



Lorentzian compact manifolds: Isometries and geodesics



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ABSTRACT

In this work we investigate families of compact Lorentzian manifolds in dimension four. We show that every lightlike geodesic on such spaces is periodic, while there are closed and non-closed spacelike and timelike geodesics. Also their isometry groups are computed. We also show that there is a non trivial action by isometries of $H_3(\mathbb{R})$ on the nilmanifold $S^1 \times (\Gamma_k \backslash H_3(\mathbb{R}))$ for Γ_k , a lattice of $H_3(\mathbb{R})$.

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1. Introduction

Due to their relations with general relativity Lorentzian manifolds, that is manifolds endowed with metric tensors of index 1, play a special role in pseudo-Riemannian geometry. Timelike and null geodesics represent, respectively, free falling particles and light rays. Isometric actions and the existence problem of closed geodesics are two of the most popular topics of research in the last time. In this work by a closed geodesic we mean a periodic geodesic.

The known results developed in the field have made use of several techniques including variational and topological methods, Lie theory, etc. (See for instance [1–5] and references therein.) After the classification of simply connected Lie groups acting locally faithfully by isometries on a compact Lorentz manifold [6,7] some other questions concerning the geometric implications of such actions arise in a natural way, specially in the noncompact case (see [8]). In [9] Melnick investigated the isometric actions of Heisenberg groups on compact Lorentzian manifolds, showing a codimension one action of the Heisenberg Lie group $H_3(\mathbb{R})$ on the Lorentzian compact solvmanifold $M = \Gamma \backslash G$, where $G = \mathbb{R} \times H_3(\mathbb{R})$ is a solvable Lie group, called the oscillator group.

The main purpose of this work is to analyze these topics more deeply in a family of examples. We study the geometry of families of compact Lorentzian manifolds in dimension four: $M_{k,i} = G / \Lambda_{k,i}$, which are stationary, that is, they admit an everywhere timelike Killing vector field. This implies the existence of closed timelike geodesics (see [10]).

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In this work we obtain:

- Every lightlike geodesic on any compact space $M_{k,i}$ is periodic, while there are periodic and injective timelike and spacelike geodesics.
- The isometry groups of these compact spaces have a countable amount of connected components (see [11]).

As already mentioned the existence question of closed geodesics on a compact Lorentzian manifold is a classical topic in Lorentzian geometry. In this context the results above relative to null geodesics are surprising in a quite different situation of those in [12] and therefore they should induce new research in the topic.

We start with an isometric codimension one action by isometries of the Heisenberg Lie group $H_3(\mathbb{R})$ on compact nilmanifolds $\Lambda_k \backslash N$ where $N = \mathbb{R} \times H_3(\mathbb{R})$. The starting point is the existence of an isometry between the Lorentzian Lie group G which is solvable and the Lie group N which is 2-step nilpotent [13]. This reveals that the existence of actions by isometries coming from non-isomorphic groups does not distinguish the isometry class of the Lorentzian manifold. However while the Lorentzian metric on G is bi-invariant, that one on N is only left-invariant. Furthermore there is a family of groups Λ_k which are cocompact lattices of G and also of N so that every quotient $\Lambda_k \backslash N$ is diffeomorphic to $\Lambda_k \backslash G$ and the metrics induced to the quotients give rise to an isometry between the compact spaces $(\Lambda_k \backslash N, g_N)$ and $(\Lambda_k \backslash G, g_G)$. It is clear that as an ideal of G , the Heisenberg Lie group \mathbb{H}_3 acts isometrically on $\Lambda_k \backslash G$ by translations on the right. Therefore the Heisenberg Lie group also acts on $\Lambda_k \backslash N$ by isometries. The Lie group N is already known in the literature: it is related to the known Kodaira–Thurston manifold. One of the advantages of the nilmanifold model arises from Nomizu’s Theorem: the de Rham cohomology can be read off from the cohomology of the Lie algebra of N .

The solvable group G admits more cocompact lattices $\Lambda_{k,i}$ which are not isomorphic to the family above. We explicitly write the full isometry group of G which is proved to be non-compact. And making use of results which relate the isometries on the quotients with those on G we compute $\text{Iso}(M_{k,i})$ the group of isometries of the compact solvmanifolds $M_{k,i} = \Lambda_{k,i} \backslash G$.

We complete the work with the study of the periodic geodesics on the compact Lorentzian solvmanifolds. It should be noticed that all the Lorentzian manifolds here are naturally reductive spaces. We notice that together with the motivations coming from Lorentzian geometry an active research is given for g.o. spaces (see for instance [14–17]). The compact Lorentzian spaces $M_{k,i}$ constitute the first examples (known to us) of compact spaces in dimension four where every lightlike geodesic is periodic.

2. Lorentzian nilmanifolds and actions

Let $H_3(\mathbb{R})$ denote the Heisenberg Lie group of dimension three, which modeled over \mathbb{R}^3 has a multiplication map given by

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y) \right).$$

Let N denote the nilpotent Lie group $\mathbb{R} \times H_3(\mathbb{R})$, which turns into a pseudo-Riemannian manifold modeled on \mathbb{R}^4 with the following Lorentzian metric

$$g = dt \left(dz + \frac{1}{2}ydx - \frac{1}{2}xdy \right) + dx^2 + dy^2 \tag{1}$$

where (t, x, y, z) are usual coordinates for \mathbb{R}^4 . Denote $v = (x, y)$ and for each $(t_1, v_1, z_1) \in \mathbb{R}^4$ consider the following differentiable function on \mathbb{R}^4 :

$$L^N_{(t_1, v_1, z_1)}(t_2, v_2, z_2) = \left(t_1 + t_2, v_1 + v_2, z_1 + z_2 + \frac{1}{2}v_1^T J v_2 \right) \tag{2}$$

where J is the linear map on \mathbb{R}^2 given by the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Clearly L^N is the translation on the left on N by the element (t_1, v_1, z_1) and it is not hard to see that the metric g is invariant under the left-translations $L^N_{(t_1, v_1, z_1)}$. A basis of left-invariant vector fields at $p = (t, x, y, z)$ is

$$\begin{aligned} e_0(p) &= \partial_t|_p \\ e_1(p) &= \partial_x|_p - \frac{1}{2}y \partial_z \Big|_p \\ e_2(p) &= \partial_y|_p + \frac{1}{2}x \partial_z \Big|_p \\ e_3(p) &= \partial_z|_p \end{aligned}$$

and the invariant Lorentzian metric g satisfies

$$g(e_0, e_3) = g(e_1, e_1) = g(e_2, e_2) = 1.$$

Particular examples of closed subgroups are lattices. A *lattice* of a Lie group G is a discrete subgroup Γ such that the quotient space G/Γ or $\Gamma \backslash G$ is compact.

For every $k \in \mathbb{N}$ consider Λ_k the following lattice in N :

$$\Lambda_k = 2\pi\mathbb{Z} \times \Gamma_k < N \quad \text{where } \Gamma_k = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2k}\mathbb{Z} < H_3(\mathbb{R})$$

for Γ_k a lattice in $H_3(\mathbb{R})$.

The metric g on N (1) can be induced to the quotient spaces $\Lambda_k \backslash N$. In fact denote also by g the induced metric, for every $\gamma \in \Lambda_k$ one has:

$$\begin{aligned} g(Z_{\gamma x}, Y_{\gamma x})_{\gamma x} &= g(dp_{\gamma x}(Z), dp_{\gamma x}(Y))_{p(\gamma x)} \\ &= g(dp_x(Z), dp_x(Y))_{p(x)} = g(Z_x, Y_x)_x \end{aligned}$$

thus the canonical projection $p : N \rightarrow \Lambda_k \backslash N$ is a local isometry.

The following proposition shows an action of $H_3(\mathbb{R})$ on the compact nilmanifolds $\Lambda_k \backslash N$ which is not explained in [9].

Proposition 2.1. *There is an isometric action of $H_3(\mathbb{R})$ on the compact nilmanifold $\Lambda_k \backslash N$ induced by the action of $H_3(\mathbb{R})$ on N given as follows:*

$$(v', z') \cdot (t, v, z) = \left(t, v - R(t)v', z - z' - \frac{1}{2}v^\tau J R(t)v' \right) \tag{4}$$

where $R(t)$ is the linear map on \mathbb{R}^2 with matrix given by

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad t \in \mathbb{R}. \tag{5}$$

The proof follows from several computations which can be done by hand: for every $(v', t') \in H_3(\mathbb{R})$ the map above (4) defines an isometry on N which can be induced to $\Lambda_k \backslash N$. This gives rise to an action of $H_3(\mathbb{R})$ on the nilmanifold $\Lambda_k \backslash N$. In next sections we shall explain the construction of the action above (see Remark 6).

Remark 1. The action of $H_3(\mathbb{R})$ by isometries on the quotient $\Lambda_k \backslash N$ is neither induced by the translations on the left nor on the right on N .

The orbits of the action of $H_3(\mathbb{R})$ on N are parametrized by $t_0 \in \mathbb{R}$:

$$\mathcal{O}_{(t_0, v_0, z_0)} = \{(t_0, v, z) \in \mathbb{R}^4 \mid v \in \mathbb{R}^2, z \in \mathbb{R}\}$$

and they are not totally geodesic except for $t = 0$ (see geodesics in the next section).

On \mathbb{R}^4 consider the lightlike distribution

$$\mathcal{D}_p = \text{span}\{e_1, e_2, e_3\},$$

which is involutive. Integral submanifolds for \mathcal{D} are given by the orbits \mathcal{O}_p .

3. A Lorentzian solvable Lie group

Recall that if G is a connected real Lie group, its Lie algebra \mathfrak{g} is identified with the Lie algebra of left-invariant vector fields on G . Assume G is endowed with a left-invariant pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$. Then the following statements are equivalent (see [18, Chapter 11]):

1. $\langle \cdot, \cdot \rangle$ is right-invariant, hence bi-invariant;
2. $\langle \cdot, \cdot \rangle$ is $\text{Ad}(G)$ -invariant;
3. the inversion map $g \rightarrow g^{-1}$ is an isometry of G ;
4. $\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0$ for all $X, Y, Z \in \mathfrak{g}$;
5. $\nabla_X Y = \frac{1}{2}[X, Y]$ for all $X, Y \in \mathfrak{g}$, where ∇ denotes the Levi Civita connection;
6. the geodesics of G starting at the identity element e are the one-parameter subgroups of G .

By (3) the pair $(G, \langle \cdot, \cdot \rangle)$ is a pseudo-Riemannian symmetric space. Furthermore by computing the curvature tensor one has

$$R(X, Y) = -\frac{1}{4}\text{ad}([X, Y]) \quad \text{for } X, Y \in \mathfrak{g}. \tag{6}$$

Thus the Ricci tensor $\text{Ric}(X, Y) = \text{tr}(Z \rightarrow R(Z, X)Y)$ is given by

$$\text{Ric}(X, Y) = -\frac{1}{4}B(X, Y)$$

where B denotes the Killing form on \mathfrak{g} given by $B(X, Y) = \text{tr}(\text{ad}(X) \circ \text{ad}(Y))$ for all $X, Y \in \mathfrak{g}$, and tr denotes the usual trace.

Consider the Lie group homomorphism $\rho : \mathbb{R} \rightarrow \text{Aut}(\mathbb{H}_3(\mathbb{R}))$ which on vectors $(v, z) \in \mathbb{R}^2 \oplus \mathbb{R}$ has the form

$$\rho(t) = \begin{pmatrix} R(t) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{where } R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{7}$$

Let G denote the simply connected Lie group which is modeled on the smooth manifold \mathbb{R}^4 , where the algebraic structure is the resulting from the semidirect product of \mathbb{R} and $\mathbb{H}_3(\mathbb{R})$, via ρ . Thus the multiplication is given by

$$(t, v, z) \cdot (t', v', z') = \left(t + t', v + R(t)v', z + z' + \frac{1}{2}v^t J R(t)v' \right) \tag{8}$$

with J and $R(t)$ as above. The Lie group G is known as the *oscillator group*.

A basis of left-invariant vector fields at a point $p = (t, x, y, z)$ is given by

$$\begin{aligned} X_0(p) &= \partial_t|_p \\ X_1(p) &= \cos t \partial_x|_p + \sin t \partial_y|_p + \frac{1}{2}(x \sin t - y \cos t) \partial_z|_p \\ X_2(p) &= -\sin t \partial_x|_p + \cos t \partial_y|_p + \frac{1}{2}(x \cos t + y \sin t) \partial_z|_p \\ X_3(p) &= \partial_z|_p. \end{aligned}$$

These vector fields verify the Lie bracket relations:

$$[X_0, X_1] = X_2 \quad [X_0, X_2] = -X_1 \quad [X_1, X_2] = X_3 \tag{9}$$

giving rise to the Lie algebra of G , namely \mathfrak{g} . On the usual basis of $T_p G$, $\{\partial_t|_p, \partial_x|_p, \partial_y|_p, \partial_z|_p\}$ the matrix:

$$\begin{pmatrix} 0 & \frac{1}{2}y & -\frac{1}{2}x & 1 \\ \frac{1}{2}y & 1 & 0 & 0 \\ -\frac{1}{2}x & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \tag{10}$$

defines a bi-invariant metric on G . On canonical coordinates of \mathbb{R}^4 it corresponds to the pseudo-Riemannian metric:

$$g = dz dt + dx^2 + dy^2 + \frac{1}{2}(ydx dt - xdy dt),$$

which coincides with the metric g (1).

Proposition 3.1. *The Lorentzian manifold (\mathbb{R}^4, g) for g the Lorentzian metric in (1) admits simple and transitive actions of both Lie groups N and G .*

As a consequence (N, g) is isometric to (G, g) .

In fact one can see that starting at $(0, 0, 0, 0) \in \mathbb{R}^4$ the translation on the left (by N or G) gives the same Lorentzian metric at every point. See [13].

Remark 2. While the metric g is left and right-invariant on G , the metric g is only left-invariant on N . In particular (G, g) and (N, g) are symmetric spaces: geodesics through the identity are one-parameter subgroups.

Remark 3. The Lie group G is the isometry group of a left-invariant Lorentzian metric on the Heisenberg Lie group $\mathbb{H}_3(\mathbb{R})$ (see [19,20]).

3.1. Isometries

Let G be a connected Lie group with a bi-invariant metric, and let $\text{Iso}(G)$ denote the isometry group of G . This is a Lie group when endowed with the compact-open topology. Let φ be an isometry such that $\varphi(e) = x$, for $x \neq e$. Then $L_{x^{-1}} \circ \varphi$ is an isometry which fixes the element $e \in G$. Therefore $\varphi = L_x \circ f$ where f is an isometry such that $f(e) = e$. Let $F(G)$ denote the isotropy subgroup of the identity e of G and let $L(G) := \{L_g : g \in G\}$, where L_g is the translation on the left by $g \in G$. Then $F(G)$ is a closed subgroup of $\text{Iso}(G)$ and

$$\text{Iso}(G) = L(G)F(G) = \{L_g \circ f : f \in F(G), g \in G\}. \tag{11}$$

Thus $\text{Iso}(G)$ is essentially determined by $F(G)$.

The bi-invariance of the metric on G implies that it is a symmetric space. For locally symmetric spaces one has the Ambrose–Hicks–Cartan theorem (see for example [18, Theorem 17, Chapter 8]), which states that on a complete locally

symmetric pseudo-Riemannian manifold M , a linear isomorphism $A : T_pM \rightarrow T_pM$ is the differential of some isometry of M that fixes the point p if and only if it preserves the scalar product that the metric induces into the tangent space and if for every $u, v, w \in T_pM$ the following equation holds:

$$R(Au, Av)Aw = AR(u, v)w.$$

By applying this to the Lie group G equipped with a bi-invariant metric and whose curvature formula was given in (6) one gets the next result (see also [21]).

Lemma 3.2. *Let G be a simply connected Lie group with a bi-invariant pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$. Then a linear isomorphism $A : \mathfrak{g} \rightarrow \mathfrak{g}$ is the differential of some isometry in $F(G)$ if and only if for all $X, Y, Z \in \mathfrak{g}$, the linear map A satisfies the following two conditions:*

- (i) $\langle AX, AY \rangle = \langle X, Y \rangle$;
- (ii) $A[[X, Y], Z] = [[AX, AY], AZ]$.

Whenever G is simply connected, every local isometry of G extends to a unique global one. Therefore the full group of isometries of G fixing the identity is isomorphic to the group of linear isometries of \mathfrak{g} that satisfy the conditions of Lemma 3.2. By applying this to our case, one gets the next result (see [19]).

Theorem 3.3. *Let G be the simply connected solvable Lie group of dimension four $\mathbb{R} \times_{\rho} H_3(\mathbb{R})$ endowed with the bi-invariant metric g . Then the group of isometries fixing the identity element $F(G)$ is isomorphic to $(\{1, -1\} \times O(2)) \times \mathbb{R}^2$.*

In particular the connected component of the identity of $F(G)$ coincides with the group of inner automorphisms $\{\chi_g : G \rightarrow G, \chi_g(x) = gxg^{-1}\}_{g \in G}$.

The computations (see [19]) show that the differential of an isometry fixing the identity element corresponds to $A : \mathfrak{g} \rightarrow \mathfrak{g}$ having the following matricial presentation on the basis of left-invariant vector fields $\{X_0, X_1, X_2, X_3\}$

$$A = \begin{pmatrix} \pm 1 & 0 & 0 \\ w & \tilde{A} & 0 \\ \mp \frac{1}{2} \|w\|^2 & \mp w^\tau \tilde{A} & \pm 1 \end{pmatrix} \tag{12}$$

where $w \in \mathbb{R}^2$ and $\tilde{A} \in O(2)$. This gives a group isomorphic to $(\{1, -1\} \times O(2)) \times \mathbb{R}^2$ for which the identity component corresponds to those matrices of the form (12) with $a_{00} = a_{33} = 1$ and $\tilde{A} \in SO(2) = \{R(t) : t \in \mathbb{R}\}$.

On the other hand, the set of orthogonal automorphisms of \mathfrak{g} coincide with the set $Ad(G)$, that is, the matrices of the form

$$A(t, v) = \begin{pmatrix} 1 & 0 & 0 \\ Jv & R(t) & 0 \\ -\frac{1}{2} \|v\|^2 & -(Jv)^\tau R(t) & 1 \end{pmatrix}, \quad v \in \mathbb{R}^2$$

being $A(t, v) = Ad(t, v, z)$ for $v = (x, y)$. Since both subgroups are connected and have the same dimension, they must coincide.

Remark 4. In [13] more features about the isometry group of (G, g) were studied. It was proved that $N = \mathbb{R} \times H_3(\mathbb{R})$ occurs as a subgroup of $Iso(G)$ but it is not contained in the nilradical of $Iso(G)$. Furthermore the action of the nilradical on G is not transitive. This shows important differences between the Riemannian situation and the Lorentzian case, even for 2-step nilpotent Lie groups.

Now we proceed to write explicitly the isometries on G . Since $F(G)$ has four connected components, our aim is to find a representative isometry on each of them.

From Theorem 3.3, the connected component of the identity

$$F_0(G) = \{\chi_g : g \in G\} \simeq (\{1\} \times SO(2)) \times \mathbb{R}^2;$$

where if $g = (t_0, v_0, z_0)$, with $v_0 = (x_0, y_0)$, then for $v = (x, y)$

$$\chi_g(t, v, z) = \left(t, v_0 + R(t_0)v - R(t)v_0, z + \frac{1}{2} v_0^\tau J R(t_0)v - \frac{1}{2} v_0^\tau J R(t)v_0 - \frac{1}{2} (R(t_0)v)^\tau J R(t)v_0 \right). \tag{13}$$

Consider the semidirect product $G \times G$ given by conjugation: $g \cdot h = \chi_g(h)$ as above. Then $G \times G$ acts by isometries on the pseudo-Riemannian manifold G , the first factor acts by conjugation $\chi : G \rightarrow F_0(G)$ and the second one by translations on the left $L : G \rightarrow L(G)$, however this action is not effective. Since

$$\chi_g \circ L_h \circ \chi_{g^{-1}} = L_{\chi_g(h)} \tag{*}$$

the action induces the group homomorphism:

$$G \times G \rightarrow Iso(G) \quad (h, g) \mapsto L_g \circ \chi_h.$$

The homomorphism $\chi : G \rightarrow F_0(G)$ has the center of G as kernel

$$Z(G) = \{g \in G : gxg^{-1} = x \text{ for all } x \in G\}$$

and one gets

$$F_0(G) \simeq G/Z(G) \simeq SO(2) \times \mathbb{R}^2. \tag{14}$$

It is not hard to see that the center of G is the subgroup generated by the element of $(0, 0, 0, 1)$. On the other hand the subgroup $L(G)$ is normal in $ISO_0(G)$ and the group homomorphism $L : G \rightarrow L(G)$ has a trivial kernel.

Thus the connected component of the identity (isometry) is $ISO_0(G) = (SO(2) \times \mathbb{R}^2) \times G$.

Let $f_1, f_2, f_3 : G \rightarrow G$ denote the following diffeomorphisms:

$$f_1(t, v, z) = (-t, Sv, -z), \quad \text{where } S(x, y) = (-x, y) \tag{15}$$

$$f_2(t, v, z) = (-t, R(t)v, -z), \tag{16}$$

$$f_3(t, v, z) = f_1 \circ f_2(t, v, z) = (t, R(t)Sv, z). \tag{17}$$

Usual computations show that f_i is an isometry for $i = 1, 2, 3$ and they belong to different connected components of the isometry group. Thus the other three components of $F(G)$ are

$$F_0(G) \cdot f_1, \quad F_0(G) \cdot f_2 \quad \text{and} \quad F_0(G) \cdot f_3$$

where $F_0 \cdot f_i = \{gf_i : g \in F_0(G)\}$.

3.2. Geodesics

From (10) one can compute the Christoffel symbols of the Levi-Civita connection (cf. [18]) and therefore a curve $\alpha(s) = (t(s), x(s), y(s), z(s))$ is a geodesic on G if its components satisfy the second order system of differential equations:

$$\begin{cases} t''(s) = 0, \\ x''(s) = -t'(s)y'(s), \\ y''(s) = t'(s)x'(s), \\ z''(s) = \frac{1}{2} t'(s)(x(s)x'(s) + y(s)y'(s)). \end{cases}$$

On the other hand, if $X_e = \sum_{i=0}^3 a_i X_i(e) \in T_e G$, then the geodesic α through e with initial condition $\alpha'(0) = X_e$ is the integral curve of the left-invariant vector field $X = \sum_{i=0}^3 a_i X_i$. Then we should have $\alpha'(s) = X_{\alpha(s)}$.

• If $a_0 \neq 0$ the components of α must verify the following system

$$\begin{aligned} t'(s) &= a_0, \\ x'(s) &= a_1 \cos a_0 s - a_2 \sin a_0 s, \\ y'(s) &= a_1 \sin a_0 s + a_2 \cos a_0 s, \\ z'(s) &= \frac{1}{2} \left[\frac{a_1^2}{a_0} + \frac{a_2^2}{a_0} + 2a_3 - \left(\frac{a_2^2}{a_0} + \frac{a_1^2}{a_0} \right) \cos a_0 s \right] \end{aligned}$$

and so the geodesic through $e = (0, 0, 0, 0)$ with initial condition X_e satisfies:

$$\begin{aligned} t(s) &= a_0 s, \\ x(s) &= \frac{a_1}{a_0} \sin a_0 s + \frac{a_2}{a_0} \cos a_0 s - \frac{a_2}{a_0}, \\ y(s) &= -\frac{a_1}{a_0} \cos a_0 s + \frac{a_2}{a_0} \sin a_0 s + \frac{a_1}{a_0}, \\ z(s) &= \frac{1}{2} \left[\left(\frac{a_1^2}{a_0} + \frac{a_2^2}{a_0} + 2a_3 \right) s - \left(\frac{a_2^2}{a_0} + \frac{a_1^2}{a_0} \right) \sin a_0 s \right]. \end{aligned}$$

If $a_0 = 0$, it is easy to see that $\alpha(s) = (0, a_1 s, a_2 s, a_3 s)$ is the corresponding geodesic.

Therefore the exponential map $\exp : \mathfrak{g} \rightarrow G$ is

$$\exp(X) = \left(a_0, \frac{1}{a_0} (R_0(a_0)J - J)(a_1, a_2)^T, a_3 + \frac{1}{2} \left(\frac{a_1^2}{a_0} + \frac{a_2^2}{a_0} \right) \left(1 - \frac{\sin a_0}{a_0} \right) \right)$$

for $a_0 \neq 0$, while if $a_0 = 0$,

$$\exp(X) = (0, a_1, a_2, a_3).$$

The geodesic passing through the point $h \in G$, is the translation on the left by h of the one-parameter subgroup at e , that is $\gamma(s) = h \exp(sX)$ for $\exp(sX)$ given above.

4. Lorentzian compact manifolds

Let K denote a closed subgroup of G so that G/K is a differentiable manifold endowed with a G -invariant metric, that is, a metric such that the transformations $\tau_h : G/K \rightarrow G/K$ given by $\tau_h(xK) = hxK$ are isometries for all $h \in G$ and such that the natural projection $p : G \rightarrow G/K$ is a pseudo-Riemannian submersion. Thus

$$\tilde{L}(G/K) = \{\tau_h : h \in G\}$$

is a subgroup of the isometry group $\text{Iso}(G/K)$ of the quotient space.

If $f \in \text{Iso}(G)$ is an isometry of G we say that f is fiber preserving if $f(gK) = f(g)K$ for every $g \in G$. If f is a fiber preserving isometry of G , it induces an isometry \tilde{f} of G/K defined by $\tilde{f}(gK) = f(p(g))$. Observe that left-translations in G are fiber preserving and they induce the isometries τ_h in G/K .

Example 4.1. Let $\Gamma < G$ be a lattice of a Lie group (G, g) which is equipped with a bi-invariant metric. Then the metric g of G is induced to both quotients $(G/\Gamma, g)$ and $(\Gamma \backslash G, g)$ (by abuse we name the induced metrics also by g). Since the inversion map: $G \rightarrow G$ which sends $h \rightarrow h^{-1}$ is an isometry of G , one induces this map to the quotients: $x\Gamma \rightarrow \Gamma x^{-1}$ and one gets that G/Γ and $\Gamma \backslash G$ are isometric compact spaces. This isometry enables the computation of the geometry without distinguishing these spaces. Furthermore G acts by isometries on G/Γ on the left via the maps τ_h (as before); G acts isometrically on $\Gamma \backslash G$ on the right $h \cdot \Gamma x = \Gamma xh^{-1}$.

Lemma 4.2. Let G be a Lie group with a bi-invariant metric and let Γ be a lattice of G . Then G/Γ admits a G -invariant metric making it a naturally reductive pseudo-Riemannian space and consequently:

1. $p : G \rightarrow G/\Gamma$ is a pseudo-Riemannian covering;
2. The geodesics in G/Γ starting at the point $o = p(e)$ are of the form $p(\exp tX)$ with $X \in \mathfrak{g}$.

See [22, Chapter X, vol. 2], [18].

We can study the isometry group of G/Γ once one has information about the isometry group of G , $\text{Iso}(G)$ as follows.

Theorem 4.3. Let G be an arcwise-connected, simply connected Lie group with a left-invariant metric and Γ a discrete subgroup of G . Then every isometry f of G/Γ is induced to G/Γ by a fiber preserving isometry of G .

Proof. Let $f \in \text{Iso}(G/\Gamma)$ and consider $f \circ p : G \rightarrow G/\Gamma$. Since G is simply connected, from the Lifting Theorem (cf. [23, Chapter III, Theorem 4.1]), there exists a differentiable map $\phi : G \rightarrow G$ such that

$$p \circ \phi = f \circ p. \tag{18}$$

From the construction of ϕ it is not difficult to see that ϕ is a diffeomorphism of G if f is a diffeomorphism of G/Γ . Since the projection $p : G \rightarrow G/\Gamma$ is a pseudo-Riemannian covering map one gets that ϕ is a local isometry and therefore an isometry. From (18) it is immediate that ϕ is fiber preserving and f is induced by ϕ . \square

Recall that the Lie algebra of the isometry group is obtained from the Killing vector fields. The next lemma states a relationship between the Killing vector fields on G and those on G/Γ , for a lattice $\Gamma < G$.

Lemma 4.4. Let G be a Lie group with a left-invariant metric and Γ a discrete closed subgroup of G . Let X be a Killing vector field in G/Γ with monparametric subgroup $\{\Psi_t\}$. Then the horizontal lift \bar{X} to G of X (with respect to the pseudo-Riemannian submersion $p : G \rightarrow G/\Gamma$) is a Killing vector field on G whose monparametric subgroup $\{\varphi_t\}$ verifies

$$\Psi_t \circ p = p \circ \varphi_t.$$

Proof. Let $\mathfrak{iso}(G/\Gamma)$ and $\mathfrak{iso}(G)$ denote the Lie algebras of the isometry groups of G/Γ and G respectively. Since G and G/Γ are complete, the Lie algebras $\mathfrak{iso}(G/\Gamma)$ and $\mathfrak{iso}(G)$ can be identified with the corresponding Lie algebras of Killing vector fields. Therefore, if Ψ belongs to $\text{Iso}_0(G/\Gamma)$ there exist Killing fields X_1, \dots, X_n in G/Γ with monparametric subgroups $\{\Psi_t^i\}$ such that

$$\Psi = \Psi_1^1 \circ \dots \circ \Psi_1^n.$$

Let \bar{X}_i be the horizontal lift to G of X_i (with respect to the pseudo-Riemannian submersion $p : G \rightarrow G/\Gamma$), $i = 1, \dots, n$, and let $\{\varphi_t^i\}$ be the associated monparametric subgroups. Let $f = \varphi_1^1 \circ \dots \circ \varphi_1^n \in \text{Iso}_0(G)$.

Fix $q \in G/\Gamma$ and let σ_n be a local section of $p : G \rightarrow G/\Gamma$ defined on a neighborhood of q and for each $i = 1, \dots, n - 1$, let σ_i be a local section around $q_i = \Psi_1^{i+1} \circ \dots \circ \Psi_1^n(q)$, mapping q_i into $\varphi_1^{i+1} \circ \dots \circ \varphi_1^n(\sigma_n(q))$. Then, we must have

$$\Psi = p\varphi_1^1\sigma_1 \circ \dots \circ p\varphi_1^n\sigma_n = p \circ f \circ \sigma_n.$$

This decomposition is independent of the choice of the local section and in fact,

$$\Psi \circ p = p \circ f. \quad \square$$

Remark 5. By the previous lemma any isometry in $\text{Iso}_0(G/\Gamma)$ is induced to the quotient by an isometry in $\text{Iso}_0(G)$.

We concentrate our attention now to the solvable Lie group G equipped with the bi-invariant metric g given in (10). We shall construct compact manifolds and study their geometry. Consider the following lattices of G .

Set Γ_k the lattice of the Heisenberg Lie group $H_3(\mathbb{R})$ given by

$$\Gamma_k = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2k}\mathbb{Z} \quad k \in \mathbb{N}.$$

Every lattice Γ_k is invariant under the subgroups generated by $\rho(2\pi)$, $\rho(\pi)$ and $\rho(\frac{\pi}{2})$, $\rho : \mathbb{R} \rightarrow \text{Aut}(H_3(\mathbb{R}))$ as in (7). Consequently we have three families of lattices in $G = \mathbb{R} \times_{\rho} H_3(\mathbb{R})$:

$$\begin{aligned} \Lambda_{k,0} &= 2\pi\mathbb{Z} \times \Gamma_k < G \\ \Lambda_{k,\pi} &= \pi\mathbb{Z} \times \Gamma_k < G \\ \Lambda_{k,\pi/2} &= \frac{\pi}{2}\mathbb{Z} \times \Gamma_k < G, \end{aligned} \tag{19}$$

so that $\Lambda_{k,0} \triangleleft \Lambda_{k,\pi} \triangleleft \Lambda_{k,\pi/2}$, which induce the solvmanifolds

$$\begin{aligned} M_{k,0} &= \Lambda_{k,0} \backslash G \simeq G / \Lambda_{k,0}, \\ M_{k,\pi} &= \Lambda_{k,\pi} \backslash G \simeq G / \Lambda_{k,\pi}, \\ M_{k,\pi/2} &= \Lambda_{k,\pi/2} \backslash G \simeq G / \Lambda_{k,\pi/2}. \end{aligned} \tag{20}$$

Since the subgroups $\Lambda_{k,i}$ are not pairwise isomorphic (see for instance [24]), they determine non-diffeomorphic solvmanifolds (see for instance [25]).

Observe that the action of $\rho(2\pi)$ is trivial, so

- $\Lambda_{k,0} = 2\pi\mathbb{Z} \times \Gamma_k$ (a direct product) and
- $M_{k,0} = G / \Lambda_{k,0}$ is diffeomorphic to $\Lambda_{k,0} \backslash G \simeq \Lambda_k \backslash N \simeq S^1 \times H_3(\mathbb{R}) / \Gamma_k$, a Kodaira–Thurston manifold (see more details in [24]).

Moreover every compact space in the family $M_{k,0}$ admits a symplectic but non-Kähler structure, but any compact space $M_{k,i}$ $i = \pi, \pi/2$ admits no symplectic structure since the second Betty number vanishes (see [24]).

Proposition 4.5. *The compact solvmanifolds $M_{k,i}$ for $k \in \mathbb{N}$ and $i = 0, \pi, \pi/2$ are pseudo-Riemannian naturally reductive spaces, hence complete.*

The solvable Lie group $G = \mathbb{R} \times_{\rho} H_3(\mathbb{R})$ acts by isometries on each of the compact spaces $M_{k,i}$ for $k \in \mathbb{N}$ and $i = 0, \pi, \pi/2$. As a consequence the Heisenberg Lie group $H_3(\mathbb{R}) < G$ also acts on each of the compact spaces $M_{k,i}$ for $k \in \mathbb{N}$ and $i = 0, \pi, \pi/2$.

Both actions are locally faithful.

Remark 6. The action of $H_3(\mathbb{R})$ on $\Lambda_k \backslash N$ of Proposition 2.1 is induced by the right action of G on $M_{k,0} \simeq \Lambda_{k,0} \backslash G \simeq \Lambda_k \backslash N$:

$$(v', z') \cdot \Lambda_{k,0}(t, v, z) = \Lambda_{k,0}((t, v, z)(0, v', z')^{-1})$$

where on the right side we are considering the multiplication map of G . Since the metric is bi-invariant the right-translation is also an isometry.

4.1. Isometries of the compact spaces $M_{k,s}$

Our goal now is to study the isometry groups of the compact spaces $M_{k,s}$.

Notice that all translations on the left L_h for $h \in G$ are fiber preserving isometries. Direct computations show that the only isometries in $F(G)$ that are fiber preserving are the inner homomorphisms χ_h with $h \in \mathcal{N}_G(\Lambda_{k,s})$, the normalizer of $\Lambda_{k,s}$ in G .

Lemma 4.6. *Consider the lattices $\Lambda_{k,s}$ defined in (19), and set $M_{k,s} = G / \Lambda_{k,s}$ for every $k \in \mathbb{N}$.*

- *The only isometries in $F(G)$ that are fiber preserving are the inner homomorphisms χ_h with $h \in \mathcal{N}_G(\Lambda_{k,s})$.*
- *The normalizers in G of these lattices are given by*
 1. $\mathcal{N}_G(\Lambda_{k,0}) = \frac{\pi}{2}\mathbb{Z} \times (\frac{1}{2k}\mathbb{Z} \times \frac{1}{2k}\mathbb{Z} \times \mathbb{R})$,
 2. $\mathcal{N}_G(\Lambda_{k,\pi}) = \frac{\pi}{2}\mathbb{Z} \times (\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \mathbb{R})$,
 3. *Set $\mathcal{W} = \{(m, n) \in \mathbb{Z}^2 : m \equiv n \pmod{2}\}$ then*

$$\mathcal{N}_G\left(\Lambda_{k,\frac{\pi}{2}}\right) = \begin{cases} \frac{\pi}{2}\mathbb{Z} \times (\mathcal{W} \times \mathbb{R}) & \text{for } k = 1, \\ \frac{\pi}{2}\mathbb{Z} \times \left(\frac{1}{2}\mathcal{W} \times \mathbb{R}\right) & \text{for } k \geq 2. \end{cases}$$

Proof. Let $\Lambda_{k,0}$ be the lattice of G given in (19). Let $g = (t_0, v_0, z_0) \in G$ with $v_0 = (x_0, y_0) \in \mathbb{R}^2$, be an element in the normalizer of $\Lambda_{k,0}$. Let $\gamma = (t, v, z) \in \Lambda_{k,0}$ where $v = (x, y)$. Thus from the formulas in (13) the condition $\chi_h(\gamma) \in \Lambda_{k,0}$ gives

$$v_0 + R(t_0)v - R(t)v_0 \in \mathbb{Z} \times \mathbb{Z} \quad (21)$$

$$z + \frac{1}{2}v_0^t J R(t_0)v - \frac{1}{2}v_0^t J R(t)v_0 - \frac{1}{2}(v^t R(-t_0)J R(t)v_0) \in \frac{1}{2k}\mathbb{Z}. \quad (22)$$

Since $t \in 2\pi\mathbb{Z}$, we have $R(t) \equiv Id$, thus $R(t_0)v \in \mathbb{Z} \times \mathbb{Z}$ for $v \in \mathbb{Z} \times \mathbb{Z}$ which implies

$$t_0 = \frac{\pi}{2}r \quad \text{for some } r \in \mathbb{Z}. \quad (23)$$

Now using this in (22) one gets

$$v_0 \in \frac{1}{2k}\mathbb{Z} \times \frac{1}{2k}\mathbb{Z}. \quad (24)$$

Canonical computations show that $g = (\frac{\pi}{2}r, \frac{1}{2k}p, \frac{1}{2k}q, s) \in \mathcal{N}_G(\Lambda_{k,0})$ for all $r, p, q \in \mathbb{Z}$ and $s \in \mathbb{R}$.

For $\Lambda_{k,\pi}$ an element $h = (t_0, v_0, z_0) \in G$ which belongs to $\mathcal{N}_G(\Lambda_{k,\pi})$ must satisfy Eqs. (21) and (22). Observe that elements of the form $\gamma = (2\pi s, m, n, \frac{1}{2k}z) \in \Lambda_{k,\pi}$. Therefore h must satisfy the conditions above (23) and (24).

For $t = \pi s$ with $s \equiv 1 \pmod{2}$ the condition (21) implies that $v_0 \in \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. Finally usual computations give $\mathcal{N}_G(\Lambda_{k,\pi}) = \frac{\pi}{2}\mathbb{Z} \times (\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \mathbb{R})$.

For the lattice $\Lambda_{k,\frac{\pi}{2}}$ notice that we can use conditions obtained for the other two families of lattices. Thus assume that $h \in \mathcal{N}_G(\Lambda_{k,\frac{\pi}{2}})$ has the form $g = (\frac{\pi}{2}r, \frac{1}{2}p, \frac{1}{2}q, z_0)$ for $r, p, q \in \mathbb{Z}, z_0 \in \mathbb{R}$. Thus we should analyze Eqs. (21) and (22) for $t \in \pm\frac{\pi}{2} + 2\pi\mathbb{Z}$.

Condition (21) implies $p \equiv q \pmod{2}$. Imposing this together with condition (22) accounts to $v_0 \in \frac{1}{2}(\mathbb{Z} \times \mathbb{Z})$ for $k \geq 2$ or $v_0 \in \mathbb{Z} \times \mathbb{Z}$ for $k = 1$. \square

Once one knows which isometries of G are fiber preserving, to study the isometry group of $M_{k,i}$ one should determine, among others, which of these isometries act effectively on $M_{k,i}$ for $i = 0, \pi, \pi/2$.

Thus to determine the isometry group of the compact space $M_{k,s}$ we need to find the kernel of the following homomorphisms:

$$\tilde{\chi} : \mathcal{N}_G(\Lambda_{k,s}) \rightarrow \text{Iso}(M_{k,s}), \quad h \mapsto \tilde{\chi}_h$$

$$\tilde{\tau} : G \rightarrow \tilde{\text{L}}(M_{k,s}), \quad h \mapsto \tau_h,$$

where $\tilde{\chi}_h(g \Lambda_{k,s}) = \chi(g) \Lambda_{k,s}$ and $\tau_h(g \Lambda_{k,s}) = hg \Lambda_{k,s}$.

Set $\tilde{\text{F}}(M_{k,s}) := \text{Im}(\tilde{\chi})$ and $\tilde{\text{L}}(M_{k,s}) := \text{Im}(\tilde{\tau})$. Then, by the Isomorphism Theorem one obtains

$$\tilde{\text{F}}(M_{k,s}) \simeq \mathcal{N}_G(\Lambda_{k,s}) / \ker(\tilde{\chi}) \quad (25)$$

where $\ker(\tilde{\chi}) = \{h \in \mathcal{N}_G(\Lambda_{k,s}) : h = (2\pi l, 0, r) \text{ with } l \in \mathbb{Z}, r \in \mathbb{R}\}$ and $\mathcal{N}_G(\Lambda_{k,s})$ as in Lemma 4.6; and

$$\tilde{\text{L}}(M_{k,s}) \simeq G / \ker(\tilde{\tau}) \quad (26)$$

where $\ker(\tilde{\tau}) = \{h \in G : h = (2\pi l, 0, z) \text{ with } l \in \mathbb{Z}, z \in \frac{1}{2k}\mathbb{Z}\}$.

Theorem 4.7. Let $M_{k,s}$ denote the solvmanifolds of dimension four as in (20) equipped with the naturally reductive metric induced by the bi-invariant metric of G given by g (1). Then the isometry group of $M_{k,s}$ is given by

$$\text{Iso}(M_{k,s}) = \tilde{\text{F}}(M_{k,i}) \cdot \tilde{\text{L}}(M_{k,s})$$

where $\tilde{\text{F}}(M_{k,i})$ is the group in (25) and $\tilde{\text{L}}(M_{k,s})$ is the group in (26).

Moreover

- $\tilde{\text{L}}(M_{k,s})$ is a normal subgroup and
- $\tilde{\text{N}}(M_{k,s}) \cap \tilde{\text{L}}(M_{k,s}) = \{\tau_Z \circ \tilde{\chi}_\gamma, \text{ where } Z := (0, 0, 0, z) z \in \mathbb{R}, \gamma \in \Lambda_{k,s}\}$.

Remark 7. Notice that $\text{Iso}_0(M_{k,i})$ has G as universal covering.

Also note that $\mathbb{R} \times \text{H}_3(\mathbb{R})$ does not act by isometries on the quotients $M_{k,i}$ for any k, i .

Since the projection of the left-invariant vector field $X_0 - X_3$ to $M_{k,s}$ gives a timelike Killing vector field one gets the following fact.

Corollary 4.8. All of the compact spaces $M_{k,s}$ are stationary.

Remark 8. Theorem 4.1 in [8] states that when the identity component of the isometry group is non-compact and it has some timelike orbit, then it must contain a non-trivial factor locally isomorphic to $SL(2, \mathbb{R})$ or to an oscillator group.

4.2. Geodesics on $M_{k,s}$

Our aim here is to study the geodesics on the quotient spaces $M_{k,s} = G/\Lambda_{k,s}$ for $s = 0, \pi, \pi/2$. Since $M_{k,s}$ is a naturally reductive space the geodesics starting at $p(e)$ are precisely the projections of the geodesics of G through the identity element e (see Chapter 11 [18]). Any other geodesic of G is the translation on the left of a geodesic through e , giving rise to every geodesic on the quotient.

Let $\bar{\gamma}(t) = p \circ \gamma(t)$ denote a curve on $M_{k,s}$ with initial velocity

$$\bar{v} = \bar{\gamma}'(0) = dp_e(\gamma'(0)).$$

The tangent vector \bar{v} is called

- *lightlike or null* if it has null norm.
- *spacelike* if it has positive norm.
- *timelike* if it has negative norm.

The curve $\bar{\gamma}$ is called *lightlike* (resp. *spacelike*, *timelike*) if its tangent vector is lightlike (resp. spacelike, timelike) at every point.

Observe first that a tangent vector X of G of the form $X = \sum_{i=0}^3 a_i X_i$ for the left-invariant vector fields X_i , is null if it satisfies the condition:

$$a_1^2 + a_2^2 + 2a_0a_3 = 0, \tag{27}$$

while other tangent vectors on G satisfying $a_1^2 + a_2^2 + 2a_0a_3 > 0$ or < 0 are either spacelike or timelike respectively.

Let α denote a curve on G . Its projection will be denoted by $\bar{\alpha} = p \circ \alpha$. Observe that $\bar{\alpha}$ is self-intersecting if and only if there exist $t_0, t_1 \in \mathbb{R}$ such that $\alpha(t_1)^{-1}\alpha(t_0) \in \Lambda_{k,s}$.

Lemma 4.9. *Let G denote a Lie group, let $K < G$ be a subgroup of G and $\alpha : \mathbb{R} \rightarrow G$ a one-parameter subgroup. Denote by $p : G \rightarrow G/K$ the canonical projection. Then, either $p \circ \alpha : \mathbb{R} \rightarrow G/K$ is injective, or it is periodic.*

Proof. Assume that there exist $t_0, t_1 \in \mathbb{R}$ such that $\bar{\alpha}(t_0) = \bar{\alpha}(t_1)$. Thus $\alpha(t_1)^{-1}\alpha(t_0) \in K$. Since α is a one-parameter subgroup it holds $\alpha(t_0 - t_1) \in K$. Set $T = t_1 - t_0$ then $\alpha(s + T) = \alpha(s)\alpha(T)$ and so $\bar{\alpha}(t + T) = \bar{\alpha}(t)$ for all $t \in \mathbb{R}$. \square

Corollary 4.10. *Let G/K be a naturally reductive pseudo-Riemannian space. Then every self-intersecting geodesic in G/K is periodic.*

The next step is to apply this result to study periodic geodesics on the quotient spaces $M_{k,s}$, $s = 0, \pi, \pi/2$. Geodesics on $M_{k,s}$ are induced by one-parameter subgroups of G since the metric of G is bi-invariant.

Indeed a geodesic α on G through e with tangent vector $X = \sum_{i=0}^3 a_i X_i$ gives rise to a closed geodesic on $M_{k,0}$ if and only if there exists $T \in \mathbb{R}$ such that $\alpha(T) \in \Lambda_{k,0}$, which

- for $a_0 \neq 0$ gives the following conditions
 - $a_0 T \in 2\pi\mathbb{Z}$
 - $a_0^{-1}(R(a_0 T)J - J)(a_1, a_2)^t \in \mathbb{Z} \times \mathbb{Z}$
 - $\left(\frac{a_1^2 + a_2^2}{2a_0} + a_3\right)T - \frac{a_1^2 + a_2^2}{a_0^2} \sin(a_0 T) \in \frac{1}{2k}\mathbb{Z}$.

$$\tag{28}$$

Notice that if the first condition holds then $R(a_0 T)$ is the identity map so that $R(a_0 T)J - J = 0$ and the second condition is satisfied for all $a_1, a_2 \in \mathbb{R}$. Since $a_0 T \in 2\pi\mathbb{Z}$ then $\sin(a_0 T) = 0$ and the third condition reduces to

$$\frac{\|X\|^2}{2a_0} T = \left(\frac{a_1^2 + a_2^2}{2a_0} + a_3\right) T \in \frac{1}{2k}\mathbb{Z}. \tag{29}$$

Hence if $a_0 \neq 0$ the condition of $p \circ \alpha$ being closed on $M_{k,0}$ reduces to (29).

For spacelike or timelike geodesics, that is $\|X\|^2 > 0$ or $\|X\|^2 < 0$ respectively, where $\|X\|^2 = \langle X, X \rangle$ closed geodesics on $M_{k,0}$ are determined by the conditions

$$a_0 T = 2\pi l \quad \text{and} \quad \frac{\|X\|^2}{2a_0} T = \frac{m}{2k} \quad \text{for } m, l \in \mathbb{Z}.$$

- For $a_0 = 0$ notice the geodesic $\bar{\alpha}$ is closed if there exists $T \in \mathbb{R}$ such that

$$\begin{aligned} (a_1 T, a_2 T)^t &\in \mathbb{Z} \times \mathbb{Z} \\ a_3 T &\in \frac{1}{2k}\mathbb{Z}. \end{aligned} \tag{30}$$

Thus on G a null geodesic is $\alpha(v) = (0, 0, 0, a_3 v)$ which gives rise to a periodic geodesic on $M_{k,s}$ if and only if $a_3 T \in \frac{1}{2k}\mathbb{Z}$. Therefore

- every lightlike geodesic on $M_{k,0}$ is periodic.
- there are periodic and injective timelike and spacelike geodesics on $M_{k,0}$.

Theorem 4.11. Let $M_{k,i}$ denote the solvmanifolds as in (20).

- Every null geodesic is periodic on $M_{k,i}$ for $i = 0, \pi, \pi/2$.
- There are periodic and injective timelike and spacelike geodesics on $M_{k,i}$ for $i = 0, \pi, \pi/2$.

For the other families of lattices $\Lambda_{k,\pi}$ and $\Lambda_{k,\pi/2}$ one should modify the equations in (28) and (30) to get the condition for $\bar{\alpha}$ to be periodic. Analogous arguments prove all the assertions of the theorem. One should notice that the analysis in these cases gives some extra geodesics once $a_0T = \pi m$ or $a_0T = \frac{\pi m}{2}$ for some $m \in \mathbb{Z}$.

Remark 9. Every compact manifold $M_{k,i}$ is even-dimensional and orientable. Compare with Theorem 2 in [26].

The Ricci tensor on G verifies

$$\text{Ric}(X, X) = \frac{1}{2}a_0^2 \geq 0 \quad \text{for } X = a_0\partial_t + V, \quad V \in \text{span}\{\partial_z, \partial_x, \partial_y\}$$

and since p is a local isometry, G so as their quotients satisfy the lightlike and timelike convergence conditions.

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