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On the facets of lift-and-project relaxations under graph operations[☆]

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ABSTRACT

We study the behavior of lift-and-project procedures for solving combinatorial optimization problems as described by Lovász and Schrijver (1991) [6] in the context of the stable set problem on graphs. Following the work of Wolsey (1976) [10], Lipták and Lovász (2001) [4] and Lipták and Tunçel (2003) [5], we investigate how to generate facets of the relaxations obtained by these procedures from facets of the relaxations of the original graph, after applying fundamental graph operations. We show our findings for the odd and the star subdivision, the stretching of a node and a new operation defined herein called the clique subdivision of an edge.

1. Introduction

In a seminal paper, Lovász and Schrijver [6] introduced two lift-and-project operators, N_0 and N, with which – starting from a (convex) polytope $\mathcal{K} \subset [0,1]^n$ – we may construct a sequence of polytopes yielding in at most n steps the convex hull of the integer points in $\mathcal{K}$, $\mathcal{K}_I = \text{conv}(\mathcal{K} \cap \{0,1\}^n)$. (They introduced also a third operator, N_+, which generally does not yield a polyhedron and the results in this paper may not apply.)

In what follows, we denote by $N_\sharp$ either operator, N or N_0, and define $N_\sharp^0(\mathcal{K}) = \mathcal{K}$ and $N_\sharp^k(\mathcal{K}) = N_\sharp(N_\sharp^{k-1}(\mathcal{K}))$ for every integer $k \geq 1$ (we refer the reader to Section 2 for further basic definitions and notation).

We always have $\mathcal{K}_I \subset N_\sharp^{k+1}(\mathcal{K}) \subset N_\sharp^k(\mathcal{K})$ for any k, and $N(\mathcal{K}) \subset N_0(\mathcal{K})$, although $N^n(\mathcal{K}) = N_0^n(\mathcal{K}) = \mathcal{K}$.

This brings up the idea of the $N_\sharp$*-rank* or *index* of the convex set $\mathcal{K}$, $r_\sharp(\mathcal{K})$, defined as the smallest k for which $N_\sharp^k(\mathcal{K}) = \mathcal{K}_I$. Thus, $r(\mathcal{K}) \leq r_0(\mathcal{K}) \leq n$.

A particularly interesting case is when $K_I = \text{STAB}(G)$, the stable set polytope of a simple graph $G = (V, E)$, and $\mathcal{K} = \text{FRAC}(G)$, the fractional stable set polytope. Since we will mostly use FRAC(G) as the initial relaxation, for simplicity we write $N_\sharp^k(G) = N_\sharp^k(\text{FRAC}(G))$.

Lovász and Schrijver pointed out that STAB(G) = FRAC(G) if and only if G is bipartite, whereas $N(G) = N_0(G) = \text{CSTAB}(G)$, the polytope defined by the trivial, edge and odd cycle inequalities,

$$\sum_{u \in C} x_u \leq \frac{1}{2}(|C| - 1) \quad \text{for all odd cycles } C.$$

Thus, for bipartite graphs we have $r_\sharp(G) = 0$, and in general, $r_\sharp(G) \leq |V| - 2$, with equality attained if $G = K_n$, the complete graph on n vertices. Many other properties are shown in [6].

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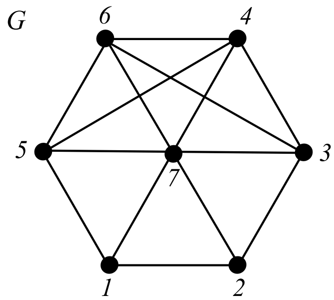

Fig. 1.1. The Au and Tunçel graph (AT-graph).

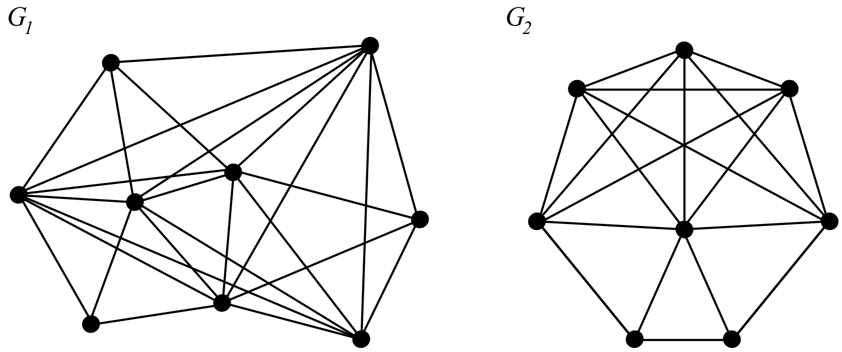

Fig. 1.2. Graphs where the operators differ at the second iteration and coincide (G_1) or not (G_2) at the third, reaching STAB(G) at the fourth.

Several questions naturally arise from the fact that $N_\sharp(G) = \text{CSTAB}(G)$. For example, are there simple characterizations of $N_0^k(G)$ or $N^k(G)$ for $k \geq 2$? Is it always the case that $N_0^k(G) = N^k(G)$? Or, at least, is it $r_0(G) = r(G)$ for all G?

The last two questions were raised by Lipták and Tunçel [5], who were among the first to study the ranks of $N_\sharp(G)$, and introduced two conjectures: the $N - N_0$ *conjecture*, stating that $N_0^k(G) = N^k(G)$ for all k, and the *rank conjecture*, stating that $r_0(G) = r(G)$.

The $N - N_0$ conjecture was shown to be false by Au and Tunçel [1], by giving an example of a graph with 7 nodes and 14 edges for which $r(G) = r_0(G) = 3$ and $N^2(G) \neq N_0^2(G)$. This graph – which we will call the AT-graph – is shown in Fig. 1.1. They also showed that adding a suitable edge yields a perfect graph with the same properties.

Another counter-example to the $N - N_0$ conjecture is given by the web graph W_8^2, having 8 nodes and 16 edges (u is adjacent to v if $|u - v| \in \{1, 2\}$, where the difference is taken modulo 8). As with the AT-graph, for $G = W_8^2$ we have $r(G) = r_0(G) = 3$ and $N^2(G) \neq N_0^2(G)$ (it can be seen that $x = \frac{1}{5}(1, 1, 1, 2, 1, 1, 2, 2) \in N_0^2(G) \setminus N^2(G)$). Unlike the AT-graph, W_8^2 is planar.

Obviously, the $N - N_0$ conjecture holds for graphs G with $r_0(G) \leq 2$, which includes bipartite graphs ($r_0(G) = 0$) and t-perfect graphs (defined by $\text{STAB}(G) = \text{CSTAB}(G)$), and in particular, series–parallel graphs. It also holds for the complete graphs. Nevertheless, as seen by the previous examples, neither perfect graphs nor planar graphs are properly contained in the family of graphs for which the $N - N_0$ conjecture is true.

On the other hand, the rank conjecture remains unsettled. It holds for perfect graphs (for which $\text{STAB}(G) = \text{QSTAB}(G)$, the clique polytope associated with G), h-perfect graphs (defined by $\text{STAB}(G) = \text{QSTAB}(G) \cap \text{CSTAB}(G)$), and many other graphs (see, e.g., [1,5]).

Thus, it is important to extend the known families of graphs for which either conjecture holds: in these cases we would not need to study $N^k(G)$, and just consider $N_0^k(G)$, which in a sense is "easier" since it can be obtained as $\bigcap_j P_j(N_0^{k-1}(G))$, where P_j denotes the Balas et al. lift-and-project operator defined by $P_j(\mathcal{K}) = \text{conv}\big((\mathcal{K} \cap \{x : x_j = 0\}) \cup (\mathcal{K} \cap \{x : x_j = 1\})\big)$ (see [2,6]).

A normal technique for studying the N_+, N and N_0 ranks is to bound them by the *disjunctive rank* associated to the operator P_j. Still, the disjunctive rank may be strictly greater than the N and N_0 ranks, as shown by the web W_7^2, having $N_\sharp$-rank 2 but disjunctive rank 3. Of course, the N_+-rank may be strictly smaller than either the N or N_0 ranks: for $n > 3$ we have $r_\sharp(K_n) = n - 2$ whereas $r_+(K_n) = 1$ [6].

The complexity of the study is illustrated by the fact that we could have $N^2(G) \neq N_0^2(G)$, $r(G) = r_0(G) = 4$ and either $N^3(G)$ equal to or different from $N_0^3(G)$ (respective examples are given by the graphs G_1 and G_2 of Fig. 1.2). That is, the operators may differ at some step but be equal later (and before reaching STAB(G)).

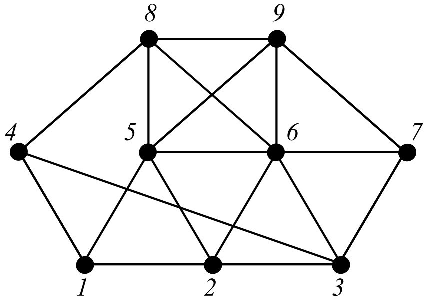

Fig. 1.3. A graph whose full support facets have different ranks.

Also, although $N^k(G) \subset N_0^k(G)$ always, it is the case that sometimes all the facets of $N_0^k(G)$ are facets of $N^k(G)$ as for the AT-graph and W_8^2, and sometimes this is not true.

One approach to these studies is to compare the $N_\sharp$-ranks of just the facets of STAB(G), and in particular to consider only those with full support. In this regard, it is interesting to notice that the rank (N or N_0) of a facet with full support need not be equal to the rank of G. For example,

$$x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + x_7 + x_8 + x_9 \le 3,$$
$$x_1 + 2x_2 + 2x_3 + x_4 + 2x_5 + 3x_6 + x_7 + x_8 + x_9 \le 4,$$

are two of the facets of the stable set polytope of the graph in Fig. 1.3 (a variant of the AT-graph), the first one has $N_\sharp$-rank 2 and the last one has $N_\sharp$-rank 3.

Yet another common technique – which we follow here – is to consider different operations on a graph and study how these operations influence the behavior of the N and N_0 operators.

For instance, the AT-graph may be obtained from the wheel W_5 (a center node joined to every node in the cycle C_5) by replicating one of the rim nodes. It is easy to see that $r_\sharp(W_5) = 2$, and by replicating the node the $N_\sharp$-rank has increased by 1. For general G, if we replicate a node in G to obtain G', the facets of STAB(G') are completely characterized in terms of those of STAB(G), and $r_\sharp(G) \le r_\sharp(G') \le r_\sharp(G) + 1$, but either inequality could be strict.

Another important operation is the (complete) join of the disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to obtain $G = (V, E)$ with $V = V_1 \cup V_2$ and $e \in E$ if either $e \in E_1$, or $e \in E_2$, or $e = [v_1, v_2]$ with $v_i \in V_i$ and $i \in \{1, 2\}$. To wit, we could think of the AT-graph as the join of the center with the 6 rim nodes. For general G, the non-trivial facets of STAB(G) are precisely those obtained by combining the non-trivial facets of STAB(G_i). As in the case of the replication of nodes, this simplicity does not carry over to the behavior of the ranks: $r_\sharp(G) \ge \max\{r_\sharp(G_1), r_\sharp(G_2)\}$ but we could have strict inequality (as in the AT-graph case) or equality (e.g., by completely joining the graph in Fig. 1.3 with one node). Even more, $r_\sharp(G)$ may be as large as $r_\sharp(G_1) + r_\sharp(G_2) + 2$, as seen by taking G_1 and G_2 to be complete graphs.

When G is obtained by replication of a node or the join of two graphs, the facets of STAB(G) are completely characterized, but there are many operations for which some facets of STAB(G) may be obtained from those of the original graph (see, e.g., [9,8,10,3,11]).

With a view to understanding the Lipták and Tunçel conjectures, here we extend their work on the $N_\sharp$-ranks of FRAC(G) by studying the relationship between the facets of $N_\sharp^k(G)$ and those of its induced subgraphs. After reviewing some notation and some elementary results, in Sections 3 and 4 we study the *odd subdivision of an edge* and the *stretching of a node* operations introduced by Wolsey [10]; in Section 5 we study the *star subdivision*, introduced by Lipták and Lovász [4], and in Section 6 we study what we believe is a new operation, the *clique subdivision of an edge*, motivated by the AT-graph. We end the discussion by applying our results to show that the $N - N_0$ conjecture remains false if we take QSTAB(G) instead of FRAC(G) as the initial relaxation, and stating a general result along this line.

2. Preliminaries

In this section we review some of the nomenclature we use, and state some simple results.

We will not differentiate between row and column vectors, since the orientation of the vectors employed in this paper should be clear from the context. We denote by $\mathbf{e}_i$ the i-th unit vector and by $\mathbf{0}$ the null vector, in any case of appropriate dimension. If x and y are vectors of the same dimension, their inner product is indicated by $x \cdot y$, and $x \ge y$ indicates $x_i \ge y_i$ for all i. If $I \subset \{1, 2, \dots, n\}$ we write $x(I) = \sum_{i \in I} x_i$. An inequality of the form $a \cdot x \le b$ is *valid for the set* $S \subset \mathbb{R}^n$ if it is true for all $x \in S$. If $\pi : a \cdot x \le b$ is valid for the polytope $\mathcal{K}$, and $\{x \in \mathcal{K} : a \cdot x = b\}$ is a facet of $\mathcal{K}$, we will say that π *defines a facet*, and sometimes we will say simply that π *is a facet* of $\mathcal{K}$.

2.1. The N_0 and N operators

As already mentioned, the N_0 and N operators were introduced by Lovász and Schrijver [6], who considered convex cones $\tilde{\mathcal{K}} \subset \mathbb{R}^{n+1}$ (with components indexed by $0, 1, \ldots, n$) such that

$$\{x \in \mathbb{R}^n : (1, x) \in \tilde{\mathcal{K}}\} \subset [0, 1]^n.$$

Denoting by $\operatorname{diag}(Y)$ the diagonal of the matrix Y, for a convex cone $\tilde{\mathcal{K}}$ as above, we let $M_0(\tilde{\mathcal{K}})$ be the cone of matrices $Y \in \mathbb{R}^{(n+1)\times(n+1)}$ such that $\operatorname{diag}(Y) = Y^{\mathrm{T}}\mathbf{e}_0 = Y\mathbf{e}_0$, and $Y\mathbf{e}_i \in \tilde{\mathcal{K}}$ and $Y(\mathbf{e}_0 - \mathbf{e}_i) \in \tilde{\mathcal{K}}$ for $i = 1, \ldots, n$. Projecting $M_0(\tilde{\mathcal{K}})$ to $\mathbb{R}^{n+1}$, we get the cone

$$N_0(\tilde{\mathcal{K}}) = \{Y\mathbf{e}_0 : Y \in M_0(\tilde{\mathcal{K}})\}.$$

By requiring the matrices in $M_0(\tilde{\mathcal{K}})$ to be symmetric also, we obtain the cone $M(\tilde{\mathcal{K}})$ and its projection

$$N(\tilde{\mathcal{K}}) = \{Y\mathbf{e}_0 : Y \in M(\tilde{\mathcal{K}})\}.$$

The *cone generated* by the convex set $\mathcal{K} \subset [0, 1]^n$ is

$$\operatorname{cone}(\mathcal{K}) = \{(t, tx) \in \mathbb{R}^{n+1} : t \geq 0, x \in \mathcal{K}\}.$$

For simplicity, when we say we are *applying* the N_0 or the N operator to a convex set $\mathcal{K} \subset [0, 1]^n$, we mean that we consider $\operatorname{cone}(\mathcal{K})$, apply the corresponding operator to it, then take the intersection of this cone with $x_0 = 1$ and project it back onto $\mathbb{R}^n$. Thus, in the sequel, $N_0(\mathcal{K})$ and $N(\mathcal{K})$ stand for these final subsets of $[0, 1]^n$. We say that Y *represents* a point $x \in N_\sharp(\mathcal{K}) \subset \mathbb{R}^n$ if $Y \in M_\sharp(\operatorname{cone}(\mathcal{K})) \subset \mathbb{R}^{(n+1)\times(n+1)}$ and $Y\mathbf{e}_0 = (1, x)$.

The next lemma establishes that if a valid inequality is tight at some point $x \in N_\sharp(\mathcal{K})$, then it is also tight for the columns of a matrix representing it in the higher dimensional space.

Lemma 2.1. *If the inequality $\pi : a \cdot x \leq b$ is valid for $\mathcal{K}$, and $a \cdot x = b$ for some $x \in N_\sharp(\mathcal{K})$, then for every representation Y of x and $X^i \in \mathbb{R}^n$ such that $Y\mathbf{e}_i = (x_i, X^i)$ we have $a \cdot X^i = x_i b$ and $a \cdot (x - X^i) = (1 - x_i)\, b$ for all $i = 1, \ldots, n$.*

Proof. Since (x_i, X^i) and $(1 - x_i, x - X^i)$ both belong to $\operatorname{cone}(\mathcal{K})$ and π is valid for $\mathcal{K}$, $a \cdot X^i \leq x_i b$ and $a \cdot (x - X^i) \leq (1 - x_i)\, b$. Adding these two inequalities we obtain

$$a \cdot x = a \cdot X^i + a \cdot (x - X^i) \leq x_i b + (1 - x_i)\, b = b.$$

The result follows now from $a \cdot x = b$. □

2.2. Graphs, stable sets and related polytopes

We will work with simple undirected graphs $G = (V, E)$, usually letting $n = |V|$. If $v \in V$, the graph obtained *by deletion of* v, denoted by $G - v$, is the subgraph of G induced by the nodes in $V \setminus \{v\}$. Similarly, $G - [v_1, v_2]$ stands for the graph obtained after removing the edge $[v_1, v_2]$ from G.

A *stable set* in G is a subset of mutually nonadjacent nodes and a *clique* is a subset of pairwise adjacent nodes.

The *stable set polytope* of the graph G is the convex hull of the incidence vectors χ^S of the stable sets S of G,

$$\operatorname{STAB}(G) = \operatorname{conv}\{\chi^S : S \text{ stable set in } G\},$$

the *fractional stable set polytope* is defined by the trivial ($0 \leq x_v \leq 1$ for $v \in V$) and edge inequalities,

$$\operatorname{FRAC}(G) = \{x \in [0, 1]^V : x_u + x_v \leq 1, [u, v] \in E\},$$

and the *clique polytope* is defined by the trivial inequalities and the clique inequalities,

$$\operatorname{QSTAB}(G) = \{x \in [0, 1]^V : x(K) \leq 1, K \text{ clique in } G\}.$$

The maximal clique inequalities always define facets of $\operatorname{STAB}(G)$.

If an inequality π is valid for $\operatorname{STAB}(G)$, we define its $N_\sharp$-rank, $r_\sharp(\pi)$, as the minimum k such that π is valid for $N_\sharp^k(G)$.

3. Odd subdivision of an edge

Wolsey [10] introduced the *odd subdivision of an edge* as follows: given the graph G and an edge $[v_1, v_2]$ of G, construct the graph G' from G by deleting the edge $[v_1, v_2]$, adding two new nodes, v_{n+1} and v_{n+2}, and the edges $[v_1, v_{n+1}]$, $[v_{n+1}, v_{n+2}]$ and $[v_{n+2}, v_2]$.

Let $a \cdot x \le b$, with $a \ge \mathbf{0}$ and $a \ne \mathbf{e}_1 + \mathbf{e}_2$, define a facet of STAB(G). Wolsey proved [10, Proposition 2] that

$$a \cdot x + b' (x_{n+1} + x_{n+2}) \le b + b'$$

defines a facet of STAB(G') if

$$b' = \max \{a \cdot x - b : x \in \text{STAB}(G - [v_1, v_2])\} > 0. \tag{3.1}$$

Our purpose is to generalize this result to the $N_\sharp$ context. Given a valid inequality for $N^k{}_\sharp(G)$,

$$\pi : a \cdot x \le b, \tag{3.2}$$

with $a \gneq \mathbf{0}$ and $a \ne \mathbf{e}_1 + \mathbf{e}_2$, we look for a valid inequality for $N_\sharp^k(G')$ of the form

$$\bar{\pi} : a \cdot x + b' (x_{n+1} + x_{n+2}) \le b + b', \tag{3.3}$$

with $b' > 0$.

Notice that the inequality $x_1 + x_2 \le 1$, valid for FRAC(G), is replaced for FRAC(G') by the three inequalities

$$x_1 + x_{n+1} \le 1, \qquad x_{n+1} + x_{n+2} \le 1 \quad \text{and} \quad x_{n+2} + x_2 \le 1, \tag{3.4}$$

which define facets of $N_\sharp^k(G')$ for every k.

Lipták and Tunçel [5, Theorem 16 and Lemma 17] showed the following.

$$\text{If } x \notin \text{STAB}(G) \quad \text{then } \bar{x} = (x, 1 - x_1, x_1) \notin \text{STAB}(G'). \tag{3.5}$$

$$\text{If } x \in N_\sharp^k(G) \quad \text{then } \bar{x} = (x, 1 - x_1, x_1) \in N_\sharp^k(G'). \tag{3.6}$$

$$r_\sharp(G) \le r_\sharp(G'). \tag{3.7}$$

They also gave an example where the inequality (3.7) is strict.

For $\bar{x} \in \mathbb{R}^{n+2}$ we write $\bar{x} = (x, x_{n+1}, x_{n+2})$ with $x \in \mathbb{R}^n$, and set

$$H = \{\bar{x} \in \mathbb{R}^{n+2} : x_{n+1} + x_{n+2} = 1\}.$$

We establish now a partial converse and a more precise version of (3.5) and (3.6).

Lemma 3.1. *Let G' be obtained from G by the odd subdivision of $[v_1, v_2]$. If $x \in \mathbb{R}^n$, let $\bar{x}^1 = (x, 1 - x_1, x_1)$ and $\bar{x}^2 = (x, x_2, 1 - x_2)$.*

1. *If $\bar{x} = (x, x_{n+1}, x_{n+2}) \in N_\sharp^k(G') \cap H$, then $x \in N_\sharp^k(G)$ and $\bar{x}$ is a convex combination of $\bar{x}^1$ and $\bar{x}^2$. In particular, $x_1 + x_2 \le 1$, and if $\bar{x}$ is an extreme point of $N_\sharp^k(G')$ then $\bar{x} = \bar{x}^1$ or $\bar{x} = \bar{x}^2$.*
2. *If x is an extreme point of $N_\sharp^k(G)$, then $\bar{x}^1$ and $\bar{x}^2$ are extreme points of $N_\sharp^k(G')$.*

Proof. 1. Given $\bar{x} = (x, x_{n+1}, x_{n+2}) \in N_\sharp^k(G') \cap H$, we prove that $x \in N_\sharp^k(G)$ by induction on k.

For $k = 0$ we only have to check that $x_1 + x_2 \le 1$. Since the inequalities in (3.4) hold and $x \in H$, it follows that

$$x_1 + x_{n+1} + x_2 + x_{n+2} = 1 + x_1 + x_2 \le 2,$$

and then $x_1 + x_2 \le 1$.

Suppose $k > 0$ and let $\bar{Y}$ be a representation of $\bar{x} \in N_\sharp^k(G')$,

$$\bar{Y} = \left[\begin{array}{c|ccccc|cc}
1 & x_1 & \cdots & x_i & \cdots & x_n & x_{n+1} & x_{n+2} \\ \hline
x_1 & x_1 & & x_{1,i} & & x_{1,n} & 0 & x_{1,n+2} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
x_i & x_{i,1} & \cdots & x_i & \cdots & x_{i,n} & x_{i,n+1} & x_{i,n+2} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
x_n & x_{n,1} & & x_{n,i} & & x_n & x_{n,n+1} & x_{n,n+2} \\ \hline
x_{n+1} & 0 & & x_{n+1,i} & & x_{n+1,n} & x_{n+1} & 0 \\
x_{n+2} & x_{n+2,1} & & x_{n+2,i} & & x_{n+2,n} & 0 & x_{n+2}
\end{array}\right],$$

where, e.g., $x_{n+1,n+2} = x_{n+2,n+1} = 0$ since $[v_{n+1}, v_{n+2}] \in E(G')$ [6].

Lemma 2.1 and $x_{n+1} + x_{n+2} = 1$ imply

$$x_{n+1,i} + x_{n+2,i} = x_i, \qquad (x_{n+1} - x_{n+1,i}) + (x_{n+2} - x_{n+2,i}) = 1 - x_i,$$

for $i = 0, \dots, n + 2$, so that, by the inductive hypothesis, the submatrix of $\bar{Y}$ with rows and columns in $\{0, \dots, n\}$ is a representation of x as a point in $N_\sharp^k(G)$.

2. If $\bar{x}^1$ were not an extreme point of $N_\sharp^k(G')$, then $\bar{x}^1 = \lambda\bar{z}+(1-\lambda)\bar{w}$ with $\lambda \in (0,1)$ and $\bar{z}, \bar{w} \in N_\sharp^k(G')$. If $\bar{z} = (z, z_{n+1}, z_{n+2})$ and $\bar{w} = (w, w_{n+1}, w_{n+2})$ it is easy to check that $x = \lambda z + (1-\lambda)w$. Moreover, $z_{n+1} + z_{n+2} = 1$ and $w_{n+1} + w_{n+2} = 1$. According to the first part, $z, w \in N_\sharp^k(G)$, contradicting that x is an extreme point of $N^k{}_\sharp(G)$. □

The next result follows immediately from the previous lemma.

Lemma 3.2. *If G' is obtained from G by the odd subdivision of $[v_1, v_2]$ and π in (3.2) is valid for $N_\sharp^k(G)$, then $\bar{\pi}$ in (3.3) is valid for $N_\sharp^k(G') \cap H$ for every b'.*

Thus, in order to define b' for $\bar{\pi}$ as in (3.3), we only need to consider points outside H. To this end, let W be the set of extreme points of $N_\sharp^k(G')$ not in H ($W \neq \emptyset$ since $\mathbf{0} \in W$), and for $\bar{x} = (x, x_{n+1}, x_{n+2}) \in W$ let

$$\beta(\bar{x}) = \min\{\gamma \geq 0 : \gamma\,(1 - x_{n+1} - x_{n+2}) \geq a \cdot x - b\}.$$

For every $\bar{x} \in W$ we will have that if $b' \geq \beta(\bar{x})$ then

$$a \cdot x + b'\,(x_{n+1} + x_{n+2}) \leq b + b',$$

so it is natural to define

$$b' = \max\{\beta(\bar{x}) : \bar{x} \in W\}. \tag{3.8}$$

Remark 3.3. If $N_\sharp^k(G') = \mathrm{STAB}(G')$, the value of b' in (3.8) coincides with that of (3.1), and is called *strength of the edge* $[v_1, v_2]$ in [4].

The above definitions and Lemma 3.2 imply the following.

Theorem 3.4. *If G' is obtained from G by the odd subdivision of $[v_1, v_2]$ and π in (3.2) is valid for $N^k{}_\sharp(G)$, then $\bar{\pi}$ defined in (3.3) with b' as in (3.8), is valid for $N_\sharp^k(G')$.*

When π defines a facet of $N^k{}_\sharp(G)$, we may say more.

Theorem 3.5. *If G' is obtained from G by the odd subdivision of $[v_1, v_2]$, π in (3.2) defines a facet of $N_\sharp^k(G)$ different from $x_1 + x_2 \leq 1$ and b' given in (3.8) is positive, then $\bar{\pi}$ given in (3.3) defines a facet of $N_\sharp^k(G')$.*

Proof. We know from Theorem 3.4 that $\bar{\pi}$ is a valid inequality for $N_\sharp^k(G')$. Let $x^1, \ldots, x^n \in N_\sharp^k(G)$ be affinely independent points satisfying π with equality, and define

$$\bar{x}^i = (x^i, 1 - x_1^i, x_1^i) \quad \text{for } i = 1, \ldots, n.$$

In addition, there must exist $j \in \{1, \ldots, n\}$ such that $x_1^j + x_2^j < 1$ (otherwise, π would define the same facet as $x_1 + x_2 \leq 1$). Let us set $\bar{x}^{n+1} = (x^j, 1 - x_2^j, x_2^j)$. From (3.6), $\bar{x}^i \in N_\sharp^k(G')$ for $i = 1, \ldots, n+1$. Finally, let $\bar{x}^{n+2} \in W$ be an optimal solution of (3.8).

It is not difficult to see that $\bar{x}^1, \ldots, \bar{x}^{n+2}$ are affinely independent points satisfying $\bar{\pi}$ with equality. Therefore, $\bar{\pi}$ defines a facet of $N_\sharp^k(G')$. □

Corollary 3.6. *Under the hypotheses of Theorem 3.4, both inequalities, π and $\bar{\pi}$, have the same $N_\sharp$-rank, i.e., $r_\sharp(\pi) = r_\sharp(\bar{\pi})$.*

Proof. Let r and $\bar{r}$ be the $N_\sharp$-ranks of π and $\bar{\pi}$, respectively. Theorem 3.4 (with $k = r$) implies $r \geq \bar{r}$.

On the other hand, let $x \in N_\sharp^{\bar{r}}(G)$. Using (3.6), $\bar{x} = (x, 1 - x_1, x_1) \in N_\sharp^{\bar{r}}(G')$ and then

$$a \cdot x + b'(1 - x_1 + x_1) \leq b + b'.$$

Therefore, π is a valid inequality for $N_\sharp^{\bar{r}}(G)$ and $r \leq \bar{r}$. □

We conclude this section by commenting on similarities and differences between the $N_\sharp$ relaxations and the stable set polytope after odd subdivision of an edge.

Mahjoub [7] presents nice structural results complementing Wolsey's result. More precisely, if $\bar{a} = (a, a_{n+1}, a_{n+2})$ and $\bar{a} \cdot \bar{x} \leq b$ is a facet defining inequality of $\mathrm{STAB}(G')$ different from those in (3.4), then we have the following.

1. If both a_{n+1} and a_{n+2} are positive, we must have $a_{n+1} = a_{n+2}$.
2. We cannot have $a_{n+1} > 0$ and $a_{n+2} = 0$ (and vice versa).

The first of these two statements is no longer true when we consider $N_\sharp^k(G')$ instead of $\mathrm{STAB}(G')$. A counter-example is the graph obtained by the odd subdivision of [1, 7] in the AT-graph (Fig. 1.1). However, we may extend the second statement to the intermediate $N_\sharp$ relaxations, using that if $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$, then $N_\sharp^k(G)$ is defined by the facets of the polytopes $N_\sharp^k(G_1)$ and $N_\sharp^k(G_2)$ [5, Theorem 6].

Lemma 3.7. *Let G' be obtained from G by the odd subdivision of $[v_1, v_2]$ and let $\bar{a} \cdot \bar{x} \leq b$ be a facet defining inequality of $N_\sharp^k(G')$ different from those in (3.4). Then we cannot have $a_{n+1} > 0$ and $a_{n+2} = 0$ (and vice versa).*

Proof. If $a_{n+1} > 0$ and $a_{n+2} = 0$, $\bar{a} \cdot \bar{x} \leq b$ defines a facet of $N_\sharp^k(G' - v_{n+2})$. By the above mentioned result, the facet defining inequalities of $N_\sharp^k(G' - v_{n+2})$ also define facets of $N_\sharp^k(G' - \{v_{n+1}, v_{n+2}\})$. Since $a_{n+1} > 0$, $\bar{a} \cdot \bar{x} \leq b$ must coincide with the facet $x_1 + x_{n+1} \leq 1$, contradicting the hypothesis. □

4. Stretching of a node

Wolsey [10] presented also a generalization of the odd subdivision of an edge, the *stretching of a node*: given the graph G and a selected node v_n, we obtain G' by separating the adjacent nodes of v_n into two non-empty subsets V_1 and V_2, introducing two new nodes v_{n+1} and v_{n+2} so that each vertex of V_ℓ is joined to $v_{n+\ell}$, $\ell = 1, 2$, and finally joining v_n to v_{n+1} and v_{n+2} only.

This operation is named the stretching operation of type I in [5] and is also analyzed in [3] for obtaining facets of the stable set polytope.

Let us recall some results in [5,10], analogous to the ones presented in the previous section.

If $a \cdot x \leq b$ with $a \geq \mathbf{0}$ defines a facet of STAB(G) such that

$$\max\left\{a \cdot x : x \in \text{STAB}(G), x_j = 0, j \in V_\ell \cup \{n\}\right\} = b$$

for $\ell = 1, 2$, then [10, Proposition 3],

$$\bar{\pi} : a \cdot x + a_n (x_{n+1} + x_{n+2}) \leq b + a_n, \tag{4.1}$$

defines a facet of STAB(G').

The main purpose of this section is to obtain a similar result for the intermediate $N_\sharp$ relaxations.

If $\bar{z} = (z_1, \ldots, z_{n+p}) \in \mathbb{R}^{n+p}$ for some $p \geq 0$ we write $\hat{z} = (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1}$, so that $\bar{z} = (\hat{z}, z_n, \ldots, z_{n+p})$. We also consider

$$H_\ell = \{\bar{x} \in \mathbb{R}^{n+2} : x_n + x_{n+\ell} = 1\} \quad \text{for } \ell = 1, 2.$$

We have, by [5, Lemma 26 and Theorem 25], the following.

$$\text{If } x = (\hat{x}, x_n) \notin \text{STAB}(G), \quad \text{then } \bar{x} = (\hat{x}, 1 - x_n, x_n, x_n) \notin \text{STAB}(G'). \tag{4.2}$$

$$\text{If } x = (\hat{x}, x_n) \in N_\sharp^k(G), \quad \text{then } \bar{x} = (\hat{x}, 1 - x_n, x_n, x_n) \in N_\sharp^k(G'). \tag{4.3}$$

$$r_\sharp(G) \leq r_\sharp(G'). \tag{4.4}$$

Lemma 4.1. *Let G' be obtained from G by stretching the node v_n. If $\bar{x} = (\hat{x}, x_n, x_{n+1}, x_{n+2}) \in N_\sharp^k(G') \cap H_1$, then $(\hat{x}, x_{n+2}) \in N_\sharp^k(G)$.*

Proof. We prove that $(\hat{x}, x_{n+2}) \in N_\sharp^k(G)$ by induction on k.

For $k = 0$ we only need to check that the edge inequalities $x_i + x_n \leq 1$ for $i \in V_1 \cup V_2$ are satisfied. If $i \in V_2$, this is true since $x_i + x_{n+2} \leq 1$ for every point in FRAC(G'). Let $i \in V_1$. Combining the inequalities $x_n + x_{n+2} \leq 1$, $x_i + x_{n+1} \leq 1$ and the fact that $\bar{x} \in H_1$ we have

$$x_i + x_{n+2} \leq x_i + 1 - x_n = x_i + x_{n+1} \leq 1.$$

Hence, $x \in$ FRAC(G).

Suppose $k > 0$ and let $\bar{Y}$ be a representation of $\bar{x}$ in $N_\sharp^k(G')$,

$$\bar{Y} = \left[\begin{array}{c|ccccc|c|cc}
1 & x_1 & \cdots & x_i & \cdots & x_n & x_{n+1} & x_{n+2} \\
\hline
x_1 & x_1 & & x_{1,i} & & x_{1,n} & x_{1,n+1} & x_{1,n+2} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
x_i & x_{i,1} & \cdots & x_i & \cdots & x_{i,n} & x_{i,n+1} & x_{i,n+2} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
\hline
x_n & x_{n,1} & & x_{n,i} & & x_n & 0 & 0 \\
\hline
x_{n+1} & x_{n+1,1} & & x_{n+1,i} & & 0 & x_{n+1} & x_{n+1,n+2} \\
x_{n+2} & x_{n+2,1} & & x_{n+2,i} & & 0 & x_{n+2,n+1} & x_{n+2}
\end{array}\right].$$

Since $x_n + x_{n+1} = 1$, Lemma 2.1 implies

$$x_{n,i} + x_{n+1,i} = x_i, \qquad (x_n - x_{n,i}) + (x_{n+1} - x_{n+1,i}) = 1 - x_i,$$

for $i = 0, \ldots, n+2$. Using the inductive hypothesis we obtain that

$$Y = \left[\begin{array}{c|ccccc|c} 1 & x_1 & \cdots & x_i & \cdots & & x_{n+2} \\ \hline x_1 & x_1 & & x_{1,i} & & & x_{1,n+2} \\ \vdots & \vdots & & \vdots & & & \vdots \\ x_i & x_{i,1} & \cdots & x_i & \cdots & & x_{i,n+2} \\ \vdots & \vdots & & \vdots & & & \vdots \\ \hline x_{n+2} & x_{n+2,1} & & x_{n+2,i} & & & x_{n+2} \end{array}\right]$$

is a representation of x as a point in $N_\sharp^k(G)$. $\square$

Let $W_{i,j}$ be the set of extreme points of $N_\sharp^k(G')$ in $H_i \setminus H_j$ for $i, j = 1, 2$ (clearly, $W_{i,j} \neq \emptyset$).

Theorem 4.2. *Let G' be obtained from G by stretching v_n. If $\pi : a \cdot x \leq b$ defines a facet of $N_\sharp^k(G)$ such that $\bar{\pi}$ as in (4.1) is valid for $N_\sharp^k(G')$, and*

$$\max\{\hat{a} \cdot \hat{x} + a_n x_{n+2} : \bar{x} \in W_{1,2}\} = \max\{\hat{a} \cdot \hat{x} + a_n x_{n+1} : \bar{x} \in W_{2,1}\} = b, \tag{4.5}$$

then $\bar{\pi}$ defines a facet of $N_\sharp^k(G')$.

Proof. Let $x^1, \ldots, x^n \in N_\sharp^k(G)$ be affinely independent points satisfying π with equality, and for $i \in \{1, \ldots, n\}$, consider $\bar{x}^i = (\hat{x}^i, 1 - x_n^i, x_n^i, x_n^i)$.

Using (4.3), $\bar{x}^i \in N_\sharp^k(G') \cap H_1 \cap H_2$ for every $i \in \{1, \ldots, n\}$.

Let $\bar{x}^{n+1} \in W_{1,2}$ and $\bar{x}^{n+2} \in W_{2,1}$ be optimal solutions to (4.5), i.e.,

$$\hat{a} \cdot \hat{x}^{n+1} + a_n x_{n+2}^{n+1} = b \quad \text{and} \quad \hat{a} \cdot \hat{x}^{n+2} + a_n x_{n+1}^{n+2} = b.$$

Then $\bar{x}^1, \ldots, \bar{x}^{n+2}$ are affinely independent points satisfying $\bar{\pi}$ with equality, and $\bar{\pi}$ defines a facet of $N_\sharp^k(G')$. $\square$

Remark 4.3. Let us observe the following.

- After Lemma 4.1, if $\bar{x} \in N^k{}_\sharp(G') \cap H_1$, then $\hat{a} \cdot \hat{x} + a_n x_{n+2} \leq b$ provided that $a \cdot x \leq b$ is valid for $N^k{}_\sharp(G)$.
- Theorem 4.2 generalizes Wolsey's result [10] when $N^k{}_\sharp(G') = \mathrm{STAB}(G')$.
- The hypotheses of Theorem 4.2 clearly imply that $r_\sharp(\bar{\pi}) \leq r_\sharp(\pi)$.

5. Subdivision of a star

Given G and a selected node v_n with neighbors $v_1, \ldots, v_s$, Lipták and Lovász [4] define the graph G' obtained by *subdivision of a star on v_n* as follows: for every $i \in \{1, \ldots, s\}$, delete the edge $[v_i, v_n]$, add a new node v_{n+i} and the edges $[v_{n+i}, v_i]$ and $[v_{n+i}, v_n]$.

If $a \cdot x \leq b$ with $a \geq \mathbf{0}$ defines a facet of STAB(G) and

$$c_{i,n} = \max\{a \cdot x - b : x \in \mathrm{STAB}(G - [v_i, v_n])\} > 0, \quad \text{for } i = 1, \ldots, s,$$

it is proved in [4] that

$$\bar{\pi} : \hat{a} \cdot \hat{x} + \left(\sum_{i=1}^{s} c_{i,n} - a_n\right) x_n + \sum_{i=1}^{s} c_{i,n} x_{n+i} \leq b + \sum_{i=1}^{s} c_{i,n} - a_n, \tag{5.1}$$

defines a facet of STAB(G').

In [5, Lemmas 16 and 17] the authors show the following.

$$\text{If } x \notin \mathrm{STAB}(G), \quad \text{then } \bar{x} = (\hat{x}, 1 - x_n, x_n, \ldots, x_n) \notin \mathrm{STAB}(G'). \tag{5.2}$$

$$\text{If } x \in N_\sharp^k(G), \quad \text{then } \bar{x} = (\hat{x}, 1 - x_n, x_n, \ldots, x_n) \in N_\sharp^k(G'). \tag{5.3}$$

$$r_\sharp(G) \leq r_\sharp(G'). \tag{5.4}$$

Following the ideas of the previous sections we look for a facet $\bar{\pi}$ of $N_\sharp^k(G')$ of the form (5.1) which could be derived from a facet of $N_\sharp^k(G)$. In order to do so, we define for $i = 1, \ldots, s$,

$$H_i = \{\bar{x} \in \mathbb{R}^{n+s} : x_{n+i} + x_n = 1\}, \qquad T_i = \bigcap_{\substack{j=1 \\ j \neq i}}^{s} H_j \setminus H_i, \qquad H = \bigcap_{i=1}^{s} H_i. \tag{5.5}$$

The following simple result helps us to construct a point in $N_\sharp^k(G)$ from a point in $N_\sharp^k(G')$ that also lies in H.

Lemma 5.1. *Let G' be obtained from G by subdivision of a star on v_n. If $\bar{x} = (\hat{x}, x_n, x_{n+1}, \dots, x_{n+s}) \in N_\sharp^k(G') \cap H$, with H given in (5.5), then $x = (\hat{x}, 1 - x_n) \in N_\sharp^k(G)$.*

Proof. The proof is by induction on k.

For $k = 0$ it only remains to check that $x_i + (1 - x_n) \le 1$ for every $i \in \{1, 2, \dots, s\}$. Since $\bar{x} \in H$, $x_n + x_{n+i} = 1$ for every i, so that $x_i + (1 - x_n) = x_i + x_{n+i} \le 1$ given that $\bar{x} \in \text{FRAC}(G')$.

Assume that $k > 0$ and let

$$\bar{Y} = \left[\begin{array}{c|ccccc|c|ccc}
1 & x_1 & \cdots & x_j & \cdots & & x_n & x_{n+1} & & x_{n+s} \\
\hline
x_1 & x_1 & & x_{1,j} & & & x_{1,n} & x_{1,n+1} & \cdots & x_{1,n+s} \\
\vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\
x_j & x_{j,1} & \cdots & x_j & \cdots & & x_{j,n} & x_{j,n+1} & \cdots & x_{j,n+s} \\
\vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\
\hline
x_n & x_{n,1} & & x_{n,j} & & & x_n & x_{n,n+1} & & x_{n,n+s} \\
\hline
x_{n+1} & x_{n+1,1} & & x_{n+1,j} & & & x_{n+1,n} & x_{n+1} & & x_{n+1,s} \\
\vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\
x_{n+s} & x_{n+s,1} & & x_{n+s,j} & & & x_{n+s,n} & x_{n+s,n+1} & & x_{n+s}
\end{array}\right]$$

be a representation of $\bar{x} \in N_\sharp^k(G')$.

Using Lemma 2.1, we know that for $i = 1, \dots, s$ and $j = 1, \dots, n + s$,

$$x_{n,j} + x_{n+i,j} = x_j, \qquad (x_n - x_{n,j}) + (x_{n+i} - x_{n+i,j}) = 1 - x_j.$$

Hence, the inductive hypothesis implies that

$$Y = \left[\begin{array}{c|ccccc|c}
1 & x_1 & \cdots & x_j & \cdots & & 1 - x_n \\
\hline
x_1 & x_1 & & x_{1,j} & & & x_1 - x_{1,n} \\
\vdots & \vdots & & \vdots & & & \vdots \\
x_j & x_{j,1} & \cdots & x_j & \cdots & & x_j - x_{j,n} \\
\vdots & \vdots & & \vdots & & & \vdots \\
\hline
1 - x_n & x_1 - x_{n,1} & & x_i - x_{n,j} & & & 0
\end{array}\right]$$

is a representation of x as a point of $N_\sharp^k(G')$. $\square$

When the points belong to H, there are no restrictions on the coefficients c_i. More precisely, we have the following.

Lemma 5.2. *Let G' be obtained from G by subdivision of a star on v_n. If $\pi : a \cdot x \le b$ is valid for $N_\sharp^k(G)$, then*

$$\bar{\pi} : \hat{a} \cdot \hat{x} + \left(\sum_{i=1}^{s} c_i - a_n \right) x_n + \sum_{i=1}^{s} c_i x_{n+i} \le b + \sum_{i=1}^{s} c_i - a_n, \tag{5.6}$$

is valid for $N_\sharp^k(G') \cap H$ for every c_i, $i = 1, \dots, s$.

Proof. Let us first rewrite $\bar{\pi}$ as

$$\hat{a} \cdot \hat{x} + a_n(1 - x_n) - \sum_{i=1}^{s} c_i (1 - x_n - x_{n+i}) \le b.$$

If $\bar{x} \in N^k{}_\sharp(G') \cap H$, from Lemma 5.1, $(\hat{x}, 1 - x_n) \in N^k{}_\sharp(G)$. Since $a \cdot x \le b$ is valid for $N^k(G)$,

$$\hat{a} \cdot \hat{x} + a_n(1 - x_n) \le b. \tag{5.7}$$

Finally, $1 - x_n - x_{n+i} = 0$ for every i since $\bar{x} \in H$, and using (5.7), the result follows. $\square$

Let us now rewrite $\bar{\pi}$ in (5.6) as

$$\bar{\pi} : \hat{a} \cdot \hat{x} + a_n(1 - x_n) - b \le \sum_{i=1}^{s} c_i (1 - x_n - x_{n+i}).$$

As is easily seen, $\bar{x} \in T_i$ satisfies $\bar{\pi}$ if and only if

$$\frac{\hat{a} \cdot \hat{x} + a_n(1 - x_n) - b}{(1 - x_n - x_{n+i})} \le c_i.$$

Let W_i be the set of extreme points of $N_\sharp^k(G')$ which are in T_i, and for $\bar{x} \in W_i$, let

$$\beta_i(\bar{x}) = \min\{\gamma \ge 0 : \hat{a} \cdot \hat{x} + a_n(1 - x_n) - b \le \gamma(1 - x_n - x_{n+i})\},$$

and

$$c_i = \max\{\beta_i(\bar{x}) : \bar{x} \in W_i\}. \tag{5.8}$$

Theorem 5.3. *Let G' be obtained from G by subdivision of a star on v_n. If $\pi : a \cdot x \le b$ defines a facet of $N_\sharp^k(G)$ such that $\bar{\pi}$ in (5.6) is valid for $N_\sharp^k(G')$, then $\bar{\pi}$ defines a facet of $N_\sharp^k(G')$.*

Proof. Let $x^1, \ldots, x^n \in N_\sharp^k(G)$ be affinely independent points satisfying π with equality. For $j \in \{1, \ldots, n\}$ let

$$\bar{x}^j = (\hat{x}^j, 1 - x_n^j, x_n^j, \ldots, x_n^j) \in \mathbb{R}^{n+s}.$$

By (5.3), $\bar{x}^j \in N_\sharp^k(G') \cap H$ for all $j \in \{1, \ldots, n\}$.

Now let $\bar{x}^{n+i} \in W_i$ be an optimal solution of (5.8) for $i \in \{1, \ldots, s\}$.

Then, $\bar{x}^\ell \in N_\sharp^k(G')$ for every $\ell \in \{1, \ldots, n+s\}$, are affinely independent and satisfy $\bar{\pi}$ with equality. This shows that $\bar{\pi}$ defines a facet of $N_\sharp^k(G')$. □

Remark 5.4. Let us observe the following.

- The coefficients c_i in (5.8) generalize the ones obtained by Lipták and Lovász for the stable set polytope. That is, if $N_\sharp^k(G') = \mathrm{STAB}(G')$ then $c_i = c_{i,n}$ with $c_{i,n}$ defined as in [4].
- The hypotheses of Theorem 5.3 clearly imply that $r_\sharp(\bar{\pi}) \le r_\sharp(\pi)$.

6. Clique subdivision of an edge

In the introduction we mentioned that the AT-graph (Fig. 1.1) could be obtained from the graph-join of W_5 and a point, or as the replication of a rim node in W_5. It may be seen also as obtained from K_5 consisting of the nodes $\{3, 4, 5, 6, 7\}$ in which we make an odd subdivision of the edge $[3, 5]$, obtaining the nodes 1 and 2, and then connecting these with 7. As a matter of fact, the reverse operation of contracting the edges $[1, 5]$ and $[2, 3]$ shows that the AT-graph is not planar.

Thus, given the graph G with nodes $\{1, \ldots, n\}$ and the clique (not necessarily maximal) $K = \{v_1, \ldots, v_s\}$ $(2 \le s \le n)$ in G, it seems natural to define the *clique subdivision of the edge* $[v_1, v_2]$ *in* K as follows: G' is obtained from G by deleting the edge $[v_1, v_2]$, adding the nodes v_{n+1} and v_{n+2} together with the edges $[v_1, v_{n+1}]$, $[v_{n+1}, v_{n+2}]$, $[v_{n+2}, v_2]$ and $[v_{n+i}, v_j]$ for $i = 1, 2$ and $j = 3, \ldots, s$. (In Fig. 6.1 we show the clique subdivision of the edge $[1, 2]$ in the clique $K = \{1, 2, 3\}$.)

Notice that if the clique is $K = \{v_1, v_2\}$, this operation reduces to the odd subdivision of $[v_1, v_2]$.

We know of no results on the facets of the stable set polytope after applying this operation, and we start by following the ideas of Wolsey [10] and Mahjoub [7].

Let us consider the sets

$$\hat{K} = K \setminus \{v_1, v_2\} = \{v_3, \ldots, v_s\}, \qquad \bar{K} = \hat{K} \cup \{v_{n+1}, v_{n+2}\}, \qquad K_i = \hat{K} \cup \{v_i, v_{n+i}\} \quad \text{for } i = 1, 2, \tag{6.1}$$

and let us note that K, the original clique in G, is no longer a clique in G', but every set in (6.1) is.

Remark 6.1. The above defined cliques $\bar{K}$, K_1 and K_2 are maximal in G'. Hence, the inequalities

$$x(\bar{K}) \le 1, \qquad x(K_i) \le 1 \quad \text{for } i = 1, 2, \tag{6.2}$$

define facets of $\mathrm{STAB}(G')$ and of $N_\sharp^k(G')$ for every $k \ge s - 2$.

Consider the valid inequality for $\mathrm{STAB}(G)$

$$\pi : a \cdot x \le b, \tag{6.3}$$

where $b > 0$, $a \ge \mathbf{0}$ and such that there is $j \in \{4, \ldots, s\}$ with $a_j \ne 0$. For $\bar{x} \in \mathbb{R}^{n+2}$, we denote $\bar{x} = (x, x_{n+1}, x_{n+2})$, and look for a valid inequality of $\mathrm{STAB}(G')$ of the form

$$\bar{\pi} : a \cdot x + b'\bar{x}(\bar{K}) \le b + b', \tag{6.4}$$

with $b' > 0$.

Proposition 6.2. *Let G' be obtained by the clique subdivision of $[v_1, v_2]$ in the clique K of G. If π as in (6.3) is valid for $\mathrm{STAB}(G)$ different from the clique inequality $x(K) \le 1$, and*

$$b' = \max\{a \cdot x - b : x \in \mathrm{STAB}(G - [v_1, v_2])\} > 0, \tag{6.5}$$

then $\bar{\pi}$ defined in (6.4) is a valid inequality for $\mathrm{STAB}(G')$.

Moreover, if π defines a facet of $\mathrm{STAB}(G)$, $\bar{\pi}$ defines a facet of $\mathrm{STAB}(G')$.

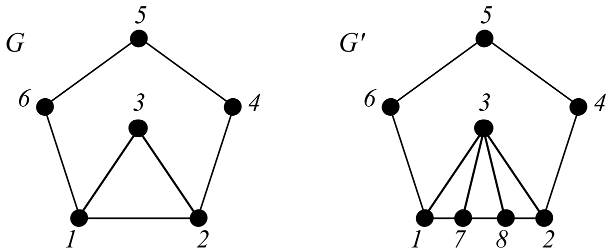

Fig. 6.1. The graphs G and G' of Example 6.7.

Proof. We first study the validity of $\bar{\pi}$ and it is enough to do so for the binary points in STAB(G').

If $\bar{x} \in$ STAB(G'), it is clear that $x \in$ STAB$(G-[v_1, v_2])$. Now, if $x \notin$ STAB(G) then $x_1 = x_2 = 1$ and $\bar{x}(\bar{K}) = 0$, and therefore,

$$a \cdot x + b'\bar{x}(\bar{K}) = a \cdot x \leq b + b'.$$

On the other hand, if $x \in$ STAB(G) then $a \cdot x \leq b$ and $\bar{x}(\bar{K}) \leq 1$. Summing the two inequalities we have the validity of $\bar{\pi}$.

If π defines a facet of STAB(G), there exist binary affinely independent points $x^1, \ldots, x^n \in$ STAB(G) satisfying π with equality.

If $i \in \{1, \ldots, n\}$, let us define

$$\bar{y}^i = \begin{cases} (x^i, 1, 0) & \text{if } x_1^i + x^i(\hat{K}) = 0, \\ (x^i, 0, 1) & \text{if } x_1^i = 1 \text{ (then } x_1^i + x^i(\hat{K}) = 1 \text{ and } x_2^i + x^i(\hat{K}) = 0), \\ (x^i, 0, 0) & \text{if } x_t^i = 1 \text{ for some } 3 \leq t \leq s \text{ (then } x_1^i + x^i(\hat{K}) = 1). \end{cases}$$

Note that all these cases are mutually disjoint and that each one is possible.

Since this facet does not coincide with $x(K) \leq 1$, there exists j such that $x^j(K) = 0$, and let $\bar{y}^{n+1} = (x^j, 0, 1)$. Finally, let $x^* \in$ STAB$(G - [v_1, v_2])$ be an optimal solution to (6.5). Since $x_1^* = x_2^* = 1$ and $x_t^* = 0$ for $t = 3, \ldots, s$, we define $\bar{y}^{n+2} = (x^*, 0, 0)$, which satisfies $\bar{y}^{n+2}(\bar{K}) = 0$.

It is not hard to check that $\bar{y}^1, \ldots, \bar{y}^{n+2}$ satisfy (6.4) with equality and are affinely independent points. □

Let us now study necessary conditions for the coefficients of the facets of STAB(G') for this operation (cf. [7]).

Lemma 6.3. *Let G' be obtained from G by the clique subdivision of $[v_1, v_2]$ in the clique K. If $\bar{a} \geq \mathbf{0}$ and $\bar{b} > 0$ are such that*

$$\bar{\pi} : \bar{a} \cdot \bar{x} \leq \bar{b} \tag{6.6}$$

is a facet defining inequality for STAB(G'), *then we have the following.*

1. *If this inequality is different from $\bar{x}(K_1) \leq 1$ in (6.2), then we cannot have both $a_{n+1} > 0$ and $a_{n+2} = 0$. Similarly for $\bar{x}(K_2)$, $a_{n+2} > 0$ and $a_{n+1} = 0$.*
2. *If this inequality is different from those in (6.2), then $a_{n+1} = a_{n+2}$.*

Proof. 1. Let us assume that $a_{n+1} > 0 = a_{n+2}$, and let $\mathcal{S}$ be a set of affinely independent stable sets in G' for which (6.6) is satisfied with equality.
Let $S \in \mathcal{S}$. Note that $S \cap K_1 \neq \emptyset$ since if $n + 2 \in S$ then $S' = S \setminus \{n + 2\} \cup \{n + 1\}$ would be a stable set for which $\bar{a} \cdot \chi^{S'} > b$. Hence, $\chi^S(K_1) = 1$ for every $S \in \mathcal{S}$. This implies that $\bar{\pi}$ defines the same facet as $\bar{x}(K_1) \leq 1$.

2. Again, suppose that $a_{n+1} > 0$ and consider $\mathcal{S}$ as before. We will prove the existence of a set $S_0 \in \mathcal{S}$ such that $n + 2 \in S_0$ and $1 \notin S_0$.
If this were not the case, for every $S \in \mathcal{S}$ we would have that $n + 2 \in S$ implies $1 \in S$, and then $S \cap K_1 \neq \emptyset$. If $n + 2 \notin S$, again $S \cap K_1 \neq \emptyset$ for $S \in \mathcal{S}$, since otherwise we could add the node $n + 1$ to the stable set S obtaining $\bar{a} \cdot \chi^S > b$ (recall that $a_{n+1} > 0$). In any case, we have $\chi^S(K_1) = 1$ for every $S \in \mathcal{S}$, contradicting the fact that $\bar{\pi}$ is different from $\bar{x}(K_1) \leq 1$.
Let $S' = S_0 \cup \{n + 1\} \setminus \{n + 2\}$. S' is a stable set in G' and $\bar{a} \cdot \chi^{S'} \leq \bar{b} = \bar{a} \cdot \chi^{S_0}$, proving $a_{n+1} \leq a_{n+2}$. By symmetry, $a_{n+2} \leq a_{n+1}$. □

We end this section by analyzing the relationship between the $N_\sharp$ ranks of G and G'. For this purpose we first establish the following.

Lemma 6.4. *Let G' be obtained from G by the clique subdivision of edge $[v_1, v_2]$ in the clique K. Then, for $x \in \mathbb{R}^n$, we have the following.*

1. *If $x \notin$ STAB(G), then $\bar{x} = (x, x_2, x_1) \notin$ STAB(G').*
2. *If $x \in N_\sharp^k(G)$, then $\bar{x} = (x, x_2, x_1) \in N_\sharp^k(G')$.*
Moreover, $\bar{x}(\bar{K}) = \bar{x}(K_1) = \bar{x}(K_2) = \bar{x}(K)$.

Proof. 1. Once again, it is enough to consider only binary points. If $\bar{x} \in \mathrm{STAB}(G') \cap \{0, 1\}^{n+2}$, we cannot have $\bar{x}_{n+1} = \bar{x}_{n+2} = 1$ or, equivalently, $x_1 = x_2 = 1$. Since $[v_1, v_2]$ is the only edge in G not in G', the result follows.

2. Let $x \in N^k{}_\sharp(G)$. The proof is by induction on k.
If $k = 0$, the edge inequalities

$$x_{n+1} + x_{n+2} = x_1 + x_2 \le 1, \qquad x_{n+i} + x_j = x_i + x_j \le 1$$

for $i = 1, 2, j \in \hat{K}$, are clearly satisfied. Assume $k > 0$ and let

$$\left[\begin{array}{c|cc|ccc} 1 & x_1 & x_2 & \cdots & x_i & \cdots \\ \hline x_1 & x_1 & 0 & \cdots & x_{1,i} & \cdots \\ x_2 & 0 & x_2 & \cdots & x_{2,i} & \cdots \\ \hline \vdots & & & & & \\ x_i & x_{i,1} & x_{i,2} & \cdots & x_i & \cdots \\ \vdots & & & & & \end{array}\right]$$

be a representation of $x \in N^k{}_\sharp(G)$.
Define

$$\left[\begin{array}{c|cc|ccc|cc} 1 & x_1 & x_2 & \cdots & x_i & \cdots & x_{n+1} = x_2 & x_{n+2} = x_1 \\ \hline x_1 & x_1 & 0 & \cdots & x_{1,i} & \cdots & 0 & x_1 \\ x_2 & 0 & x_2 & \cdots & x_{2,i} & \cdots & x_2 & 0 \\ \hline \vdots & & & & & & & \\ x_i & x_{i,1} & x_{i,2} & \cdots & x_i & \cdots & x_{i,2} & x_{i,1} \\ \vdots & & & & & & & \\ \hline x_2 & 0 & x_2 & \cdots & x_{2,i} & \cdots & x_2 & 0 \\ x_1 & x_1 & 0 & \cdots & x_{1,i} & \cdots & 0 & x_1 \end{array}\right].$$

By the inductive hypothesis, column i belongs to $x_i\mathrm{cone}(N_\sharp^{k-1}(G'))$ and similarly for the difference between the first and the ith column, for every i. This matrix is also symmetric if the original one is, proving that $\bar{x} \in N^k{}_\sharp(G')$. □

We may present now the main result of this section.

Theorem 6.5. *If G' is obtained from G by the clique subdivision of $[v_1, v_2]$ in the clique K, then $r_\sharp(G') \ge r_\sharp(G)$.*

Proof. If $r_\sharp(G) = k + 1$, there exists $x \in N_\sharp^k(G) \setminus \mathrm{STAB}(G)$. By Lemma 6.4, $\bar{x} = (x, x_2, x_1) \in N_\sharp^k(G')$ and $\bar{x} \notin \mathrm{STAB}(G')$, so $r_\sharp(G') \ge k + 1$. □

The following examples show that the $N_\sharp$-ranks can either strictly increase or remain unchanged by the subdivision of a clique.

Example 6.6. Consider a clique K of size 3 in $G = K_4$. By the clique subdivision of any edge with nodes in K, we obtain $G' = W_5$, and $r_\sharp(G) = r_\sharp(G') = 2$.

Example 6.7. The graph G in Fig. 6.1 is t-perfect and $r_\sharp(G) = 1$. The rank inequality $x_1 + \cdots + x_8 \le 3$ defines a facet of $\mathrm{STAB}(G')$, and has $N_\sharp$-rank at least 2 since it is neither a trivial, nor an edge nor an odd cycle inequality. Therefore, $r_\sharp(G') \ge 2$. In fact, it can be seen that it is 2.

7. A counter-example to the $N - N_0$ conjecture starting from the clique relaxation

In this section we show that the $N - N_0$ conjecture is still false if we substitute $\mathrm{QSTAB}(G)$ for $\mathrm{FRAC}(G)$ as the initial relaxation.

To do so, we will apply some of our previous results to the AT-graph, for which we know the original conjecture does not hold. So let G be the AT-graph shown in Fig. 1.1, and consider the graph G'' (to the right in Fig. 7.1) obtained from G by the odd-subdivision of [4, 6] followed by the star subdivision on node 7. As is easily seen, $\mathrm{FRAC}(G'') = \mathrm{QSTAB}(G'')$.

Claim 1. $N_0^2(\mathrm{QSTAB}(G'')) \ne N^2(\mathrm{QSTAB}(G''))$.

Proof. We will make use of the point

$$x = \frac{1}{5}(2, 2, 1, 2, 1, 1, 1) \in N_0^2(G) \setminus N^2(G),$$

considered by Au and Tunçel [1] to disprove the $N - N_0$ conjecture.

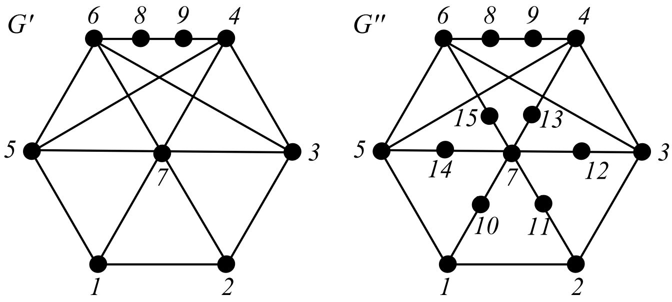

Fig. 7.1. The graphs G' and G'' in Claim 1.

Let G' be the graph obtained from the AT-graph G by the odd subdivision of [4, 6] (shown to the left in Fig. 7.1). Using (3.6) for x we have

$$x' = \frac{1}{5}(2, 2, 1, 2, 1, 1, 1, 4, 1) \in N_0^2(G').$$

Now, by Lemma 3.1 and since $x \notin N^2(G)$, we must have $x' \notin N^2(G')$.

Let G'' be obtained by the star subdivision on node 7 of G'. As before, (5.3) and Lemma 5.1 imply

$$x'' = \frac{1}{5}(2, 2, 1, 2, 1, 1, 4, 4, 1, 1, 1, 1, 1, 1, 1) \in N_0^2(G'') \setminus N^2(G''),$$

proving the claim. □

The arguments used in the previous proof can be extended to any graph G not verifying the $N - N_0$ conjecture.

Theorem 7.1. *Let G be a graph such that $N_0^k(G) \neq N^k(G)$ for some k. If G' is obtained from G by the odd subdivision of an edge, the star subdivision of a node or the stretching operation, then $N_0^k(G') \neq N^k(G')$.*

Remark 7.2. A related example is given by $G = W_8^2$, for which we have $\text{FRAC}(G) \neq \text{QSTAB}(G)$, $r(\text{QSTAB}(G)) = r_0(\text{QSTAB}(G)) = 2$ and $N(\text{QSTAB}(G)) \subsetneq N_0(\text{QSTAB}(G))$.

For instance, $\frac{1}{4}(1, 1, 1, 1, 1, 1, 1, 2)$ is a vertex of $N_0(\text{QSTAB}(G))$ and does not satisfy

$$x_1 + x_2 + x_3 + x_4 + x_5 + 4x_6 + 4x_7 + 4x_8 \leq 5,$$

a facet defining inequality of $N(\text{QSTAB}(G))$.

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