INTEGRAL TRANSFORMS AND SPECIAL FUNCTIONS, 2017
VOL. 28, NO. 5, 403–420
http://dx.doi.org/10.1080/10652469.2017.1296835

Review Article

Riesz transforms on variable Lebesgue spaces with Gaussian measure

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ABSTRACT
We give sufficient conditions on variable exponent functions $p : \mathbb{R}^n \to [1, \infty)$ for which the higher-order Riesz transforms, associated with the Ornstein–Uhlenbeck semigroup, are bounded on $L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$, where γ denotes the Gaussian measure.

ARTICLE HISTORY
Received 24 October 2016
Accepted 11 February 2017

KEYWORDS
Riesz transforms; Ornstein–Uhlenbeck semigroup; Gaussian measure; variable Lebesgue spaces

AMS SUBJECT CLASSIFICTION
Primary: 42B20; Secondary: 42B25; 42B35; 46E30; 47G15

1. Introduction

Let $\mathcal{L}$ be the differential operator given by

$$\mathcal{L} = -\tfrac{1}{2}\Delta + x \cdot \nabla,$$

where Δ and ∇ denote the Laplacian and the gradient, respectively. When the particles of a Brownian motion are attached to an elastic force, the process obtained is the well-known Ornstein–Uhlenbeck process, whose infinitesimal generator is the operator $\mathcal{L}$ given above, and the acting force is described by the term $x \cdot \nabla$ (see [1]). $\mathcal{L}$ also is important from the point of view of hypoellipticity (see [2]).

The eigenfunctions of $\mathcal{L}$, that is, those functions u that solve the eigenvalue problem

$$\begin{aligned} \mathcal{L}u &= \lambda u \\ u(x) &= O(|x|^k), \quad \text{for some } k \geq 0 \text{ when } |x| \to \infty, \end{aligned}$$

are the Hermite polynomials of degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, and the corresponding eigenvalues have the form $\lambda = |\alpha|$. These polynomials are

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defined by

$$H_\alpha(x_1,\dots,x_n) = H_{\alpha_1}(x_1)\cdots H_{\alpha_n}(x_n),$$

where H_{α_i} are the one-dimensional Hermite polynomials given by Rodrigues' formulas

$$H_0(x) = 1, \quad H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{d^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}, \quad n \geq 1.$$

It is well-known that the multidimensional polynomials are orthogonal on $L^2(\mathbb{R}^n, d\gamma)$ with respect to the Gaussian measure $d\gamma = \mathrm{e}^{-|x|^2}\,\mathrm{d}x$ and they can be normalized in order to get an orthonormal basis of $L^2(\mathbb{R}^n, d\gamma)$. Such measure makes the operator $\mathcal{L}$ self-adjoint, so it is the natural measure for studying properties for a large class of operators related with the Ornstein–Uhlenbeck process.

As in the case of the Laplacian, there is a concept of semigroup associated with $\mathcal{L}$, called the Ornstein–Uhlenbeck semigroup, which solves the diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\mathcal{L}u \\ u(x,0) &= f(x), \end{aligned}$$

with initial data $f \in L^2(\mathbb{R}^n, d\gamma)$. That is, the solution $u(\cdot, t)$ is determined by the functionals $\{\mathrm{e}^{-t\mathcal{L}}\}_{t>0}$, given by

$$u(x,t) = \mathrm{e}^{-t\mathcal{L}} f(x) = \pi^{-n/2} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-|y-\mathrm{e}^{-t}x|^2/(1-\mathrm{e}^{-2t})}}{(1-\mathrm{e}^{-2t})^{n/2}} f(y)\,\mathrm{d}y.$$

There exist several operators that arise in connection with this semigroup. A classical example are the Gaussian Riesz transforms, which have been widely studied in different contexts. This article is devoted to the study of the behaviour of those transforms on variable Lebesgue spaces with respect to the Gaussian measure $d\gamma = \mathrm{e}^{-|x|^2}\,\mathrm{d}x$. Before introducing these spaces, let us recall the definition of the Gaussian Riesz transforms. It is well-known that, for the Laplacian case, $\mathcal{L} = -\Delta$, the eigenvalue problem given above has as solutions all the numbers $\lambda \geq 0$, corresponding to the eigenvectors $\mathrm{e}^{\mathrm{i}y\cdot x}$, being $\lambda = -|y|^2$. Thus, the jth Riesz transform associated with this problem is defined by means of the eigenfunctions in the following way

$$R_j(\mathrm{e}^{\mathrm{i}y\cdot\cdot})(x) = -\frac{1}{|y|}\frac{\partial \mathrm{e}^{\mathrm{i}y\cdot x}}{\partial x_j} = -\mathrm{i}\frac{y_j}{|y|}\mathrm{e}^{\mathrm{i}y\cdot x}.$$

The classical definition is often given through the Fourier transform, that is, if $\hat{f}$ denotes the Fourier transform of f, then

$$R_jf(x) = \int_{\mathbb{R}^n} R_j(\mathrm{e}^{\mathrm{i}y\cdot\cdot})(x)\hat{f}(y)\,\mathrm{d}y = -\mathrm{i}\int_{\mathbb{R}^n} \frac{y_j}{|y|}\mathrm{e}^{\mathrm{i}y\cdot x}\hat{f}(y)\,\mathrm{d}y, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

On the other hand, when $\mathcal{L}$ is the Ornstein–Uhlenbeck operator, the generalization of the Riesz transforms is obtained by using the Hermite polynomials, since they are now the eigenfunctions of $\mathcal{L}$. Given $1 \leq j \leq n$ and H_α an n-dimensional Hermite polynomial, the jth Gaussian Riesz transform of first-order verifies

$$\mathcal{R}_j(H_\alpha)(x) = -\frac{1}{|\alpha|}\frac{\partial}{\partial x_j}H_\alpha(x).$$

Higher-order Riesz transforms are also known. Following the same ideas, for a given multi-index α, and each multi-index β, the n-dimensional Riesz transforms of order α verify

$$\mathcal{R}_\alpha(H_\beta)(x) = \frac{(-1)^{|\alpha|}}{|\beta|^{|\alpha|/2}}D^\alpha H_\beta(x) = \frac{(-1)^{|\alpha|}}{|\beta|^{|\alpha|/2}}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}H_\beta(x).$$

In the classical context, it is well-known that the Riesz transforms R_j and the higher-order Riesz transforms are bounded on $L^p(\mathbb{R}^n, dx)$ for every $1 < p < \infty$ (see, for instance, [3]). Since they are singular integrals that can be controlled, in some sense, by the Hardy–Littlewood maximal operator, it is also known that they are bounded on variable Lebesgue spaces under certain conditions on the exponents $p(x)$ (see [4]). For the Gaussian Riesz transforms, $\mathcal{R}_j$, the study of their continuity properties on $L^p(\mathbb{R}^n, d\gamma)$, as we said before, has a long history and it began with Muckenhoupt's work, [5], for the one-dimensional case and the first-order Riesz transform. Later, Meyer [6] and Gundy [7] gave probabilistic proofs for any order and dimension. From the analytical point of view, and using several and different techniques, we can cite, chronologically, the works of Pisier [8], Urbina [9], Gutiérrez [10], Gutiérrez et al. [11], Forzani et al. [12], and Pérez [13]. Other works that include this boundedness are [14,15]. The variable exponents case is an open problem and it is the main aim of this article.

2. Preliminaries

Given a measure μ over $\mathbb{R}^n$, we shall denote by $\mathcal{P}(\mathbb{R}^n, \mu)$ the set of exponents, that is, the set of μ-measurable and bounded functions $p : \mathbb{R}^n \to [1, \infty)$. When μ is the Lebesgue measure, we write $\mathcal{P}(\mathbb{R}^n)$. We will write

$$p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

Associated with each exponent $p \in \mathcal{P}(\mathbb{R}^n, \mu)$, we have another exponent $p' \in \mathcal{P}(\mathbb{R}^n, \mu)$, which is the generalization to the variable context of the Hölder's conjugate exponent given by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

almost everywhere.

Given $p \in \mathcal{P}(\mathbb{R}^n, \mu)$, we say that a μ-measurable function f belongs to $L^{p(\cdot)}(\mathbb{R}^n, d\mu)$ if the modular

$$\varrho_{p(\cdot),\mu}\left(\frac{f}{\lambda}\right) := \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mu(x) < \infty$$

for some $\lambda > 0$. In this case, we define the Luxemburg norm on $L^{p(\cdot)}(\mathbb{R}^n, d\mu)$ by

$$\|f\|_{p(\cdot),\mu} = \inf\left\{\lambda > 0 : \varrho_{p(\cdot),\mu}\left(\frac{f}{\lambda}\right) \le 1\right\},$$

which is the usual norm $\|f\|_{p,\mu} = (\int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x))^{1/p}$ when $p(x) \equiv p$. It is also well-known that $(L^{p(\cdot)}(\mathbb{R}^n, d\mu), \|\cdot\|_{p(\cdot),\mu})$ is a Banach space.

When μ is the classical Lebesgue measure, we simply write $L^{p(\cdot)}(\mathbb{R}^n)$, $\varrho_{p(\cdot)}$ for the modular and $\|\cdot\|_{p(\cdot)}$ for the norm. The measure we shall be dealing with is the Gaussian measure, which is a non-doubling, upper Ahlfors n-regular measure. From now on, $\mu = \gamma$.

A very useful and well-known inequality on these spaces is the generalization of Hölder's inequality, that is, given a measure μ, for every pair of functions $f \in L^{p(\cdot)}(\mathbb{R}^n, d\mu)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n, d\mu)$,

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, d\mu(x) \le 2\|f\|_{p(\cdot),\mu}\|g\|_{p'(\cdot),\mu}. \tag{2.1}$$

Another important property is the norm conjugate formula: for any μ-measurable function f, the following inequalities

$$\frac{1}{2}\|f\|_{p(\cdot),\mu} \le \sup_{\|g\|_{p'(\cdot),\mu} \le 1} \int_{\mathbb{R}^n} |f(x)g(x)| \, d\mu(x) \le 2\|f\|_{p(\cdot),\mu} \tag{2.2}$$

hold (see [16, Corollary 3.2.14]). For more information about $L^{p(\cdot)}$ spaces, see [4,16] or [17].

The exponents we will consider are not arbitrary, but we may allow them to have some continuity properties. The following conditions on the exponent arise related with the boundedness of the Hardy–Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$ (see, e.g. [18] or [19]):

(1) We will say that $p \in \mathrm{LH}_0(\mathbb{R}^n)$ if there exists $C_{\log}(p) > 0$ such that, for any pair $x, y \in \mathbb{R}^n$ with $0 < |x - y| < \frac{1}{2}$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{-\log(|x - y|)}. \tag{2.3}$$

(2) We will say that $p \in \mathrm{LH}_\infty(\mathbb{R}^n)$ if there exist constants $C_\infty > 0$ and $p_\infty \ge 1$ for which

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n. \tag{2.4}$$

Conditions (2.3) and (2.4) are usually called the local log-Hölder condition and the decay log-Hölder condition, respectively. When p satisfies both conditions, we shall denote it by $p \in \mathrm{LH}(\mathbb{R}^n)$. It is well-known that for $1 < p^- \le p^+ < \infty$, $\mathrm{LH}(\mathbb{R}^n)$ is sufficient for the Hardy–Littlewood maximal operator $\mathcal{M}_{HL}$ to be bounded on variable Lebesgue spaces (see [18]). However, it is known that while these are the sharpest possible pointwise conditions, they are not necessary (see Examples 4.1 and 4.43 in [4]). The authors in [16] gave a necessary and sufficient condition for the $L^{p(\cdot)}$-boundedness of $\mathcal{M}_{HL}$, but it is not easy to work with it, from a practical point of view.

The $\mathrm{LH}(\mathbb{R}^n)$ class is also sufficient for the boundedness on $L^{p(\cdot)}$ spaces of singular integrals of Calderón–Zygmund type (see [4, Theorem 5.39]). We include this result and the corresponding one about $\mathcal{M}_{HL}$ in the following theorem.

Theorem 2.1 ([4,18]): *Let $p \in \mathrm{LH}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$. Then, the Hardy–Littlewood maximal operator and singular integrals of Calderón–Zygmund type are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

An easy consequence of condition (2.4) is given in the following lemma, which says that sometimes we can replace a variable exponent p for the constant exponent p_∞, and viceversa, adding an integrable error (see, for instance, [4, Lemma 3.26]).

Lemma 2.2 ([4]): *Let $p \in \mathrm{LH}_\infty(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$. Then, there exists a constant C, depending on n and C_∞, such that for any measurable set E and any function G with $0 \le G(y) \le 1$ for $y \in E$,*

$$\int_E G(y)^{p(y)}\,\mathrm{d}y \le C \int_E G(y)^{p_\infty}\,\mathrm{d}y + \int_E (e+|y|)^{-np^-}\,\mathrm{d}y, \tag{2.5}$$

$$\int_E G(y)^{p_\infty}\,\mathrm{d}y \le C \int_E G(y)^{p(y)}\,\mathrm{d}y + \int_E (e+|y|)^{-np^-}\,\mathrm{d}y. \tag{2.6}$$

We will introduce now a new class of exponents that is more restrictive than the above $\mathrm{LH}_\infty(\mathbb{R}^n)$, but is related with the underlying measure γ.

Definition 2.3: Given $p \in \mathcal{P}(\mathbb{R}^n, \gamma)$, we will say that $p \in \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$ if there exist constants $C_\gamma > 0$ and $p_\infty \ge 1$ such that

$$|p(x) - p_\infty| \le \frac{C_\gamma}{|x|^2}, \quad \forall\, x \in \mathbb{R}^n \setminus \{(0,\ldots,0)\}. \tag{2.7}$$

Easy examples of this kind of exponents are those of the form

$$p(x) = p_\infty + \frac{A}{(e+|x|)^q},$$

for any $p_\infty \ge 1$, $A \ge 0$ and $q \ge 2$.

Remark 2.4: Clearly, if $p \in \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$, then $p \in \mathrm{LH}_\infty(\mathbb{R}^n)$ by virtue of the inequality $\log(e+|x|) \le C|x|^2$. Moreover, if $p^- > 1$, also $p' \in \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$ with $(p')_\infty = (p_\infty)' < \infty$, which will be simply denoted by p'_∞. An easy consequence is that $p_\infty = \lim_{|x|\to\infty} p(x)$, which gives $p_\infty > 1$ whenever $p^- > 1$.

A very useful characterization of the class $\mathcal{P}_\gamma^\infty(\mathbb{R}^n)$ will be used. The proof is immediate so we shall omit it.

Lemma 2.5: *If $1 < p^- \le p^+ < \infty$, the next two statements are equivalent.*

(i) $p \in \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$;

(ii) *There exists* $p_\infty > 1$ *such that*

$$C_1^{-1} \leq e^{-|x|^2(p(x)/p_\infty - 1)} \leq C_1 \quad \text{and} \quad C_2^{-1} \leq e^{-|x|^2(p'(x)/p'_\infty - 1)} \leq C_2, \tag{2.8}$$

for every $x \in \mathbb{R}^n$, *where* $C_1 = e^{C_\gamma/p_\infty}$ *and* $C_2 = e^{C_\gamma (p^-)'/p_\infty}$.

The following definition was first introduced by Berezhnoĭ in [20] for ideal Banach spaces (see also [21]), defined for families of disjoint balls or cubes. In the context of variable exponent spaces was considered in [16], allowing the family to have bounded overlap. If $\mathcal{B}$ is a family of balls (or cubes) of $\mathbb{R}^n$, we say that it is N-finite if $\sum_{B\in\mathcal{B}} \chi_B(x) \leq N$ for every $x \in \mathbb{R}^n$; that is, no more than N balls (resp. cubes) can intersect at the same time. In what follows, we will use this notation: given two functions f and g, by the symbols $\lesssim$ and $\gtrsim$ we will mean that there exists a positive constant c such that $f \leq cg$ and $cf \geq g$, respectively. When both inequalities hold, that is, $f \lesssim g \lesssim f$, we will write it as $f \approx g$.

Definition 2.6: Given an exponent $p \in \mathcal{P}(\mathbb{R}^n)$, we say that $p \in \mathcal{G}$ if, for every N-finite family of balls (or cubes) $\mathcal{B}$,

$$\sum_{B\in\mathcal{B}} \|f\chi_B\|_{p(\cdot)} \|g\chi_B\|_{p'(\cdot)} \lesssim \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}$$

for every pair of functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$. The constant may depend on N.

For certain kind of exponents, we can guarantee the validity of the $\mathcal{G}$-condition.

Lemma 2.7 ([16]): *If* $p \in \mathrm{LH}(\mathbb{R}^n)$, *then* $p \in \mathcal{G}$.

By virtue of Remark 2.4, the following corollary holds.

Corollary 2.8: *If* $p \in \mathcal{P}_\gamma^\infty(\mathbb{R}^n) \cap \mathrm{LH}_0(\mathbb{R}^n)$, *then* $p \in \mathcal{G}$.

3. Main results

The first-order Riesz transforms introduced before can be written as a principal value

$$\mathcal{R}_j f(x) = p.v. \int_{\mathbb{R}^n} k_j(x, y) f(y)\, dy,$$

where the kernels k_j are defined by

$$k_j(x, y) = c_n \int_0^1 \left(\frac{1 - r^2}{-\log r}\right)^{1/2} \frac{y_j - rx_j}{(1 - r^2)^{(n+3)/2}} e^{-|y-rx|^2/(1-r^2)}\, dr, \quad 1 \leq j \leq n.$$

In the case of the higher-order Riesz transforms, they can also be expressed as a principal value. For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \setminus \{(0, \ldots, 0)\}$, on functions having mean value zero with respect to the Gaussian measure, they are defined by

$$\mathcal{R}_\alpha f(x) = p.v. \int_{\mathbb{R}^n} k_\alpha(x,y) f(y)\, \mathrm{d}y,$$

with

$$k_\alpha(x,y) = c_n \int_0^1 r^{|\alpha|-1} \left(\frac{-\log r}{1-r^2}\right)^{|\alpha|/2-1} H_\alpha\left(\frac{y-rx}{(1-r^2)^{1/2}}\right) \frac{\mathrm{e}^{-|y-rx|^2/(1-r^2)}}{(1-r^2)^{n/2+1}}\, \mathrm{d}r.$$

More generally, one can consider linear operators T_F with kernels of the form

$$K_F(x,y) = c_n \int_0^1 r^{m-1} \left(\frac{-\log r}{1-r^2}\right)^{m/2-1} F\left(\frac{y-rx}{(1-r^2)^{1/2}}\right) \frac{\mathrm{e}^{-|y-rx|^2/(1-r^2)}}{(1-r^2)^{n/2+1}}\, \mathrm{d}r, \quad m \in \mathbb{N},$$

being F a function in $C^1(\mathbb{R}^n)$ (differentiable with continuous first-order derivatives) that is orthogonal to γ, that is, $\int_{\mathbb{R}^n} F(z)\mathrm{e}^{-|z|^2}\,\mathrm{d}z = 0$. These type of operators were first introduced by Urbina [9] (see also [13]).

It is easy to see that, when $F(z) = H_\alpha(z)$ and $m = |\alpha|$, the operator T_F is the n-dimensional Gaussian Riesz transform of order α.

Our main interest is, then, to study the behaviour of this more general singular integral T_F on variable Lebesgue spaces with respect to the Gaussian measure, which will give us the boundedness of $\mathcal{R}_\alpha$ as a particular case.

In order to do so, we will allow the function F to have some exponential growth (see [9,13]). That is, let us assume that F satisfies that for every $\epsilon > 0$ there exists a positive constant C_ϵ such that

(i) $|F(z)| \leq C_\epsilon \mathrm{e}^{\epsilon|z|^2}$,
(ii) $|\nabla F(z)| \leq C_\epsilon \mathrm{e}^{\epsilon|z|^2}$.

Let $F_1(z) = F(-z)$, $\forall z \in \mathbb{R}^n$. Then, F_1 has the same properties as F.

Let us define, then, the operator

$$T_F f(x) = p.v. \int_{\mathbb{R}^n} K_F(x,y)\, f(y)\, dy, \tag{3.1}$$

where the kernel considered above can be written in the form

$$\begin{aligned} K_F(x,y) &= \int_0^1 \varphi_m(r) F\left(\frac{y-rx}{\sqrt{1-r^2}}\right) \frac{\mathrm{e}^{-|y-rx|^2/(1-r^2)}}{(1-r^2)^{n/2+1}}\, \mathrm{d}r \\ &= \int_0^1 \psi_m(\sqrt{1-t}) F_1\left(\frac{\sqrt{1-t}x-y}{\sqrt{t}}\right) \frac{\mathrm{e}^{-u(t)}}{t^{n/2+1}}\, \mathrm{d}t \end{aligned}$$

being $\varphi_m(r) = r^{m-1}(-\log r/(1-r^2))^{(m-2)/2}$, $\psi_m(r) = \varphi_m(r)/r$, for any $m \in \mathbb{N}$, and where we have used the change of variables $t = 1-r^2$, and

$$u(t) = \frac{|\sqrt{1-t}x - y|^2}{t}.$$

We shall analyse the 'local' and 'global' parts of T_F separately, for which we may define the hyperbolic balls

$$B(x) := \{y \in \mathbb{R}^n : |x-y| \leq n(1 \wedge 1/|x|)\}, \quad x \in \mathbb{R}^n.$$

3.1. The local part

Let us define

$$K_1(x,y) = \int_0^1 F_1\left(\frac{\sqrt{1-t}x-y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{n/2+1}}\,dt.$$

From Pérez [13, p.503] we know that

$$K_F(x,y) = \psi_m(1)K_1(x,y) + H(x,y)$$

with $\psi_m(1) = \psi_m(1^+) = 2^{-(m-2)/2}$ and

$$|H(x,y)| \leq \int_0^1 \frac{e^{-\delta(|x-y|^2/t)}}{t^{n/2}}\frac{dt}{\sqrt{1-t}} := \tilde{H}(x-y). \tag{3.2}$$

Set

$$K_2(x) = \int_0^\infty F_1\left(\frac{x}{t^{1/2}}\right)e^{-|x|^2/t}\frac{dt}{t^{n/2+1}} = \frac{\Omega(x')}{|x|^n}, \tag{3.3}$$

with $x' = x/|x|$, $\Omega(x') = 2\int_0^\infty F_1(\rho x')e^{-\rho^2}\rho^{n-1}\,d\rho$ and, clearly,

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 2\int_{\mathbb{R}^n} F_1(z)e^{-|z|^2}\,dz = 0.$$

Again, according to Pérez [13], over the local region, that is, for $y \in B(x)$, we have

$$|K_1(x,y) - K_2(x-y)| \leq C\frac{1+|x|^{1/2}}{|x-y|^{n-1/2}} := K_3(x,y). \tag{3.4}$$

Thus, by considering

$$Tf(x) = p.v.\int_{\mathbb{R}^n} K_2(x-y)f(y)\,dy, \tag{3.5}$$

we get

$$|T_F(f\chi_{B(x)})(x)| \lesssim |Sf(x)| + \int_{B(x)} K_3(x,y)|f(y)|\,dy + (\tilde{H} * |f\chi_{B(x)}|)(x), \tag{3.6}$$

where $Sf(x) = T(f\chi_{B(x)})(x)$.

In order to estimate these terms, we will describe a very useful tool that gives us a decomposition of the space into a family of balls which has bounded overlap but, more importantly, on each of these balls, all the values of the Gaussian function are equivalent. This technique leads us to use the boundedness properties of T on variable Lebesgue spaces but with respect to the Lebesgue measure. This decomposition was used in [13], and the proof follows similar arguments to those given in [22], so we shall omit it.

Lemma 3.1: *We define the sequence $x_k = \sqrt{k}$ for $k \in \mathbb{N}$. By means of this strictly increasing sequence of positive real numbers, we can construct a family of disjoint balls B_j^k, for $k \in \mathbb{N}$ and $1 \le j \le N_k$ that satisfies the following properties:*

- (i) *if $\widetilde{B_j^k} = 2B_j^k$, the collection $\mathcal{F} = \{B(0,1), \{\widetilde{B_j^k}\}_{j,k}\}$ is a covering of $\mathbb{R}^n$;*
- (ii) *$\mathcal{F}$ has bounded overlap;*
- (iii) *the centre y_j^k of B_j^k verifies $|y_j^k| = (x_{k+1} + x_k)/2$;*
- (iv) $\mathrm{diam}(B_j^k) = x_{k+1} - x_k = 1/(2|y_j^k|)$;
- (v) *For every ball $B \in \mathcal{F}$, and every pair $x, y \in B$, $\gamma(x) \approx \gamma(y)$ with constants independent of B;*
- (vi) *There exists an uniform positive constant C_n such that, if $x \in B \in \mathcal{F}$, then $B(x) \subset C_nB := \hat{B}$. Moreover, the collection $\hat{\mathcal{F}} = \{\hat{B}\}_{B\in\mathcal{F}}$ also verifies the properties* (ii) *and* (v).

Lemma 3.2: *Let $\mathcal{F}$ and $\hat{\mathcal{F}}$ be given by Lemma* 3.1. *If T_F is as in* (3.1), *K_3 and $\tilde{H}$ are the kernels introduced in* (3.4) *and* (3.2), *and T,S are defined as in* (3.5) *and* (3.6), *respectively, then*

$$|Sf(x)| \le \sum_{B\in\mathcal{F}} (|T(f\chi_{\hat{B}})(x)| + \mathcal{M}_{HL}(f\chi_{\hat{B}})(x))\chi_B(x), \tag{3.7}$$

$$\int_{B(x)} K_3(x,y)|f(y)|\,\mathrm{d}y \lesssim \sum_{B\in\mathcal{F}} \mathcal{M}_{HL}(f\chi_{\hat{B}})(x)\chi_B(x) \tag{3.8}$$

and

$$\int_{B(x)} \tilde{H}(x-y)|f(y)|\,\mathrm{d}y \lesssim \sum_{B\in\mathcal{F}} \mathcal{M}_{HL}(f\chi_{\hat{B}})(x)\chi_B(x), \tag{3.9}$$

being $\mathcal{M}_{HL}$ the classical non-centred Hardy–Littlewood maximal function with respect to the Lebesgue measure. Consequently,

$$|T_F(f\chi_{B(x)})(x)| \lesssim \sum_{B\in\mathcal{F}} (|T(f\chi_{\hat{B}})(x)| + \mathcal{M}_{HL}(f\chi_{\hat{B}})(x))\chi_B(x). \tag{3.10}$$

Proof: We shall first prove (3.7). We take $x \in B$ and we denote by r_B the radius of B and by $R_x = n(1 \wedge 1/|x|)$ the radius of $B(x)$. Since the operator is not positive, we must split the principal value into two parts, one over $\hat{B}$ and the other over $\hat{B} \setminus B(x)$. Then,

$$\begin{aligned}
|Sf(x)| &= \left| p.v. \int_{\hat{B}} K_2(y-x)f(y)dy - p.v. \int_{\hat{B}\setminus B(x)} K_2(y-x)f(y)\,\mathrm{d}y \right| \\
&\le \left| p.v. \int_{\mathbb{R}^n} K_2(y-x)f(y)\chi_{\hat{B}}(y)\,\mathrm{d}y \right| \\
&\quad + \int_{R_x\le|x-y|<(C_n+1)r_B} |K_2(y-x)||f(y)|\chi_{\hat{B}}(y)\,\mathrm{d}y \\
&\lesssim |T(f\chi_{\hat{B}})(x)| + \int_{r_B\le|y-x|<(C_n+1)r_B} \frac{1}{|y-x|^n}|f(y)|\chi_{\hat{B}}(y)\,\mathrm{d}y \\
&\lesssim |T(f\chi_{\hat{B}})(x)| + \mathcal{M}_{HL}(f\chi_{\hat{B}})(x),
\end{aligned}$$

where we have used that K_2 is a Calderón–Zygmund kernel (see (3.3)) and that $R_x \geq r_B$ for every $x \in B$. By adding the terms over all $B \in \mathcal{F}$, we get the desired inequality (3.7).

Now, let us show (3.8). For each $x \in B \in \mathcal{F}$, since $B(x) \subset \hat{B}$, we obtain the following estimates

$$\begin{aligned}\int_{|x-y|<R_x} K_3(x,y)|f(y)|\,\mathrm{d}y &= (1+|x|^{1/2})\sum_{k=0}^{\infty}\int_{2^{-(k+1)}R_x\leq|x-y|<2^{-k}R_x}\frac{|f(y)|\chi_{\hat{B}}(y)}{|x-y|^{n-1/2}}\,\mathrm{d}y\\ &\leq 2^n\mathcal{M}_{HL}(f\chi_{\hat{B}})(x)(1+|x|^{1/2})R_x^{1/2}\sum_{k=0}^{\infty}2^{-(k+1)/2}\\ &\lesssim \mathcal{M}_{HL}(f\chi_{\hat{B}})(x)\chi_B(x).\end{aligned}$$

Finally, we shall prove (3.9). Consider the function $\phi(z) = C_\delta \mathrm{e}^{-\delta|z|^2}$, where C_δ is a constant such that $\int_{\mathbb{R}^n}\phi(z)\,\mathrm{d}z = 1$. Given $t>0$, we rescale this function as $\phi_{\sqrt{t}}(x) = t^{-n/2}\phi(z/\sqrt{t})$, and, since $0 \leq \phi \in L^1(\mathbb{R}^n)$, $\{\phi_{\sqrt{t}}\}_{t>0}$ is an approximate identity. Then, since $\int_0^1 (1/\sqrt{1-t})\,\mathrm{d}t < \infty$,

$$\begin{aligned}\int_{B(x)}\tilde{H}(x-y)|f(y)|\,\mathrm{d}y &= \int_{B(x)}\left(\int_0^1\phi_{\sqrt{t}}(x-y)\frac{1}{\sqrt{1-t}}\,\mathrm{d}t\right)|f(y)|\,\mathrm{d}y\\ &\leq \int_{B(x)}\left(\sup_{t>0}\phi_{\sqrt{t}}(x-y)\right)|f(y)|\left(\int_0^1\frac{1}{\sqrt{1-t}}\,\mathrm{d}t\right)\mathrm{d}y\\ &\leq C\int_{B(x)}\left(\sup_{t>0}\phi_{\sqrt{t}}(x-y)\right)|f(y)|\,\mathrm{d}y.\end{aligned}$$

In a similar way as we did before, we can show that, if $x \in B \in \mathcal{F}$, then

$$\int_{B(x)}\tilde{H}(x-y)|f(y)|\,\mathrm{d}y \leq \int_{\mathbb{R}^n}\left(\sup_{t>0}\phi_{\sqrt{t}}(x-y)\right)|f(y)|\chi_{\hat{B}}(y)\,\mathrm{d}y,$$

which yields

$$\int_{B(x)}\tilde{H}(x-y)|f(y)|\,\mathrm{d}y \leq \sum_{B\in\mathcal{F}}\chi_B(x)\sup_{t>0}|(\phi_{\sqrt{t}} * |f\chi_{\hat{B}}|)(x)| \leq \sum_{B\in\mathcal{F}}\chi_B(x)\mathcal{M}_{HL}(f\chi_{\hat{B}})(x),$$

where we have used a classical result due to E. M. Stein (see [3]), since ϕ is non-increasing.

Inequality (3.10) now follows from (3.6), together with (3.7)–(3.9). ■

The main result of this section holds for a large class of singular integrals, thanks to the above decomposition (3.7), and it gives us the boundedness of the local part of T_F on variable Lebesgue spaces with respect to the Gaussian measure.

Theorem 3.3: *Let $p \in \mathrm{LH}_0(\mathbb{R}^n) \cap \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. Then, there exists a positive constant C such that*

$$\|T_F(f\chi_{B(\cdot)})\|_{p(\cdot),\gamma} \leq C\|f\|_{p(\cdot),\gamma}$$

for every $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$.

Proof: Let $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$. We will use the norm on the dual space $L^{p'(\cdot)}(\mathbb{R}^n, d\gamma)$:

$$\|T_F(f\chi_{B(\cdot)})\|_{p(\cdot),\gamma} \le 2 \sup_{\|g\|_{p'(\cdot),\gamma}\le 1} \int_{\mathbb{R}^n} |T_F(f\chi_{B(x)})(x)\|g(x)|\,\mathrm{d}\gamma(x).$$

We split the integral in the following way, according to the pointwise inequality (3.10)

$$\begin{aligned}\int_{\mathbb{R}^n} |T_F(f\chi_{B(x)})(x)||g(x)|\,\mathrm{d}\gamma(x) &\lesssim \sum_{B\in\mathcal{F}} \int_B \left|T\left(f\chi_{\hat{B}}\right)(x)\right| |g(x)|\mathrm{e}^{-|x|^2}\,\mathrm{d}x \\ &\quad + \sum_{B\in\mathcal{F}} \int_B \mathcal{M}_{HL}(f\chi_{\hat{B}})(x)|g(x)|\mathrm{e}^{-|x|^2}\,\mathrm{d}x \\ &\approx \sum_{B\in\mathcal{F}} \mathrm{e}^{-|c_B|^2} \int_B |T(f\chi_{\hat{B}})(x)||g(x)|\,\mathrm{d}x \\ &\quad + \sum_{B\in\mathcal{F}} \mathrm{e}^{-|c_B|^2} \int_B \mathcal{M}_{HL}(f\chi_{\hat{B}})(x)|g(x)|\,\mathrm{d}x,\end{aligned}$$

where c_B is the centre of B and $\hat{B}$ and we have used the fact that over each ball of the family $\mathcal{F}$, the values of γ are all equivalent. Now, we apply Hölder's inequality with $p(\cdot)$ and $p'(\cdot)$ with respect of the Lebesgue measure in each integral, the boundedness of T and $\mathcal{M}_{HL}$ on $L^{p(\cdot)}(\mathbb{R}^n)$, which hold from the properties on p, Remark 2.4 and Theorem 2.1, obtaining

$$\begin{aligned}\int_{\mathbb{R}^n} |T_F(f\chi_{B(x)})(x)||g(x)|\,\mathrm{d}\gamma(x) &\lesssim \sum_{B\in\mathcal{F}} \mathrm{e}^{-|c_B|^2} \|T(f\chi_{\hat{B}})\chi_B\|_{p(\cdot)}\|g\chi_B\|_{p'(\cdot)} \\ &\quad + \sum_{B\in\mathcal{F}} \mathrm{e}^{-|c_B|^2} \|\mathcal{M}_{HL}(f\chi_{\hat{B}})\chi_B\|_{p(\cdot)}\|g\chi_B\|_{p'(\cdot)} \\ &\lesssim \sum_{B\in\mathcal{F}} \mathrm{e}^{-|c_B|^2} \|f\chi_{\hat{B}}\|_{p(\cdot)}\|g\chi_B\|_{p'(\cdot)} \\ &= \sum_{B\in\mathcal{F}} \mathrm{e}^{-|c_B|^2/p_\infty} \|f\chi_{\hat{B}}\|_{p(\cdot)} \mathrm{e}^{-|c_B|^2/p'_\infty} \|g\chi_{\hat{B}}\|_{p'(\cdot)}. \quad (3.11)\end{aligned}$$

Since $p \in P_\gamma^\infty(\mathbb{R}^n)$ and $p^- > 1$, $p' \in P_\gamma^\infty(\mathbb{R}^n)$. Thus, from Lemma 2.5, for every $x \in \mathbb{R}^n$,

$$\mathrm{e}^{-|x|^2(p(x)/p_\infty - 1)} \le C_1 \quad \text{and} \quad \mathrm{e}^{-|x|^2(p'(x)/p'_\infty - 1)} \le C_2. \quad (3.12)$$

Moreover, since the values of γ are all equivalent on each ball $\hat{B}$, we have

$$\begin{aligned}\int_{\hat{B}} \left(\frac{|f(y)|}{\mathrm{e}^{|c_B|^2/p_\infty}\|f\chi_{\hat{B}}\|_{p(\cdot),\gamma}}\right)^{p(y)} \mathrm{d}y &\lesssim \int_{\hat{B}} \left(\frac{|f(y)|}{\|f\chi_{\hat{B}}\|_{p(\cdot),\gamma}}\right)^{p(y)} \mathrm{e}^{-|y|^2(p(y)/p_\infty - 1)}\,\mathrm{d}\gamma(y) \\ &\lesssim \int_{\hat{B}} \left(\frac{|f(y)|}{\|f\chi_{\hat{B}}\|_{p(\cdot),\gamma}}\right)^{p(y)} \mathrm{d}\gamma(y) \lesssim 1,\end{aligned}$$

which yields

$$\mathrm{e}^{-|c_B|^2/p_\infty}\|f\chi_{\hat{B}}\|_{p(\cdot)} \lesssim \|f\chi_{\hat{B}}\|_{p(\cdot),\gamma}.$$

Similarly, by applying the second inequality in (3.12), we get

$$e^{-|c_B|^2/p'_\infty}\|g\chi_{\hat{B}}\|_{p'(\cdot)} \lesssim \|g\chi_{\hat{B}}\|_{p'(\cdot),\gamma}.$$

Replacing both estimates in (3.11), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |T_F(f\chi_{B(\cdot)})(x)||g(x)|\,\mathrm{d}\gamma(x) &\lesssim \sum_{B\in\mathcal{F}} \|f\chi_{\hat{B}}\|_{p(\cdot),\gamma}\|g\chi_{\hat{B}}\|_{p'(\cdot),\gamma} \\ &= \sum_{B\in\mathcal{F}} \|f\chi_{\hat{B}}e^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)}\|g\chi_{\hat{B}}e^{-|\cdot|^2/p'(\cdot)}\|_{p'(\cdot)}.\end{aligned}$$

Since the family of balls $\hat{\mathcal{F}} = \{\hat{B}\}_{B\in\mathcal{F}}$ has bounded overlap, from Corollary 2.8 applied to $f(\cdot)e^{-|\cdot|^2/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g(\cdot)e^{-|\cdot|^2/p'(\cdot)} \in L^{p'(\cdot)}(\mathbb{R}^n)$, it follows that

$$\int_{\mathbb{R}^n} |T_F(f\chi_{B(\cdot)})(x)||g(x)|\,\mathrm{d}\gamma(x) \lesssim \|f\|_{p(\cdot),\gamma}\|g\|_{p'(\cdot),\gamma}.$$

Taking the supremum over all functions g with $\|g\|_{p'(\cdot),\gamma} \leq 1$, the thesis holds. ■

3.2. The global part

In order to study the global part of T_F, we will follow the spirit of [13]. To that end, we might recall the following estimates obtained in that article.

Lemma 3.4 ([13]): *Let us consider the kernel $K_F(x,y)$ in the global part, that is, for $y \in B^c(x)$. If $a = |x|^2 + |y|^2$ and $b = 2\langle x,y\rangle$, we have the following inequalities*

(i) *If $b \leq 0$, for each $0 < \epsilon < 1$, there exists $C_\epsilon > 0$ such that*

$$|K_F(x,y)| \leq C_\epsilon e^{-(1-\epsilon)|y|^2};$$

(ii) *If $b > 0$, for each $0 < \epsilon < 1/n$ there exists $C_\epsilon > 0$ such that*

$$|K_F(x,y)| \leq C_\epsilon \frac{e^{-(1-\epsilon)u_0}}{t_0^{n/2}},$$

where $t_0 = 2\sqrt{a^2-b^2}/(a+\sqrt{a^2-b^2})$ and $u_0 = \frac{1}{2}(|y|^2 - |x|^2 + |x+y||x-y|)$.

Theorem 3.5: *Let $p \in \mathrm{LH}_0(\mathbb{R}^n) \cap \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. Then, there exists a positive constant C such that*

$$\|T_F(f\chi_{B^c(\cdot)})\|_{p(\cdot),\gamma} \leq C\|f\|_{p(\cdot),\gamma}$$

for every $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$.

Proof: We shall consider two cases, based on Lemma 3.4, for which we will denote by $E = \{(x,y) : b(x,y) > 0\}$.

If $b \le 0$, we take $0 < \epsilon < 1/(p')^+ = 1/(p^-)' < 1$ in Lemma 3.4(i) and write $1 - \epsilon = \tilde{\epsilon} + 1/p^-$ so $\tilde{\epsilon} > 0$. Then, for any $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$ with $\|f\|_{p(\cdot),\gamma} = 1$, by applying Holder's inequality, we have

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \left(\int_{B^c(x)\cap E^c} |K_F(x,y)||f(y)|\,\mathrm{d}y \right)^{p(x)} \mathrm{d}\gamma(x) \\ &\lesssim \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathrm{e}^{-\left(\tilde{\epsilon}+\frac{1}{p^-}\right)|y|^2} |f(y)|\,\mathrm{d}y \right)^{p(x)} \mathrm{d}\gamma(x) \\ &\lesssim \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2} |f(y)|^{p^-}\,\mathrm{d}y \right)^{p(x)/p^-} \left(\int_{\mathbb{R}^n} \mathrm{e}^{-\tilde{\epsilon}(p^-)'|y|^2}\,\mathrm{d}y \right)^{p(x)/(p^-)'} \mathrm{d}\gamma(x) \\ &\lesssim \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2} |f(y)|^{p^-}\,\mathrm{d}y \right)^{p(x)/p^-} \mathrm{d}\gamma(x). \end{aligned}$$

Since $\|f\|_{p(\cdot),\gamma} = 1$, we know that $\int_{\mathbb{R}^n} |f(y)|^{p(y)}\,\mathrm{d}\gamma(y) \le 1$. Then,

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2} |f(y)|^{p^-}\,\mathrm{d}y \le \int_{|f|>1} |f(y)|^{p(y)}\,\mathrm{d}\gamma(y) + \int_{|f|\le 1} 1\,\mathrm{d}\gamma(y) \le 1 + C_n.$$

Hence,

$$I \lesssim \int_{\mathbb{R}^n} (1 + C_n)^{p^+/p^-}\,\mathrm{d}\gamma(x) \le C_{n,p},$$

which means that $\|T_F(f\chi_{B^c(\cdot)\cap E^c})\|_{p(\cdot),\gamma} \le C_{n,p}$ for every $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$ with norm $\|f\|_{p(\cdot),\gamma} = 1$. By homogeneity, we get the same result for every $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$.

Now, if $b > 0$, from Lemma 3.4(ii), with $0 < \epsilon < 1/n$ and for every $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$ with $\|f\|_{p(\cdot),\gamma} = 1$, we obtain that

$$\begin{aligned} II &= \int_{\mathbb{R}^n} \left(\int_{B^c(x)\cap E} |K_F(x,y)||f(y)|\,\mathrm{d}y \right)^{p(x)} \mathrm{d}\gamma(x) \\ &\lesssim \int_{\mathbb{R}^n} \left(\int_{B^c(x)\cap E} \frac{\mathrm{e}^{-(1-\epsilon)u_0}}{t_0^{n/2}} |f(y)|\,\mathrm{d}y \right)^{p(x)} \mathrm{d}\gamma(x) \\ &= \int_{\mathbb{R}^n} \left(\int_{B^c(x)\cap E} \frac{\mathrm{e}^{-(1-\epsilon)u_0}\mathrm{e}^{|y|^2/p(y)-|x|^2/p(x)}}{t_0^{n/2}} |f(y)|\mathrm{e}^{-|y|^2/p(y)}\,\mathrm{d}y \right)^{p(x)} \mathrm{d}x, \end{aligned}$$

where $t_0 = 2\sqrt{a^2 - b^2}/(a + \sqrt{a^2 - b^2})$ and $u_0 = \frac{1}{2}(|y|^2 - |x|^2 + |x+y||x-y|)$.

Since $p \in \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$, from Lemma 2.5 we have that $\mathrm{e}^{|y|^2/p(y)-|x|^2/p(x)} \approx \mathrm{e}^{(|y|^2-|x|^2)/p_\infty}$. Notice that, on one hand, we have that $||y|^2 - |x|^2| \le |x+y||x-y|$; on the other hand, for $b > 0$, $|x+y||x-y| \ge n$ whenever $y \in B^c(x)$. Moreover, $t_0 \approx |x-y||x+y|/(|x|^2 + |y|^2)$

(see [13, p.499]). Thus, since $|x|^2 + |y|^2 = a < a + b = |x + y|^2$, we have that

$$t_0 \geq C\frac{|x-y||x+y|}{|x|^2+|y|^2} \geq C\frac{n}{|x+y|^2}.$$

Thus, we can bound part of the integrand in the following way

$$\frac{e^{-(1-\epsilon)u_0}}{t_0^{n/2}} e^{(|y|^2-|x|^2/p_\infty)} = \frac{e^{(|y|^2-|x|^2)(1/p_\infty-(1-\epsilon)/2)}}{t_0^{n/2}} e^{-((1-\epsilon)/2)|x+y||x-y|}$$
$$\lesssim |x+y|^n e^{-\alpha_\infty|x+y||x-y|}$$

where the constant

$$\alpha_\infty := \frac{1-\epsilon}{2} - \left|\frac{1}{p_\infty} - \frac{1-\epsilon}{2}\right|$$

is positive if we choose $\epsilon < 1/p'_\infty$. Thus, we take $0 < \epsilon < \min\{1/n, 1/p'_\infty\}$. As we can see, we have obtained a kernel $P(x,y) := |x+y|^n e^{-\alpha_\infty|x+y||x-y|}$ as in [13], and as it was proved there, $P(x,y)$ is integrable in each variable (since it is symmetric), with constant independent of x and y.

So, we have

$$\int_{B^c(x)\cap E} \frac{e^{-(1-\epsilon)u_0} e^{|y|^2/p(y)-|x|^2/p(x)}}{t_0^{n/2}} |f(y)| e^{-|y|^2/p(y)}\, dy$$
$$\leq C\int_{B^c(x)} P(x,y)|f(y)| e^{-|y|^2/p(y)}\, dy.$$

Let us define $A_x = \{y : n/|x| < |y-x| < \frac{1}{2}\}$ and $C_x = B^c(x, \frac{1}{2})$, so that $B^c(x) = A_x \cup C_x$. Then, we will show that the above operator restricted to A_x and to C_x is bounded by constants independent of x, in order to replace the variable exponent $p(x)$ with a suitable constant exponent related with p. Set

$$J_1 = \int_{A_x} P(x,y)|f(y)| e^{-|y|^2/p(y)}\, dy, \quad J_2 = \int_{C_x} P(x,y)|f(y)| e^{-|y|^2/p(y)}\, dy.$$

In order to estimate J_1, we shall notice first that $|y| \approx |x|$ whenever $n/|x| < |x-y| \leq 1/2$. Indeed, it is easy to see that

$$\tfrac{3}{4}|x| \leq |y| \leq \tfrac{5}{4}|x|.$$

On the other hand, $|x+y| \leq |x| + |y| \leq 5|x|$, and from the parallelogram law, we have

$$3|x|^2 \leq 2|x|^2 + \tfrac{16}{9}|y|^2 \leq 2(|x|^2 + |y|^2) \leq |x+y|^2 + \tfrac{1}{4} \leq |x+y|^2 + 2|x|^2,$$

so $|x+y| \geq |x|$. Thus, $|x+y| \approx |x|$. By applying this fact to J_1, we obtain

$$
\begin{aligned}
J_1 &\lesssim \int_{n/|x|<|x-y|} |x|^n \mathrm{e}^{-\alpha_\infty |x||x-y|} f(y) \mathrm{e}^{-|y|^2/p(y)}\, \mathrm{d}y \\
&= \sum_{k=1}^{\infty} \int_{kn/|x|<|x-y|\leq (k+1)(n/|x|)} |x|^n \mathrm{e}^{-\alpha_\infty |x||x-y|} f(y) \mathrm{e}^{-|y|^2/p(y)}\, \mathrm{d}y \\
&\leq C \sum_{k=1}^{\infty} |x|^n \left((k+1)\frac{n}{|x|} \right)^n \mathrm{e}^{-\alpha_\infty nk} \mathcal{M}_{HL}(f(\cdot)\mathrm{e}^{-|\cdot|^2/p(\cdot)})(x) \\
&= \mathcal{M}_{HL}(f(\cdot)\mathrm{e}^{-|\cdot|^2/p(\cdot)})(x) \sum_{k=1}^{\infty} ((k+1)n)^n \mathrm{e}^{-\alpha_\infty nk} \lesssim \mathcal{M}_{HL}(f(\cdot)\mathrm{e}^{-|\cdot|^2/p(\cdot)})(x),
\end{aligned}
$$

since $\alpha_\infty > 0$. The constant in the above inequality only depends on n and p. From the hypotheses on the exponent p, we know that

$$
\|\mathcal{M}_{HL}(f(\cdot)\mathrm{e}^{-|\cdot|^2/p(\cdot)})\|_{p(\cdot)} \lesssim \|f(\cdot)\mathrm{e}^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)} = \|f\|_{p(\cdot),\gamma} = 1.
$$

We shall now analyse $J_2 = \int_{|y-x|\geq \frac{1}{2}} P(x,y) f(y) \mathrm{e}^{-|y|^2/p(y)}\, \mathrm{d}y$. In this case, we can apply Hölder's inequality to obtain

$$
J_2 \leq \|P(x,\cdot)\chi_{C_x}\|_{p'(\cdot)} \|f\mathrm{e}^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)} = \|P(x,\cdot)\chi_{C_x}\|_{p'(\cdot)},
$$

and we estimate the remaining norm. We shall see that the corresponding modular, that is, $\varrho_{p'(\cdot)}(P(x,\cdot)\chi_{C_x})$, is smaller than a constant, independent of x, which implies that the norm is also finite. Indeed,

$$
\begin{aligned}
\int_{C_x} P(x,y)^{p'(y)}\, \mathrm{d}y &\leq \int_{\mathbb{R}^n} |x+y|^{np'(y)} \mathrm{e}^{-(\alpha_\infty/2)|x+y|p'(y)}\, \mathrm{d}y \\
&\leq \int_{|x+y|\leq 1} |x+y|^n \mathrm{e}^{-(\alpha_\infty/2)|x+y|}\, \mathrm{d}y \\
&\quad + \int_{|x+y|>1} |x+y|^{n(p')^+} \mathrm{e}^{-(\alpha_\infty/2)|x+y|}\, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} (|z|^n + |z|^{n(p')^+}) \mathrm{e}^{-(\alpha_\infty/2)|z|}\, \mathrm{d}z \lesssim \int_{\mathbb{R}^n} \mathrm{e}^{-(\alpha_\infty/4)|z|}\, \mathrm{d}z \leq C_{n,p}.
\end{aligned}
$$

Thus, $\|P(x,\cdot)\chi_{C_x}\|_{p'(\cdot)} \leq C_{n,p}$.

Combining the estimates on A_x and C_x, we have proved that there exists a constant $D>0$, independent of x, such that

$$
\int_{B^c(x)} P(x,y)|f(y)|\mathrm{e}^{-|y|^2/p(y)}\, \mathrm{d}y \leq D.
$$

We split the function $g(y) = |f(y)|e^{-|y|^2/p(y)} = g_1(y) + g_2(y)$, where $g_1 = g\chi_{\{g\geq 1\}}$. Then,

$$II \lesssim \int_{\mathbb{R}^n} \left(\frac{1}{D} \int_{B^c(x)} P(x,y) g_1(y)\, dy \right)^{p(x)} dx + \int_{\mathbb{R}^n} \left(\frac{1}{D} \int_{B^c(x)} P(x,y) g_2(y)\, dy \right)^{p(x)} dx$$
$$:= II_1 + II_2.$$

The exponent $p(x)$ in II_1 can be now replaced by p^-. For II_2, we will use Lemma 2.2 with $G(x) = (1/D) \int_{B^c(x)} P(x,y) g_2(y)\, dy$. Hence, we get

$$II_2 \leq \int_{\mathbb{R}^n} \left(\frac{1}{D} \int_{B^c(x)} P(x,y) g_2(y)\, dy \right)^{p_\infty} dx + \int_{\mathbb{R}^n} \frac{1}{(e+|x|)^{np^-}}\, dx$$

and this last term is finite since $p^- > 1$. We get, then, that

$$II \lesssim \int_{\mathbb{R}^n} \left(\int_{B^c(x)} P(x,y) g_1(y)\, dy \right)^{p^-} dx + \int_{\mathbb{R}^n} \left(\int_{B^c(x)} P(x,y) g_2(y)\, dy \right)^{p_\infty} dx + C.$$

We will proceed to estimate the above integrals. In the first one, we apply Hölder's inequality with p^-, splitting $P(x,y) = P(x,y)^{1/(p^-)'} P(x,y)^{1/p^-}$ and recalling that P is symmetric and integrable in each variable with constant independent of x and y, to get

$$\int_{\mathbb{R}^n} \left(\int_{B^c(x)} P(x,y) g_1(y)\, dy \right)^{p^-} dx \lesssim \int_{\mathbb{R}^n} g_1(y)^{p^-}\, dy \leq \int_{\mathbb{R}^n} g_1(y)^{p(y)}\, dy$$
$$\lesssim \int_{\mathbb{R}^n} |f(y)|^{p(y)} e^{-|y|^2}\, dy \leq 1,$$

since $g_1 \geq 1$ or $g_1 = 0$, and $\|f\|_{p(\cdot),\gamma} = 1$.

Similarly, for the integral involving g_2, with p_∞ instead of p^-, we obtain

$$\int_{\mathbb{R}^n} \left(\int_{B^c(x)} P(x,y) g_2(y)\, dy \right)^{p_\infty} dx \lesssim \int_{\mathbb{R}^n} g_2(y)^{p_\infty}\, dy.$$

Since $0 \leq g_2 \leq 1$, we apply again Lemma 2.2 to get

$$\int_{\mathbb{R}^n} \left(\int_{B^c(x)} P(x,y) g_2(y)\, dy \right)^{p_\infty} dx \lesssim \int_{\mathbb{R}^n} g_2(y)^{p(y)}\, dy + \int_{\mathbb{R}^n} \frac{1}{(e+|y|)^{np^-}}\, dy$$
$$\leq \int_{\mathbb{R}^n} |f(y)|^{p(y)} e^{-|y|^2}\, dy + C \leq 1 + C.$$

Therefore, we have shown that for any function $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$ with $\|f\|_{p(\cdot),\gamma} = 1$,

$$\int_{\mathbb{R}^n} \left(\int_{B^c(x)\cap E} |K_F(x,y)||f(y)|\, dy \right)^{p(x)} d\gamma(x) \leq C,$$

which yields $\|T_F(f\chi_{B^c(\cdot)\cap E})\|_{p(\cdot),\gamma} \leq C$ and, from the homogeneity of the norm, the result holds for every function $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$. Combining both cases, $b \leq 0$ and $b > 0$, we get the thesis. ■

Finally, we can deduce the boundedness result for the higher-order Gaussian Riesz transforms, taking $F(z) = H_\alpha(z)$ and $m = |\alpha|$.

Theorem 3.6: *Let* $p \in \mathrm{LH}_0(\mathbb{R}^n) \cap \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$. *Then, given any multi-index* α, *there exists a positive constant C such that*

$$\|\mathcal{R}_\alpha f\|_{p(\cdot),\gamma} \leq C\|f\|_{p(\cdot),\gamma}$$

for every $f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma)$.

Acknowledgements

The authors wish to thank Liliana Forzani for her encouragement that brought them together to work on this problem. They would also like to thank the reviewer for his/her comments, suggestions and corrections which have been really helpful to improve this manuscript.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by Consejo Nacional de Investigaciones Científicas y Técnicas [grant number PIP 2012-196], Universidad Nacional del Litoral [grant numbers CAID 77-163 and CAID 77-250] and Fondo para la Investigación Científica y Tecnológica [grant number PICT 2013-1873].

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