

The following theorem shows that the above expression remains valid for $f \in \mathbb{L}_\sigma^p(X, \delta, \mu)$. Moreover, we will give a characterization of the Sobolev space $\mathbb{L}_\sigma^p$ in terms of the Haar system.

THEOREM 4.1. *Let $0 < \sigma < 1$, $1 < p < \infty$ and $f \in \mathbb{L}_\sigma^p(X, \delta, \mu)$, then*

$$D^\sigma f(x) = \sum_{h \in \mathcal{H}} m_h \mu(Q(h))^{-\sigma} \langle f, h \rangle h(x), \tag{4.5}$$

where the series converge in $\mathbb{L}^p$. Hence the space $\mathbb{L}_\sigma^p(X, \delta, \mu)$ coincides with the set of function of $\mathbb{L}^p$ such that

$$\left(\sum_{h \in \mathcal{H}} \mu(Q(h))^{-2\sigma} |\langle f, h \rangle|^2 |h|^2 \right)^{\frac{1}{2}} \in \mathbb{L}^p.$$

Moreover, given $\mathcal{H}_1 = \{h \in \mathcal{H} : \mu(Q(h)) < 1\}$ we have that

$$\|f\|_{L^p} + \left\| \left(\sum_{h \in \mathcal{H}_1} \mu(Q(h))^{-2\sigma} |\langle f, h \rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

is equivalent to the norm $\|f\|_{L_\sigma^p}$.

P r o o f. Since every Haar function $h \in \mathcal{H}$ is a Lipschitz function with respect to δ and has bounded support we have $h \in \mathcal{D}(X, \delta, \mu)$. Hence for every $f \in \mathbb{L}^p$ we have that

$$\langle D^\sigma f, h \rangle = \langle f, D^\sigma h \rangle.$$

By Theorem 3.1 we know that $D^\sigma h = m_h \mu(Q(h))^{-\sigma} h$, and thereby

$$\langle D^\sigma f, h \rangle = m_h \mu(Q(h))^{-\sigma} \langle f, h \rangle. \tag{4.6}$$

Suppose that $f \in \mathbb{L}^p_\sigma(X, \delta, \mu)$ then $D^\sigma f \in \mathbb{L}^p$. Since $\mathcal{H}$ is an unconditional basis of $\mathbb{L}^p$ we know that

$$
\begin{aligned}
D^\sigma f &= \sum_{h \in \mathcal{H}} \langle D^\sigma f . h \rangle h \\
&= \sum_{h \in \mathcal{H}} m_h \mu(Q(h))^{-\sigma} \langle f . h \rangle h,
\end{aligned}
$$

which proves (4.5). On the other hand, from (2.1) we can assure that

$$\|D^\sigma f\|_{L^p} \simeq \left\| \left(\sum_{h \in \mathcal{H}} |\langle D^\sigma f, h \rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.$$

Hence by (4.6) we get that

$$\|D^\sigma f\|_{L^p} \simeq \left\| \left(\sum_{h \in \mathcal{H}} \mu(Q(h))^{-2\sigma} |\langle f, h \rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.$$

On the other hand, since $f \in \mathbb{L}^p$ we know that the $\mathbb{L}^p$ norm of the function

$$\left(\sum_{h \in \mathcal{H} \setminus \mathcal{H}_1} \mu(Q(h))^{-2\sigma} |\langle f, h \rangle|^2 |h|^2 \right)^{\frac{1}{2}}$$

is bounded by the $\mathbb{L}^p$ norm of f and so the proof is complete. $\square$

We are now in position to formulate and prove the statement (2) of our main result stated in Section 1.

THEOREM 4.2. Let $0 < \sigma < 1$, $1 < p < \infty$ and $f \in \mathbb{L}^p$ be given. Then, the function u defined on $\mathbb{R}^+$ by

$$u(t) := \sum_{h \in \mathcal{H}} e^{-m_h \mu(Q(h))^{-\sigma} t} \langle f, h \rangle h,$$

(1) belongs to $\mathbb{L}^p_\sigma(X, \delta, \mu)$ for every $t > 0$;
(2) solves the problem

$$
\begin{cases}
u_t &= -D^\sigma u, \qquad t > 0, \\
u(0) &= f, \qquad\quad\ \text{on } X,
\end{cases}
\tag{4.7}
$$

in the Fréchet sense for $\mathbb{L}^p$. Precisely u_t is the Fréchet derivative of u in $\mathbb{L}^p$ and $\lim_{t\to 0} u(t) = f$ in the $\mathbb{L}^p$ norm.

P r o o f. Since $m_h\mu(Q(h))^{-\sigma}t$ is positive then $|e^{-m_h\mu(Q(h))^{-\sigma}t}| \le 1$. Hence by (2.1) we know that $\|u(\cdot, t)\|_{L^p} \le \|f\|_{L^p}$ for all $t > 0$. Furthermore, since $0 < m_1 < m_h$ then

$$0 \le \mu(Q(h))^{-2\sigma} e^{-2m_h\mu(Q(h))^{-\sigma}t} \le (em_1 t)^{-2}. \tag{4.8}$$

So for each $t > 0$ we have that

$$\left\| \left(\sum_{h\in\mathcal{H}} \mu(Q(h))^{-2\sigma}|\langle u(\cdot,t), h\rangle|^2|h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$= \left\| \left(\sum_{h\in\mathcal{H}} \mu(Q(h))^{-2\sigma} e^{-2m_h\mu(Q(h))^{-\sigma}t}|\langle f, h\rangle|^2|h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$\lesssim \left\| \left(\sum_{h\in\mathcal{H}} |\langle f, h\rangle|^2|h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$\lesssim \|f\|_{L^p}. \tag{4.9}$$

Therefore, by Theorem 4.1, $u(\cdot, t) \in \mathbb{L}^p_\sigma(X, \delta, \mu)$ for each $t > 0$ and thus (1) is proved.

For the second part of the theorem, let us start by rewriting the first equality of (4.7) as

$$\left\| \frac{u(x, t+s) - u(x, t)}{s} + D^\sigma u(x, t) \right\|_{L^p} \longrightarrow 0,$$

when $s \to 0$. By (2.1) it is enough to see that for every $t > 0$,

$$\sup_{h\in\mathcal{H}} \left| \frac{e^{-m_h\mu(Q(h))^{-\sigma}(t+s)} - e^{-m_h\mu(Q(h))^{-\sigma}t}}{s} + m_h\mu(Q(h))^{-\sigma}e^{-m_h\mu(Q(h))^{-\sigma}t} \right| \to 0,$$

when $s \to 0$. This is equivalent to show that

$$\sup_{h\in\mathcal{H}} \left| \frac{e^{-m_h\mu(Q(h))^{-\sigma}t}}{s} \left[e^{-m_h\mu(Q(h))^{-\sigma}s} - 1 + m_h\mu(Q(h))^{-\sigma}s \right] \right| \longrightarrow 0, \tag{4.10}$$

when $s \to 0$. However by Taylor's theorem,

$$\left| \frac{e^{-m_h\mu(Q(h))^{-\sigma}t}}{s} \left[e^{-m_h\mu(Q(h))^{-\sigma}s} - 1 + m_h\mu(Q(h))^{-\sigma}s \right] \right|$$

$$\leq \left| \frac{e^{-m_h\mu(Q(h))^{-\sigma}t}}{s} \left[s^2 \max_{0 \leq \zeta \leq s} \left| (m_h\mu(Q(h))^{-\sigma})^2 e^{-m_h\mu(Q(h))^{-\sigma}\zeta} \right| \right] \right|$$

$$= \left| \frac{m_h^2}{\mu(Q(h))^{2\sigma}} e^{-m_h\mu(Q(h))^{-\sigma}t} s \right|$$

$$\leq \left| \frac{m_2^2}{\mu(Q(h))^{2\sigma}} e^{-m_1 t \mu(Q(h))^{-\sigma}} \right| |s|$$

$$\leq \left(\frac{2m_2}{m_1 te} \right)^2 |s|,$$

which proves (4.10) and the first equality of (2) is done.

Finally in order to prove that

$$\|u(\cdot,t) - f\|_{L^p} \longrightarrow 0, \quad \text{as } t \to 0, \tag{4.11}$$

we shall use the rearrangement of $\mathcal{H}$ presented in Section 1 by considering for each $j \in \mathbb{Z}$ the family $\mathcal{H}_\mu^j := \{ h \in \mathcal{H} : 2^{-j} \leq \mu(Q(h)) \leq 2^{-j+1} \}$. Since $f \in \mathbb{L}^p$ and we know that $\mathcal{H}$ is an unconditional base of $\mathbb{L}^p$, then for each $\epsilon > 0$ we can chose $i \in \mathbb{Z}$ large enough to have that

$$\left\| \left(\sum_{j>i} \sum_{h \in \mathcal{H}_\mu^j} |\langle f, h \rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p} < \epsilon. \tag{4.12}$$

Notice that $\mu(Q(h)) \geq 2^{-i}$ if $h \in \mathcal{H}_\mu^j$ for any $j \leq i$. So we can chose t_0 small enough such that

$$|e^{-m_h\mu(Q(h))^{-\sigma}t} - 1| \leq 1 - e^{-m_2 2^{i\sigma}t} < \epsilon, \tag{4.13}$$

for all $t < t_0$ and for all $h \in \mathcal{H}_\mu^j$, with $j \leq i$. Hence

$$\|u(\cdot,t) - f\|_{L^p} \lesssim \left\| \left(\sum_{h \in \mathcal{H}} |e^{-m_h\mu(Q(h))^{-\sigma}t} - 1| |\langle f, h \rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$\leq \left\| \left(\sum_{j \leq i} \sum_{h \in \mathcal{H}_\mu^j} |e^{-m_h \mu(Q(h))^{-\sigma}t} - 1||\langle f, h\rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$+ \left\| \left(\sum_{j > i} \sum_{h \in \mathcal{H}_\mu^j} |e^{-m_h \mu(Q(h))^{-\sigma}t} - 1||\langle f, h\rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.$$

Therefore, from (4.12) and (4.13), for every $t < t_0$ we have that

$$\|u(\cdot, t) - f\|_{L^p} \lesssim \epsilon \left\| \left(\sum_{j \leq i} \sum_{h \in \mathcal{H}_\mu^j} |\langle f, h\rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$+ 2 \left\| \left(\sum_{j > i} \sum_{h \in \mathcal{H}_\mu^j} |\langle f, h\rangle|^2 |h|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$\lesssim \epsilon \|f\|_{L^p} + 2\epsilon.$$

Thereby (4.11) is proven, concluding the proof of the theorem. $\qquad \square$

Let us remark that even when Theorem (4.2) shows that the solution u belongs to $\mathbb{L}_\sigma^p(X, \delta, \mu)$ as a function of $x \in X$ for each $t > 0$, this can be improved. Indeed, the estimation made in (4.8) can be generalized for every $\eta \geq \sigma$ in the following way:

$$0 \leq \mu(Q(h))^{-2\eta} e^{-2m_h \mu(Q(h))^{-\sigma}t} \leq \left(\frac{\eta}{e\sigma m_1 t} \right)^{\frac{2\eta}{\sigma}}.$$

Hence, repeating the steps in (4.9) and following the constants we can show that

$$\|u(\cdot, t)\|_{L_\eta^p} \lesssim \left(1 + \frac{C}{t^{\eta/\sigma}} \right) \|f\|_{L^p}.$$

Therefore u belongs to $\mathbb{L}_\eta^p(X, \delta, \mu)$ for every $\eta > \sigma$ as a function of $x \in X$ for each $t > 0$. Furthermore, it can be shown that $u_t = -D^\sigma u$ is satisfied in $\mathbb{L}_{\eta-\sigma}^p$ sense, for every $\eta > \sigma$.

5. Pointwise convergence to the initial data

As usual the pointwise convergence relies on the identification and properties of the heat kernel associated to the solution

$$u(x, t) = \sum_{h \in \mathcal{H}} e^{-m_h \mu(Q(h))^{-\sigma}t} \langle f, h\rangle h(x). \tag{5.1}$$

First of all, in order to identify the kernel we need to consider f in the linear span of the Haar system, i.e. $S(\mathcal{H})$, which is dense in $\mathbb{L}^p$ for $1 < p < \infty$. Then we study the basic properties of integrability, radiality and monotonicity of the kernel in order to perform the classical methods of harmonic analysis.

For $f \in S(\mathcal{H})$ let us start rewriting as an integral the inner product in (5.1) and changing the integration and summation orders to obtain

$$u(x,t) = \int_X \left[\sum_{h \in \mathcal{H}} e^{-m_h \mu(Q(h))^{-\sigma} t} h(y) h(x) \right] f(y) d\mu(y).$$

We shall use $k_t(x, y)$ to denote the kernel in the above equation. More precisely,

$$k_t(x, y) = \sum_{h \in \mathcal{H}} e^{-m_h \mu(Q(h))^{-\sigma} t} h(y) h(x). \tag{5.2}$$

Then, if K_t denotes the operator with kernel k_t, we have that

$$u(x,t) = \int_X k_t(x, y) f(y) d\mu(y) =: K_t f(x),$$

at least for functions in $S(\mathcal{H})$. If we prove that for fixed t the kernel $k_t(x, y)$ belongs to L^1 of each variable uniformly on the other, then the series coincides with the integral representation for every $f \in \mathbb{L}^p$ with $1 < p < \infty$. Those integrability properties of k_t will follow from Lemma 5.2.

In this section we aim to prove that

$$K^* f(x) := \sup_{t > 0} |K_t f(x)| \leq \sup_{t > 0} |u(x, t)| \leq C M_{dy} f(x), \tag{5.3}$$

for every $f \in \mathbb{L}^p$, where M_{dy} denotes the dyadic Hardy-Littlewood maximal operator. In order to do this, we shall prove two lemmas. The first one is a consequence of the basic character of $\mathcal{H}$ and it will be used in the proof of the second one, which establish that the kernel k_t is bounded above by the rescaling of an integrable radial function.

The next lemma contains a proof of the formal fact that $\sum_{h \in \mathcal{H}} h(x) h(y)$ coincides with the Dirac delta on the diagonal of $X \times X$. It shall be useful when dealing with the pointwise convergence for the unbounded case. Let us remind that given a cube $Q \in \mathscr{D}^i$, Q^j denotes the only cube in $\mathscr{D}^{i-j}$ such Q is contained in Q^j. Observe that Q^0 is in fact Q. Furthermore, let us define $\mathcal{H}_Q^j$ as the family of Haar function based on Q^j, i.e. $\mathcal{H}_Q^j = \{h \in \mathcal{H} : Q(h) = Q^j\}$.

LEMMA 5.1. Let $Q \in \mathscr{D}$ be such that $\#\mathcal{L}_Q > 1$. Assume that $x \in Q'$, $y \in Q''$ with $Q' \neq Q''$ and both Q' and Q'' belong to $\mathcal{L}_Q$, then

$$\sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} h(x)h(y) = 0. \tag{5.4}$$

Moreover,

$$-\sum_{h \in \mathcal{H}_Q^0} h(x)h(y) = \sum_{j \geq 1} \sum_{h \in \mathcal{H}_Q^j} h(x)h(y) = \mu(Q)^{-1}. \tag{5.5}$$

P r o o f. Since $x \in Q'$ and $y \in Q''$, then

$$0 = \chi_{Q''}(x) = \sum_{h \in \mathcal{H}} \langle \chi_{Q''}, h \rangle h(x)$$

$$= \sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} \langle \chi_{Q''}, h \rangle h(x)$$

$$= \sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} \left(\int_{Q''} h(z) d\mu(z) \right) h(x)$$

$$= \sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} \left(\int_{Q''} h(y) d\mu(z) \right) h(x)$$

$$= \mu(Q'') \sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} h(y)h(x).$$

So the first part is proved. In order to show (5.5) let us first notice that

$$\chi_Q(x) = \sum_{j \geq 1} \sum_{h \in \mathcal{H}_Q^j} \langle \chi_Q, h \rangle h(x).$$

Hence, by Parseval's identity we have that

$$\mu(Q) = \|\chi_Q\|_2^2 = \sum_{j \geq 1} \sum_{h \in \mathcal{H}_Q^j} |\langle \chi_Q, h \rangle|^2$$

$$= \sum_{j \geq 1} \sum_{h \in \mathcal{H}_Q^j} \left(\int_Q h(z) d\mu(z) \right)^2$$

$$= [\mu(Q)]^2 \sum_{j \geq 1} \sum_{h \in \mathcal{H}_Q^j} h(y)h(x).$$

Therefore, the proof is complete. $\qquad \square$

LEMMA 5.2. *There exists a decreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\psi \in L^1(0, \infty)$ *and*

$$|k_t(x, y)| \leq \frac{1}{t^{1/\sigma}} \psi \left(\frac{\delta(x, y)}{t^{1/\sigma}} \right).$$

P r o o f. Notice first that for fixed x and y in X, only remains in (5.2) the terms in which Q contains both x and y. We shall denote Q^0 the first common ancestor of x and y, and let $i \in \mathbb{Z}$ be such that $Q^0 \in \mathscr{D}^i$. As before, we shall denote Q^j the dyadic cube in $\mathscr{D}^{i-j}$ containing Q^0. Then

$$k_t(x, y) = \sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} e^{-m_h \mu(Q^j)^{-\sigma} t} h(y) h(x)$$

Since for every $h \in \mathcal{H}$ we have that $\|h\|_{L^2} = 1$ and the measure of a any cube and his father are proportional uniformly in the scale, it is easy to check that $\|h\|_{L^\infty} \leq C\mu \left(Q(h) \right)^{-1/2}$, then

$$|k_t(x, y)| \leq C \sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} e^{-m_h \mu(Q^j)^{-\sigma} t} \mu(Q^j)^{-1}.$$

From (3.2) we know that there exist a positive constant $C_2 > 1$ such that $\mu(Q^j)^{-1} \leq C_2^{-j} \mu(Q^0)^{-1}$. Let us recall that m_h is also bounded above by a positive constant m_1. For that reason, $e^{-m_h \mu(Q^j)^{-\sigma} t} \leq e^{-m_1 C_2^{-j\sigma} \mu(Q^0)^{-\sigma} t}$. Hence,

$$|k_t(x, y)| \leq \frac{C}{\mu(Q^0)} \sum_{j \geq 0} \sum_{h \in \mathcal{H}_Q^j} C_2^{-j} e^{-m_1 C_2^{-j\sigma} \mu(Q^0)^{-\sigma} t}.$$

We also know that the number of functions on each $\mathcal{H}_Q^j$ is uniformly bounded, then

$$|k_t(x, y)| \leq \frac{C}{\mu(Q^0)} \sum_{j \geq 0} C_2^{-j} e^{-m_1 C_2^{-j\sigma} \mu(Q^0)^{-\sigma} t}.$$

Finally, since Q^0 is the smallest dyadic cube containing x and y we know that $\delta(x, y) = \mu(Q^0)$, so

$$|k_t(x, y)| \leq \frac{C}{\delta(x, y)} \sum_{j \geq 0} C_2^{-j} e^{-m_1 C_2^{-j\sigma} \delta(x,y)^{-\sigma} t}$$

Therefore, defining $\varphi : \mathbb{R}^+ \to \mathbb{R}$ as

$$\varphi(r) = \frac{1}{r} \sum_{j \geq 0} C_2^{-j} e^{-m_1 (C_2^j r)^{-\sigma}},$$

we see that

$$|k_t(x,y)| \leq \frac{\bar{C}}{t^{1/\sigma}} \varphi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right).$$

Now we shall prove that ϕ is locally integrable. For $0 < r < 1$, fix $0 < \epsilon < 1$ and define $n(r) := \lceil -\epsilon \log_{C_2} r \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. Then we can split the series defining φ in the following way

$$\varphi(r) = \frac{1}{r} \sum_{j=0}^{n(r)} C_2^{-j} e^{-m_1(C_2^j r)^{-\sigma}} + \frac{1}{r} \sum_{j>n(r)} C_2^{-j} e^{-m_1 s(C_2^j r)^{-\sigma}}.$$

The first term is bounded since for $j \leq n(r)$ we can easily prove that $e^{-m_1(C_2^j r)^{-\sigma}} \leq C e^{-m_1 r^{-(1-\epsilon)\sigma}}$. Then

$$\frac{1}{r} \sum_{j=0}^{n(r)} C_2^{-j} e^{-m_1(C_2^j r)^{-\sigma}} \leq \frac{C}{r} e^{-m_1 r^{-(1-\epsilon)\sigma}} \sum_{j=1}^{n(r)} C_2^{-j} \leq \frac{C e^{-m_1 r^{-(1-\epsilon)\sigma}}}{r} \leq \bar{C}.$$

For the second term we see that

$$\frac{1}{r} \sum_{j>n(r)} C_2^{-j} e^{-m_1(C_2^j r)^{-\sigma}} \leq \frac{1}{r} \sum_{j>n(r)} C_2^{-j} \leq C \frac{1}{r} C_2^{-n(r)} \leq \frac{C}{r^{1-\epsilon}}.$$

Therefore, for $0 < r < 1$ we know that $\varphi(r) \leq \frac{C}{r^{1-\epsilon}}$. Since φ is bounded for all $r > 1$, then φ is locally integrable.

On the other hand, getting back to the definition of the kernel k_t, we can see that

$$k_t(x,y) = \sum_{h \in \mathcal{H}_Q^0} e^{-m_h \mu(Q^0)^{-\sigma} t} h(y)h(x) + \sum_{j\geq 1} \sum_{h \in \mathcal{H}_Q^j} e^{-m_h \mu(Q^j)^{-\sigma} t} h(y)h(x)$$

$$\leq e^{-m_h \mu(Q^0)^{-\sigma} t} \sum_{h \in \mathcal{H}_Q^0} h(y)h(x) + \sum_{j\geq 1} \sum_{h \in \mathcal{H}_Q^j} h(y)h(x).$$

Since $m_h \leq m_2$ and by Lemma 5.1 we can obtain that

$$k_t(x,y) \leq -e^{-m_2 \mu(Q^0)^{-\sigma} t} \mu(Q^0)^{-1} + \mu(Q^0)^{-1} = \frac{1}{\delta(x,y)}\left[1 - e^{-m_2 \delta(x,y)^{-\sigma} t}\right].$$

Defining $\phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\phi(r) = \frac{1}{r}\left[1 - e^{-m_2 r^{-\sigma}}\right],$$

we conclude that

$$|k_t(x,y)| \leq \frac{\bar{C}}{t^{1/\sigma}} \phi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right).$$

Notice that $\phi(r) \lesssim r^{-1-\sigma}$, for $r > 1$. Hence ϕ is integrable outside the ball center at the origin and with radius 1.

Therefore, there exist a positive constant C such the function

$$\psi(r) = C \begin{cases} r^{-(1-\epsilon)}, & \text{if } 0 < r < 1, \\ r^{-(1+\sigma)}, & \text{if } r \geq 1, \end{cases}$$

satisfies the properties stated in the lemma. □

THEOREM 5.1. Let $0 < \sigma < 1, 1 < p < \infty, f \in \mathbb{L}^p$ and u be given by (5.1). Then we have for some constant C that

$$\sup_{t>0} |u(x,t)| \leq C M_{dy} f(x),$$

where M_{dy} is the dyadic Hardy-Littlewood maximal operator associated to the dyadic family $\mathscr{D}$. Hence $u(x,t)$ tends to $f(x)$ for almost every $x \in X$ as $t \to 0^+$.

P r o o f. By Lemma 5.2 we have that

$$\begin{aligned} |K_t f(x)| &\leq \int_X |k_t(x,y)||f(y)| \, d\mu(y) \\ &\leq \int_X \frac{1}{t^{1/\sigma}} \psi \left(\frac{\delta(x,y)}{t^{1/\sigma}} \right) |f(y)| \, d\mu(y) \\ &= \sum_{j=-\infty}^{\infty} \frac{1}{t^{1/\sigma}} \int_{\{y : t^{1/\sigma} 2^j \leq \delta(x,y) < t^{1/\sigma} 2^{j+1}\}} \psi \left(\frac{\delta(x,y)}{t^{1/\sigma}} \right) |f(y)| \, d\mu(y) \\ &\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) \frac{1}{t^{1/\sigma} 2^{j+1}} \int_{B_\delta(x, t^{1/\sigma} 2^{j+1})} |f(y)| \, d\mu(y). \end{aligned}$$

Since $\mu(B_\delta(x,r)) < r$ and each B_δ is a dyadic cube, we obtain that

$$\begin{aligned} |K_t f(x)| &\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) \frac{1}{\mu(B_\delta(x, t^{1/\sigma} 2^{j+1})|)} \int_{B_\delta(x, t^{1/\sigma} 2^{j+1})} |f(y)| \, d\mu(y) \\ &\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) M_{dy} f(x) \\ &= 4 M_{dy} f(x) \sum_{j=-\infty}^{\infty} \int_{\{r : 2^{j-1} \leq r < 2^j\}} \psi(2^j) \, dr \\ &\leq 4 M_{dy} f(x) \int_{\mathbb{R}^+} \psi(r) \, dr \\ &\leq 4 \|\psi\|_{L^1} M_{dy} f(x). \end{aligned}$$

Therefore, taking supremum in t we obtain

$$\sup_{t>0} |K_t f(x)| \le 4\|\psi\|_{L^1} M_{dy} f(x). \tag{5.6}$$

Finally, as usual, the pointwise convergence to the initial data is an immediate consequence of the boundedness properties of the maximal operator associated to u and the pointwise convergence in $S(\mathcal{H})$, which is a dense subset of $\mathbb{L}^p$ $(1 < p < \infty)$. We will sketch a brief proof for the sake of completeness.

Since we already know that $K_t f \to f$ in the $\mathbb{L}^p$ sense as $t \to 0^+$, in order to prove the pointwise convergence, define

$$E = \{f \in \mathbb{L}^p : \lim_{t \to 0^+} K_t f \text{ exists for almost every } x \in \mathbb{R}^+\}.$$

Notice that $S(\mathcal{H}) \subseteq E \subseteq \mathbb{L}^p$. Since $S(\mathcal{H})$ is dense in $\mathbb{L}^p$, then we only need to prove that E is a closed subset of $\mathbb{L}^p$. Let $\{f_n\}$ be a sequence contained in E such that f_n converges in $\mathbb{L}^p$ to a function f. To see that $f \in E$ it is enough to prove that for all $\epsilon > 0$ we have $|E_\epsilon| = 0$ where

$$E_\epsilon := \left\{x : \limsup_{t \to 0^+} K_t f(x) - \liminf_{t \to 0^+} K_t f(x) > \epsilon\right\}. \tag{5.7}$$

For every n we can write

$$|E_\epsilon| \le \left|\left\{x : \lim_{t \to 0^+} K_t f_n(x) - \lim_{t \to 0^+} K_t f_n(x) > \frac{\epsilon}{3}\right\}\right|$$
$$+ \left|\left\{x : \limsup_{t \to 0^+} K_t(f_n - f)(x) > \frac{\epsilon}{3}\right\}\right|$$
$$+ \left|\left\{x : \liminf_{t \to 0^+} K_t(f_n - f)(x) > \frac{\epsilon}{3}\right\}\right|.$$

The first term is zero since $f_n \in E$. For the other two terms we use the weak type inequality boundedness on $\mathbb{L}^p$ of the maximal operator K^* which follows from (5.6). Notice that for every function g we have that

$$\left|\liminf_{t \to 0^+} K_t g(x)\right| \le \left|\limsup_{t \to 0^+} K_t g(x)\right| \le K^* g(x).$$

Then, since K^* is weakly bounded on $\mathbb{L}^p$, we obtain

$$\left|\left\{x : \limsup_{t \to 0^+} K_t(f_n - f)(x) > \frac{\epsilon}{3}\right\}\right| \le \frac{C}{\epsilon^p}\|f_n - f\|_{L^p}^p.$$

Hence,

$$|E_\epsilon| \le \frac{C}{\epsilon^p}\|f_n - f\|_{L^p}^p.$$

When n tends to infinity we have (5.7). Then E is closed and therefore $E = \mathbb{L}^p$. This means that for every $f \in \mathbb{L}^p$ we have that

$$\lim_{t \to 0^+} u(x,t) = \lim_{t \to 0^+} K_t f \quad \text{exists.}$$

But we already know that $u(x,t) \to f(x)$ when $t \to 0^+$ in $\mathbb{L}^p$, then the pointwise convergence follows, which completes the proof. $\square$

6. **Example**

Aside from the classical Euclidean spaces or smooth manifolds, the general theory becomes relevant when no algebraic structure is available. This section is devoted to provide a non classical example where the above results can be applied. Namely the Sierpinski quadrant: an unbounded version of the Sierpinski triangle.

Let S be the Sierpinski triangle in the Euclidean plane. The Sierpinski quadrant $\widehat{S}$ can be obtained by dyadic dilations of the Sierpinski triangle, i.e.

$$\widehat{S} = \bigcup_{j=0}^{\infty} S_j,$$

where $S_j = 2^j S$. The set $\widehat{S}$ with the inherited Euclidean distance and with the normalized Hausdorff measure $\mu = \mathcal{H}_s$ of order $s = \log 3 / \log 2$ is a space of homogeneous type. Moreover (see [19]) it is an unbounded s-Ahlfors regular metric measure space.

For each integer j we have that

$$\widehat{S} = \bigcup_{k=1}^{\infty} T_k^j,$$

where each T_k^j is a translation of the contraction of S by a factor 2^{-j}. Set $T_1^0 = S$. Let $\mathscr{D}^j = \{T_k^j : k \in \mathbb{N}\}$ and $\mathscr{D} = \cup_{j \in \mathbb{Z}} \mathscr{D}^j$. It is easy to check that $\mu(T_k^j) = 3^{-j}$ for every j and every k.

For a given $T \in \mathscr{D}^j$ set $T(1), T(2)$ and $T(3)$ to denote the three pieces in $\mathscr{D}^{j+1}$ such that

$$T = \bigcup_{i=1}^{3} T(i).$$

The space of those real functions on T which are constant on each $T(i)$ has dimension three and has as non-orthogonal basis the set $\{\mathcal{X}_T, \mathcal{X}_{T(1)}, \mathcal{X}_{T(2)}\}$, where $\mathcal{X}_A$ denotes the characteristic function on the set $A \subseteq S$. By ortho-normalization with the inner product of $L^2(d\mu)$ and preserving $3^{j/2} \mathcal{X}_T$ as one of its elements we get a basis of the form $\{3^{j/2} \mathcal{X}_T, h_T^1, h_T^2\}$. To fix

ideas and, because of the self similarity of $\widehat{S}$, we can take h_T^1 and h_T^2 as translations of rescalings of the ones obtained when $T = S$. Precisely, if $S = S(1) \cup S(2) \cup S(3)$,

$$h_S^1(x) = \sqrt{\tfrac{3}{2}} \left(\mathcal{X}_{S(1)} - \mathcal{X}_{S(2)} \right),$$

$$h_S^2(x) = \sqrt{\tfrac{1}{2}} \left(\mathcal{X}_{S(1)} + \mathcal{X}_{S(2)} - 2\mathcal{X}_{S(3)} \right).$$

And for $T_k^j \in \mathscr{D}^j$,

$$h_{T_k^j}^1(x) = 3^{j/2} \sqrt{\tfrac{3}{2}} \left(\mathcal{X}_{T_k^j(1)} - \mathcal{X}_{T_k^j(2)} \right),$$

$$h_{T_k^j}^2(x) = 3^{j/2} \sqrt{\tfrac{1}{2}} \left(\mathcal{X}_{T_k^j(1)} + \mathcal{X}_{T_k^j(2)} - 2\mathcal{X}_{T_k^j(3)} \right).$$

Let $\mathcal{H} = \{h_{T_k^j}^1, h_{T_k^j}^2 : \text{ for } j \in \mathbb{Z} \text{ and } k \in \mathbb{N}\}$. Then the system $\mathcal{H}$ is an orthonormal basis for $L^2(\widehat{S}, d\mu)$.

In this particular setting our main results throughout this paper read as follows:

- For every $j \in \mathbb{Z}$ and every $k \in \mathbb{N}$ we have

$$D^\sigma h_{T_k^j}^i(x) = m_\sigma 3^{j\sigma} h_{T_k^j}^i(x), \qquad \text{for } i = 1, 2,$$

  where $m_\sigma = 1 + \tfrac{1}{2}\tfrac{1}{3^\sigma - 1}$.

- For $f \in L^p(\widehat{S}, d\mu)$, with $1 < p < \infty$, the function

$$u(x,t) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} e^{-m_\sigma 3^{j\sigma} t} \left\langle f, h_{T_j^k} \right\rangle h_{T_j^k}(x)$$

  solves the problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) &= -D^\sigma u(x,t), \quad x \in \widehat{S}, t > 0, \\ u(x,0) &= f(x), \qquad x \in \widehat{S}. \end{cases}$$

- $u(x,t)$ can be written as an integral operator with the positive and finite kernel

$$k_t(x,y) = \frac{1}{t^{1/\sigma}} \varphi \left(\frac{\delta(x,y)}{t^{1/\sigma}} \right),$$

  where

$$\varphi(s) = \frac{1}{s} \left[-e^{-m_\sigma s^{-\sigma}} + \sum_{j \geq 1} 2^{-j} e^{-m_\sigma (2^j s)^{-\sigma}} \right],$$

  for $t > 0$. In other words,

$$u(x,t) = \int_{\widehat{S}} k_t(x,y) f(y) \, d\mu.$$

- $\lim_{t \to 0^+} u(x,t) = f(x)$ for almost every $x \in \widehat{S}$.

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