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HAAR TYPE SYSTEMS AND BANACH FUNCTION SPACES ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. In this note we prove that the Haar type systems defined on spaces of homogeneous type are unconditional bases for a wide family of Banach function spaces. Also, we give a characterization of these spaces via Haar coefficients. The main tool used in the proof is a generalization of the technique of extrapolation of Rubio de Francia in Banach function spaces defined on spaces of homogeneous type.

1. INTRODUCTION AND MAIN RESULT

In the setting of $L^2(\mathbb{R}^n)$ one of the main properties of the orthonormal wavelets bases is that they constitute unconditional bases for many spaces of functions that arise in Harmonic Analysis. For example, if Ψ is such a basis with certain given properties then the system Ψ is an unconditional basis for the weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$ with $1 < p < \infty$, where the weight w belongs to the classical Muckenhoupt A_p class. The same is valid when we consider the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ with $1 < p,q < \infty$. The characterization of these functional spaces is given in terms of the coefficients of wavelets in the representation of the function (see for example [27]). Moreover if we suppose certain regularity properties on the functions $\psi \in \Psi$, then it is known that the system Ψ is also an unconditional basis for a large class of spaces that include the mentioned above as well as Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ and Hardy spaces $H_p(\mathbb{R}^n)$ ([21, 10]).

In the Euclidean context, Izuki proved in [17] that the bases of wavelets with certain conditions of regularity are unconditional bases for the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. These spaces seem to be the most adequate context in order to describe the behaviour of certain fluids, called electrorheologic fluids ([26]). In order to prove the result in [17], the author used the extrapolation technique due to Rubio de Francia ([24, 25]), by virtue of the fact that the boundedness of Hardy-Littlewood maximal operators on $L^{p(\cdot)}(\mathbb{R}^n)$ is fulfilled.

In the setting of variable Lebesgue spaces, several authors studied the continuity properties of the operators that arise in connection with the partial differential equations modelling many situations. Particularly, in [11] and [8] the authors proved the boundedness of the Hardy-Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$ by requiring certain continuity properties on the exponent $p(\cdot)$ and in [15] the same result is proved in the context of metric spaces.

The systems Ψ mentioned above characterize the functional spaces in the sense that the norm of the function f involved can be given in terms of the coefficients of wavelets $\langle \psi, f\rangle$ in the representation formula $\sum\langle \psi, f\rangle \psi$. In this paper we shall adopt this point of view in the framework of metric spaces with a doubling measure. More precisely, we will prove that the Haar type systems are unconditional bases for certain Banach function spaces (B.F.S.). As an application of our main result we obtain the characterization of several spaces via the

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coefficient of wavelets such as Lebesgue and Lorentz spaces on spaces of homogeneus type and their variable versions.

In order to state our main result we give some definitions and properties that we shall be working with.

We denote L_c^∞ as the set

$$L_c^\infty = L_c^\infty(X,\mu) = \{f \in L^\infty(X) \ / \ \mathrm{supp}(f) \subset B(x_0,r) \text{ for some } x_0 \in X, r>0\}.$$

We shall say that $\mathbb{B}$ is a B.F.S. with the bounded maximal property (B.M.P.) if L_c^∞ is dense in $\mathbb{B}$ and there exists a real number $p_1>1$ such that $\mathbb{B}^{1/p_1}$ is a B.F.S. with

$$\|M_{\mathscr{D}} f\|_{\left(\mathbb{B}^{1/p_1}\right)'} \le c\,\|f\|_{\left(\mathbb{B}^{1/p_1}\right)'},$$

for every $f \in \left(\mathbb{B}^{\frac{1}{p_1}}\right)'$, where $M_{\mathscr{D}}$ is the dyadic Hardy-Littlewood maximal operator defined in Section §2, where $\mathbb{B}^{1/p_1}$ is also defined.

We are now in a position to state our main result.

Theorem 1.1. *Let (X,d,μ) be a space of homogeneous type and let $\tilde{\mathscr{H}}$ be a Haar type system. Let $\mathbb{B}$ be a B.F.S. with the B.M.P. Then $\tilde{\mathscr{H}}$ is an unconditional basis for $\mathbb{B}$. Moreover, there exist two positive constants c_1 and c_2 such that for every $f \in \mathbb{B}$ the following inequalities hold*

$$c_1\,\|f\|_{\mathbb{B}} \le \|Sf\|_{\mathbb{B}} \le c_2\,\|f\|_{\mathbb{B}}, \tag{1.1}$$

where

$$Sf(x) = \left(\sum_{h\in\tilde{\mathscr{H}}} |\langle f,h\rangle|^2\,|h(x)|^2\right)^{1/2}.$$

For the definition of unconditional bases see [2, Section 7].

Before describing the organization of the paper, we give some comments on the previous theorem. At first, given an arbitrary function f in $\mathbb{B}$, the definition of Sf may not make sense. But as we shall be working in $\mathbb{B}$ with B.M.P., the operator S can be defined by using an extension argument.

The paper is organized as follows. In Section §2 we give some preliminaries. In Section §3 we define the Haar type system and recall some useful properties in the context of weighted Lebesgue spaces defined on spaces of homogeneous type. In Section §4 we give the extrapolation theorem in the context of B.F.S. and the proof of the main result. In Section §5 we show some applications of Theorem 1.1.

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2. Preliminaries. Dyadic analysis on spaces of homogeneous type

Let us first recall the definition and basic properties of the spaces of homogeneous type. If X is a set, a nonnegative symmetric function d defined on $X\times X$ is called a quasi-distance in X if there exists a constant $K\ge 1$ such that

$$d(x,y)=0 \quad \text{if and only if } x=y$$
$$d(x,y)=d(y,x)$$
$$d(x,y)\le K[d(x,z)+d(z,y)],$$

for every $x,y,z\in X$.

We shall say that (X,d,μ) is a space of homogeneous type if d is a quasi-distance on X, and μ is a positive Borel measure defined on a σ-algebra of subsets of X which contains the d-balls that satisfies a doubling condition, that is, there exists a positive constant C such that the inequalities

$$0 < \mu(B(x,2r)) \leq C\,\mu(B(x,r)) < \infty$$

hold for every $x \in X$ and every $r > 0$.

It is well known that the d-balls are generally not open sets. Nevertheless in [20], R. Macías and C. Segovia proved that if d is a quasi-distance on X, then there exists a distance ρ and a number $\alpha \geq 1$ such that d is equivalent to ρ^{α}, and the ρ^{α}-balls are open sets. This remark allows us to consider (X,d,μ) to be a space of homogeneous type where d is a distance on X. In order to apply the Lebesgue Differentiation Theorem we shall also suppose that continuous functions are dense in $L^1(X,\mu)$ if $\mu(X) = \infty$.

The construction of dyadic type families in metric or quasi-metric spaces with some inner and outer metric control of the sizes of the dyadic sets is given by M. Christ in [5]. We now introduce a special family of such dyadic subsets. These families satisfy all the relevant properties of the usual dyadic cubes in $\mathbb{R}^n$. Actually, the only properties of Christ's cubes needed in our context of spaces of homogeneous type are contained in the next definition given in [3].

Definition 2.1. We say that $\mathscr{D} = \bigcup_{j\in\mathbb{Z}} \mathscr{D}^j$ is a dyadic family on X with parameter $\delta \in (0,1)$, briefly that $\mathscr{D}$ belongs $\mathfrak{D}(\delta)$, if each $\mathscr{D}^j$ is a family of open subsets Q of X, such that

- *(d.1) For every $j \in \mathbb{Z}$ the cubes in $\mathscr{D}^j$ are pairwise disjoint.*
- *(d.2) For every $j \in \mathbb{Z}$ the family $\mathscr{D}^j$ covers almost all X in the sense that $\mu(X - \bigcup_{Q\in\mathscr{D}^j} Q) = 0$.*
- *(d.3) If $Q \in \mathscr{D}^j$ and $i < j$, then there exists a unique $\tilde{Q} \in \mathscr{D}^i$ such that $Q \subseteq \tilde{Q}$.*
- *(d.4) If $Q \in \mathscr{D}^j$ and $\tilde{Q} \in \mathscr{D}^i$ with $i \leq j$, then either $Q \subseteq \tilde{Q}$ or $Q \cap \tilde{Q} = \emptyset$.*
- *(d.5) There exist two constants a_1 and a_2 such that for each $Q \in \mathscr{D}^j$ there is a point $x \in Q$ for which $B(x, a_1\delta^j) \subseteq Q \subseteq B(x, a_2\delta^j)$.*

The main properties of the dyadic family $\mathscr{D}$ defined above are given in the following result (see [2]).

Proposition 1. *Let $\mathscr{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. Then*

- *(d.6) There exists a positive integer N depending only on the doubling constant such that for every $j \in \mathbb{Z}$ and all $Q \in \mathscr{D}^j$ the inequalities $1 \leq \#(\mathscr{O}(Q)) \leq N$ hold, where $\mathscr{O}(Q) = \{Q' \in \mathscr{D}^{j+1} : Q' \subseteq Q\}$.*
- *(d.7) X is bounded if and only if there exists a dyadic cube Q in $\mathscr{D}$ such that $X = Q$.*
- *(d.8) The families $\tilde{\mathscr{D}}^j = \{Q \in \mathscr{D}^j : \#(\{Q' \in \mathscr{D}^{j+1} : Q' \subseteq Q\}) > 1\}$, $j \in \mathbb{Z}$ are pairwise disjoint.*

The dyadic Hardy-Littlewood maximal operator associated to $\mathscr{D}$ is defined, for a locally integrable function f, by

$$M_{\mathscr{D}} f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f|\, d\mu,$$

where the supremum is taken over the dyadic cubes $Q \in \mathscr{D}$ containing x.

Given a dyadic family $\mathscr{D}$ we say that a nonnegative and locally integrable function w is a Muckenhoupt type dyadic weight, $w \in A_p^{\mathrm{dy}}$, $1 < p < \infty$, if the inequality

$$\left(\frac{1}{\mu(Q)}\int_Q w(x)\,d\mu(x)\right)\left(\frac{1}{\mu(Q)}\int_Q w(x)^{-\frac{1}{p-1}}\,d\mu(x)\right)^{p-1} \leq C \tag{2.1}$$

holds for some constant C and every dyadic cube $Q \in \mathscr{D}$.

When $p = 1$ we say that $w \in A_1^{\mathrm{dy}}$ if the inequality

$$\frac{w(Q)}{\mu(Q)} \leq Cw(x) \tag{2.2}$$

holds for almost every $x \in Q$ and for every dyadic cube $Q \in \mathscr{D}$. The class A_∞^{dy} is defined as

$$A_\infty^{\mathrm{dy}} = \bigcup_{p\geq 1} A_p^{\mathrm{dy}}. \tag{2.3}$$

Let us now introduce the basic notions of Banach function spaces. We refer to [4] for complete details. Let (X,μ) be a σ-finite measure space. We shall write $\mathscr{M}_\mu$ and $\mathscr{M}_\mu^+$ to denote the set of all μ-measurable functions $f : X \longrightarrow [-\infty,+\infty]$ and the subset of $\mathscr{M}_\mu$ whose values lie in $[0,\infty]$, respectively. A function norm is a mapping $\rho : \mathscr{M}_\mu^+ \longrightarrow [0,\infty]$ such that for all f,g and f_n in $\mathscr{M}_\mu^+$, $n \in \mathbb{Z}$, the following statements hold:

- $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.,
- for all $a > 0$ we have that $\rho(af) = a\rho(f)$,
- $\rho(f+g) \leq \rho(f)+\rho(g)$,
- if $0 \leq g \leq f$ μ-a.e., then $\rho(g) \leq \rho(f)$,
- if $0 \leq f_n \nearrow f$ μ-a.e., then $\rho(f_n) \nearrow \rho(f)$,
- if $E \subseteq X$ with $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$,
- for each $E \subseteq X$ with $\mu(E) < \infty$, there exists a positive constant C such that $\int_E f\,d\mu \leq C\rho(f)$, for every f.

The space $\mathbb{B} = \{f \in \mathscr{M}_\mu : \|f\|_\mathbb{B} < \infty\}$ is a normed Banach space where the norm is given by $\|f\|_\mathbb{B} = \rho(|f|)$. Such space is called a Banach function space.

If $\mathbb{B}$ denotes a Banach function space defined on X with norm given by $\|\cdot\|_\mathbb{B}$, we define the scale space $\mathbb{B}^r$, $0 < r < \infty$, as the μ-measurable functions f such that $|f|^r \in \mathbb{B}$, with "norm" given by $\|f\|_{\mathbb{B}^r} = \||f|^r\|_\mathbb{B}^{\frac{1}{r}}$. If $r \geq 1$, then $\mathbb{B}^r$ is a B.F.S. However, if $r < 1$, then $\mathbb{B}^r$ is not necessarily a B.F.S.; the simplest example is the case $\mathbb{B} = L^1(X)$.

The associated space $\mathbb{B}'$ is the set of the μ-measurable functions $f : X \to [-\infty,+\infty]$ such that the quantity

$$\sup\left\{\int_X |f(x)g(x)|\,d\mu(x),\ g \in \mathbb{B},\ \|g\|_\mathbb{B} \leq 1\right\}$$

is finite. This space $\mathbb{B}'$ is a B.F.S. and the following generalized Hölder inequality holds

$$\int_X |f(x)g(x)|\,d\mu(x) \leq \|f\|_\mathbb{B}\|g\|_{\mathbb{B}'},$$

for all $f \in \mathbb{B}$ and $g \in \mathbb{B}'$.

3. Haar systems and weighted Lebesgue spaces

The underlying algebraic structure in the Euclidean context is crucial for the construction of systems of wavelets. In fact, they can be obtained by translations and dilations of some given function. In the general setting of spaces of homogeneous type this structure is not given and must be replaced by geometric arguments.

For a given dyadic family $\mathscr{D}$ in the class $\mathfrak{D}(\delta)$ it was proved in [1] and [2] that we can always construct Haar type bases $\mathscr{H}$, of Borel measurable simple real functions h, satisfying the following properties:

(h.1) For each $h \in \mathscr{H}$ there exists a unique $j \in \mathbb{Z}$ and a cube $Q = Q(h) \in \tilde{\mathscr{D}}^j$ such that $\{x \in X : h(x) \neq 0\} \subseteq Q$, and this property does not hold for any cube in $\mathscr{D}^{j+1}$.

(h.2) For every $Q \in \tilde{\mathscr{D}} = \bigcup_{j\in\mathbb{Z}} \tilde{\mathscr{D}}^j$ there exist exactly $M_Q = \#(\mathscr{O}(Q)) - 1 \geq 1$ functions $h \in \mathscr{H}$ such that (h.1) holds. We shall write $\mathscr{H}_Q$ to denote the set of all these functions h.

(h.3) For each $h \in \mathscr{H}$ we have that $\int_X h\, d\mu = 0$.

(h.4) For each $Q \in \tilde{\mathscr{D}}$ let V_Q denote the vector space of all functions on Q which are constant on each $Q' \in \mathscr{O}(Q)$. Then the system $\{\chi_Q/\mu(Q)^{1/2}\} \bigcup \mathscr{H}_Q$ is an orthonormal basis for V_Q.

By $\tilde{\mathscr{H}}$ we shall denote the Haar type system $\mathscr{H}$ when $\mu(X) = \infty$ and $\mathscr{H} \cup \{\mu(X)^{-\frac{1}{2}}\}$ when $\mu(X) < \infty$.

As an easy consequence of the properties above, in [2] the authors obtained the following results. In this context $L^2_w := \{f : \int_X |f|^2\, w\, d\mu < \infty\}$.

Theorem 3.1. *Let $\mathscr{D}$ be a dyadic family on X such that $\mathscr{D}$ belongs to the class $\mathfrak{D}(\delta)$. Then every Haar type system $\tilde{\mathscr{H}}$ associated to $\mathscr{D}$ is an orthonormal basis in $L^2(X,\mu)$.*

In [2] the authors prove that the Haar type systems associated to Christ's dyadic cubes are unconditional bases for the spaces $L^{p_0}_w$ with $w \in A^{\mathrm{dy}}_{p_0}$ and $1 < p_0 < \infty$. More specifically,

Theorem 3.2. *Let (X,d,μ) be a space of homogeneous type and let $\tilde{\mathscr{H}}$ be a Haar system associated to a dyadic family $\mathscr{D}$ and let S be the square function. If $1 < p_0 < \infty$ and $w \in A^{dy}_{p_0}$ then there exist two positive constants C_1 and C_2 depending on the $A^{dy}_{p_0}$ constant of w such that for all $f \in L^{p_0}_w(X,\mu)$ we have that*

$$C_1 \|f\|_{L^{p_0}_w} \leq \|Sf\|_{L^{p_0}_w} \leq C_2 \|f\|_{L^{p_0}_w}.$$

Moreover, $\tilde{\mathscr{H}}$ is an unconditional basis for $L^{p_0}_w(X,\mu)$.

4. Extrapolation: Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. An important tool used in this proof is an extrapolation theorem in B.F.S. that generalizes the well known result due to Rubio de Francia. In order to state the extrapolation theorem we introduce the following definition.

Definition 4.1. ***($\mathbb{B}$-admissible family)*** Let $\mathfrak{F}$ be a family of ordered pairs (f,g) of non negative, measurable functions on X. We say that $\mathfrak{F}$ is $\mathbb{B}$-admissible if given $(f,g) \in \mathfrak{F}$ the following statements hold

- $\left(\int_X f(x)^{p_0} w(x)\, d\mu(x)\right)^{\frac{1}{p_0}} < \infty$ for some $p_0 > 1$ and for every $w \in A^{\mathrm{dy}}_{p_0}$.
- $\|f\|_{\mathbb{B}} < \infty$.

We are now in a position to establish the extrapolation theorem.

Theorem 4.2. (Extrapolation) *Let $\mathscr{D}$ be a dyadic family on X such that $\mathscr{D}$ belongs to $\mathfrak{D}(\delta)$. Let $\mathbb{B}$ be a B.F.S. on X. Let $\mathfrak{F} = \{(f,g)\}$ be a $\mathbb{B}$-admissible family. Suppose that there exists $p_1 > 1$ such that $\mathbb{B}^{1/p_1}$ is a B.F.S. and*

$$\|M_{\mathscr{D}} v\|_{\left(\mathbb{B}^{\frac{1}{p_1}}\right)'} \leq C \|v\|_{\left(\mathbb{B}^{\frac{1}{p_1}}\right)'} \tag{4.1}$$

for every function $v \in \left(\mathbb{B}^{\frac{1}{p_1}}\right)'$, *with* $C = C(\mathbb{B})$.

If there exists $p_0 \in (1,\infty)$ *such that for all* $w \in A^{dy}_{p_0}$

$$\|f\|_{L^{p_0}_w(X)} \leq C_0 \|g\|_{L^{p_0}_w(X)} \quad \forall (f,g) \in \mathfrak{F} \tag{4.2}$$

with $C_0 = C_0(w, p_0)$ *independent of* (f,g), *then the inequality*

$$\|f\|_{\mathbb{B}^q} \leq C_1 \|g\|_{\mathbb{B}^q} \quad \forall (f,g) \in \mathfrak{F} \tag{4.3}$$

holds with $C_1 = C_1(\mathbb{B}, q)$ *independent of* (f,g), *and* $1/p_1 < q$.

In the Euclidean context, this result was proved in [7] and [9]; see also [14]. Also, it can be obtained as a corollary of the following extrapolation result (see Theorem 5.1 in [23]).

Theorem 4.3. *Let* $\mathscr{D}$ *be a dyadic family on* X *such that* $\mathscr{D}$ *belongs to class* $\mathfrak{D}(\delta)$ *and let* $\mathbb{B}$ *be a B.F.S. on* X. *Let* $\mathscr{F}$ *be a* $\mathbb{B}$*-admissible family of pairs* (f,g). *Suppose that for some* p_0, $0 < p_0 < \infty$, *and every* $w \in A^{dy}_1$,

$$\int_X f(x)^{p_0} w(x)\, d\mu(x) \leq C \int_X g(x)^{p_0} w(x)\, d\mu(x). \tag{4.4}$$

If there exists q_0, $p_0 \leq q_0 < \infty$, *such that* $\mathbb{B}^{1/q_0}$ *is a B.F.S. and* $M_{\mathscr{D}}$ *is bounded on* $(\mathbb{B}^{1/q_0})'$, *then*

$$\|f\|_{\mathbb{B}} \leq C \|g\|_{\mathbb{B}}. \tag{4.5}$$

In $\mathbb{R}^n$ the theorem above was proved in ([9, Theorem 4.6]) for a more general family than the classic dyadic cubes, and the same proof can be adapted to the context of spaces of homogeneous type.

Now we prove the main result of this paper.

Proof of Theorem 1.1. Let $f \in L^\infty_c$ and $F \subset \tilde{\mathscr{H}}$ with $\sharp(F) < \infty$ then

$$S_F f = \left(\sum_{h \in F} |\langle f, h\rangle|^2 |h|^2\right)^{\frac{1}{2}} \in L^\infty_c \subset \mathbb{B}.$$

Then, defining $\mathfrak{F}_1 = \{(|f|, Sf) : f \in L^\infty_c\}$ and $\mathfrak{F}_{2,F} = \{(S_F f, |f|) : f \in L^\infty_c\}$, we obtain two $\mathbb{B}$-admissible families and, by Theorem 3.2, it is easy to check that the following inequalities also hold

$$C_1 \|f\|_{L^{p_0}_w} \leq \|Sf\|_{L^{p_0}_w}, \quad \|S_F f\|_{L^{p_0}_w} \leq C_2 \|f\|_{L^{p_0}_w}, \qquad \forall w \in A^{\mathrm{dy}}_{p_0}. \tag{4.6}$$

Now, by the hypothesis and applying the extrapolation Theorem 4.2 we obtain that there exist two positive constants C_1' and C_2' such that

$$\|f\|_{\mathbb{B}} \leq C_1' \|Sf\|_{\mathbb{B}} \tag{4.7}$$

and

$$\|S_F f\|_{\mathbb{B}} \leq C_2' \|f\|_{\mathbb{B}}. \tag{4.8}$$

It is important to note that C_2' does not depend on $F \subset \tilde{\mathscr{H}}$.

Let $\{F_n\}$ be a sequence such that $F_n \subset F_{n+1}$ and $\cup F_n = \tilde{\mathscr{H}}$. Then for every function $f \in L^\infty_c$ and $x \in X$ we obtain that $S_{F_n} f(x) \nearrow Sf(x)$, and thus, by the Banach function space properties and (4.8),

$$\|Sf\|_{\mathbb{B}} = \lim_{n\to\infty} \|S_{F_n} f\|_{\mathbb{B}} \leq C_2 \|f\|_{\mathbb{B}} \quad \forall f \in L^\infty_c(X). \tag{4.9}$$

Then, from (4.7) and (4.9) we can conclude that

$$C_1 \|f\|_{\mathbb{B}} \le \|Sf\|_{\mathbb{B}} \le C_2 \|f\|_{\mathbb{B}} \quad \forall f \in L^\infty_c(X). \tag{4.10}$$

The general result for $f \in \mathbb{B}$ follows by applying a density argument, as follows.

Let us first see that $\|Sf\|_{\mathbb{B}} \le c_2 \|f\|_{\mathbb{B}}\ \forall f \in \mathbb{B}$.
Let $\{f_k\} \subset L^\infty_c$ be such that $\|f_k - f\|_{\mathbb{B}} \to 0$ when $k \to \infty$. Using a discrete version of Fatou's Lemma we can write

$$Sf(x) \le \liminf_{k\to\infty} Sf_k(x)$$

and consequently, by Fatou's Lemma for B.F.S. (see [4, Lemma 1.5]) and (4.10),

$$\|Sf\|_{\mathbb{B}} \le \liminf_{k\to\infty} \|Sf_k\|_{\mathbb{B}} \le C_2 \liminf_{k\to\infty} \|f_k\|_{\mathbb{B}} \le C_2 \|f\|_{\mathbb{B}} \quad \forall f \in \mathbb{B}. \tag{4.11}$$

From the fact that $Sf_k(x) \le 2\,[S(f-f_k)(x) + Sf(x)]$, the other inequality can be obtained from the previous case in the following way:

$$\begin{aligned} \|f\|_{\mathbb{B}} &= \lim_{k\to\infty} \|f_k\|_{\mathbb{B}} \le c \liminf_{k\to\infty} \|Sf_k\|_{\mathbb{B}} \\ &\le c\left(\liminf_{k\to\infty} \|S(f_k - f)\|_{\mathbb{B}} + \|Sf\|_{\mathbb{B}}\right) \\ &\le c\left(\liminf_{k\to\infty} \|f_k - f\|_{\mathbb{B}} + \|Sf\|_{\mathbb{B}}\right) \\ &\le c\,\|Sf\|_{\mathbb{B}} \quad \forall f \in \mathbb{B}. \end{aligned} \tag{4.12}$$

This concludes the proof of (1.1).

Let us now see that the Haar system is an unconditional basis for $\mathbb{B}$.

Given $h \in \tilde{\mathscr{H}} \subset \mathbb{B}'$, we define $h^* : \mathbb{B} \to \mathbb{R}$ by $h^*(f) := \int_X hf\,d\mu = \langle f, h\rangle$. These operators satisfy

$$h^*(g) = \begin{cases} 1 & \text{if} \quad h = g \\ 0 & \text{if} \quad h \neq g \end{cases}$$

for every $g \in \tilde{\mathscr{H}}$.

For $F \subset \tilde{\mathscr{H}}$ with $\sharp(F) < \infty$ we define

$$T_F f = \sum_{h\in F} \langle f, h\rangle\, h.$$

We use the previous result in order to see that these operators are bounded in $\mathbb{B}$ with a constant independent of F. Indeed, given $f \in \mathbb{B}$ and $g \in \tilde{\mathscr{H}}$ we have that

$$\langle T_F f, g\rangle = \sum_{h\in F} \langle h, f\rangle \langle h, g\rangle = \begin{cases} \langle f, g\rangle & \text{if} \quad g \in F \\ 0 & \text{if} \quad g \notin F \end{cases}$$

and then we obtain that

$$S(T_F f)(x) = \left(\sum_{g\in\tilde{\mathscr{H}}} |\langle T_F f, g\rangle|^2\, |g(x)|^2\right)^{\frac{1}{2}} = \left(\sum_{g\in F} |\langle f, g\rangle|^2\, |g(x)|^2\right)^{\frac{1}{2}} \le S(f)(x) \tag{4.13}$$

Finally, by (4.12), (4.13) and (4.11) we get

$$\|T_F f\|_{\mathbb{B}} \le C\,\|S(T_F f)\|_{\mathbb{B}} \le C\,\|S(f)\|_{\mathbb{B}} \le C\,\|f\|_{\mathbb{B}}.$$

Now we will prove that $\operatorname{span}(\tilde{\mathscr{H}})$ is dense in $\mathbb{B}$.

From the hypothesis on $\mathbb{B}$, it is enough to prove that $\operatorname{span}(\tilde{\mathscr{H}})$ is dense in L^∞_c with norm $\|\cdot\|_{\mathbb{B}}$.

Let $\varepsilon > 0$ be given and $g \in L_c^\infty(X)$. From the fact that span$(\tilde{\mathscr{H}})$ is dense in $L_w^{p_0}(X)$ for $1 < p_0 < \infty$ and $w \in A_{p_0}^{\mathrm{dy}}$ (see [2, Theorem 9.1]) and $g \in L_w^{p_0}(X)$, there exists $f \in$ span$(\tilde{\mathscr{H}})$ such that

$$\|f-g\|_{L_w^{p_0}(X)} \leq \frac{\varepsilon w(B)^{\frac{1}{p_0}}}{C_1\,\|\chi_B\|_{\mathbb{B}}} = \left\| \frac{\varepsilon w(B)^{\frac{1}{p_0}}}{C_1\,\|\chi_B\|_{\mathbb{B}}}\,\frac{\chi_B}{w(B)^{\frac{1}{p_0}}} \right\|_{L_w^{p_0}(X)},$$

where B is a fixed ball in X and C_1 is the constant in (4.3).

On the other hand, $f-g \in L_c^\infty(X) \subset \mathbb{B}$, $f-g \in L_w^{p_0}$, and $\|f-g\|_{\mathbb{B}} < \infty$.

Let us now consider the family $\mathfrak{F} := \{(f-g, \varepsilon \frac{\chi_B}{C_1\|\chi_B\|_{\mathbb{B}}})\}$. From the fact that $\mathbb{B}$ has the B.M.P. we can apply Theorem 4.2 in order to obtain that

$$\|f-g\|_{\mathbb{B}} \leq C_1 \left\| \varepsilon \frac{\chi_B}{C_1\,\|\chi_B\|_{\mathbb{B}}} \right\|_{\mathbb{B}} = \varepsilon,$$

and thus, the density result is proved. □

5. Applications

In this section we show some examples of B.F.S. which satisfy the hypothesis of Theorem 1.1.

- When $\mathbb{B} = L^p(X,\mu)$, $1 < p < \infty$, Theorem 1.1 is nothing but Theorem 3.2 (which is Theorem 7.1 in [2]). In fact, in [2] the authors proved the boundedness of $M_{\mathscr{D}}$ in $L^p(X,\mu)$ for every $1 < p < \infty$ and the density of $L_c^\infty(X,\mu)$ in $L^p(X,\mu)$, that is the B.M.P. in this case.

- Let us now consider the Lorentz spaces $\mathbb{B} = L^{p,q}(X,\mu)$ with $1 < p,q < \infty$, given by the measurable function f defined on X such that

  $$\|f\|_{p,q}^* := \left(\frac{q}{p} \int_0^\infty (t^p f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

  where f^* is the non increasing rearrangement of f. It is well known that $\|\cdot\|_{p,q}^*$ is not a norm in $L^{p,q}(X,\mu)$ but we can always define an equivalent norm on these spaces, (see for example [16] or [19]).

  In this context, Theorem 1.1 is contained in Theorem 6.1 in [23], since in the same article, the author proved that M_D is bounded on $L^{p,q}(X,\mu)$ for every $1 < p,q < \infty$.

- We now introduce some basic facts about $L^{p(\cdot)}(X,\mu)$ in order to establish the result in this environment.

  Let (X,d,μ) be a locally compact space of homogeneous type and let $p : X \to [1,\infty)$ be a measurable function. For $A \subset X$ we define

  $$p_A^- := \inf_{x\in A} p(x), \qquad p_A^+ := \sup_{x\in A} p(x).$$

  For simplicity we denote $p^+ = p_X^+$, $p^- = p_X^-$. We shall also suppose that $1 < p^- \leq p(x) \leq p^+ < \infty$ for $x \in X$.

  The variable exponent Lebesgue spaces $L^{p(\cdot)}(X) = L^{p(\cdot)}(X,\mu)$ are defined by the μ-measurable functions f defined on X such that, for some positive λ, the functional convex modular $\rho(f/\lambda) := \int_X |f(x)/\lambda|^{p(x)}\,d\mu(x)$ is finite. A Luxemburg norm can be defined in $L^{p(\cdot)}(X)$ by taking

  $$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 \,/\, \rho(f/\lambda) \leq 1\}.$$

These are special cases of Musielak-Orlicz spaces (see [22]), and generalize the classical Lebesgue spaces. Actually, the most up to date reference for Lebesge spaces with variable exponents happens to be the monograph recently published by David Cruz-Uribe and Alberto Fiorenza [6].

Some of the following results were obtained in the Euclidean context (see for example [18]), but it can be proved that they also hold in the context of homogeneous type spaces (for metric spaces see [15]). For example, it can be seen that $(L^{p(\cdot)}(X), \|\cdot\|_{p(\cdot)})$ is a B.F.S. On the other hand the dual space of $L^{p(\cdot)}(X)$ is $L^{p'(\cdot)}(X)$ and the Hölder inequality holds, that is

$$\int_X |f(x)g(x)|\, d\mu(x) \leq C\,\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)}\,.$$

It was proved in [15] that $C_0(X,\mu)$, and consequently $L_c^\infty(X,\mu)$, is dense in $L^{p(\cdot)}(X,\mu)$ when X is a locally compact space. Thus Theorem 1.1 holds if we ask M to be bounded on $L^{p(\cdot)}$ with $1 < p^- \leq p(x) \leq p^+ < \infty$. In [15] the authors proved that if X is an α-Ahlfors space with finite measure and $p(\cdot)$ satisfies the log-Hölder condition

$$|p(x) - p(y)| \leq \frac{c}{-\log(d(x,y))}$$

when $d(x,y) < \frac{1}{2}$, for some positive constant c, then M is bounded on $L^{p(\cdot)}(X,\mu)$.

- Finally, if Ω is an open subset of $\mathbb{R}^n$ with $|\Omega| = l$, in [13] the authors define the weighted variable exponent Lorentz spaces, as the set $\mathscr{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ of the measurable functions f such that the norm $\|f\|_{\mathscr{L}_w^{p(\cdot),q(\cdot)}(\Omega)} := \left\| w(t)t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t)\right\|_{L^{q(\cdot)}([0,l])}$ is finite.

  In the same article the authors proved that this is a B.F.S. with the B.M.P. (see [13, Theorem 3.12]), when $w(t) = t^{\gamma(t)}$, for certain function γ, and for $p,q \in \mathbb{P}_1([0,l])$, where

$$\mathbb{P}_1([0,l]) := \{r \in L^\infty([0,l]) \;/\; \exists \lim_{t\to 0} r(t) = r(0),\; \exists \lim_{t\to\infty} r(t) = r(\infty),\; 1 < r^- \leq r^+ < \infty\}.$$

REFERENCES

[1] Aimar, H. *Construction of Haar type bases on quasi-metric spaces with finite Assouad dimension*, Anal. Acad. Nac. Cs. Ex. Fís. Nat., Buenos Aires 54 (2002).

[2] Aimar, Hugo; Bernardis, Ana; Iaffei, Bibiana. *Multiresolution approximations and unconditional bases on weighted Lebesgue spaces on spaces of homogeneous type.* J. Approx. Theory 148 (2007), no. 1, 12–34. MR 2356573.

[3] Aimar, Hugo; Bernardis, Ana; Nowak, Luis. *Equivalence of Haar bases associated with different dyadic systems.* J. Geom. Anal. 21 (2011), no. 2, 288–304. MR 2772074.

[4] Bennett, Colin; Sharpley, Robert. *Interpolation of Operators.* Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988. MR 0928802.

[5] Christ, M. *A T(b) theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. 60/61 (1990), no. 2, 601–628. MR 1096400.

[6] Cruz-Uribe, David V.; Fiorenza, Alberto. *Variable Lebesgue spaces. Foundations and harmonic analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. x+312 pp. MR 3026953.

[7] Cruz-Uribe, D.; Fiorenza, A.; Martell, J. M.; Pérez, C. *The boundedness of classical operators on variable L^p spaces.* Ann. Acad. Sci. Fenn. Math. 31 (2006), no. 1, 239–264. MR 2210118.

[8] Cruz-Uribe, D.; Fiorenza, A. and Neugebauer, C.J. *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn 28 (2003), 223–238. MR 1976842.

[9] Cruz-Uribe, D.; Martell, J.M.; Perez, C. *Weights, Extrapolation and the Theory of Rubio de Francia.* Operator theory: Advances and Applications, 215. Birkhuser/Springer Basel AG, Basel, 2011. MR 2797562.

[10] Daubechies, I.: *Ten lectures of wavelets.* CBMS-NFS regional conference series in applied mathematics, 61, 1992. MR 1162107.
[11] Diening, L.: *Maximal function on generalized Lebesgue spaces $L^{p(x)}$.* Math. Inequal. Appl. 7 (2004), no. 2, 245–253. MR 2057643.
[12] Edmunds, D. E.; Lang, J.; Nekvinda, A.: *On $L^{p(x)}$ norms.* R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 455 (1999), no. 1981, 219–225. MR 1700499.
[13] Ephremidze, Lasha; Kokilashvili, Vakhtang; Samko, Stefan.: *Fractional, maximal and singular operators in variable exponent Lorentz spaces.* Fract. Calc. Appl. Anal. 11 (2008), no. 4, 407–420. MR 2459733.
[14] García-Cuerva, José; Rubio de Francia, José L.: *Weighted norm inequalities and related topics.* North-Holland Mathematics Studies, 116. Notas de Matemática [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985. MR 0848136.
[15] Harjulehto, Petteri; Hästö, Peter; Pere, Mikko: *Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator.* Real Anal. Exchange 30 (2004/05), no. 1, 87–103. MR 2126796.
[16] Hunt, R.: *On $L(p,q)$ spaces.* Enseignement Math. (2) 12 (1966), 249–276. MR 0223874.
[17] Izuki, Mitsuo: *Wavelets and modular inequalities in variable L^p spaces.* Georgian Math. J. 15 (2008), no. 2, 281–293. MR 2428471.
[18] Kováčik, Ondrej; Rákosník, Jiří: *On spaces $L^{p(x)}$ and $W^{k,p(x)}$.* Czechoslovak Math. J. 41(116) (1991), no. 4, 592–618. MR 1134951.
[19] Lorentz, G.: *On the theory of spaces* Λ. Pacific J. Math. 1 (1951), 411–429. MR 0044740.
[20] Macías, Roberto A.; Segovia, Carlos: *Lipschitz functions on spaces of homogeneous type.* Adv. in Math. 33 (1979), no. 3, 257–270. MR 0546295.
[21] Meyer, Yves: *Wavelets and operators.* Translated from the 1990 French original by D. H. Salinger. Cambridge Studies in Advanced Mathematics, 37. Cambridge University Press, Cambridge, 1992. MR 1228209.
[22] Musielak, Julian: *Orlicz spaces and modular spaces.* Lecture Notes in Mathematics, 1034. Springer-Verlag, Berlin, 1983. MR 0724434.
[23] Nowak, Luis: *Haar type bases in Lorentz spaces via extrapolation.* Rev. Un. Mat. Argentina 53 (2012), no. 1, 71–83. MR 2987157.
[24] Rubio de Francia, José Luis: *Factorization and extrapolation of weights.* Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 2, 393–395. MR 0663793.
[25] Rubio de Francia, José Luis: *A new technique in the theory of A_p weights.* Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982), 571–579, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983. MR 0748875.
[26] Ružička, M.: *Electrorheological fluids: modeling and mathematical theory.* Mathematical analysis of liquids and gases (Japanese) (Kyoto, 1999). Sūrikaisekikenkyūsho Kōkyūroku no. 1146 (2000), 16–38. MR 1788852.
[27] Soardi, P.: *Wavelets bases in rearrangement invariant function spaces.* Proc. Amer. Math. Soc. 125 (1997), no. 12, 3669–3673. MR 1443168.

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