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The packing coloring problem for lobsters and partner limited graphs☆

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ABSTRACT

A *packing k-coloring* of a graph G is a k-coloring such that the distance between two vertices having color i is at least $i+1$.

To compute the *packing chromatic number* is NP-hard, even restricted to trees, and it is known to be polynomial time solvable only for a few graph classes, including cographs and split graphs.

In this work, we provide upper bounds for the packing chromatic number of lobsters and we prove that it can be computed in polynomial time for an infinite subclass of them, including caterpillars.

In addition, we provide superclasses of split graphs where the packing chromatic number can be computed in polynomial time. Moreover, we prove that the packing chromatic number can be computed in polynomial time for the class of partner limited graphs, a superclass of cographs, including also P_4-sparse and P_4-tidy graphs.

1. Introduction

A *packing k-coloring* of a graph G is a k-coloring using colors in $\{1, \ldots, k\}$ such that the distance between two vertices having color i is at least $i+1$. The *packing chromatic number* of G, denoted by $\chi_\rho(G)$, is the minimum k such that G admits a packing k-coloring. This concept was originally introduced by Goddard et al. in [5] under the name *broadcast chromatic number* as one of its applications involves frequency planning in wireless networks, and renamed as packing chromatic number by Brešar et al. [2].

In this work we consider the following decision problem:

PACKING COLORING (PackCol)

Instance: $G = (V, E)$, $k \in \mathbb{N}$

Question: Is there a packing k-coloring of G?

Goddard et al. [5] proved that PackCol is NP-complete for general graphs and Fiala and Golovach [3] proved that it is NP-complete even for trees. Then, it would be worth it to determine maximal (minimal) subclasses of trees for which PackCol is solvable in polynomial time (NP-complete).

In addition, PackCol is solvable in polynomial time for graphs whose treewidth and diameter are both bounded [3] and for cographs and split graphs [5].

The task of this work is to enlarge the family of graphs where PackCol is polynomial.

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This paper is organized as follows: in Section 2 we state the notation, definitions and previous results we need in this work. In Section 3, we provide an upper bound for the packing chromatic number of lobsters. This bound allows us to find families of lobsters, including caterpillars, where PackCol is solvable in polynomial time. Finally, in Section 4 we analyze the problem for some families of neighborhood modules graphs, including split and spider graphs, and these results allow us to prove that PackCol is polynomial time solvable for partner limited graphs.

2. Definitions and preliminary results

All the graphs in this paper are finite and simple. Given a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively, and $\overline{G}$ denotes its complement.

For any positive integer m, we denote by K_m, S_m and P_m the graphs with m vertices corresponding to the complete graph, the complement of a complete graph and a path, respectively.

For any $v \in V(G)$, $N(v)$ is the set of its neighbors, and if $U \subseteq V(G)$, then $N(U) = \cup_{v\in U} N(v)$. The degree of v in G is $\deg(v)$. We denote by $L(G)$ the set of nodes of degree 1 in G.

Given a graph G and $U \subseteq V(G)$, $G - U$ denotes the graph obtained from the *deletion* of the vertices in U, i.e., the subgraph with vertex set $V(G) - U$ and edge set $E(G) - \{vw : v \in U\}$. An *induced* subgraph of G is a graph obtained from G by the deletion of a subset of vertices. Given $R \subseteq V(G)$, $G[R]$ denotes the subgraph $G - (V(G) - R)$. We simply refer as *subgraphs* for induced subgraphs and, when it is not necessary to identify the subset of deleted vertices, we simply use the notation $G' \subseteq G$.

Let v be a vertex of a graph G and let G' be a graph such that $V(G) \cap V(G') = \emptyset$. The graph obtained by *replacing* v *by* G' is the graph whose vertex set is $(V(G) - \{v\}) \cup V(G')$ and whose edges are $E(G - \{v\}) \cup E(G')$ together with all the edges connecting a vertex in $V(G')$ with a vertex in $N(v)$.

A *complete set* in a graph G is a set of pairwise adjacent vertices and a *stable set* in G is a set of pairwise nonadjacent vertices. The *stability number* of G is the size of a maximum stable set in a graph G and it is denoted by $\alpha(G)$. The *Stable Set Problem* (SSP) is that of finding a maximum stable set in a graph.

We denote by $dist_G(v, u)$ the distance between vertices v and u in G and the *diameter* of G is $diam(G) = \max\{dist_G(v, u) : v, u \in V(G)\}$.

A *caterpillar* is a tree T in which all the vertices are at distance at most 1 of a central path P of T. A *lobster* is a tree T in which all the vertices are at distance at most 2 of a central path P of T. In both cases, we assume that no node in $L(T)$ belongs to P.

We generalize packings colorings in the following way:

Given a graph G, $U \subseteq V(G)$ and two positive integers s and k with $s \leq k$, a *packing* (k, s)*-coloring of* U *in* G is a function $f : U \to \{s, \dots, k\}$ such that if $u \neq v$ and $f(u) = f(v) = i$ then $dist_G(u, v) \geq i + 1$. We define the *s-packing chromatic number of* U (*in* G), and denote $\chi_\rho^s(U)$, as the minimum k such that U admits a packing (k, s)-coloring in G. In particular, if $U = V$, we denote $\chi_\rho^s(V) = \chi_\rho^s(G)$ and if $s = 1$, $\chi_\rho^1(U) = \chi_\rho(U)$.

The following remarks are immediate:

Remark 2.1. For every graph G, $\chi_\rho^s(G) \leq |V(G)| + s - 1$, with equality if $diam(G) \leq s$.

Remark 2.2. Let $G' \subseteq G$, then $\chi_\rho(G') \leq \chi_\rho(G)$.

Remark 2.3. If $U \subset W \subseteq V(G)$ and $\chi_\rho^s(U) \leq h$, then

$$\chi_\rho^s(W) \leq \chi_\rho^{h+1}(W - U).$$

The stability number and the packing chromatic number of a graph G are related, as shows the following result:

Lemma 2.4 ([5]). *For every graph G, $\chi_\rho(G) \leq |V(G)| + 1 - \alpha(G)$, with equality if $diam(G) \leq 2$. Moreover, if $diam(G) \leq 2$, for each maximum stable set S of G there is a packing $\chi_\rho(G)$-coloring of G where the vertices in S have color* 1.

3. PackCol for caterpillar and lobsters

Let us consider the following decision problem arising from PackCol:

PACKING k-COLORING (k-PackCol)

Instance: $G = (V, E)$

Question: Is there a packing k-coloring of G?

Goddard et al. [5] showed that 4-PackCol is NP-complete for general graphs.

As we have mentioned before, PackCol is NP-complete for trees. However, Fiala and Golovach [3] observed that k-PackCol is solvable in polynomial time for graphs with bounded treewidth. In particular, k-PackCol is solvable in polynomial time for trees.

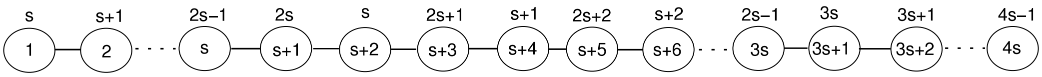

Fig. 1. Packing $(4s-1, s)$-coloring of P_{4s} provided by f.

Then, it holds the following immediate result:

Lemma 3.1. *Let $\mathcal{F}$ be a family of trees such that, for some integer M, $\chi_\rho(T) \le M$, for all $T \in \mathcal{F}$. Then,* PackCol *is solvable in polynomial time on $\mathcal{F}$.*

In particular, Sloper proved in [7] that $\chi_\rho(T) \le 7$ if T is a caterpillar. Hence, from Lemma 3.1, we have:

Corollary 3.2. PackCol *is solvable in polynomial time for caterpillars.*

It is known that there is no M such that $\chi_\rho(T) \le M$ for every lobster T [5]. Our scope is to find suitable bounds for the packing chromatic number of lobsters that allow us to determine a subclass of lobsters where PackCol is polynomial time solvable, by using Lemma 3.1.

Observe that if T is a lobster (caterpillar), $T - L(T)$ is a caterpillar (resp., path). Then, considering Remark 2.3, we study the s-packing chromatic number of paths and caterpillars.

From Remark 2.1, if $m \le s+1$, $\chi_\rho^s(P_m) = m + s + 1$.

For $m \ge s+2$, Fiala et al. provided in [4] a packing $(4s-1, s)$-coloring of P_m which gives the following upper bound for its s-packing chromatic number:

Theorem 3.3 (*[4]*)**.** *For all $m \ge 2$, $\chi_\rho^s(P_m) \le 2s - 1 + \min\left\{2s, \left\lfloor \frac{m}{2} \right\rfloor\right\}$.*

In particular, when $m \le 4s$ the theorem above states that $\chi_\rho^s(P_m) \le 2s - 1 + \left\lfloor \frac{m}{2} \right\rfloor$. This bound can be improved by considering an alternative packing $(4s-1, s)$-coloring of P_{4s}. We have:

Lemma 3.4. *If $s+2 \le m \le 3s$, then $\chi_\rho^s(P_m) \le 2s - 1 + \left\lfloor \frac{1}{2}(m - s + 1) \right\rfloor$. If $3s+1 \le m \le 4s$, then $\chi_\rho^s(P_m) \le m - 1$.*

Proof. Consider the packing $(4s-1, s)$-coloring of P_{4s} defined by $f : \{1, \dots, 4s\} \to \{s, \dots, 4s-1\}$ such that

$$f(i) = \begin{cases} s + i - 1 & \text{if } 1 \le i \le s+1, \\ s - 1 + k & \text{if } i = s + 2k \text{ and } k \in \{1, \dots, s\}, \\ 2s + k & \text{if } i = s + 2k + 1 \text{ and } k \in \{1, \dots, s\}, \\ i - 1 & \text{if } 3s + 2 \le i \le 4s. \end{cases} \tag{1}$$

The packing $(4s-1, s)$-coloring of P_{4s} provided by f is represented in Fig. 1.

Clearly, by using the same sequence of colors in a path P_m with $m \le 4s$ we obtain a packing (r, s)-coloring of P_m, for some $r \le 4s - 1$.

It can be easily checked that if $3s+1 \le m \le 4s$, $r = m - 1$. Besides, if $m = s + j$ with $2 \le j \le 2s$, $r = f(m)$ if j is odd and $r = f(m-1)$ otherwise.

It only remains to prove that if j is odd (even) $f(m)$ ($f(m-1)$) is at most $2s - 1 + \left\lfloor \frac{1}{2}(m - s + 1) \right\rfloor$.

If j is odd, $m = s + j = s + (2k+1)$ for some $1 \le k \le s$. Then,

$$f(m) = 2s + k = 2s + \frac{m - s - 1}{2} = 2s - 1 + \frac{m - s + 1}{2} = 2s - 1 + \left\lfloor \frac{1}{2}(m - s + 1) \right\rfloor.$$

If j is even, $m = s + j = s + 2k$ for some $1 \le k \le s$. Then,

$$f(m-1) = 2s + \frac{j}{2} - 1 = 2s - 1 + \left\lfloor \frac{j+1}{2} \right\rfloor = 2s - 1 + \left\lfloor \frac{1}{2}(m - s + 1) \right\rfloor. \quad \square$$

Our goal now is to provide bounds for the s-packing chromatic number of caterpillars. Observe that if T is a caterpillar with central path P_m and $m \le s - 1$, then $diam(T) \le s$ and from Remark 2.1 we have $\chi_\rho^s(T) = |V(T)| + s - 1$. Then we are interested in the cases where $m \ge s$.

Let us first present the following technical lemma.

Lemma 3.5. *Let T be a caterpillar with central path P_m, $s \ge 3$ and $U \subseteq L(T)$ such that $|U \cap N(v)| = 1$ for all $v \in V(P_m)$. Then, $\chi_\rho^s(U) \le 4s - 7$.*

Moreover,

(1) *if $s \le m \le 3s - 6$,*

$$\chi_\rho^s(U) \le \left\lfloor \frac{1}{2}(m + 3s - 3) \right\rfloor,$$

(2) *if $3s - 5 \le m \le 4s - 9$, $\chi_\rho^s(U) \le m + 1$.*

Proof. For each $v \in P_m$, let $\{v_U\} = U \cap N(v)$.

Let $k \geq 3$ and let g be a packing $(k-2, s-2)$-coloring of P_m. We define $f : U \to N$ such that $f(v_U) = g(v) + 2$. It is clear that f is a packing (k, s)-coloring of U.

Then, $\chi_\rho^s(U) \leq \chi_\rho^{s-2}(P_m) + 2$. From Theorem 3.3, we obtain

$$\chi_\rho^s(U) \leq 4s - 7. \tag{2}$$

The bounds for the cases when $s \leq m \leq 4s-9$ can be easily derived by considering the bounds for the packing chromatic numbers for paths given in Lemma 3.4. □

Given a caterpillar T with central path P_m, we denote

$$e_T = \max\{|N(v) \cap L(T)| : v \in V(P_m)\}.$$

Theorem 3.6. *Let T be a caterpillar with central path P_m. Then, $\chi_\rho^2(T) \leq 7 + 4^{e_T}$ and if $s \geq 3$,*

$$\chi_\rho^s(T) \leq 7 + (s-2)4^{e_T+1}. \tag{3}$$

Moreover,

(1) *if* $s \leq m \leq 3s-6$ *then*

$$\chi_\rho^s(T) \leq \left\lfloor \frac{3}{2}(s-1) + \left(e_T + \frac{1}{2}\right) m \right\rfloor,$$

(2) *if* $3s-5 \leq m \leq 4s-9$,

$$\chi_\rho^s(T) \leq (m+1)e_T + 1.$$

Proof. From Remark 2.2 we can assume that $|L(T) \cap N(v)| = e_T$ for all $v \in V(P_m)$.

If $s \geq 3$ the proof is by induction on e_T.

If $e_T = 1$, considering $U = L(T)$ in Lemma 3.5 we have $\chi_\rho^s(U) \leq 4s - 7$. Then, from Remark 2.3 and Theorem 3.3,

$$\chi_\rho^s(T) \leq \chi_\rho^{4s-6}(P_m) \leq 16s - 25 = 7 + (s-2)4^2.$$

Assume that the thesis holds for every caterpillar T' such that $1 \leq e_{T'} \leq n$ and consider T, a caterpillar with $e_T = n+1$.

Let $U \subseteq L(T)$ such that $|U \cap N(v)| = 1$ for all $v \in V(P_m)$. From Lemma 3.5, $\chi_\rho^s(U) \leq 4s - 7$. Then, from Remark 2.3 we have

$$\chi_\rho^s(T) \leq \chi_\rho^{4s-6}(T - U).$$

Since $T' = T - U$ is a caterpillar with $e_{T'} = n$, from inductive hypothesis

$$\chi_\rho^s(T) \leq 7 + ((4s-6)-2)4^{n+1} = 7 + (4s-8)4^{n+1} = 7 + (s-2)4^{n+2}$$

and the result follows.

For the case $s = 2$, let us observe that, if $U \subseteq L(T)$ such that $|U \cap N(v)| = 1$ for all $v \in V(P_m)$ then $\chi_\rho^2(U) = 2$. From Remark 2.3, we have that $\chi_\rho^2(T) \leq \chi_\rho^3(T - U)$.

Then, applying (3) for $T' = T - U$ and $s = 3$ and considering that $e_{T'} = e_T - 1$, we obtain $\chi_\rho^2(T) \leq 7 + 4^{e_T}$.

The bounds given in (1) and (2) can be obtained from Lemma 3.5 and observing that for any graph G and $U \subseteq V(G)$, $\chi_\rho^s(G) \leq \chi_\rho^s(U) + |V(G) - U|$. □

The bounds for the s-packing chromatic number of a caterpillar T can be improved when e_T is sufficiently large.

Lemma 3.7. *Let $s \geq 3$ and T be a caterpillar with central path P_m with $m \geq 4s - 8$. If $e_T \geq \left\lceil \log_4 \frac{m+1}{4s-8} \right\rceil + 1$, then*

$$\chi_\rho^s(T) \leq \left(e_T - \left\lceil \log_4 \frac{m+1}{4s-8} \right\rceil + 2\right) m + 2.$$

Proof. Again, from Remark 2.2 we can assume that $|L(T) \cap N(v)| = e_T$ for all $v \in V(P_m)$. Let $M = \left\lceil \log_4 \frac{m+1}{4s-8} \right\rceil$ and $U \subseteq L(T)$ such that $|U \cap N(v)| = M$ for all $v \in V(P_m)$.

Following the same reasoning that in Theorem 3.6 we have that

$$\chi_\rho^s(U) \leq (s-2)4^M + 1 \leq m + 2.$$

Then,

$$\chi_\rho^s(T) \leq \chi_\rho^s(U) + |V(T) - U| \leq (m+2) + (e_T - M + 1)m,$$

and the thesis holds. □

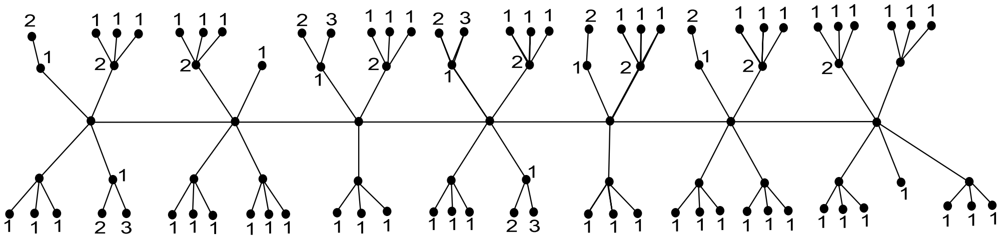

Fig. 2. Pattern.

The previous results for the s-packing chromatic number of caterpillars can be used in order to obtain bounds for the packing chromatic number of lobsters.

Let T be a lobster with central path P. For each $v \in V(P)$, let us denote by $N_4(v)$ the set of vertices in $N(v) - V(P)$ having degree at least 4 and let $c_T = \max\{|N_4(v)| : v \in V(P)\}$.

Theorem 3.8. *Let T be a lobster with central path P_m. Then, if $c_T \leq 1$, $\chi_\rho(T) \leq 15$, else*

$$\chi_\rho(T) \leq 7 + 2^{2c_T+1}.$$

Moreover, when $c_T \geq 2$ we have:

(1) *if* $1 \leq m \leq 3$,

$$\chi_\rho(T) \leq c_T m + 3,$$

(2) *if* $4 \leq m \leq 6$,

$$\chi_\rho(T) \leq \left\lceil \frac{m(2c_T - 1)}{2} \right\rceil + 4,$$

(3) *if* $m = 7$,

$$\chi_\rho(T) \leq 8c_T + 1,$$

(4) *if* $m \geq 8$ *and* $c_T \geq \lceil \log_4 \frac{m+1}{8} \rceil + 2$, *then*

$$\chi_\rho(T) \leq \left(c_T - \left\lceil \log_4 \frac{m+1}{8} \right\rceil + 1 \right) m + 2.$$

Proof. For each $v \in V(P_m)$, let $N_3(v) = \{u \in N(v) - V(P_m) : \deg(u) \leq 3\}$.

Assign color 1 to the vertices in $N_3(P_m) = \bigcup_{v \in V(P_m)} N_3(v)$ and colors 2 and 3 to the vertices in $L(T)$ which are adjacent to some vertex in $N_3(P_m)$. For the remaining vertices in $L(T)$, we assign color 1.

Finally, for each v such that $N_4(v) \neq \emptyset$, assign color 2 to one vertex in $N_4(v)$.

We have a packing 3-coloring of a subset X of $V(T)$ including $L(T)$, $N_3(P_m)$ and one vertex in $N_4(v)$, for each v with $N_4(v) \neq \emptyset$ (see Fig. 2).

Let $T' = T - X$. From Remark 2.3 we have that $\chi_\rho(T) \leq \chi_\rho^4(T')$.

Observe that, if $c_T \leq 1$, T' is a path. Then, from Theorem 3.3, we obtain $\chi_\rho(T) \leq \chi_\rho^4(T') \leq 15$.

Otherwise, when $c_T \geq 2$, T' is a caterpillar with central path P_m and $e_{T'} = c_T - 1$. Then, the bounds follow from Theorem 3.6 and Lemma 3.7. □

Although the complexity of PackCol for lobsters and trees with bounded degree is still unknown, from Lemma 3.1 and Theorem 3.8 we obtain the main result of this section.

Theorem 3.9. *Let M be a fixed positive integer. Then* PackCol *is solvable in polynomial time for lobsters T with $c_T \leq M$. In particular,* PackCol *is solvable in polynomial time for lobsters with bounded maximum degree.*

4. PackCol for well labelled spider and partner limited graphs

A graph G is *neighborhood module* if G and $\bar{G}$ are both connected. Given a family of graphs $\mathcal{F}$, we denote by $NM(\mathcal{F})$ the set of neighborhood module graphs in $\mathcal{F}$.

Let us observe that, if a graph G is not connected its packing chromatic number is the maximum of the packing chromatic numbers of its connected components. In addition, if its complement $\overline{G}$ is not connected, G has diameter at most two and from Lemma 2.4, $\chi_\rho(G) = |V(G)| + 1 - \alpha(G)$.

Hence, the following lemma provides a strategy to prove the polynomiality of PackCol for families of graphs for which the stable set problem is polynomial time solvable.

Lemma 4.1. *Let $\mathcal{F}$ be a graph class such that the connected components of a graph in $\mathcal{F}$ also belong to $\mathcal{F}$. If the Stable Set problem is solvable in polynomial time for $\mathcal{F}$ and* PackCol *is solvable in polynomial time for $NM(\mathcal{F})$, then* PackCol *is solvable in polynomial time for $\mathcal{F}$.*

The above result leads us to study PackCol for some families of neighborhood module graphs. Firstly, we introduce a graph class that includes split graphs.

Definition 4.2. A graph G is a hypersplit graph if $V(G)$ can be partitioned into S, C and R, where S is a stable set, C is a non empty complete set such that $N(S) = C$ and $R \subseteq N(v)$ for all $v \in C$.

The triple (S, C, R) is called the hypersplit partition of G. It is easy to see that the hypersplit partition of a hypersplit graph can be found in linear time using its modular decomposition.

If $R = \emptyset$, G is a split graph and $\chi_\rho(G) = |C| + 1$ [5]. The following theorem generalizes this result for hypersplit graphs.

Theorem 4.3. *Let G be a hypersplit graph with vertex partition (S, C, R) and $R \neq \emptyset$. Then, $\chi_\rho(G) = |C| + |R| + 1 - \alpha(G[R])$.*

Proof. Let $G' = G[C \cup R]$. Since $diam(G') \leq 2$, from Lemma 2.4,

$$\chi_\rho(G') = |C| + |R| + 1 - \alpha(G') = |C| + |R| + 1 - \alpha(G[R]).$$

Clearly, any maximum stable set I of $G[R]$ is a maximum stable set of G'. Then, from Lemma 2.4, there exists a packing $\chi_\rho(G')$-coloring of G' where the vertices in I are at color 1. Since $I \cap S = \emptyset$ and there are no edges between these sets, this packing coloring can be extended to G by assigning color 1 to every vertex in S. Thus, we obtain a packing $\chi_\rho(G')$-coloring of G. Then, from Remark 2.2, $\chi_\rho(G) = \chi_\rho(G')$. □

Given a family $\mathcal{F}$ of graphs, a $\mathcal{F}$-hypersplit graph is a hypersplit graph such that $G[R]$ belongs to $\mathcal{F}$ or $R = \emptyset$.

Hence, from Theorem 4.3 we have the following result:

Corollary 4.4. *Let $\mathcal{F}$ be a graph class for which the Stable Set problem is polynomial time solvable. Then* PackCol *is polynomial time solvable for $\mathcal{F}$-hypersplit graphs.*

Spider graphs are particular cases of hypersplit graphs. Following [1] a *spider graph* is a hypersplit graph with partition (S, C, R), where $S = \{s_1, \ldots, s_r\}$, $C = \{c_1, \ldots, c_r\}$ with $r \geq 2$ and one of the following conditions holds:

1. *thin spider*: s_i is adjacent to c_j if and only if $i = j$,
2. *thick spider*: s_i is adjacent to c_j if and only if $i \neq j$.

Observe that the complement of a thin spider is a thick spider, and vice-versa. Edges with one endpoint in S are called *legs* of the spider and R is called its *head*.

In [6], Roussel et al. introduced the class of *well labelled spider graphs*. Given a thin spider graph with partition (S, C, R), and a vector L of $|S| + |C|$ positive integer components, a *labelled spider graph* (S, C, R, L) is any graph obtained by replacing every vertex $v \in S \cup C$ by a graph G_v such that $|V(G_v)| = L(v)$. Given a labelled spider graph (S, C, R, L) and an edge uv of the spider graph (S, C, R), the label of uv is $L(u) + L(v)$.

A *well labelled spider graph* is a labelled spider graph (S, C, R, L) such that $L(v) \in \{1, 2, 3\}$ for all v and

i. either, exactly one leg has label 4 and all the other legs have label 2, or
ii. every leg has label at most 3.

Observe that thin spider graphs are well labelled spider graphs with $L(v) = 1$ for all $v \in S \cup C$.

We restate the definition in [6] in terms of the following operations over a thin spider graph (S, C, R):

R1: replace one node $v \in S \cup C$ by a graph G with three vertices.
R2: replace both endpoints of one leg of (S, C, R) by graphs with two vertices.
R3: replace at most one endpoint of every leg of (S, C, R) by a graph with two vertices.

Observe that operations R1 and R2 correspond with the possible substitutions verifying item i in Roussel et al.'s definition and operation R3 corresponds with the substitutions verifying item ii.

Then, the following lemma provides an alternative definition of well labelled spider graphs.

Lemma 4.5. *A graph W is a well labelled spider graph if and only if W is obtained from a thin spider (S, C, R) by performing one of the operations* R1, R2, R3 *once.*

A well labelled spider graph obtained from operation R1 will be called 3-*well labelled spider graph* and it will be identified with its vertex partition denoted by $(S \hookleftarrow G, C, R)$ or $(S, C \hookleftarrow G, R)$ if the replaced vertex v belongs to S or C, respectively and G is the graph replacing v.

A well labelled spider graph obtained from operation R2 will be called $(2, 2)$-*well labelled spider graph* and it will be identified with its vertex partition denoted by $(S \hookleftarrow G_1, C \hookleftarrow G_2, R)$, where G_1 and G_2 are the graphs replacing the endpoints of one leg.

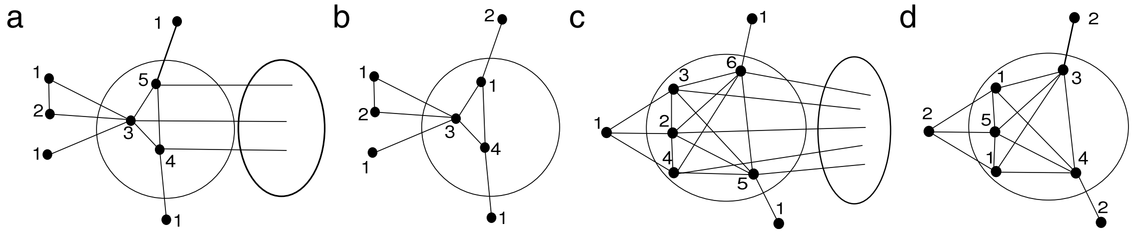

Fig. 3. Scheme of packing colorings for (a) $(S \hookleftarrow \overline{P_3}, C, R)$, (b) $(S \hookleftarrow \overline{P_3}, C, \emptyset)$, (c) $(S, C \hookleftarrow P_3, R)$ and (d) $(S, C \hookleftarrow P_3, \emptyset)$.

For those obtained from operation R3, we will say that we have a *multiple well labelled spider graph* and it will be identified with its vertex partition denoted by $(S_{\sigma,\kappa}, C_{\gamma,\eta}, R)$, where σ, κ, γ and η denote the number of vertices in S replaced by S_2, the number of vertices in S replaced by K_2, the number of vertices in C replaced by S_2 and the number of vertices in C replaced by K_2, respectively.

In addition, the vertex partition of a well labelled spider graph and the replaces performed can be obtained in linear time (see [6]).

Observe that $(S \hookleftarrow S_3, C, R)$, $(S, C \hookleftarrow K_3, R)$, $(S \hookleftarrow S_2, C \hookleftarrow K_2, R)$ and $(S_{\sigma,0}, C_{0,\eta}, R)$ are hypersplit graphs and their packing chromatic number is given by Theorem 4.3.

For the remaining cases, we have the following results:

Lemma 4.6. *Let W be a 3-well labelled spider graph. Then, W has a subgraph H of diameter two which can be obtained in linear time and such that $\chi_\rho(W) = \chi_\rho(H)$.*

Proof. Let W be a 3-well labelled spider graph obtained from the thin spider graph (S, C, R). Observe that $(S \hookleftarrow S_3, C, R)$ and $(S, C \hookleftarrow K_3, R)$ are hypersplit graphs. Then, the proof of the claim can be found in the proof of Theorem 4.3.

If $W = (S \hookleftarrow G, C, R)$ with $G \neq S_3$, let H be the subgraph of W induced by $V(G) \cup C \cup R$. Clearly, H has diameter two and it can be obtained in linear time.

If $R \neq \emptyset$, it is not hard to see that the union of a maximum stable set of $W[R]$ and a maximum stable set of G is a maximum stable set I of H such that $I \cap C = \emptyset$. From Lemma 2.4 there is a packing $\chi_\rho(H)$-coloring of H that assigns color 1 to the vertices in I. Then, this packing coloring can be extended to a $\chi_\rho(H)$-coloring of W by assigning color 1 to the vertices in $V(W) - V(H)$ (see the case $(S \hookleftarrow \overline{P_3}, C, R)$ in Fig. 3(a)).

If $R = \emptyset$, we can consider I as the union of a maximum stable set of G and one vertex $v \in C - N(V(G))$. Then, a packing $\chi_\rho(H)$-coloring of H that assigns color 1 to the vertices in I and color 2 to one vertex $u \in V(G) - I$ can be extended to a $\chi_\rho(H)$-coloring of W by assigning color 2 to the neighbour of v in S and 1 to the remaining vertices in $V(W) - V(H)$ (see the case $(S \hookleftarrow \overline{P_3}, C, \emptyset)$ in Fig. 3(b)).

Now, let $W = (S, C \hookleftarrow G, R)$ with $G \neq K_3$ and let s' be the neighbor of G in S.

Consider $H = W - (S - \{s'\})$. Observe that H has diameter two and it can be obtained in linear time.

If $\alpha(W[R]) \geq 2$, we can follow the same strategy as in the previous cases by identifying a maximum stable set I of H such that $I \cap C = \emptyset$ and extending a packing $\chi_\rho(H)$-coloring of H assigning color 1 to the remaining vertices of W. It is not hard to see that the union of a maximum stable set of $W[R]$ and s' is the required maximum stable set of H (see the case $(S, C \hookleftarrow P_3, R)$ in Fig. 3(c)).

When R is a complete set (probably empty, as it is shown in the case $(S, C \hookleftarrow P_3, \emptyset)$ in Fig. 3(d)), we consider a packing $\chi_\rho(H)$-coloring of H where the vertices in a maximum stable set I of G have color 1 and s' has color 2. Clearly, this packing coloring can be extended to a $\chi_\rho(H)$-coloring of W by assigning color 2 to the vertices in $S - \{s'\}$. □

Lemma 4.7. *Let $W = (S \hookleftarrow G_1, C \hookleftarrow G_2, R)$ be a $(2, 2)$-well labelled spider graph obtained from a thin spider (S, C, R). If $W \neq (S \hookleftarrow S_2, C \hookleftarrow S_2, \emptyset)$ then W has a subgraph H of diameter two which can be obtain in linear time and such that $\chi_\rho(W) = \chi_\rho(H)$.*

Proof. If $G_1 = S_2$ and $G_2 = K_2$, W is an hypersplit graph and the result follows from the proof of Theorem 4.3. Hence, we assume that $G_1 = K_2$ or $G_2 = S_2$. In both cases we consider $H = W - (S \hookleftarrow G_1 - V(G_1))$. Clearly, H has diameter two and can be obtained in linear time.

If $R \neq \emptyset$, let I be the union of a maximum stable set of G_1 and a maximum stable set of $W[R]$. Clearly, I is a maximum stable set of H. From an optimal packing coloring of H such that the class of color 1 is I, we obtain a packing $\chi_\rho(H)$-coloring of W by assigning 1 to the remaining vertices in S (see cases $(S \hookleftarrow S_2, C \hookleftarrow S_2, R)$ in Fig. 4(a), and $(S \hookleftarrow K_2, C \hookleftarrow S_2, R)$ in Fig. 4(b)).

If $R = \emptyset$ and $G_1 = K_2$, let I be the maximum stable set of H containing one vertex of G_1 and one vertex $v \in C \hookleftarrow G_2$ such that v is non adjacent to any vertex of G_1. We consider an optimal packing coloring of H for which I is the class of color 1 and the vertex in $V(G_1) - I$ has color 2. Then, assigning color 2 to the remaining vertices in S we have a packing $\chi_\rho(H)$-coloring of W (see case $(S \hookleftarrow K_2, C \hookleftarrow S_2, \emptyset)$ in Fig. 4(c)). □

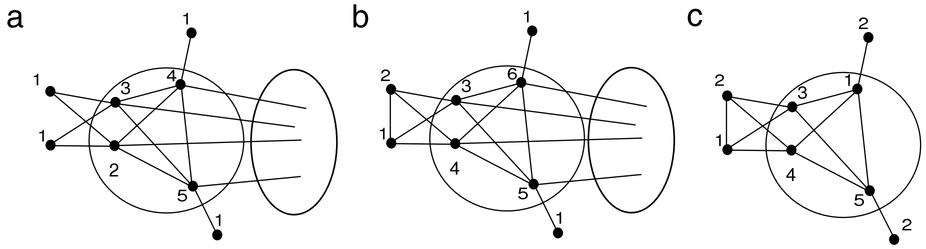

Fig. 4. Scheme of packing colorings for (a) $(S \hookleftarrow S_2, C \hookleftarrow S_2, R)$, (b) $(S \hookleftarrow K_2, C \hookleftarrow S_2, R)$ and (c) $(S \hookleftarrow K_2, C \hookleftarrow S_2, \emptyset)$.

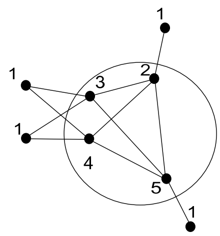

Fig. 5. Scheme of packing colorings for $(S \hookleftarrow S_2; C \hookleftarrow S_2; \emptyset)$.

The $(2, 2)$-well labelled spider graphs $(S \hookleftarrow S_2; C \hookleftarrow S_2; \emptyset)$ do not have the property stated in the lemma above. However, we can prove the following result:

Lemma 4.8. *Let W be a $(2, 2)$-well labelled spider $(S \hookleftarrow S_2; C \hookleftarrow S_2; \emptyset)$. Then, $\chi_\rho(W) = |C| + 2$.*

Proof. Firstly, let us prove that $\chi_\rho(W) \geq |C| + 2$. Observe that if at least one vertex $v \in C \hookleftarrow S_2$ has color 1 in a packing k-coloring, every neighbor of v must have different colors, greater than 1. Then, since $\deg(v) \geq |C| + 1$, we have $k \geq |C| + 2$. Moreover, if W admits a packing $(|C| + 1)$-coloring of W, at least one vertex $v \in C \hookleftarrow S_2$ must have color 1, a contradiction. Then, $\chi_\rho(W) \geq |C| + 2$.

To prove that $\chi_\rho(W) \leq |C| + 2$, we consider the packing $(|C| + 2)$-coloring of W obtained by assigning color 1 to every vertex in $S \hookleftarrow S_2$ and the remaining colors to the vertices in $C \hookleftarrow S_2$ (see Fig. 5). □

For multiple well labelled spiders we have similar results.

Lemma 4.9. *Let $W = (S_{\sigma,\kappa}, C_{\gamma,\eta}, R)$ be a multiple well labelled spider graph such that*

- *if R is a complete set then $\gamma \leq 1$,*
- *$W \neq (S_{\sigma,|S|-\sigma}, C_{\gamma,\eta}, \emptyset)$.*

Then W has a subgraph of diameter two which can be obtained in linear time and such that $\chi_\rho(W) = \chi_\rho(H)$.

Proof. Recall that multiple well labelled spider graphs with $\kappa = \gamma = 0$ are hypersplit graphs and the result follows from Theorem 4.3. Then, let us assume that $\kappa + \gamma \geq 1$.

Case 1: R is a complete set and $\gamma = 1$.

Let V_2 be the vertex set of a graph S_2 in $C_{\gamma,\eta}$ and let s be the vertex of $S_{\sigma,\kappa}$ adjacent to V_2. Let H be the subgraph of W induced by $\{s\} \cup C_{\gamma,\eta} \cup R$. Clearly, H has diameter two and can be obtained in linear time.

Consider $I = V_2$ and an optimal packing coloring of H where the class of color 1 is I and s receives color 2. It is clear that any vertex in $V(W) - V(H) = S_{\sigma,\kappa} - \{s\}$ is at distance 3 from s and it is not adjacent to a vertex in V_2. Then we can assign color 1 and 2 to these vertices (see Fig. 6) and we obtain $\chi_\rho(W) = \chi_\rho(H)$.

Case 2: R is a non complete set and $\gamma, \kappa \geq 1$, or $R \neq \emptyset$ and $\gamma = 0$.

Let V_2 be the set of vertices of a K_2 in $S_{\sigma,\kappa}$. We define H as the subgraph induced by $V_2 \cup C_{\gamma,\eta} \cup R$. It is clear that the diameter of H is two and it can be obtained in linear time. Now consider the maximum stable set I of H, obtained by adding one vertex $s \in V_2$ to a maximum stable set of $W[R]$. We obtain an optimal packing coloring of H such that I is the class at color 1 and $\{v\} = V_2 \setminus I$ is the class at color 2. Hence, we have a packing $\chi_\rho(H)$-coloring of W assigning colors 1 and 2 to $S_{\sigma,\kappa}$ (see Fig. 6). Then, $\chi_\rho(W) = \chi_\rho(H)$.

Case 3: R is not a complete set and $\kappa = 0$.

We define H as the subgraph induced by $\{s\} \cup C_{\gamma,\eta} \cup R$ with $s \in S_{\sigma,\kappa}$ and the maximum stable set $I = \{s\} \cup I_R$, where I_R a maximum stable set of $W[R]$. Considering an optimal packing coloring of H where I is the class at color 1, we obtain a packing $\chi_\rho(H)$-coloring of W assigning color 1 to $S_{\sigma,\kappa}$ (see Fig. 6).

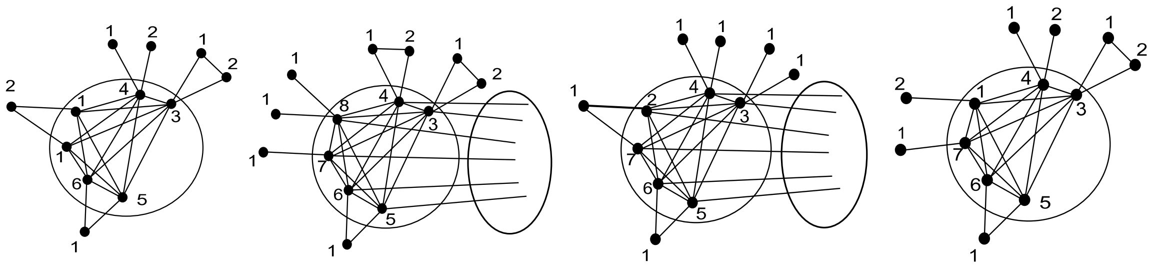

Fig. 6. Scheme of packing colorings for Cases 1–4.

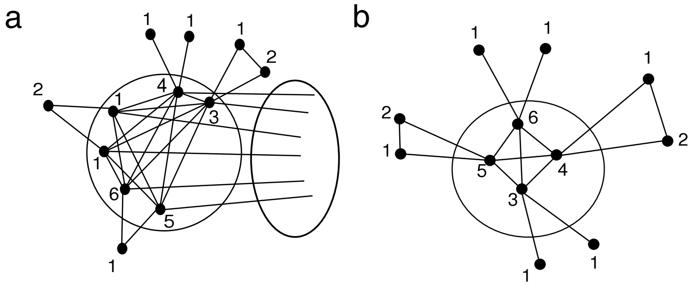

Fig. 7. Scheme of packing colorings for Lemmas 4.10 and 4.11.

Case 4: $R = \emptyset$, $\gamma = 0$ and $\kappa + \sigma \leq |S| - 1$.

Let H be the subgraph with diameter two of W induced by the subset $V_2 \cup C_{\gamma,\eta}$, with V_2 the set of vertices of a K_2 in $S_{\sigma,\kappa}$. Consider $I = \{c, s\}$ a maximum stable set of H, where $c \in C_{\gamma,\eta}$ has only one neighbor in $S_{\sigma,\kappa}$ and s belongs to V_2. We can obtain an optimal packing coloring of H such that I is the class at color 1 and the vertex in V_2 different to s has color 2.

Now, we can assign color 2 to the neighbor of c in $S_{\sigma,\kappa}$ and colors 1 and 2 to the remaining vertices in $S_{\sigma,\kappa}$ and obtain a packing $|C_{\gamma,\eta}| + 1$-coloring of W (see Fig. 6). □

For the remaining multiple labelled spider graphs we have the following results:

Lemma 4.10. *Let $W = (S_{\sigma,\kappa}, C_{\gamma,\eta}, R)$ be a multiple well labelled spider graph such that R is a complete set and $\gamma \geq 2$. Then, $\chi_\rho(W) = |R| + |C_{\gamma,\eta}|$.*

Proof. Let $v \in R$. Since $\deg(v) = |R| - 1 + |C_{\gamma,\eta}|$, if we assign color 1 to v we need at least $|R| + |C_{\gamma,\eta}|$ colors. Analogously if $v \in C_{\gamma,\eta}$, $\deg(v) \geq |R| + |C_{\gamma,\eta}| - 1$, then we need at least $|C| + 2$ colors.

Suppose now that color 1 is not used in $R \cup C_{\gamma,\eta}$. If $v \in R \cup C_{\gamma,\eta}$ is at color 2, we need at least $|R| + |C_{\gamma,\eta}|$ colors, since the diameter of $W[R \cup C_{\gamma,\eta}]$ is 2. Obviously if color 1 and 2 are not used in $R \cup C_{\gamma,\eta}$ we need more than $|R| + |C_{\gamma,\eta}|$ colors. Therefore $\chi_\rho(W) \geq |R| + |C_{\gamma,\eta}|$.

Let V_2 be the vertices of one S_2 in $C_{\gamma,\eta}$. Then we consider the packing coloring that assigns color 1 to V_2, colors 1 and 2 to the vertices in $S_{\sigma,\kappa}$ and the remaining colors for the vertices in $R \cup C_{\gamma,\eta}$ as it is shown in Fig. 7(a), then we obtain that $\chi_\rho(W) = |R| + |C_{\gamma,\eta}|$. □

Lemma 4.11. *Let $W = (S_{\sigma,\kappa}, C_{\gamma,\eta}, \emptyset)$ be a multiple well labelled spider graph with $\kappa + \sigma = |S|$. Then, $\chi_\rho(W) = |C| + 2$.*

Proof. Let $v \in C$. If we assign color 1 to v, since $\deg(v) = |C| + 1$, we need at least $|C| + 2$ colors.

Suppose now that v is at color 2, we need at least $|C|$ colors (color 1 is not used in $C_{\gamma,\eta}$) and since there exists at least one substitution in S for K_2, then two more colors are necessary.

Now, if colors 1 and 2 are not used for the vertices in C, it holds that $\chi_\rho(W) \geq |C| + 2$.

Then we consider the packing coloring that assigns colors 1 and 2 to the vertices in $V(W) - C$ and the remaining colors for the vertices in C, as we show in Fig. 7(b), then we obtain that $\chi_\rho(W) = |C| + 2$. □

From Lemmas 4.6–4.11 and 2.4, we obtain the following general result.

Theorem 4.12. *Let W be a well labelled spider graph obtained from a thin spider graph (S, C, R). Then, $\chi_\rho(W) = c(W) - \alpha(W[R])$, where $c(W)$ can be computed in linear time.*

It is not hard to see that if a graph G is the complement of a well labelled spider graph obtained from a thin spider different from P_4, G has diameter 2. Moreover, if the thin spider graph is P_4, G is also a well labelled spider obtained from P_4.

Given a family of graphs $\mathcal{F}$, an *$\mathcal{F}$-well labelled spider graph* is a well labelled spider graph whose head belongs to $\mathcal{F}$. As a consequence of the theorem above we have:

Theorem 4.13. *Let $\mathcal{F}$ be a self complementary family of graphs where the Stable Set problem is solvable in polynomial time. Then,* PackCol *is solvable in polynomial time for $\mathcal{F}$-well labelled spider graphs and their complements.*

4.1. PackCol *for partner limited graphs*

Let U be a subset of vertices inducing a P_4 in G. A *partner* of U is a vertex $v \in G - U$ such that $U \cup \{v\}$ induces at least two P_4 in G. In [6], Roussel et al. called *partner limited graph* (*PL* graphs for short) the graphs G for which any P_4 in G has at most two partners. *PL* graphs generalize cographs and P_4-tidy graphs, and constitute a hereditary and self complementary family of graphs for which the Stable Set problem is solvable in linear time [6].

From Theorem 1 in [6] it can be showed that non trivial graphs in $NM(PL)$ are one of the graphs in a self complementary class of graphs called *ZOO*, consisting of paths and cycles with at least 5 vertices, their complements and a set of graphs with at most nine vertices, a subclass of *PL*-hypersplit graphs, *PL*-well labelled spiders and their complements.

Since the packing chromatic number can be linearly obtained for paths, cycles and their complements [5] and the remaining graphs in *ZOO* form a finite set, using Lemma 4.1 and Theorem 4.13 we obtain the main result of this section:

Theorem 4.14. PackCol *is solvable in linear time for partner limited graphs.*

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