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Nonlocal diffusions on fractals. Qualitative properties and numerical approximations.

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We propose a numerical method to approximate the solution of a nonlocal diffusion problem on a general setting of metric measure spaces. These spaces include, but are not limited to, fractals, manifolds and Euclidean domains. We obtain error estimates in $L^\infty(L^p)$ for $p = 1,\infty$ under the sole assumption of the initial datum being in L^p. An improved bound for the error in $L^\infty(L^1)$ is obtained when the initial datum is in L^2. We also derive some qualitative properties of the solutions like stability, comparison principles and study the asymptotic behavior as $t \to \infty$. We finally present two examples on fractals: the Sierpinski gasket and the Sierpinski carpet, which illustrate on the effect of nonlocal diffusion for piecewise constant initial datum.

Keywords: nonlocal diffusions, discretizations, space of homogeneous type, fractals.

1. Introduction and Main Result

Many results from classical harmonic analysis have been developed on more general metric measure spaces, containing typical fractals and manifolds. However, the study of differential equations in such a primitive context are under development (see Bell *et al.*, 2014; Li & Strichartz, 2014; Spicer *et al.*, 2013; Ionescu *et al.*, 2013; Qiu & Strichartz, 2013; Begué *et al.*, 2013; Owen & Strichartz, 2012, and references therein). Kigami (1989) defined a Laplacian on the Sierpinki gasket, and later extended his construction to a wider class of fractals in Kigami (1993). This set the stage for an analytic study of the analogs of some of the classical partial differential equations on these fractals, which are a particular case of metric measure spaces.

Linear nonlocal diffusion equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^n} J(x-y)[u(y,t)-u(x,t)]\,dy,$$

have been widely used to model diffusion problems (Fife, 2003), and can be generalized as follows (Actis, 2014; Rodríguez-Bernal & Sastre-Gómez, 2014). Let (X,d,μ) be a metric measure space. Given

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$T \in \mathbb{R}^+$ fixed, $f \in L^1(X,\mu)$ and $J : X \times X \to \mathbb{R}^+$ we can consider the following nonlocal diffusion problem:

$$\begin{cases} u_t(x,t) = \displaystyle\int_X J(x,y)[u(y,t)-u(x,t)]\,d\mu(y), & x \in X,\, t \in (0,T), \\ u(x,0) = f(x), & x \in X. \end{cases} \tag{1.1}$$

where the equalities are understood in the sense of $L^1(X,\mu)$. The well posedness of (1.1) has been addressed by Actis (2014) and Rodríguez-Bernal & Sastre-Gómez (2014) for the metric measure space setting; see Cortazar *et al.* (2009) for the Euclidean case. It has been proved that for each $f \in L^1(X,\mu)$ there exists a unique function u belonging to

$$\mathscr{B}_T := C([0,T];L^1(X,\mu)) \cap C^1((0,T);L^1(X,\mu)),$$

which solves problem (1.1). Here $C([0,T];L^1(X,\mu))$ denotes the space of continuous functions from $[0,T]$ to $L^1(X,\mu)$, i.e., $u(\cdot,t) \in L^1(X,\mu)$ for each $t \in [0,T]$ and $\|u(\cdot,t)-u(\cdot,t+h)\|_{L^1} \to 0$ when $h \to 0$; and $C^1\left((0,T);L^1(X,\mu)\right)$ denotes the space of functions with continuous Frechet's derivative in L^1, i.e., there exists $v \in C((0,T);L^1(X,\mu))$ such that

$$\left\| \frac{u(\cdot,t+h)-u(\cdot,t)}{h} - v(\cdot,t) \right\|_{L^1} \longrightarrow 0,$$

when $h \to 0$, for each $t \in (0,T)$. In such case we write $u_t = v$.

Nevertheless, no explicit form of the solution is known. The goal of this article is to propose a general method for the approximation of this solution in metric measure spaces, solving discrete problems, and to provide error estimates, analogous to those in Pérez-Llanos & Rossi (2011) which hold in domains of $\mathbb{R}^n$. Also, as in Pérez-Llanos & Rossi (2011), we study the asymptotic behavior as $t \to \infty$ of the solutions of (1.1).

In order to define the discrete solutions, let us assume that we can decompose X into a union of K pairwise disjoint measurable subsets, i.e. we can write $X = \bigcup_{k=1}^K X_k$, with $X_k \cap X_j = \emptyset$ if $k \neq j$. We shall refer to these sets X_k as the *components* of the space X.

For each k let us fix a point $x_k \in X_k$, that we shall call the *representative point* of the component X_k. Let $\mathscr{X}$ be the set of all the representative points, i.e. $\mathscr{X} = \{x_k \in X_k : 1 \leqslant k \leqslant K\}$, and let ν be the measure defined on $\mathscr{X}$ by $\nu(\{x_k\}) = \mu(X_k)$. Then $(\mathscr{X},d,\nu)$ is also a metric measure space.

Problem (1.1) considered on $(\mathscr{X},d,\nu)$, with a preassigned initial condition $\boldsymbol{f} = [f_1,f_2,\ldots,f_K] \in \mathbb{R}^K$, can be equivalently written as

$$\begin{cases} u_t(x_i,t) = \displaystyle\sum_{j=1}^K J(x_i,x_j)[u(x_j,t)-u(x_i,t)]\mu(X_j), & i \in I_K,\, t \in (0,T), \\ u(x_i,0) = f_i, & i \in I_K, \end{cases} \tag{1.2}$$

where $I_K := \{1,2,\ldots,K\}$. Notice that (1.2) is a homogeneous first-order linear system of ordinary differential equations. Indeed, if we denote $u_i(t) := u(x_i,t)$, $\boldsymbol{u}(t) = [u_1(t),u_2(t),\ldots,u_K(t)]$ and $A = (a_{ij})_{i,j=1}^K$ the matrix given by

$$a_{ij} = \begin{cases} -\sum_{\substack{k=1 \\ k\neq i}}^K J(x_i,x_k)\mu(X_k), & \text{if } i = j. \\ J(x_i,x_j)\mu(X_j), & \text{if } i \neq j, \end{cases} \tag{1.3}$$

then (1.2) can be rewritten as

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}, \qquad \boldsymbol{u}(0) = \boldsymbol{f}. \tag{1.4}$$

Therefore, $\boldsymbol{u}(t) = e^{At}\boldsymbol{f}$ is the unique solution which belongs to $C^\infty(\mathbb{R}^+)$. Moreover, given a vector norm $\|\cdot\|$ in $\mathbb{R}^K$, if we denote with $\|\cdot\|$ also the induced matrix norm, all the time derivatives of $\boldsymbol{u}$ can be bounded as follows:

$$\left\| \frac{d^k}{dt^k}\boldsymbol{u} \right\| \leqslant \|A\|^k \|\boldsymbol{u}\|, \qquad k = 1, 2, \ldots, \quad t \geqslant 0. \tag{1.5}$$

We now extend $\boldsymbol{u}$ and $\boldsymbol{f}$ to $X \times (0,T)$ as follows

$$U(x,t) := u_k(t) \quad \text{and} \quad F(x) := f_k \quad \text{for every } x \in X_k.$$

In other words, if $\mathbb{I}_A(x)$ denotes the indicator function on the set A,

$$U(x,t) = \sum_{k=1}^{K} u_k(t)\mathbb{I}_{X_k}(x) \quad \text{and} \quad F = \sum_{k=1}^{K} f_k\, \mathbb{I}_{X_k}(x), \tag{1.6}$$

so both are constant on each component X_k. We shall refer to F as *the extension of* $\boldsymbol{f}$ and U as *the extended solution associated to* $\boldsymbol{f}$.

The following error estimate between U and u is the first main result of this article:

Main Result 1 Let u be the solution of (1.1) for a given $f \in L^1(X,\mu)$, and let U be the extended solution associated to a given $\boldsymbol{f} \in \mathbb{R}^K$. Then

$$|||u - U|||_1 := \max_{0 \leqslant t \leqslant T} \|u(\cdot,t) - U(\cdot,t)\|_{L^1} \leqslant C\xi + \|f - F\|_{L^1},$$

where F is the extension of $\boldsymbol{f}$ and ξ depends on $\max\{\text{diam}(X_k) : k \in I_K\}$ and regularity properties of J. Moreover, if $f \in C(X)$, then

$$|||u - U|||_\infty := \max_{0 \leqslant t \leqslant T} \|u(\cdot,t) - U(\cdot,t)\|_{L^\infty} \leqslant C\xi + \|f - F\|_{L^\infty}.$$

In both cases C denotes a constant which depends on J but is otherwise independent of the particular decomposition of X.

We want to point out the following remarks concerning the above result.

- The approximation U of u is as good as the approximation F of f, except for the term ξ measuring the approximation of J by piecewise constant kernels. This term will have the form δ^r, where $\delta = \max\{\text{diam}(X_k) : k \in I_K\}$ and $r > 0$ is the Hölder regularity of J. In the particular case of f and J Lipschitz continuous, we obtain $|||u - U|||_\infty \leqslant C\delta^r$.
- In every bounded metric space with finite Assouad dimension, and in particular in every bounded space of homogeneous type, we can decompose the space in such a way that δ is as small as desired (see Christ, 1990; Aimar *et al.*, 2007; Hytönen & Kairema, 2012).
- In non-atomic spaces of homogeneous type, such as manifolds and typical fractals, the aforementioned decomposition can be obtained such that $\max\{\mu(X_k) : k \in I_K\}$ is small, allowing the elementary function F to be as close to f as desired, choosing $f_k = \frac{1}{\mu(X_k)} \int_{X_k} f\, d\mu$. Moreover, if $f \in C(X)$, F can be constructed using $f_k = f(x_k)$.

- The first numerical method for computing approximate solutions of this kind of nonlocal diffusion problems was developed by Pérez-Llanos & Rossi (2011) for domains of $\mathbb{R}^n$. We generalize that result to metric measure spaces and provide a different proof, by considering the approximations as solutions to problem (1.1) for piecewise constant kernels $\bar{J}$, rather than looking at the solution at points. Indeed, the function U is the unique solution in $\mathscr{B}_T$ of problem (1.1) with kernel $\bar{J}$ and initial datum F (see Lemma 3.1). So that u and U satisfy the same qualitative properties (see Section 2) without having to prove a discrete version of the results.

- The study of differential equations on fractals and their approximation, is not new. Finite element methods based on piecewise harmonic and biharmonic splines have been developed by Gibbons *et al.* (2001); Coletta *et al.* (2004). They studied certain classes of fractal differential equations on the Sierpinski gasket associated to the Kigami Laplacian, such us the heat equation, the wave equation, Schrödinger type equations, etc. Also, boundary value problems with fractal boundaries have been considered in (Mosco, 2013; Evans, 2011) and the references therein. However, as far as we know, the study of evolutionary problems involving integrable nonlocal operators in this context has never been done.

The second main result is a bound for the error corresponding to the fully discretized problem.

Main Result 2 Let $\Delta t > 0$ denote a time discretization parameter, and let $\bar{U}_n$, $n = 0, 1, 2, \ldots$, denote the approximations of $U(\cdot, t_n)$ obtained by a Runge-Kutta method of order k with step-size Δt, where $t_n = n\Delta t$. Then, for $p = 1$ or $p = \infty$,

$$\|u(\cdot,t_n) - \bar{U}_n\|_{L^p} \leqslant C\xi + \|f - F\|_{L^p} + \bar{C}_k^{\mathrm{RK}} C_p^{k+1} \|F\|_{L^p} T \Delta t^k, \qquad n = 0, 1, 2, \ldots, \lceil T/\Delta t \rceil,$$

where C, F, ξ are as before, C_k^{RK} depends on the Runge-Kutta method and C_p is defined as follows:

$$C_1 = 2 \max_{x,y \in X} J(x,y), \qquad C_\infty = 2 \max_{x \in X} \int_X J(x,y)\, d\mu(y).$$

The paper is organized as follows. In Section 2 we present the setting and we prove some qualitative properties of the solution. We shall use these results to show our first main result, which is precisely stated in Theorem 3.1 and proved in Section 3. In Section 4 we improve the given error estimation for the particular case that the initial datum $f \in L^2(X, \mu)$. The second main result is stated in Theorem 5.2, which is proved in Section 5. Section 6 is devoted to apply the results on the Sierpinski gasket and the Sierpinski carpet. Finally in Section 7 we state some conclusions and remarks.

2. Setting and qualitative properties

Let X be a set. A *quasi-distance* on X is a non-negative symmetric function d defined on $X \times X$ such that $d(x,y) = 0$ if and only if $x = y$, and there exists a constant $K \geqslant 1$ such that

$$d(x,y) \leqslant K(d(x,z) + d(z,y)), \quad \forall x, y, z \in X.$$

A quasi-distance d on X induces a topology through the neighborhood system given by the family of all subsets of X containing a d-ball $B(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$ (Coifman & Weiss, 1971; Macías & Segovia, 1979).

Throughout this paper (X, d, μ) shall be a compact quasi-metric measure space such that the d-balls are open sets with positive μ-measure, and μ is a finite non-negative Borel measure on X.

Also, $J : X \times X \to \mathbb{R}^+$ shall be a measurable function with respect to the product σ-algebra in $X \times X$ having the following properties:

(J1) $J(x,y) = J(y,x)$ for all $x,y \in X$.

(J2) The integral $\int_X J(x,y)\,d\mu(x)$ is positive and uniformly bounded in $y \in X$.

It is worth mentioning that assumptions (J1) and (J2) guarantee that problem (1.1) has a unique solution in $\mathscr{B}_T$ for each $f \in L^1(X,\mu)$, and it belongs to $C([0,T];C(X)) \cap C^1((0,T);C(X))$ if $f \in C(X)$ (Actis, 2014, Thm. 8.2.2 and Lemma 8.3.1); see also Rodríguez-Bernal & Sastre-Gómez (2014). In this section we shall analyze some qualitative properties of this solution: conservation of the total mass, a comparison principle, stability and asymptotic behavior as $t \to \infty$. These properties are analogous to well known properties in the Euclidean case.

PROPOSITION 2.1 (Conservation of total mass) Let $f \in L^1(X,\mu)$ and let u be the solution of (1.1). Then

$$\int_X u(x,t)\,d\mu(x) = \int_X f(x)\,d\mu(x), \qquad \text{for all } t > 0.$$

Proof. Notice that for each $t > 0$ we have

$$u(x,t) = f(x) + \int_0^t \int_X J(x,y)[u(y,s) - u(x,s)]\,d\mu(y)\,ds, \qquad \text{a.e. } x \in X.$$

The assertion follows after integrating on x over X, applying Fubini's theorem and using the symmetry of J. □

In order to state the stability of the problem, which is contained in Proposition 2.2, we shall first prove some preliminary results.

LEMMA 2.1 If $u \in \mathscr{B}_T$ then the scalar function $g : \mathbb{R} \to \mathbb{R}$, $g(t) = \|u^+(\cdot,t)\|_{L^1}$ is weakly differentiable on $[0,T]$ and

$$\frac{d}{dt}\|u^+(\cdot,t)\|_{L^1} = \int_X u_t(x,t)\mathbb{I}_{\{u(\cdot,t)>0\}}(x)\,d\mu(x),$$

where $u^+(x,t) = \max\{u(x,t),0\}$ is the positive part of u.

REMARK 2.1 Notice that if $u^-(x,t) = \max\{-u(x,t),0\}$ denotes the negative part of u, then we have that $u^-(x,t) = (-u)^+(x,t)$, so that Lemma 2.1 yields

$$\frac{d}{dt}\|u^-(\cdot,t)\|_{L^1} = -\int_X u_t(x,t)\mathbb{I}_{\{u(\cdot,t)<0\}}(x)\,d\mu(x).$$

Proof of Lemma 2.1. For $\varepsilon > 0$, let $u_\varepsilon = \varphi_\varepsilon \circ u$ with

$$\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad \varphi_\varepsilon(s) = \begin{cases} \sqrt{s^2+\varepsilon^2} - \varepsilon, & \text{if } s > 0, \\ 0, & \text{if } s \leqslant 0. \end{cases}$$

Notice that $0 \leqslant \varphi_\varepsilon(s) \leqslant \max\{s,0\}$ for all $s \in \mathbb{R}$, and moreover

$$\varphi_\varepsilon(s) = 0 = \max\{s,0\} \text{ if } s \leqslant 0,$$

$$|\varphi_\varepsilon(s) - \max\{s,0\}| = s - \sqrt{s^2+\varepsilon^2} + \varepsilon \leqslant \varepsilon \text{ if } s > 0,$$

so that $\varphi_\varepsilon(s) \to \max\{s,0\}$ uniformly in $s \in \mathbb{R}$. Therefore, the dominated convergence theorem in X yields, as $\varepsilon \to 0$,

$$\|u_\varepsilon(\cdot,t)\|_{L^1} = \int_X u_\varepsilon(x,t) d\mu(x) \to \int_X \max\{u(x,t),0\} d\mu(x) = \|u^+(\cdot,t)\|_{L^1}.$$

Let $\psi \in C_0^\infty(0,T)$. Then, on the one hand, as $\varepsilon \to 0$

$$\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^1} \psi'(t)\,dt \to \int_0^T \|u^+(\cdot,t)\|_{L^1} \psi'(t)\,dt, \tag{2.1}$$

by the dominated convergence theorem on $[0,T]$.

On the other hand, since $\varphi_\varepsilon \in C^1(\mathbb{R})$ and $0 \leqslant \varphi'_\varepsilon(s) \leqslant 1$ for all $s \in \mathbb{R}$ we have that

$$\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|_{L^1} = \frac{d}{dt}\int_X \varphi_\varepsilon(u(x,t))\,d\mu(x) = \int_X \varphi'_\varepsilon(u(x,t))u_t(x,t)\,d\mu(x),$$

so that

$$\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^1} \psi'(t)\,dt = -\int_0^T \left(\int_X \varphi'_\varepsilon(u(x,t))u_t(x,t)\,d\mu(x)\right)\psi(t)\,dt$$

and thus, as $\varepsilon \to 0$,

$$\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^1} \psi'(t)\,dt \to -\int_0^T \left(\int_X \mathbb{I}_{\{u(\cdot,t)>0\}}(x)u_t(x,t)\,d\mu(x)\right)\psi(t)\,dt. \tag{2.2}$$

Here we have used that $\varphi'_\varepsilon(s) \to \mathbb{I}_{(0,+\infty)}(s)$ and the dominated convergence theorem twice, once on X for each $t \in [0,T]$ and once on $[0,T]$. Finally, (2.1) and (2.2) imply that

$$\int_0^T \|u^+(\cdot,t)\|_{L^1} \psi'(t)\,dt = -\int_0^T \left(\int_X \mathbb{I}_{\{u(\cdot,t)>0\}}(x)u_t(x,t)\,d\mu(x)\right)\psi(t)\,dt$$

and the assertion follows. □

Let us recall that $u \in \mathscr{B}_T$ is a *supersolution* of (1.1) if it satisfies

$$\begin{cases} u_t(x,t) \geqslant Lu(x,t), & \text{in } X \times (0,T), \\ u(x,0) \geqslant f(x), & \text{in } X, \end{cases}$$

where

$$Lu(x,t) = \int_X J(x,y)[u(y,t) - u(x,t)]\,d\mu(y).$$

We define *subsolutions* in a similar way, with $\leqslant$ instead of $\geqslant$.

LEMMA 2.2 If $u \in \mathscr{B}_T$ is a supersolution of (1.1), then

$$\frac{d}{dt}\|u^-(\cdot,t)\|_{L^1} \leqslant 0.$$

Analogously, if u is a subsolution, we obtain $\frac{d}{dt}\|u^+(\cdot,t)\|_{L^1} \leqslant 0$.

Proof. Since u is a supersolution of (1.1), Lemma 2.1 (see Remark 2.1) yields

$$\begin{aligned}\frac{d}{dt}\|u^-(\cdot,t)\|_{L^1} &= -\int_X u_t(x,t)\mathbb{I}_{\{u(\cdot,t)<0\}}(x)\,d\mu(x)\\ &\leqslant \int_X -Lu(x,t)\mathbb{I}_{\{u(\cdot,t)<0\}}(x)\,d\mu(x).\end{aligned}$$

By the definition of L,

$$\begin{aligned}\frac{d}{dt}\|u^-(\cdot,t)\|_{L^1} &\leqslant \int_{\{x:u(x,t)<0\}}\left(\int_X J(x,y)[-u(y,t)+u(x,t)]\,d\mu(y)\right)d\mu(x)\\
&= \int_{\{x:u(x,t)<0\}}\int_{\{y:u(y,t)<0\}} J(x,y)[-u(y,t)]\,d\mu(y)\,d\mu(x)\\
&\quad+\int_{\{x:u(x,t)<0\}}\int_{\{y:u(y,t)\geqslant 0\}} J(x,y)[-u(y,t)]\,d\mu(y)\,d\mu(x)\\
&\quad+\int_{\{x:u(x,t)<0\}}\int_{\{y:u(y,t)<0\}} J(x,y)u(x,t)\,d\mu(y)\,d\mu(x)\\
&\quad+\int_{\{x:u(x,t)<0\}}\int_{\{y:u(y,t)\geqslant 0\}} J(x,y)u(x,t)\,d\mu(y)\,d\mu(x)\\
&\leqslant \int_{\{x:u(x,t)<0\}}\int_{\{y:u(y,t)<0\}} J(x,y)[-u(y,t)]\,d\mu(y)\,d\mu(x)\\
&\quad+\int_{\{x:u(x,t)<0\}}\int_{\{y:u(y,t)<0\}} J(x,y)u(x,t)\,d\mu(y)\,d\mu(x).\end{aligned}$$

Since J is symmetric the last terms cancel out and we obtain $\frac{d}{dt}\|u^-(\cdot,t)\|_{L^1}\leqslant 0$. □

COROLLARY 2.1 (Comparison principle) If $u\in\mathscr{B}_T$ is a supersolution of (1.1) and $f\geqslant 0$, then $u(\cdot,t)\geqslant 0$ for every t.

Proof. From Lemma 2.2, the non-negative function $g(t):=\|u^-(\cdot,t)\|_{L^1}$ satisfies $g'(t)\leqslant 0$ and $g(0)=0$, because $u(x,0)\geqslant 0$ implies $u^-(\cdot,0)=0$. Then $g(t)=0$ for every t, and therefore $u(x,t)\geqslant 0$ for almost every x, for every t. □

We shall use Lemma 2.2 and Corollary 2.1 to prove the following result concerning the stability of problem (1.1).

PROPOSITION 2.2 (Stability) Let $f,g\in L^1(X,\mu)$ and let u and v denote the solutions of problem (1.1) with initial conditions f and g, respectively. Then

$$|||u-v|||_1=\max_{0\leqslant t\leqslant T}\|u(\cdot,t)-v(\cdot,t)\|_{L^1}=\|f-g\|_{L^1}. \tag{2.3}$$

Moreover, if $f,g\in L^\infty(X,\mu)$,

$$|||u-v|||_\infty=\max_{0\leqslant t\leqslant T}\|u(\cdot,t)-v(\cdot,t)\|_{L^\infty}=\|f-g\|_{L^\infty}. \tag{2.4}$$

Proof. In order to prove (2.3), let $e=u-v$ and observe that

$$\|e(\cdot,t)\|_{L^1}=\|e^+(\cdot,t)\|_{L^1}+\|e^-(\cdot,t)\|_{L^1}.$$

Since $e_t(x,t) = Le(x,t)$, e is a subsolution and a supersolution to (1.1), so that Lemma 2.2 yields

$$\frac{d}{dt}\|e(\cdot,t)\|_{L^1} = \frac{d}{dt}\|e^+(\cdot,t)\|_{L^1} + \frac{d}{dt}\|e^-(\cdot,t)\|_{L^1} \leqslant 0.$$

Therefore $\|e(\cdot,t)\|_{L^1} \leqslant \|e(\cdot,0)\|_{L^1} = \|f-g\|_{L^1}$ and (2.3) follows, because the maximum is achieved in $t=0$.

To prove (2.4) let $\ell = \|f-g\|_{L^\infty}$. Then $\overline{w} = u - v + \ell$ satisfies $\overline{w}_t = L\overline{w}$ and $\overline{w}(0,t) \geqslant 0$, so that from Corollary 2.1 we have $\overline{w}(x,t) \geqslant 0$ for almost every x and every t. Similarly, if we define $\underline{w} = \ell - (u-v)$ we obtain $\underline{w}(x,t) \geqslant 0$. Then

$$-\ell \leqslant u(x,t) - v(x,t) \leqslant \ell,$$

and (2.4) is proved. □

REMARK 2.2 Notice that as a consequence of the above proposition we have that if $u \in \mathscr{B}_T$ is the solution of (1.1) with $f \in L^\infty(X,\mu)$, then $u(\cdot,t) \in L^\infty(X,\mu)$ for each $t \in [0,T]$. Moreover, $|||u|||_\infty = \|f\|_{L^\infty}$.

Finally we shall study the asymptotic behavior of the solutions. Throughout the rest of this section, we shall assume:

- (X,d) is connected,
- $J(x,x) > 0$ for every x and $J(x,y)$ is continuous in x for each y.

We shall first consider the corresponding stationary problem:

$$Lu(x) = \int_X J(x,y)[u(y)-u(x)]\,d\mu(y) = 0, \qquad x \in X. \tag{2.5}$$

LEMMA 2.3 Every solution in $L^1(X,\mu)$ of the stationary problem is constant in X.

Proof. We shall first prove that if $u \in L^1(X,\mu)$ is a solution of (2.5), then u is a continuous function. Indeed, for almost every $x \in X$ we have that

$$u(x) = \int_X \frac{J(x,y)}{I(x)} u(y)\,d\mu(y),$$

where $I(x) := \int_X J(x,y)\,d\mu(y) > 0$ due to (J2). Since J is continuous and X is compact, there exists $I_0 > 0$ such that $I(x) \geqslant I_0$ for all $x \in X$. Then the function $\tilde{J}(\cdot,y) = \frac{J(\cdot,y)}{I(\cdot)}$ is continuous and thus uniformly continuous for each y, which immediately implies that u is continuous.

Let $M = \max\{u(x) : x \in X\}$, and consider the set

$$\mathscr{M} = \{x \in X : u(x) = M\}.$$

Then the set $\mathscr{M}$ is nonempty and closed. Since the only subsets of a connected space X which are both open and closed are X and the empty set, the result is proved if we show that $\mathscr{M}$ is also open. Fix $x_0 \in \mathscr{M}$. Since $J(x_0,x_0) > 0$ and $J(x_0,\cdot)$ is continuous, there exists $r_0 > 0$ such that $B(x_0,r_0) \subseteq \operatorname{supp} J(x_0,\cdot)$. Assume that $B(x_0,r_0) \nsubseteq \mathscr{M}$, so that there exists $z \in B(x_0,r_0)$ with $u(z) < M$. Hence $u(y) < M$ for each y in some ball B centered in z and contained in $\operatorname{supp} J(x_0,\cdot)$. Then

$$\begin{aligned} M = u(x_0) &= \int_{X\setminus B} \tilde{J}(x_0,y)u(y)\,d\mu(y) + \int_B \tilde{J}(x_0,y)u(y)\,d\mu(y) \\ &< M\int_X \tilde{J}(x_0,y)\,d\mu(y) = M, \end{aligned}$$

which is absurd. Hence $\mathscr{M}$ is open, so that $u(x) = M$ for every $x \in X$. □

PROPOSITION 2.3 (Asymptotic behavior) If $u \in \mathscr{B}_T$ is the solution of (1.1) for $f \in L^2(X,\mu)$, then there exists $\beta > 0$ such that

$$\left\| u(\cdot,t) - \fint_X f\,d\mu \right\|_{L^2} \leqslant e^{-\beta t} \left\| f - \fint_X f\,d\mu \right\|_{L^2},$$

where $\fint_X f\,d\mu := \frac{1}{\mu(X)} \int_X f\,d\mu$ denotes the average value of f.

The proof of this proposition is analogous to the one in Andreu-Vaillo *et al.* (2010) but we decided to include it here for the sake of completeness.

Proof. From the linearity of the problem, we can assume $\int_X f\,d\mu = 0$. Proposition 2.1 implies that also $\int_X u\,d\mu = 0$ for all $t > 0$. From the results of Rodríguez-Bernal & Sastre-Gómez (2014), since $f \in L^2(X,\mu)$ we have that $u(\cdot,t) \in L^2(X,\mu)$ for each t. Being u a solution of problem (1.1), we have that

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2 = \langle u_t, u\rangle = \langle Lu, u\rangle = \frac{\langle Lu, u\rangle}{\|u(\cdot,t)\|_{L^2}^2}\|u(\cdot,t)\|_{L^2}^2 \leqslant -\beta\|u(\cdot,t)\|_{L^2}^2,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L^2(X,\mu)$, and

$$\beta := \inf_{v \in L_0^2} -\frac{\langle Lv, v\rangle}{\|v\|_{L^2}^2} = \inf_{v \in L_0^2,\, \|v\|_{L^2}=1} \langle -Lv, v\rangle, \tag{2.6}$$

with $L_0^2 = \{v \in L^2(X,\mu) : \int_X v\,d\mu = 0\}$. Hence, if we denote

$$H(t) = \frac{1}{2}\|u(\cdot,t)\|_{L^2}^2,$$

we have proved that $H'(t) \leqslant -2\beta H(t)$, and using Gronwall's inequality we obtain

$$H(t) \leqslant e^{-2\beta t} H(0).$$

Therefore, the assertion will be proved if we show that $\beta > 0$. Notice that

$$\beta = \inf_{v \in L_0^2} \frac{\frac{1}{2}\int_X\int_X J(x,y)[v(y)-v(x)]^2 d\mu(y) d\mu(x)}{\|v\|_{L^2}^2},$$

hence $\beta \geqslant 0$. To prove that β is strictly positive, consider the operator $-L : L_0^2 \to L_0^2$, and notice that it is self-adjoint, so that β belongs to its spectrum $\sigma(-L)$ (Brezis, 1983). If $\beta = 0$, we have that $0 \in \sigma(-L)$, then $-L$ is not invertible. But notice that

$$-Lv(x) = Av(x) - Kv(x) = [A(I - A^{-1}K)]v(x),$$

with

$$Av(x) = v(x)\int_X J(x,y)\,d\mu(y), \qquad \text{and} \qquad Kv(x) = \int_X J(x,y)v(y)\,d\mu(y),$$

so that A is invertible and K is compact (Rodríguez-Bernal & Sastre-Gómez, 2014, Prop. 3.6). Then $I - A^{-1}K$ is not invertible, and Fredholm's alternative yields the existence of a nontrivial $u \in L_0^2$ such that $(I - A^{-1}K)u = 0$, or equivalently, $Lu = 0$. From Lemma 2.3 u must be constant, and thus $\int_X u\,d\mu \neq 0$, which is a contradiction. □

COROLLARY 2.2 If $u \in \mathscr{B}_T$ is the solution of (1.1) for a given $f \in L^1(X,\mu)$, then

$$\lim_{t\to\infty} \left\| u(\cdot,t) - \fint_X f\,d\mu \right\|_{L^1} = 0.$$

Proof. Let $f \in L^1(X)$, and as before, assume without loss of generality, that $\int_X f\,d\mu = 0$. Given $\varepsilon > 0$, let $g \in L^2(X)$ be such that $\int_X g\,d\mu = 0$ and $\|f-g\|_{L^1} \leqslant \varepsilon/2$. Let v be the solution of (1.1) with initial datum g, so that Proposition 2.3 yields

$$\|v(\cdot,t)\|_{L^2} \leqslant e^{-\beta t}\|g\|_{L^2}$$

for some $\beta > 0$. Then, by Proposition 2.2 and Hölder inequality

$$\begin{aligned}\|u(\cdot,t)\|_{L^1} &\leqslant \|u(\cdot,t)-v(\cdot,t)\|_{L^1} + \|v(\cdot,t)\|_{L^1}\\ &\leqslant \|f-g\|_{L^1} + \mu(X)^{1/2}\|v(\cdot,t)\|_{L^2}\\ &\leqslant \varepsilon/2 + \mu(X)^{1/2}e^{-\beta t}\|g\|_{L^2}.\end{aligned}$$

Choosing $t_* > 0$ such that $\mu(X)^{1/2}e^{-\beta t_*}\|g\|_{L^2} = \varepsilon/2$ we have that

$$\|u(\cdot,t)\|_{L^1} \leqslant \varepsilon,$$

for all $t \geqslant t_*$ and the claim follows. □

REMARK 2.3 The assumption of X being connected is used only in the proof of Lemma 2.3 and can be weakened. Assuming X to be R-connected as in Rodríguez-Bernal & Sastre-Gómez (2014) is sufficient for the assertion. A weaker assumption, stated in Lemma 2.2 of Gilboa & Osher (2007) also implies the assertion of Lemma 2.3. It reads as follows: given two points $x,y \in X$ there exists a finite sequence $x_1,x_2,\dots,x_k \in X$ such that $J(x,x_1)J(x_1,x_2)\dots J(x_{k-1},x_k)J(x_k,y) > 0$. We kept the stronger assumption of X being connected to simplify the presentation.

3. Error estimation for the space discretization

From now on we shall assume:

(J3) There exists a constant $\lambda > 0$ and $r \in (0,1]$ such that

$$|J(x,y) - J(x,z)| \leqslant \lambda\, d(y,z)^r, \quad \forall x,y,z \in X. \tag{3.1}$$

Notice that this condition implies condition (J2) stated in page 5.

In order to state our main result, fix a decomposition $\{X_1,\dots,X_K\}$ of X and a set of representative points $\{x_1,\dots,x_k\}$. From now on

$$\delta := \max\{\mathrm{diam}(X_k) : k = 1,\dots,K\}$$

is called the *size* of the decomposition. Let $\bar{J}$ be the kernel defined on $X \times X$ which is constant on each $X_i \times X_k$, taking the value of J in the representative pair (x_i,x_k), i.e.

$$\bar{J}(x,y) := J(x_i,x_k), \quad \text{if } x \in X_i \text{ and } y \in X_k.$$

Finally, given a discrete initial condition $\boldsymbol{f} = [f_1,\dots,f_K] \in \mathbb{R}^K$ let U be the extended solution associated to $\boldsymbol{f}$ and let F be the extension of $\boldsymbol{f}$ (see (1.6)). The main error estimate reads as follows:

THEOREM 3.1 Let u be the solution of (1.1) for a given $f \in L^p(X,\mu)$, for $p=1$ or $p=\infty$. Then

$$|||u-U|||_p \leqslant 4\lambda\mu(X)T\|f\|_{L^p}\delta^r + \|f-F\|_{L^p},$$

where λ and r denote the Lipschitz constants of J from (3.1).

To prove this theorem we need the following lemmas. We first show that the function U also solves problem (1.1) with kernel $\bar{J}$, and initial datum F.

LEMMA 3.1 The function U is the unique solution in $\mathscr{B}_T$ of the problem

$$\begin{cases} U_t(x,t) = \displaystyle\int_X \bar{J}(x,y)[U(y,t)-U(x,t)]d\mu(y), & x\in X, t\in(0,T), \\ U(x,0)=F(x), & x\in X. \end{cases} \tag{3.2}$$

Proof. Notice first that $U \in C([0,T];C(X_k)) \cap C^\infty((0,T);C(X_k))$ for every k, so that $U \in \mathscr{B}_T$. In order to see that U solves (3.2), fix $x \in X$ and $t\in(0,T)$. Then there exists a unique i such that $x \in X_i$, so that

$$\begin{aligned} \int_X \bar{J}(x,y)[U(y,t)-U(x,t)]d\mu(y) &= \sum_{k=1}^K \int_{X_k} \bar{J}(x,y)[U(y,t)-U(x,t)]d\mu(y) \\ &= \sum_{k=1}^K J(x_i,x_k)[u_k(t)-u_i(t)]\mu(X_k) \\ &= \frac{d}{dt}u_i(t) = U_t(x,t), \end{aligned}$$

and $U(x,0)=u_i(0)=F(x)$. Since $\bar{J}$ satisfies (J1) and (J2) (see page 5), problem (3.2) has a unique solution and the assertion follows. □

REMARK 3.1 Lemma 3.1 allows us to view the discrete solution U as a solution to problem (1.1) with a different kernel. Therefore, Proposition 2.2 and Remark 2.2 allow us to conclude that

$$|||U|||_1 = \|F\|_{L^1}, \qquad |||U|||_\infty = \|F\|_{L^\infty}. \tag{3.3}$$

The next lemma shows that

$$\bar{L}u(x,t) := \int_X \bar{J}(x,y)[u(y,t)-u(x,t)]\,d\mu(y), \tag{3.4}$$

approximates Lu in terms of the regularity of J.

LEMMA 3.2 If $u \in L^p(X,\mu)$ for $p=1$ or $p=\infty$ then

$$\|Lu-\bar{L}u\|_{L^p} \leqslant 4\lambda\mu(X)\|u\|_{L^p}\delta^r.$$

Proof. Notice that if $x\in X_i$ and $y \in X_k$, from the symmetry and the Lipschitz condition of J we have

$$\begin{aligned} |J(x,y)-J(x_i,x_k)| &\leqslant |J(x,y)-J(x_i,y)| + |J(x_i,y)-J(x_i,x_k)| \\ &\leqslant \lambda(d(x,x_i)^r + d(y,x_k)^r) \\ &\leqslant 2\lambda\delta^r. \end{aligned}$$

Therefore, for $u \in L^1(X,\mu)$ and $x \in X$ we have

$$\begin{aligned}
|Lu(x) - \bar{L}u(x)| &\leqslant \sum_{k=1}^{K} \int_{X_k} |J(x,y) - \bar{J}(x,y)|\,|u(y) - u(x)|\,d\mu(y) \\
&\leqslant \sum_{k=1}^{K} \int_{X_k} |J(x,y) - J(x_i,x_k)|\,|u(y) - u(x)|\,d\mu(y) \\
&\leqslant 2\lambda\delta^r \sum_{k=1}^{K} \int_{X_k} |u(y) - u(x)|\,d\mu(y) \\
&\leqslant 2\lambda\delta^r \int_X |u(y) - u(x)|\,d\mu(y). \qquad (3.5)
\end{aligned}$$

Thus

$$\begin{aligned}
\|Lu - \bar{L}u\|_{L^1} &= \int_X |Lu(x) - \bar{L}u(x)|\,d\mu(x) \\
&\leqslant 2\lambda\delta^r \int_X \int_X (|u(y)| + |u(x)|)\,d\mu(y)\,d\mu(x) \\
&= 4\lambda\delta^r \mu(X)\|u\|_{L^1}.
\end{aligned}$$

Also, if $u \in L^\infty(X,\mu)$,

$$\|Lu - \bar{L}u\|_{L^\infty} \leqslant 2\lambda\delta^r \int_X (|u(y)| + \|u\|_{L^\infty})\,d\mu(y) \leqslant 4\lambda\delta^r \mu(X)\|u\|_{L^\infty},$$

and the lemma is proved. □

The following result compares the solutions of problems with the same initial condition, but with different kernels J and $\bar{J}$.

LEMMA 3.3 Let $f \in L^p(X,\mu)$, for $p = 1$ or $p = \infty$. Let V be the unique solution in $\mathscr{B}_T$ of (1.1) with kernel $\bar{J}$ instead of J. Then, if u is the solution of (1.1), we have that

$$|||u - V|||_p \leqslant 4\lambda\mu(X)T\|f\|_{L^p}\delta^r,$$

with λ and r as in Theorem 3.1.

Proof. Define $w = u - V$, and notice that w solves

$$\begin{cases} w_t(x,t) = \bar{L}w(x,t) + G(x,t), & \text{in } X \times (0,T), \\ w(x,0) = 0, & \text{in } X. \end{cases}$$

where $G(x,t) = Lu(x,t) - \bar{L}u(x,t)$.

Let us first consider the case $u \in \mathscr{B}_T$. Let v be the unique solution in $\mathscr{B}_T$ of

$$\begin{cases} v_t(x,t) = \bar{L}v(x,t) + |G(x,t)|, & \text{in } X \times (0,T), \\ v(x,0) = 0, & \text{in } X. \end{cases}$$

It is worth mentioning that the exact same arguments used by Actis (2014) to prove existence of solution of the homogeneous problem (1.1) allow us to prove that this inhomogeneous problem has a unique solution in $\mathscr{B}_T$. Then $v-w$ satisfies

$$\begin{cases} (v-w)_t(x,t) \geqslant \bar{L}(v-w)(x,t), & \text{in } X\times(0,T), \\ (v-w)(x,0)=0, & \text{in } X. \end{cases}$$

From Corollary 2.1 we have that $v-w \geqslant 0$. Analogously we obtain $v+w \geqslant 0$, so that $|w| \leqslant v$. Notice also that from the symmetry of J we have that $\int_X \bar{L}v(x,t)\,d\mu(x)=0$. Then, for each t we obtain

$$\begin{aligned} \int_X |w(x,t)|\,d\mu(x) &\leqslant \int_X v(x,t)\,d\mu(x) \\ &= \int_X \int_0^t v_t(x,s)\,ds\,d\mu(x) \\ &= \int_0^t \int_X |G(x,s)|\,d\mu(x)\,ds \\ &\leqslant t\,|||G|||_1 \\ &\leqslant T 4\lambda\mu(X)\,|||u|||_1\,\delta^r, \end{aligned}$$

where the last inequality stems from Lemma 3.2. Hence

$$|||u-V|||_1 \leqslant 4\lambda\mu(X)T\,|||u|||_1\,\delta^r = 4\lambda\mu(X)T\,\|f\|_{L^1}\delta^r,$$

due to Proposition 2.2.

Let us now consider the case $f\in L^\infty(X,\mu)$. From Remark 2.2 we have that $|||u|||_\infty = \|f\|_{L^\infty} < \infty$. Define $\bar{v}(x,t) = k\delta^r t - w(x,t)$, with $k = 4\lambda\mu(X)\,|||u|||_\infty$. Notice that

$$\bar{v}_t(x,t) = k\delta^r - w_t(x,t) = k\delta^r - G(x,t) - \bar{L}w(x,t).$$

From Lemma 3.2, we have that $k\delta^r - G(x,t) \geqslant 0$. Then

$$\bar{v}_t(x,t) \geqslant -\bar{L}w(x,t) = \bar{L}\bar{v}(x,t) - \bar{L}(k\delta^r t) = \bar{L}\bar{v}(x,t).$$

Besides $\bar{v}(x,0)=0$, so that Corollary 2.1 yields $\bar{v}(x,t)\geqslant 0$, and thus $w(x,t)\leqslant k\delta^r t$.

Analogously, if we define $\underline{v}(x,t) = k\delta^r t + w(x,t)$, we can prove that $\underline{v}(x,t)\geqslant 0$, and then $w(x,t)\geqslant -k\delta^r t$. Then for almost every $x\in X$ and for every t we have

$$|u(x,t)-V(x,t)| \leqslant k\delta^r t \leqslant kT\delta^r.$$

Therefore,

$$|||u-V|||_\infty \leqslant 4\lambda\mu(X)\,|||u|||_\infty\,T\delta^r,$$

and the assertion follows from (2.4). □

Proof of Theorem 3.1. From Lemma 3.1, U is the unique solution in $\mathscr{B}_T$ of problem (3.2). If V is defined as in Lemma 3.3, then

$$|||u-V|||_1 \leqslant 4\lambda T\|f\|_{L^1}\delta^r.$$

Besides, from Proposition 2.2 applied to $\bar{J}$ and the initial conditions F and f we have

$$|||U-V|||_1 \leqslant \|f-F\|_{L^1}.$$

Hence

$$|||u-U|||_1 \leqslant |||u-V|||_1 + |||V-U|||_1 \leqslant 4\lambda T \|f\|_{L^1} \delta^r + \|f-F\|_{L^1}.$$

The case $f \in L^\infty(X,\mu)$ can be proved analogously. □

4. A sharper error estimation for initial datum in L^2

In Lemma 3.3 we proved that the error obtained approximating the solution u of problem (1.1) by the solution of the same problem but with a piecewise constant kernel $\bar{J}$, can be made as small as desired at any time provided the size of the decomposition of X is small enough. More precisely, if $f \in L^1(X,\mu)$ and u and V denote the unique solutions in $\mathscr{B}_T$ of (1.1) with kernels J and $\bar{J}$ respectively, then for each $t > 0$ we have

$$\|u(\cdot,t) - V(\cdot,t)\|_{L^1} \leqslant 4\lambda t \|f\|_{L^1} \delta^r,$$

where δ is the size of the decomposition of X, and λ and r denote the Lipschitz constants of J from (3.1). As we mentioned in Section 1, every bounded metric space with finite Assouad dimension, and in particular every bounded space of homogeneous type (such as manifolds and classical fractals), can be decomposed in such a way that δ is as small as desired. However, this bound is pessimistic for large values of t. Notice that, independently of the decomposition, for any $t > 0$ we have

$$\|u(\cdot,t) - V(\cdot,t)\|_{L^1} \leqslant \left\| u(\cdot,t) - \fint_X f\,d\mu \right\|_{L^1} + \left\| V(\cdot,t) - \fint_X f\,d\mu \right\|_{L^1},$$

which tends to zero when $t \to \infty$, due to Corollary 2.2. For the case $f \in L^2(X,\mu)$, from Proposition 2.3 and Hölder inequality we can obtain a more precise bound:

$$\|u(\cdot,t) - V(\cdot,t)\|_{L^1} \leqslant 2\mu(X)^{1/2} \left\| f - \fint_X f\,d\mu \right\|_{L^2} e^{-\beta_0 t},$$

with $\beta_0 = \min\{\beta, \bar{\beta}\} > 0$, where β and $\bar{\beta}$ are defined as in (2.6) with L and $\bar{L}$ respectively. On the other hand, using (3.5) and following the lines of the proof of Lemma 3.3, we get

$$\begin{aligned}
\|u(\cdot,t) - V(\cdot,t)\|_{L^1} &\leqslant \int_0^t \int_X |Lu(x,s) - \bar{L}u(x,s)|\,d\mu(x)ds \\
&\leqslant 2\lambda\delta^r \int_0^t \int_X \left(\int_X |u(y,s) - u(x,s)|d\mu(y) \right) d\mu(x)ds \\
&\leqslant 4\mu(X)\lambda\delta^r \int_0^t \int_X \left| u(y,s) - \fint_X f\,d\mu \right| d\mu(y)ds \\
&= 4\mu(X)\lambda\delta^r \int_0^t \left\| u(\cdot,s) - \fint_X f\,d\mu \right\|_{L^1} ds \\
&\leqslant 4\mu(X)^{3/2}\lambda\delta^r \int_0^t \left\| u(\cdot,s) - \fint_X f\,d\mu \right\|_{L^2} ds \\
&\leqslant 4\mu(X)^{3/2}\lambda\delta^r \left\| f - \fint_X f\,d\mu \right\|_{L^2} \int_0^t e^{-\beta s}\,ds \\
&\leqslant \frac{4\mu(X)^{3/2}\lambda \left\| f - \fint_X f\,d\mu \right\|_{L^2}}{\beta} \delta^r.
\end{aligned}$$

Then, for the case $f \in L^2(X,\mu)$ we obtain that there exists a constant C such that

$$\|u(\cdot,t) - V(\cdot,t)\|_{L^1} \leqslant C \min\{e^{-\beta_0 t}, \delta^r\},$$

so that

$$\|u(\cdot,t) - U(\cdot,t)\|_{L^1} \leqslant C \min\{e^{-\beta_0 t}, \delta^r\} + \|f - F\|_{L^1}.$$

Therefore, except for the initial error $\|f - F\|_{L^1}$, for large times t the approximation is very good even with a poor decomposition of X, due to the asymptotic behavior of the solutions. In order to have good approximations for the initial phase of small time t, we require that the decomposition has a small size δ.

5. Time discretization

In this section we will propose and study a time discretization for (3.2) which leads to a fully discretized scheme to approximate (1.1). Recall that by the definition (3.4) of $\bar{L}$, problem (3.2) can be written as

$$\begin{cases} U_t(x,t) = \bar{L}U(x,t), & x \in X, t \in (0,T), \\ U(x,0) = F(x), & x \in X, \end{cases} \tag{5.1}$$

and it is equivalent to (1.4) if we write $U(x,t) = \sum_{i=1}^{K} u_i(t)\mathbb{I}_{X_i}(x)$ and $F(x) = \sum_{i=1}^{K} f_i(t)\mathbb{I}_{X_i}(x)$.

We first show that (5.1) is non-stiff and then establish error estimates for explicit methods from the Runge-Kutta family.

We start observing that by Hölder inequality, if $\bar{L}$ is the operator defined in (3.4), then

$$\|\bar{L}u\|_{L^1} \leqslant \underbrace{2 \max_{x,y \in X} J(x,y)}_{C_1} \|u\|_{L^1}, \qquad \|\bar{L}u\|_{L^\infty} \leqslant \underbrace{2 \max_{x \in X} \int_X J(x,y)\, d\mu(y)}_{C_\infty} \|u\|_{L^\infty}. \tag{5.2}$$

Notice that the constants C_1, C_∞ do not depend on the decomposition $\{X_k\}_{k=1}^K$ of the space X, but only on the kernel $J(\cdot,\cdot)$. Even though C_1, C_∞ can be large for some specific kernels, they are of moderate size for the most common ones (Andreu-Vaillo *et al.*, 2010), and most importantly, they do not grow when the space discretization gets finer. We thus infer that the system of ODE (5.1) is not stiff and can be approximated using explicit methods such as those from the Runge-Kutta family.

In order to obtain precise bounds we now observe the behavior of the time derivatives of U. Using (5.2), the fact that $\partial^k U/\partial t^k = \bar{L}^k U$, and (3.3) we have that, if $p = 1$ or $p = \infty$,

$$\left\| \frac{\partial^k}{\partial t^k} U(\cdot,t) \right\|_{L^p} \leqslant C_p^k \|F\|_{L^p}, \qquad k = 1,2,\ldots, \quad t \geqslant 0. \tag{5.3}$$

We now study in more detail the structure of the operator $\bar{L}$ by making use of the concept of *logarithmic norm* applied to the matrix A from (1.3). Given a vector norm, the logarithmic norm (Hairer *et al.*, 1993, Def. 10.4, page 61) of a square matrix A is defined as

$$\eta(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h},$$

where $\|\cdot\|$ is the norm induced by the vector norm in the space of matrices.

We consider the following norms in $\mathbb{R}^K$:

$$\|v\|_1 = \sum_{i=1}^{K} \mu(X_i)|v_i|, \qquad \|v\|_\infty = \max_{1\leqslant i\leqslant K} |v_i|.$$

If V denotes the extension of v given by $V(x) = \sum_{i=1}^{K} v_i \mathbb{I}_{X_k}(x)$, then

$$\|v\|_p = \|V\|_{L^p}, \qquad p = 1, \infty. \tag{5.4}$$

As is done in Hairer *et al.* (1993, Theorem I.10.5) it is easy to see that the logarithmic norm η_p associated to the vector norms $\|\cdot\|_p$ satisfies:

$$\eta_1(A) = \max_{1\leqslant j\leqslant K}\left(a_{jj} + \sum_{i\neq j} |a_{ij}| \frac{\mu(X_i)}{\mu(X_j)}\right), \qquad \eta_\infty(A) = \max_{1\leqslant i\leqslant K}\left(a_{ii} + \sum_{j\neq i} |a_{ij}|\right).$$

LEMMA 5.1 (Logarithmic norm of A) If we consider the matrix A from (1.3), then

$$\eta_p(A) = 0, \qquad p = 1, \infty. \tag{5.5}$$

Proof. Notice that for each $j = 1, 2, \ldots, K$, the definition (1.3) yields

$$a_{jj} + \sum_{i\neq j} |a_{ij}| \frac{\mu(X_i)}{\mu(X_j)} = -\sum_{k\neq j}^{K} J(x_j, x_k)\mu(X_k) + \sum_{i\neq j} J(x_i, x_j)\mu(X_j) \frac{\mu(X_i)}{\mu(X_j)} = 0$$

due to the symmetry of $J(\cdot,\cdot)$. Then $\eta_1(A) = 0$.

Now observe that definition (1.3) leads to

$$a_{ii} + \sum_{j\neq i} |a_{ij}| = -\sum_{j\neq i}^{K} J(x_i, x_j)\mu(X_j) + \sum_{j\neq i}^{K} J(x_i, x_j)\mu(X_j) = 0,$$

which implies that $\eta_\infty(A) = 0$. □

We are now in position to estimate the error of the time discretization by explicit Runge-Kutta methods.

THEOREM 5.1 Let $\Delta t > 0$ denote the step-size and let $\bar{U}_n$, $n = 0, 1, 2, \ldots, \lceil T/\Delta t \rceil$ denote the approximations of $U(\cdot, t_n)$, the solution of (5.1) at time $t_n = n\Delta t$, obtained by a Runge-Kutta method of order k with step-size Δt. Then, for $p = 1$, or $p = \infty$,

$$\|U(\cdot, t_n) - \bar{U}_n\|_{L^p} \leqslant C_k^{\mathrm{RK}} C_p^{k+1} \|F\|_{L^p}\, T\, \Delta t^k, \qquad n = 0, 1, 2, \ldots, \lceil T/\Delta t \rceil,$$

where C_k^{RK} depends on the Runge-Kutta method being used, but is independent of Δt and the particular decomposition $\{X_i\}_{i=1}^K$ of X, and

$$C_1 = 2 \max_{x,y\in X} J(x,y), \qquad C_\infty = 2 \max_{x\in X} \int_X J(x,y)\, d\mu(y).$$

Proof. Let $\boldsymbol{u}^n = (u_1^n, u_2^n, \ldots, u_K^n)$, $n = 1, 2, \ldots, \lceil T/\Delta t \rceil$ denote the approximations of $\boldsymbol{u}(t_n)$, the solution of (1.4) at time t_n, obtained by the Runge-Kutta method under consideration. Then $\bar{U}_n = \sum_{i=1}^{K} u_i^n \mathbb{I}_{X_i}$ and by (Hairer *et al.*, 1993, Theorem II.3.4, page 160), we have that, for $p = 1, \infty$,

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|_p \leqslant C_k^{\mathrm{RK}} \max_{[0,T]} \left\| \frac{d^{k+1}\boldsymbol{u}}{dt^{k+1}} \right\|_p T\, \Delta t^k, \qquad n = 0, 1, 2, \ldots, \lceil T/\Delta t \rceil.$$

The identity of norms (5.4) yields $\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|_p = \|U(t_n) - \bar{U}_n\|_{L^p}$ and together with (5.3) implies $\left\| \frac{d^{k+1}\boldsymbol{u}}{dt^{k+1}} \right\|_p \leqslant C_p^{k+1} \|F\|_{L^p}$. The assertion thus follows. □

Combining Theorems 3.1 and 5.1 we arrive at the second main result of this article, which is the error estimation for the solution of the fully discrete problem.

THEOREM 5.2 Let $\Delta t > 0$ denote the step-size and let $\bar{U}_n$, $n = 0, 1, 2, \ldots, \lceil T/\Delta t \rceil$ denote the approximations of $U(\cdot, t_n)$, the solution of (5.1) at time $t_n = n\Delta t$, obtained by a Runge-Kutta method of order k with step-size Δt. Then, for $p = 1$, or $p = \infty$,

$$\|u(\cdot,t_n) - \bar{U}_n\|_{L^p} \leqslant \|f - F\|_{L^p} + 4\lambda\mu(X)T\|f\|_{L^p}\delta^r + C_k^{\mathrm{RK}} C_p^{k+1} \|F\|_{L^p} T\,\Delta t^k, \qquad n = 0, 1, 2, \ldots, \lceil T/\Delta t \rceil,$$

where λ, r and δ are as in Theorem 3.1, and C_p, C_k^{RK} are as in Theorem 5.1.

6. Examples

The aim of this section is to give examples of explicit spaces of homogeneous type (X, d, μ) where Theorem 5.2 can be applied in order to obtain numerical approximations of the solution of problem (1.1). As we already mentioned, every bounded space of homogeneous type can be decomposed in the required form due to the construction provided by Christ (1990). Nevertheless, in the case of the classical fractals it is more suitable to work with another decomposition of the space that exploit their self-similarity property. We consider the usual approximation induced by the associated iterated function system (IFS); see Hutchinson (1981) or Falconer (1997).

Given a metric space (Y, d) we shall consider a finite set $\Phi = \{\phi_i : Y \to Y, i = 1, 2, \ldots, H\}$ of contractive similitudes with the same contraction rate α. This means that each ϕ_i satisfies

$$d(\phi_i(x), \phi_i(y)) = \alpha d(x, y)$$

for every $x, y \in Y$ and some $0 < \alpha < 1$. Also we shall assume that the IFS Φ satisfies the *open set condition*, which means that there exists a non-empty open set $U \subset Y$ such that

$$\bigcup_{i=1}^{H} \phi_i(U) \subseteq U,$$

and $\phi_i(U) \cap \phi_j(U) = \emptyset$ if $i \neq j$. For $n \in \mathbb{N}$, let $\mathfrak{I}^n = \{1, 2, \ldots, H\}^n$ be the set of "words" of length n. Given $\boldsymbol{i} = (i_1, i_2, \ldots, i_n) \in \mathfrak{I}^n$, we denote with $\boldsymbol{\phi}_{\boldsymbol{i}}^n$ the composition $\phi_{i_n} \circ \phi_{i_{n-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}$. Then for any subset E of X we write $\boldsymbol{\phi}_{\boldsymbol{i}}^n(E) = \left(\phi_{i_n} \circ \phi_{i_{n-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}\right)(E)$.

It is well known that if E is a compact set and $X^n = \bigcup_{\boldsymbol{i} \in \mathfrak{I}^n} \boldsymbol{\phi}_{\boldsymbol{i}}^n(E)$, then the sequence of sets $\{X^n\}_n$ converges is the sense of the Hausdorff distance to a non-empty compact set X, which is called the *attractor* of the system Φ since it is the unique satisfying

$$X = \bigcup_{i=1}^{H} \phi_i(X).$$

It is also called the fractal induced by the IFS Φ, and moreover, if E satisfies $\phi_i(E) \subseteq E$ for every i, then $X = \bigcap_{n=1}^{\infty} X^n$.

There exists also a Borel probability measure μ supported on the attractor X. This measure is called *invariant* or *self-affine* since is the unique measure satisfying

$$\mu(A) = \frac{1}{H} \sum_{i=1}^{H} \mu(\phi_i^{-1}(A))$$

for every Borel set A. Moreover, the results in Mosco (1997) show that (X,d,μ) is an Ahlfors regular space of dimension $s = -\log_\alpha H$.

In what follows we will present a couple of simulations for different fractals.

Among other aspects, these numerical approximations allow us to visualize the lack of regularizing effect of the nonlocal diffusion. We can see that, even though the solution *tries to become continuous*, the jump from the initial condition is present at all times.

6.1 *Sierpinski gasket*

Let X be the Sierpinski Gasket immersed in $\mathbb{R}^2$, equipped with the usual distance d and the normalized s-dimensional Hausdorff measure μ, with $s = \log 3/\log 2$. This fractal X is induced by the following IFS $\Phi = \{\phi_1, \phi_2, \phi_3\}$ given by Falconer (1997):

$$\phi_1(x) = \frac{1}{2}x, \qquad \phi_2(x) = \frac{1}{2}x + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \qquad \phi_3(x) = \frac{1}{2}x + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix}.$$

Given a natural number n, we define

$$\Phi_n = \{\phi : \phi = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n} : i_j \in \{1,2,3\}\},$$

and number the functions of Φ_n as ϕ_k^n, $k \in I_{3^n} = \{1,2,\ldots,3^n\}$. On the one hand, $X = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{3^n} \phi_k^n(S)$, with S the triangle of vertices $(0,0)$, $(1,0)$, $(1/2,\sqrt{3}/2)$. On the other hand, for a fixed n we define $X_k = \phi_k^n(X)$, and it turns out that

$$X = \bigcup_{k=1}^{3^n} X_k.$$

The invariant measure satisfies that $\mu(X_k) = 1/3^n$, and except for a set of μ-measure zero, this sets X_k are pairwise disjoint, so that $\{X_k\}_{k \in I_{3^n}}$ is an appropriate decomposition of X. In order to apply Theorem 3.1 we only need to identify a point in each one of these components. We choose the bottom left vertex of each X_k, i.e., $x_k = \phi_k^n(0,0)$, $k \in I_{3^n}$.

We consider equation (1.1) with $J(x,y) = 100e^{-100|x-y|^2}$ and $f(x) = \mathbb{I}_{\{x_1 < x_2\}}(x)$. The solutions at time at $t = 0, 0.2, 0.5, 1, 2, 4$ for $n = 7$ are shown in Figure 1. The time discretization was done with the fourth order Runge-Kutta scheme using $\Delta t = 0.05$.

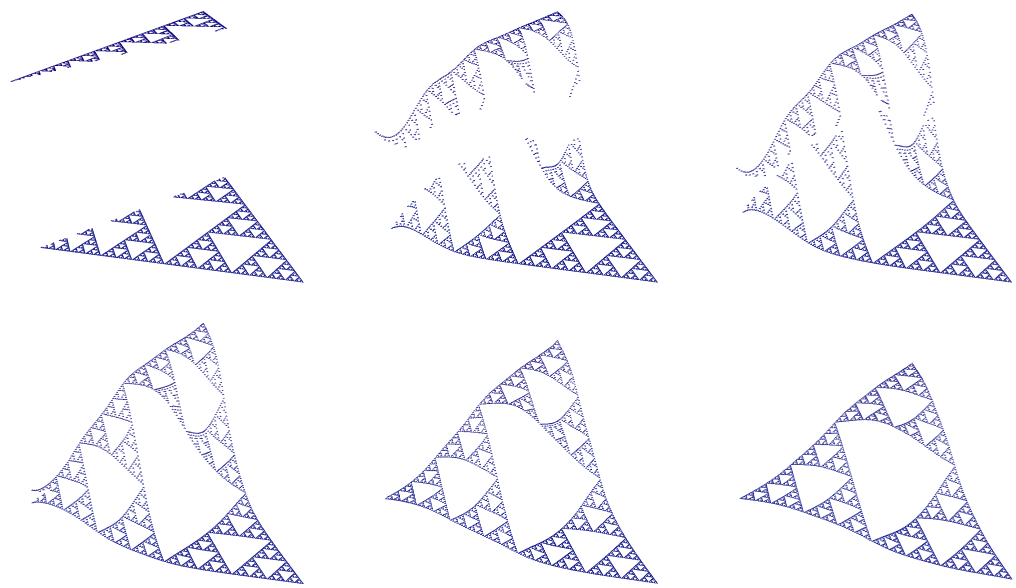

FIG. 1. Nonlocal diffusion on the Sierpinski gasket. Solution with $J(x,y) = 100e^{-100|x-y|^2}$ and $f(x) = \mathbb{I}_{\{x_1<x_2\}}(x)$. Snapshot of solution, from left to right and top to bottom, at $t = 0, 0.2, 0.5, 1, 2, 4$. The space X is decomposed into 3^7 components X_k. Each set $X_k = \phi_k(X)$ was drawn as $\phi_k(S)$ with S the triangle of vertices $(0,0)$, $(1,0)$, $(1/2, \sqrt{3}/2)$. The time discretization was done with the fourth order Runge-Kutta scheme using $\Delta t = 0.05$. The lack of regularizing effect of the non-local diffusion is apparent. Even though the solution *tries to become continuous*, the jump from the initial condition is present at all times.

6.2 *Sierpinski carpet*

In this subsection we consider the Sierpinski carpet, which is induced by IFS $\Phi = \{\phi_1, \phi_2, \ldots, \phi_8\}$ given by

$$\phi_1(x) = \frac{1}{3}x, \qquad \phi_2(x) = \frac{1}{3}x + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}, \qquad \phi_3(x) = \frac{1}{3}x + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix},$$

$$\phi_4(x) = \frac{1}{3}x + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}, \qquad \phi_5(x) = \frac{1}{3}x + \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix},$$

$$\phi_6(x) = \frac{1}{3}x + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}, \qquad \phi_7(x) = \frac{1}{3}x + \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}, \qquad \phi_8(x) = \frac{1}{3}x + \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}.$$

As before, given a natural number n, we define

$$\Phi_n = \{\phi : \phi = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n} : i_j \in \{1, 2, \ldots, 8\}\},$$

and number the functions of Φ_n as ϕ_k^n, $k \in I_{8^n} = \{1, 2, \ldots, 8^n\}$. On the one hand, $X = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{8^n} \phi_k^n(S)$, with $S = [0,1]^2$ the unit square. On the other hand, for a fixed n, $X = \bigcup_{k=1}^{8^n} X_k$ if $X_k = \phi_k^n(X)$. Also, the invariant measure satisfies that $\mu(X_k) = 1/8^n$, and except for a set of μ-measure zero, this sets X_k are pairwise disjoint. In order to apply Theorem 3.1 we choose as a representative of each component X_k the bottom left vertex, i.e., $x_k = \phi_k^n(0,0)$, $k \in I_{8^n}$.

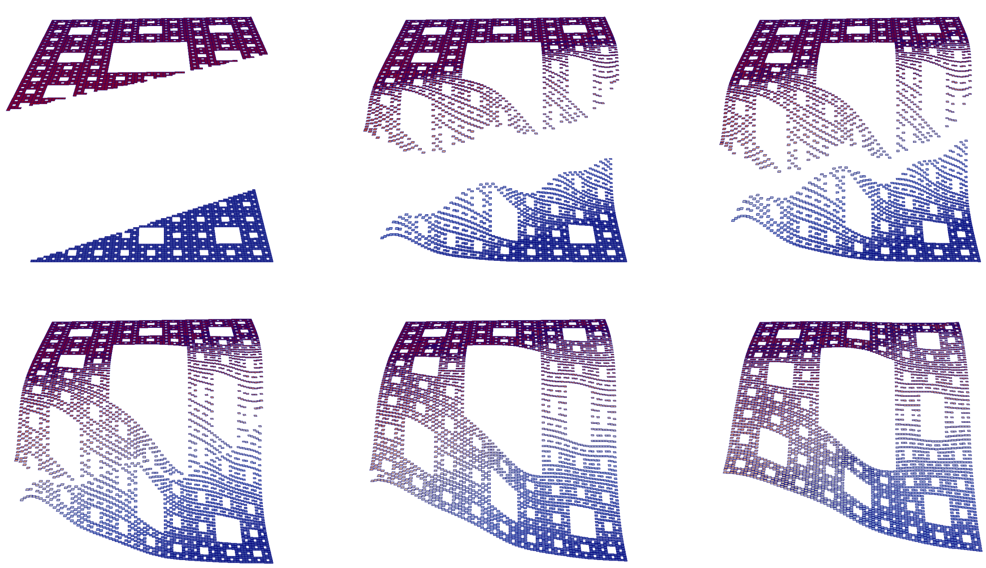

FIG. 2. Nonlocal diffusion on the Sierpinski carpet. Solution with $J(x,y) = 100e^{-100|x-y|^2}$ and $f(x) = \mathbb{I}_{\{x_2 > x_1/2\}}(x)$. Snapshot of solution, from left to right and top to bottom, at $t = 0, 0.5, 1, 2, 4, 8$. The space X is decomposed into 8^4 components X_k. Each set $X_k = \phi_k(X)$ was drawn as $\phi_k(S)$ with S the unit square. The time discretization was done with the fourth order Runge-Kutta scheme using $\Delta t = 0.05$.

We consider equation (1.1) with $J(x,y) = 100e^{-100|x-y|^2}$ and $f(x) = \mathbb{I}_{\{x_2>x_1/2\}}(x)$. The solutions at time at $t = 0, 0.2, 0.5, 1, 2, 4$ for $n = 4$ are shown in Figure 2. The time discretization was done with the fourth order Runge-Kutta scheme using $\Delta t = 0.05$.

The code was implemented in MATLAB and the graphics were produced with PARAVIEW.

7. Conclusions

We have presented a numerical method to approximate the solution of an evolutionary nonlocal diffusion problem. The theory is valid in a general setting of metric measure spaces, which include fractals, manifolds and domains of $\mathbb{R}^n$ as particular cases. We proved error estimates in $L^\infty([0,T];L^p(X,\mu))$ for $p = 1,\infty$ whenever the initial datum $f \in L^p(X,\mu)$. If the initial datum belongs to $L^2(X,\mu)$ the estimate for the error in $L^\infty([0,T];L^1(X))$ is improved and made independent of T.

Besides, we have studied some qualitative properties of the discrete and exact solutions, obtaining stability estimates, proving comparison principles and determining the asymptotic behavior as $t \to \infty$. This was done in a unified framework after noticing that the discrete solution is also the exact solution of a nonlocal diffusion problem, with piecewise constant kernel and initial datum.

We have implemented the numerical method in MATLAB and presented at the end some simulations on the Sierpinski gasket and the Sierpinski carpet, with an exponential kernel. These illustrate on the behavior of the solutions of the nonlocal diffusion problem on fractals, and sets the basis for the study of other differential equations on fractals.

The MATLAB code and some animations can be found at

`http://imal.santafe-conicet.gov.ar/pmorin/Papers/42/MATLAB`

One main disadvantage of these nonlocal diffusion problems is that the resulting matrices are fully populated and not sparse. This makes it difficult to work with very fine space discretizations, even though the resulting ODE are not stiff and can be solved with explicit time discretizations. However, we believe that for some specific fractals and kernels $J(\cdot,\cdot)$, a matrix-free implementation is possible. We did not dwell on this matter in this article, but rather on the proposal of a first numerical method for non-local diffusion problems on spaces of homogeneous type, and the proof of error estimates.

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