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Monadic MV-algebras II: Monadic implicational subreducts

Cecilia R. Cimadamore and J. Patricio Díaz Varela

Abstract. In this paper, we study the class of all monadic implicational subreducts, that is, the $\{\to, \forall, 1\}$-subreducts of the class of monadic MV-algebras. We prove that this class is an equational class, which we denote by $\mathcal{ML}$, and we give an equational basis for this variety. An algebra in $\mathcal{ML}$ is called a monadic Łukasiewicz implication algebra. We characterize the subdirectly irreducible members of $\mathcal{ML}$ and the congruences of every monadic Łukasiewicz implication algebra by monadic filters. We prove that $\mathcal{ML}$ is generated by its finite members. Finally, we completely describe the lattice of subvarieties, and we give an equational basis for each proper subvariety.

1. Introduction

Łukasiewicz implication algebras are the algebraic counterpart of the implicational fragment of Super-Łukasiewicz logic ([13], [12]). In fact, they are the class of all implicational subreducts, that is, the $\{\to, 1\}$-subreducts of MV-algebras ([10], [3]). They are also called C-algebras in [13] and Łukasiewicz residuation algebras in [2].

Monadic MV-algebras, MMV-algebras for short, were introduced and studied by Rutledge in [15] as an algebraic model for the monadic predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. He called these algebras monadic Chang algebras. Rutledge followed Halmos' study of monadic boolean algebras and represented each subdirectly irreducible MMV-algebra as a subalgebra of a functional MMV-algebra. From this representation, he proved the completeness of the monadic predicate calculus. As usual, a functional MMV-algebra is defined as follows. Let us consider the MV-algebra $\mathbf{V}^X$ of all functions from a nonempty set X to an MV-algebra $\mathbf{V}$, where the operations $\oplus$, $\neg$ and 0 are defined pointwise. If for $p \in V^X$, there exist the supremum and the infimum of the set $\{p(y) : y \in X\}$, then we define the constant functions $\exists_\vee(p)(x) = \sup\{p(y) : y \in X\}$ and $\forall_\wedge(p)(x) = \inf\{p(y) : y \in X\}$ for every $x \in X$. A *functional MMV-algebra* $\mathbf{A}'$ is an MMV-algebra whose MV-reduct is an MV-subalgebra of $\mathbf{V}^X$ and

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such that the existential and universal operators are the functions $\exists_\vee$ and $\forall_\wedge$, respectively. Observe that $\mathbf{A}'$ satisfies that

(1) if $p \in A'$, then the elements $\sup\{p(y) : y \in X\}$ and $\inf\{p(y) : y \in X\}$ exist in $\mathbf{V}$,
(2) if $p \in A'$, then the constant functions $\exists_\vee(p)$ and $\forall_\wedge(p)$ are in A'.

By a *functional representation* of an MMV-algebra $\mathbf{A}$ we simply mean a functional MMV-algebra $\mathbf{A}'$ such that $\mathbf{A}$ is isomorphic to $\mathbf{A}'$. When X is a finite set with k elements, we write $\mathbf{V}^k$ instead of $\mathbf{V}^X$.

In this paper, we study the class of all monadic implicational subreducts, that is, $\{\rightarrow, \forall, 1\}$-subreducts of monadic MV-algebras. One of the main purposes of this work is to demonstrate that the class of all monadic implicational subreducts of MMV-algebras is an equational class. We denote this class by $\mathcal{ML}$. Each algebra in $\mathcal{ML}$ is called a *monadic Łukasiewicz implication algebra*. From this, we have that there is a fundamental relationship between varieties of MMV-algebras and varieties of monadic Łukasiewicz implication algebras. Because of this relation, several results about varieties of MMV-algebras will be needed in this paper. In fact, this work can be considered as a continuation of [8], and it is the second of three. These three papers are part of the PhD. Thesis [6].

After a preliminary section, where we state the main results about Łukasiewicz implication algebras and MMV-algebras that we need for this paper, we prove in Section 3 that the class of all monadic implicational subreducts is a variety. We also give here the set of identities that characterize the variety. Next, we study general properties of the variety. We establish an order-isomorphism from the lattice of congruences of a monadic Łukasiewicz implication algebra $\mathbf{A}$ onto the lattice of monadic filters of $\mathbf{A}$ and, in addition, we prove that the lattice of all monadic filters of $\mathbf{A}$ is isomorphic to the lattice of all filters of the subalgebra $\forall\mathbf{A}$. From this, we characterize the subdirectly irreducible and the finite simple members of the variety.

In Section 4, we prove that $\mathcal{ML}$ is exactly the class of all monadic implicational subreducts of $\mathcal{MMV}$. As a first application of the relation between $\mathcal{MMV}$ and $\mathcal{ML}$, we demonstrate that the variety $\mathcal{ML}$ is generated by its subdirectly irreducible finite members, and also by the monadic implicational subreduct of the functional MMV-algebra $[\mathbf{0}, \mathbf{1}]^{\mathbb{N}}$, where we denote by $\mathbb{N}$ the set of positive integer numbers.

The last section is dedicated to the study of the lattice $\Lambda(\mathcal{ML})$ of subvarieties of $\mathcal{ML}$. First, we introduce the notion of width of an ML-algebra. We prove that if $\mathbf{A}$ is a subdirectly irreducible ML-algebra of width less than or equal to a finite positive integer k, then $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^k$, where the universal operator in $(\forall\mathbf{A})^k$ is defined as the constant function $\forall_\wedge$. The equational class of all ML-algebras of width k is generated by $[\mathbf{0}, \mathbf{1}]^k$, and the identity (α^k) characterizes the ML-algebras of width k. The main goal of this section is to prove that the width of a subdirectly irreducible monadic Łukasiewicz implication algebra **A** and the order of the Łukasiewicz implication subalgebra $\forall\mathbf{A}$ determine the subvariety generated by the algebra (Theorem 5.8). Next, we give the join-irreducible members of $\Lambda(\mathcal{ML})$, and we prove that each non-trivial proper subvariety of ML-algebras is the supremum of a finite number of join-irreducible subvarieties. From this, we describe $\Lambda(\mathcal{ML})$ completely. Moreover, we characterize each proper subvariety of $\Lambda(\mathcal{ML})$ by a single identity.

2. Preliminaries

MV-algebras were introduced by C. C. Chang in [3] as algebraic models for Łukasiewicz infinitely-valued logic. We refer the reader to [5].

An *MV-algebra* is an algebra $\mathbf{A} = \langle A; \oplus, \neg, 0\rangle$ of type $(2, 1, 0)$ satisfying the following identities:

(MV1)	$x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$,	(MV4)	$\neg\neg x \approx x$,
(MV2)	$x \oplus y \approx y \oplus x$,	(MV5)	$x \oplus \neg 0 \approx \neg 0$,
(MV3)	$x \oplus 0 \approx x$,	(MV6)	$\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

On each MV-algebra **A**, we define the constant 1 and the operations $\odot$ and $\rightarrow$ as follows: $1 := \neg 0$, $x \odot y := \neg(\neg x \oplus \neg y)$, and $x \rightarrow y := \neg x \oplus y$. For any two elements a and b of **A**, we define $a \leq b$ if and only if $a \rightarrow b = 1$. It follows that $\leq$ is a partial order, which is called the natural order of **A**. The natural order determines a lattice structure in A. Specifically, the join $a \vee b$ and the meet $a \wedge b$ of a and b are given by $a \vee b = (a \rightarrow b) \rightarrow b$ and $a \wedge b = a \odot (a \rightarrow b)$.

The real interval $[0, 1]$ enriched with the operations $a \oplus b = \min\{1, a + b\}$ and $\neg a = 1 - a$, is an MV-algebra denoted by $[\mathbf{0}, \mathbf{1}]$. Chang proved in [4] that this algebra generates the variety $\mathcal{MV}$ of MV-algebras. For every $n \in \mathbb{N}$, we denote by $\mathbf{S}_n = \langle S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}; \oplus, \neg, 0\rangle$ the finite MV-subalgebra of $[\mathbf{0}, \mathbf{1}]$ with $n + 1$ elements.

Mundici defined a functor Γ between MV-algebras and abelian ℓ-groups with strong unit, and proved that Γ is a categorical equivalence [14]. For every abelian ℓ-group $\mathbf{G}$, the functor Γ equips the unit interval $[0, u]$ with the operations $x \oplus y = u \wedge (x + y)$, $\neg x = u - x$ and $1 = u$. The resulting structure $\langle [0, u]; \oplus, \neg, 0\rangle$ is an MV-algebra. Set $\mathbf{S}_{n,\omega} = \Gamma(\mathbb{Z} \times \mathbb{Z}, (n, 0))$, where $\mathbb{Z}$ is the totally ordered additive group of integers and $\mathbb{Z}\times\mathbb{Z}$ is the lexicographic product of $\mathbb{Z}$ with itself. Let us observe that $\mathbf{S}_n$ is isomorphic to $\Gamma(\mathbb{Z}, n)$, and we write $\mathbf{S}_n \cong \Gamma(\mathbb{Z}, n)$.

Monadic MV-algebras (monadic Chang algebras in Rutledge's terminology) were introduced and studied by J. D. Rutledge in [15] as an algebraic model for the monadic predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. An algebra $\mathbf{A} = \langle A; \oplus, \neg, \exists, 0\rangle$ of type $(2, 1, 1, 0)$ is called a *monadic MV-algebra* (an MMV-algebra for short) if $\langle A; \oplus, \neg, 0\rangle$ is an MV-algebra and $\exists$ satisfies the following identities:

(MMV1) $x \leq \exists x$,
(MMV2) $\exists(x \vee y) \approx \exists x \vee \exists y$,
(MMV3) $\exists \neg \exists x \approx \neg \exists x$,
(MMV4) $\exists(\exists x \oplus \exists y) \approx \exists x \oplus \exists y$,
(MMV5) $\exists(x \odot x) \approx \exists x \odot \exists x$,
(MMV6) $\exists(x \oplus x) \approx \exists x \oplus \exists x$.

The variety of MMV-algebras is denoted by $\mathcal{MMV}$.

In each MMV-algebra $\mathbf{A}$, we define $\forall \colon A \to A$ by $\forall a = \neg \exists \neg a$, for every $a \in A$. Clearly, the following identities dual to (MMV1)–(MMV6) are satisfied:

(MMV7) $\forall x \leq x$,
(MMV8) $\forall(x \wedge y) \approx \forall x \wedge \forall y$,
(MMV9) $\forall \neg \forall x \approx \neg \forall x$,
(MMV10) $\forall(\forall x \odot \forall y) \approx \forall x \odot \forall y$,
(MMV11) $\forall(x \odot x) \approx \forall x \odot \forall x$,
(MMV12) $\forall(x \oplus x) \approx \forall x \oplus \forall x$.

For our purposes, it is more convenient to consider the operator $\forall$ instead of $\exists$. So, from now on, we consider an algebra $\mathbf{A} = \langle A; \oplus, \neg, \forall, 0\rangle$ as an MMV-algebra if $\forall$ satisfies the identities (MMV7)–(MMV12). We often write $\langle A; \forall\rangle$ for short.

The next lemma collects some basic properties of MMV-algebras.

Lemma 2.1. [15], [7] *Let* $\mathbf{A} \in \mathcal{MMV}$. *For every* $a, b \in A$ *the following properties hold:*

(MMV13) $\forall 0 = 0$,
(MMV14) $\forall 1 = 1$,
(MMV15) $\forall \forall a = \forall a$,
(MMV16) $\forall(\forall a \oplus \forall b) = \forall a \oplus \forall b$,
(MMV17) $\forall(\forall a \to \forall b) = \forall a \to \forall b$,
(MMV18) $\forall(a \to b) \leq \forall a \to \forall b$,
(MMV19) $\forall(a \vee \forall b) = \forall a \vee \forall b$.

Let us consider the set $\forall A = \{a \in A : a = \forall a\} = \{a \in A : a = \exists a\}$. From the last lemma, we have that $\forall \mathbf{A} = \langle \forall A; \oplus, \neg, 0\rangle$ is an MV-subalgebra of the MV-reduct of $\mathbf{A}$.

If $\mathbf{A}$ is a *finite* subdirectly irreducible MMV-algebra, then $\mathbf{A}$ is isomorphic to $(\forall \mathbf{A})^k$, for some positive integer k, where $\oplus$, $\neg$, and 0 are defined pointwise and $\forall_\wedge \colon (\forall A)^k \to (\forall A)^k$ is defined by

$$\forall_\wedge (\langle a_1, a_2, \dots, a_n\rangle) = \langle a_1 \wedge a_2 \wedge \cdots \wedge a_n, \dots, a_1 \wedge a_2 \wedge \cdots \wedge a_n\rangle.$$

Moreover, $\forall \mathbf{A}$ is isomorphic to the diagonal subalgebra of the product [9]. Let us observe that $\exists_\vee \colon (\forall A)^k \to (\forall A)^k$ is defined by

$$\exists_\vee (\langle a_1, a_2, \dots, a_n\rangle) = \langle a_1 \vee a_2 \vee \cdots \vee a_n, \dots, a_1 \vee a_2 \vee \cdots \vee a_n\rangle.$$

For each integer $n \geq 1$, let $\mathcal{K}_n$ be the class of MMV-algebras that satisfy the identity

$$x^n \approx x^{n+1}. \qquad (\delta_n)$$

It is easy to see that an MMV-algebra $\mathbf{A}$ satisfies (δ_n) if and only if $\mathbf{A}$ satisfies

$$x \xrightarrow{n} y \approx x \xrightarrow{n+1} y. \qquad (\epsilon_n)$$

The subvariety $\mathcal{K}_1$ is the variety of monadic boolean algebras, and it is clear that if $n \le l$ then $\mathcal{K}_n \subseteq \mathcal{K}_l$. If $\mathbf{A}$ is a finite subdirectly irreducible MMV-algebra in $\mathcal{K}_n$, then $\mathbf{A} \cong \mathbf{S}_m^k$, for some integer m such that $1 \le m \le n$ and some positive integer k [9]. Moreover, $\mathcal{K}_n = \mathcal{V}(\{\mathbf{S}_m^k : k \in \mathbb{N}, 1 \le m \le n\})$ and $\mathcal{MMV} = \mathcal{V}(\{\mathbf{S}_n^k : n, k \in \mathbb{N}\})$ [7].

Let X be an infinite set and $[\mathbf{0},\mathbf{1}]^X$ a functional MMV-algebra. Then

$$\mathcal{V}_{\mathcal{MMV}}\left([\mathbf{0},\mathbf{1}]^X\right) = \mathcal{V}_{\mathcal{MMV}}\left(\{[\mathbf{0},\mathbf{1}]^k : k \in \mathbb{N}\}\right).$$

In particular, $\mathcal{V}_{\mathcal{MMV}}(\mathbf{S}_n^{\mathbb{N}}) = \mathcal{V}_{\mathcal{MMV}}\left(\{\mathbf{S}_n^k : k \in \mathbb{N}\}\right)$. If we consider the functional MMV-algebras $\mathbf{S}_m^{\mathbb{N}}$, $1 \le m \le n$, then $\mathcal{K}_n = \mathcal{V}(\{\mathbf{S}_1^{\mathbb{N}}, \mathbf{S}_2^{\mathbb{N}}, \ldots, \mathbf{S}_n^{\mathbb{N}}\})$ [8].

The subvariety of $\mathcal{MMV}$ generated by the algebra $[\mathbf{0},\mathbf{1}]^k$ is characterized by the identity (α^k) (see [8]) where

$$x \approx \forall x, \tag{α^1}$$

and if $k \ge 2$, then

$$\bigvee_{1 \le i < j \le k+1} \left(\forall(x_i \vee x_j) \to \bigvee_{s=1}^{k+1} \forall x_s\right) \approx 1. \tag{α^k}$$

For each $\mathbf{A} \in \mathcal{MMV}$, we define the *width of* $\mathbf{A}$, which is denoted by $\operatorname{width}\mathbf{A}$, as the least integer k such that (α^k) holds in $\mathbf{A}$. If k does not exist, then we say that the width of $\mathbf{A}$ is infinite and we write $\operatorname{width}\mathbf{A} = \omega$. This definition is motivated by the following result.

Proposition 2.2. [8] *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra that satisfies* (α^k)*; then* $\mathbf{A}$ *is isomorphic to a subalgebra of* $(\forall\mathbf{A})^k$*.*

The lattice of subvarieties of the subvariety of MMV-algebras $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ generated by $[\mathbf{0},\mathbf{1}]^k$ is given in [8]. One of the most important results in this paper is that the subvariety generated by a subdirectly irreducible MMV-algebra $\mathbf{A} \in \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ depends on its width, the order and rank of $\forall\mathbf{A}$, and the partition associated to $\mathbf{A}$ of the set of antiatoms of the boolean subalgebra $\mathbf{B}(\mathbf{A})$ of its complemented elements.

Let $\mathcal{K}_n^k$ the subvariety of $\mathcal{K}_n$ generated by $\{\mathbf{S}_1^k, \mathbf{S}_2^k, \ldots, \mathbf{S}_n^k\}$. It is well known that the lattice of subvarieties of $\mathcal{K}_1$ is an $\omega + 1$-chain

$$\mathcal{K}_1^1 \subsetneq \mathcal{K}_1^2 \subsetneq \cdots \subsetneq \mathcal{K}_1^k \subsetneq \cdots \subsetneq \mathcal{K}_1.$$

More generally,

$$\mathcal{K}_n^1 \subsetneq \mathcal{K}_n^2 \subsetneq \cdots \subsetneq \mathcal{K}_n^k \subsetneq \cdots \subsetneq \mathcal{K}_n,$$

and $\mathcal{K}_n^k \subseteq \mathcal{K}_m^s$ if and only if $n \le m$ and $k \le s$ ([9], [1]). An MMV-algebra $\mathbf{A} \in \mathcal{K}_n^k$ if and only if $\mathbf{A}$ satisfies (α^k) and (δ_n) [8].

If $\mathbf{A} \in \mathcal{MMV}$ is a subdirectly irreducible algebra such that $\operatorname{ord}\forall\mathbf{A} = m$ and $\operatorname{width}\mathbf{A} = k$, then $\mathbf{A}$ is isomorphic to $\mathbf{S}_m^k$. Clearly $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_m^k)$.

The next lemma will be needed later.

Lemma 2.3. [8] *If* $\mathbf{A}$ *is an infinite subalgebra of the MV-algebra* $[\mathbf{0},\mathbf{1}]$*, then*

$$\mathcal{V}_{\mathcal{MMV}}\left(\mathbf{A}^k\right) = \mathcal{V}_{\mathcal{MMV}}\left([\mathbf{0},\mathbf{1}]^k\right).$$

A *Łukasiewicz implication algebra* is an algebra $\mathbf{A} = \langle A; \to, 1\rangle$ of type $(2,0)$ that satisfies the identities

(L1) $1 \to x \approx x$,
(L2) $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$,
(L3) $(x \to y) \to y \approx (y \to x) \to x$,
(L4) $(x \to y) \to (y \to x) \approx y \to x$.

We denote by $\mathcal{L}$ the variety of all Łukasiewicz implication algebras. The following properties are satisfied by any Łukasiewicz implication algebra:

(L5) $x \to x \approx 1$,
(L6) $x \to 1 \approx 1$,
(L7) if $x \to y \approx 1$ and $y \to x \approx 1$, then $x \approx y$,
(L8) $x \to (y \to x) \approx 1$,
(L9) $x \to (y \to z) \approx y \to (x \to z)$.

If $\mathbf{A} \in \mathcal{L}$, then the relation $a \leq b$ if and only if $a \to b = 1$ is a partial order on A, called the natural order of $\mathbf{A}$, with 1 as its greatest element. The join operation $x \vee y$ is given by the term $(x \to y) \to y$ and if $c \in A$, then the polynomial $p(x,y,c) = ((x \to c) \vee (y \to c)) \to c$ is such that $p(a,b,c) = a \wedge b = \inf\{a,b\}$ for $a, b \geq c$. The lattice operations satisfy the following properties:

(L10) $(x \to y) \vee (y \to x) \approx 1$,
(L11) $(x \vee y) \to z \approx (x \to z) \wedge (y \to z)$,
(L12) $z \to (x \vee y) \approx (z \to x) \vee (z \to y)$,

and if for $a, b \in A$ the meet $a \wedge b$ exists, then for any $c \in A$,

(L13) $(a \wedge b) \to c \approx (a \to c) \vee (b \to c)$,
(L14) $c \to (a \wedge b) \approx (c \to a) \wedge (c \to b)$.

Let us recall that an algebra $\mathbf{A} = \langle A; \to, \neg, 1\rangle$ of type $(2,1,0)$ is a *Wajsberg algebra* if

(W1) $1 \to x \approx x$,
(W2) $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$,
(W3) $(x \to y) \to y \approx (y \to x) \to x$,
(W4) $(\neg x \to \neg y) \to (y \to x) \approx 1$.

If $\mathbf{A} = \langle A; \to, 1\rangle$ is a Łukasiewicz implication algebra and $c \in A$, then $\mathbf{A}_c = \langle [c) = \{a \in A : c \leq a\}, \to_c, \neg_c, c, 1\rangle$ becomes a Wajsberg algebra by defining $\neg_c x := x \to c$ and $x \to_c y = x \to y$. Wajsberg algebras are termwise equivalent to MV-algebras. If $\mathbf{A} = \langle A; \oplus, \neg, 0\rangle$ is an MV-algebra and we define on $\mathbf{A}$ $x \to y := \neg x \oplus y$ and $1 := \neg 0$, then $\langle A; \to, \neg, 1\rangle$ is a Wajsberg algebra. Reciprocally, if $\mathbf{A} = \langle A; \to, \neg, 1\rangle$ is a Wajsberg algebra and if we define $x \oplus y := \neg x \to y$ and $0 := \neg 1$, then $\langle A; \oplus, \neg, 0\rangle$ is an MV-algebra [11].

Łukasiewicz implication algebras are exactly the class of all implicational subreducts of MV-algebras [10]. For each positive integer n, let

$$\mathbf{L}_n = \left\langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}; \rightarrow, 1 \right\rangle$$

be the Łukasiewicz implication algebra which is the $\{\rightarrow, 1\}$-reduct of the MV-algebra $\mathbf{S}_n$. By an abuse of notation, we denote by $[\mathbf{0}, \mathbf{1}]$ the Łukasiewicz implication algebra that is the $\{\rightarrow, 1\}$-reduct of the MV-algebra $[\mathbf{0}, \mathbf{1}]$.

Łukasiewicz implication algebras are congruence 1-regular. For each congruence relation θ on an algebra $\mathbf{A} \in \mathcal{L}$, $1/\theta$ is a filter, i.e., it contains the element 1 and if a, $a \rightarrow b \in 1/\theta$, then $b \in 1/\theta$. In particular, $1/\theta$ is upwardly-closed with respect to the natural order. Conversely, for any filter F of $\mathbf{A}$, the relation $\theta_F = \{(a, b) \in A^2 : a \rightarrow b, b \rightarrow a \in F\}$ is a congruence on $\mathbf{A}$ such that $F = 1/\theta_F$. In fact, the correspondence $\theta \mapsto 1/\theta$ gives an order isomorphism from the family of all congruence relations on $\mathbf{A}$ onto the family of all filters of $\mathbf{A}$, ordered by inclusion. For this reason, we often write $\mathbf{A}/F$ instead of $\mathbf{A}/\theta_F$.

Subdirectly irreducible algebras in $\mathcal{L}$ are totally ordered [13]. If $\mathbf{A} \in \mathcal{L}$ and $x \in A$, the order of x, denoted by $\operatorname{ord} x$, is the least integer n such that $x \vee (x \overset{n}{\rightarrow} y) = 1$, for every $y \in A$. If n does not exist, then $\operatorname{ord} x = \omega$. The order of $\mathbf{A}$ is $\operatorname{ord} \mathbf{A} = \sup\{\operatorname{ord} x : x \in A\}$.

Theorem 2.4. [13] *Let* $\mathbf{A}$ *be a subdirectly irreducible Łukasiewicz implication algebra.*

(a) *If* $\operatorname{ord} \mathbf{A} = m$*, then* $\mathbf{A}$ *is isomorphic to* $\mathbf{L}_m$*.*
(b) *If* $\operatorname{ord} \mathbf{A} = \omega$*, then* $\mathbf{A}$ *has a subalgebra isomorphic to* $\mathbf{L}_m$*, for each positive integer* m*.*

The lattice of subvarieties of $\mathcal{L}$ is given in [13] and it is an $\omega + 1$-chain:

$$\mathcal{V}(\mathbf{L}_0) \subsetneq \mathcal{V}(\mathbf{L}_1) \subsetneq \cdots \subsetneq \mathcal{V}(\mathbf{L}_n) \subsetneq \cdots \subsetneq \mathcal{V}([\mathbf{0}, \mathbf{1}]) = \mathcal{L}.$$

Note that $\mathcal{V}(\mathbf{L}_0)$ and $\mathcal{V}(\mathbf{L}_1)$ are the trivial subvariety and the subvariety of implication algebras, respectively. Moreover, the subvariety $\mathcal{V}(\mathbf{L}_n)$ is the subvariety of all Łukasiewicz implication algebras that satisfy the identity

$$x \overset{n}{\rightarrow} y \approx x \overset{n+1}{\longrightarrow} y, \qquad (\epsilon_n)$$

for each $n \in \mathbb{N}$.

3. Monadic Łukasiewicz implication algebras

We begin this section by studying properties of the monadic implicational reduct of an MMV-algebra. This leads us to the definition of a monadic Łukasiewicz implication algebra. Next, we show that the class of *bounded* monadic Łukasiewicz implication algebras is the class of monadic implicational reducts of $\mathcal{MMV}$. Finally, we prove that for any $\mathbf{A} \in \mathcal{ML}$, the lattice $\mathbf{Con}_{\mathcal{ML}}(\mathbf{A})$ of congruences of $\mathbf{A}$, the lattice $\mathcal{F}_M(\mathbf{A})$ of monadic filters of $\mathbf{A}$, the lattice $\mathcal{F}(\forall\mathbf{A})$ of filters of the Łukasiewicz implication algebra $\forall\mathbf{A}$, and the lattice $\mathbf{Con}_{\mathcal{L}}(\forall\mathbf{A})$ of congruences of $\forall\mathbf{A}$ are all isomorphic. From this, we characterize the subdirectly irreducible and finite simple members of $\mathcal{ML}$.

Lemma 3.1. *The $\{\to,\forall,1\}$-reduct of an MMV-algebra satisfies the following identities:*

(ML1) $\forall 1 \approx 1$,
(ML2) $\forall x \to x \approx 1$,
(ML3) $\forall((x \to \forall y) \to \forall y) \approx (\forall x \to \forall y) \to \forall y$,
(ML4) $\forall(x \to y) \to (\forall x \to \forall y) \approx 1$,
(ML5) $\forall(\forall x \to \forall y) \approx \forall x \to \forall y$,
(ML6) $\forall((y \to (y \to \forall x)) \to \forall x) \approx (\forall y \to (\forall y \to \forall x)) \to \forall x$,
(ML7) $\forall((x \to \forall y) \to x) \approx (\forall x \to \forall y) \to \forall x$.

Proof. Identities (ML1), (ML2), (ML3), (ML4), and (ML5) are immediate from (MMV14), (MMV7), (MMV19), (MMV18), and (MMV17), respectively.

Let us see that (ML6) holds:

$$\begin{aligned}\forall((b\to(b \to \forall a)) \to \forall a) &= \forall((b^2 \to \forall a) \to \forall a) = \forall(b^2 \vee \forall a) = \forall b^2 \vee \forall a \\ &= (\forall b)^2 \vee \forall a = ((\forall b)^2 \to \forall a) \to \forall a = (\forall b \to (\forall b \to \forall a)) \to \forall a.\end{aligned}$$

Let us prove that (ML7) holds in the MV-algebra $\mathbf{S}_n^t$, for all $n, t \in \mathbb{N}$. Let $a = \langle a_i \rangle_{1\le i\le t}$ and $b = \langle b_i \rangle_{1\le i\le t}$ in S_n^t. We can assume, without loss of generality, that $a_1 = \min_{1\le i\le t}\{a_i\}$ and $b_1 = \min_{1\le i\le t}\{b_i\}$. Then $\forall_\wedge a$ is the constant t-tuple $\forall_\wedge a = \langle a_1, a_1, \ldots, a_1\rangle$ and similarly $\forall_\wedge b = \langle b_1, b_1, \ldots, b_1\rangle$. So, for all j, $a_1 \le a_j$. Then, $a_j \to b_1 \le a_1 \to b_1$. Thus,

$$(a_1 \to b_1) \to a_1 \le (a_j \to b_1) \to a_1 \le (a_j \to b_1) \to a_j.$$

Hence, $\min_{1\le j\le t}\{(a_j \to b_1) \to a_j\} = (a_1 \to b_1) \to a_1$. Consequently,

$$\forall((a \to \forall b) \to a) \approx (\forall a \to \forall b) \to \forall a.$$

Since the variety $\mathcal{MMV}$ is generated by the algebras $\mathbf{S}_n^t$, we have that (ML7) holds in every MMV-algebra. □

The above lemma motivates the following definition.

Definition 3.2. An algebra $\mathbf{A} = \langle A; \to, \forall, 1\rangle$ of type $(2,1,0)$ is a *monadic Łukasiewicz implication* algebra if $\langle A; \to, 1\rangle$ is a Łukasiewicz implication algebra and if the identities (ML1)-(ML7) hold.

From the last definition, Lemma 3.1 can be stated in the following way.

Lemma 3.3. *The $\{\to,\forall,1\}$-reduct of an MMV-algebra is a monadic Łukasiewicz implication algebra.*

The variety of all monadic Łukasiewicz implication algebras is denoted by $\mathcal{ML}$.

Taking into account the definition of the order in a Łukasiewicz implication algebra, it follows that the identity (ML2) is equivalent to $\forall x \le x$. We also know that the join of a and b is given by $a \vee b = (a \to b) \to b$. Then, (ML3) can be written as $\forall(x \vee \forall y) \approx \forall x \vee \forall y$.

Lemma 3.4. *Let $\mathbf{A} \in \mathcal{ML}$. For every $a, b \in A$ the following properties hold:*

(ML8) $\forall\forall a = \forall a$,
(ML9) *if* $a \leq b$, *then* $\forall a \leq \forall b$,
(ML10) *the meet* $\forall a \wedge \forall b$ *exists if and only if the meet* $a \wedge b$ *exists,*
(ML11) $\forall(\forall a \vee \forall b) = \forall a \vee \forall b$,
(ML12) *if the meet* $\forall a \wedge \forall b$ *exists, then* $\forall(\forall a \wedge \forall b) = \forall a \wedge \forall b$,
(ML13) *if the meet* $a \wedge b$ *exists, then* $\forall(a \wedge b) = \forall a \wedge \forall b$.

Let us consider the set $\forall A = \{\forall x : x \in A\}$. From (ML1), (ML5), and (ML8), we have that $\forall \mathbf{A} = \langle \forall A; \to, \forall, 1\rangle$ is a subalgebra of $\mathbf{A}$.

Lemma 3.5. *Let $\mathbf{A} = \langle A; \to, \forall, 1\rangle$ be a monadic Łukasiewicz implication algebra, and let $c \in \forall A$. In $[c) = \{a \in A : c \leq a\}$, we define the operations $\neg_c x := x \to c$ and $x \oplus_c z := \neg_c x \to z$. Then $\mathbf{A}_c = \langle [c); \oplus_c, \neg_c, \forall, c\rangle$ is an MMV-algebra.*

Proof. Since $c \in \forall A$, so $c = \forall c$ and we know that $\langle [c); \oplus_c, \neg_c, c\rangle$ is an MV-algebra. Let us prove (MMV7)–(MMV12). Properties (MMV7) and (MMV9) are immediate from (ML2) and (ML5), respectively. Let $a, b \in [c)$. Note that $a \wedge b$ exists and $c \leq a \wedge b$. Then from (ML13), we have that $\forall(a \wedge b) = \forall a \wedge \forall b$ and (MMV8) holds. Let us see that (MMV10) holds. Indeed,

$$\begin{aligned}\forall(\forall a \odot_c \forall b) &= \forall(\neg_c(\forall a \to \neg_c \forall b)) = \forall((\forall a \to (\forall b \to \forall c)) \to \forall c) \\ &= (\forall a \to (\forall b \to \forall c)) \to \forall c = \neg_c(\forall a \to \neg_c \forall b) = \forall a \odot_c \forall b.\end{aligned}$$

Next, we prove (MMV11). From (ML6), we have that

$$\begin{aligned}\forall(a \odot_c a) &= \forall(\neg_c(a \to \neg_c a)) = \forall((a \to (a \to \forall c)) \to \forall c) \\ &= (\forall a \to (\forall a \to \forall c)) \to \forall c = \forall a \odot_c \forall a.\end{aligned}$$

Finally, by (ML7), it follows that

$$\forall(a \oplus_c a) = \forall((a \to \forall c) \to a) = (\forall a \to \forall c) \to \forall a = \forall a \oplus_c \forall a.$$

Hence, the claim is proved. □

Let us remark from the proof of the last lemma that we prove (MMV11) and (MMV12) from (ML6) and (ML7).

An ML-algebra is *bounded* if it has a first element. The following corollary is immediate from Lemma 3.5.

Corollary 3.6. *If $\mathbf{A}$ is a bounded ML-algebra with first element 0, then $\mathbf{A}_0$ is an MMV-algebra.*

Therefore, Lemma 3.3 and Corollary 3.6 imply that the class of bounded ML-algebras is the class of the $\{\to, \forall, 1\}$-reducts of the MMV-algebras.

In the following, we state the existence of isomorphisms between the lattice of monadic filters of an ML-algebra $\mathbf{A}$, the lattice of congruences of $\mathbf{A}$, the lattice of filters of the L-algebra $\forall\mathbf{A}$, and the lattice of congruences of $\forall\mathbf{A}$.

Let $\mathbf{A}$ be an ML-algebra. A subset $F \subseteq A$ is called a *monadic filter* of $\mathbf{A}$ if F is a filter of $\mathbf{A}$ and $\forall a \in F$ whenever $a \in F$.

We denote by $\mathcal{F}_M(\mathbf{A})$ the set of all monadic filters of $\mathbf{A}$, ordered by inclusion. If $F \in \mathcal{F}_M(\mathbf{A})$, then, in particular, F is an order filter, i.e., if $a \in F$ and $a \leq b$, then $b \in F$. Since $b \leq a \to b$ for any a, b in an ML-algebra, the monadic filters are always subuniverses.

If $\mathbf{A} \in \mathcal{ML}$ and $X \subseteq A, X \neq \emptyset$, the monadic filter generated by X is:

$$\mathrm{FMg}(X) = \{b \in A : \forall a_1 \to (\forall a_2 \to (\cdots (\forall a_n \to b) \cdots)) = 1 : a_1, \ldots, a_n \in X\}.$$

If $X = \{a\}$, then $\mathrm{FMg}(a) = \left\{b \in A : \forall a \stackrel{n}{\to} b = 1, \text{ for some } n \in \mathbb{N}\right\}$. Note that $\mathrm{FMg}(X) = \mathrm{Fg}(\forall X)$.

Theorem 3.7. *Let $\mathbf{A} \in \mathcal{ML}$. The map $\mathbf{Con}_{\mathcal{ML}}(\mathbf{A}) \to \mathcal{F}_M(\mathbf{A})$ defined by $\theta \to 1/\theta$ is an order-isomorphism, with inverse map $F \to \theta_F$.*

It is also straightforward to see the following.

Theorem 3.8. *Let $\mathbf{A} \in \mathcal{ML}$. The correspondence $\mathcal{F}_M(\mathbf{A}) \to \mathcal{F}(\forall\mathbf{A})$ defined by $F \to F \cap \forall A$ is an order-isomorphism, with inverse map $M \to \mathrm{FMg}(M)$.*

From the above, we have that if $\mathbf{A} \in \mathcal{ML}$, then

$$\mathbf{Con}_{\mathcal{ML}}(\mathbf{A}) \cong \mathcal{F}_M(\mathbf{A}) \cong \mathcal{F}(\forall\mathbf{A}) \cong \mathbf{Con}_{\mathcal{L}}(\forall\mathbf{A}).$$

As a consequence, the variety $\mathcal{ML}$ is congruence-distributive and it has the congruence extension property. We also have the following result.

Corollary 3.9. *Let $\mathbf{A} \in \mathcal{ML}$. Then $\mathbf{A}$ is subdirectly irreducible (simple) if and only if $\forall\mathbf{A}$ is a subdirectly irreducible (simple) L-algebra.*

We know that the subdirectly irreducible algebras in the variety $\mathcal{L}$ are totally ordered. Then the above corollary implies the following result.

Lemma 3.10. *If $\mathbf{A}$ is a subdirectly irreducible ML-algebra, then $\forall\mathbf{A}$ is totally ordered.*

Let n and k be positive integers. We denote the $\{\to, \forall, 1\}$-reduct of the MMV-algebra $\mathbf{S}_n^k$ by $\mathbf{L}_n^k$. In the next lemma, we characterize the finite simple algebras in $\mathcal{ML}$.

Lemma 3.11. *The finite simple algebras in $\mathcal{ML}$ are the algebras $\mathbf{L}_n^k$, where n and k are positive integer numbers.*

Proof. It is clear that $\mathbf{L}_n^k$ is simple. Let $\mathbf{A}$ be a finite simple algebra in $\mathcal{ML}$. From Corollary 3.9, we know that $\forall\mathbf{A}$ is a finite simple Łukasiewicz implication algebra. Then $\forall\mathbf{A} \cong \mathbf{L}_n$, for some integer n [13]. In particular, $\mathbf{A}$ has a least element 0. From Lemma 3.5, we know that $\mathbf{A}_0 = \langle [0); \to, \forall, 1\rangle$ is an MMV-algebra which it is also simple and finite. Then $\mathbf{A}_0 \cong \mathbf{S}_n^k$, for some k [9]. As a consequence, $\mathbf{A} \cong \mathbf{L}_n^k$. ☐

4. Monadic implicational subreducts of MMV-algebras

The main goal in this section is to show that every ML-algebra is isomorphic to a monadic implicational subreduct of a bounded ML-algebra. This implies, together with the results of the last section, that every ML-algebra is isomorphic to a monadic implicational subreduct of an MMV-algebra. This fact gives an important relation between the subvarieties of $\mathcal{MMV}$ and $\mathcal{ML}$. As a first application, we show in this section a characteristic algebra of $\mathcal{ML}$ and we prove that $\mathcal{ML}$ has the finite model property.

We say that an algebra $\mathbf{A} \in \mathcal{ML}$ is *directed* if for all $a, b \in A$ there exists $c \in A$ such that $c \leq a$ and $c \leq b$.

Lemma 4.1. *Every directed ML-algebra can be embedded into a bounded ML-algebra.*

Proof. Let $\mathbf{A}$ be a directed ML-algebra. For each $z \in A$, the set $[\forall z) = \{x \in A : \forall z \leq x\}$ is a bounded subuniverse of $\mathbf{A}$ and $[\forall \mathbf{z})$ is a bounded ML-algebra with first element $\forall z$. Let us see that $\mathbf{A} \in \mathrm{ISP_U}\left(\{[\forall \mathbf{z}) : z \in A\}\right)$.

For each $a \in A$, let $(a] = \{x \in A : x \leq a\}$. Let us consider the family $\{(a] : a \in A\}$. Since $\mathbf{A}$ is directed, for each $a, b \in A$, there exists $c \in A$ such that $(c] \subseteq (a] \cap (b]$. Then the family $\{(a] : a \in A\}$ has the finite intersection property. Thus, there exists an ultrafilter F in the boolean algebra $\mathbf{Su}(\mathbf{A})$ of subsets of A, containing all the members of the family. Let $\psi \colon \mathbf{A} \to (\prod_{z\in A}[\forall \mathbf{z}))/F$ be defined by $\psi(a) = (a \vee \forall z)_{z\in A}/F$. Let us prove that $\psi(\forall a) = \forall(\psi a)$, $\psi(a \to b) = \psi(a) \to \psi(b)$, and ψ is injective.

So, $\forall(\psi a) = \forall\left((a \vee \forall z)_{z\in A}/F\right) = \forall\left((a \vee \forall z)_{z\in A}\right)/F = \left(\forall(a \vee \forall z)\right)_{z\in A}/F$. Then $\psi(\forall a) = \forall(\psi a)$ if and only if $\{z \in A : \forall z \vee \forall a = \forall(a \vee \forall z)\} \in F$. From (MMV19) we have that $\{z \in A : \forall z \vee \forall a = \forall(a \vee \forall z)\} = A \in F$. Hence, $\psi(\forall a) = \forall(\psi a)$.

For each $a, b \in A$, there exists $c \in A$ such that $c \leq a, b$. Let us see that

$$(c] \subseteq \{z \in A : \forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\}.$$

Indeed, if $z \in (c]$, then $\forall z \leq z \leq c \leq a, b$. Then $\forall z \leq a \to b$. Thus,

$$\forall z \vee (a \to b) = a \to b = (\forall z \vee a) \to (\forall z \vee b).$$

Since $(c] \in F$, we have that $\{z \in A : \forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\} \in F$. As a consequence, $\psi(a \to b) = \psi(a) \to \psi(b)$.

Let $a, b \in A$ such that $\psi(a) = \psi(b)$. Then $\{z \in A : \forall z \vee a = \forall z \vee b\} \in F$. Since $(a] \in F$, we obtain that $(a] \cap \{z \in A : \forall z \vee a = \forall z \vee b\} \in F$. In particular, this intersection is not empty.

Let $w \in (a] \cap \{z \in A : \forall z \vee a = \forall z \vee b\}$. Then $a = \forall w \vee a = \forall w \vee b$, and consequently $b \leq a$. Similarly, considering $(b] \in F$, we obtain that $a \leq b$. Then $a = b$ and this proves that ψ is injective. $\square$

Lemma 4.2. *Every subdirectly irreducible ML-algebra is isomorphic to a monadic implicational subreduct of a bounded ML-algebra* $\mathbf{B}$ *where, in addition,* $\forall\mathbf{B}$ *is totally ordered.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible ML-algebra and let $a, b \in A$. Since $\forall\mathbf{A}$ is totally ordered, let $\forall z = \min\{\forall a, \forall b\}$. Clearly, $\forall z \leq a, b$. Thus, $\mathbf{A}$ is directed.

From Lemma 4.1, we know that there exists $\mathbf{B} \in \mathcal{ML}$, which is bounded, such that $\mathbf{A}$ is embedded into $\mathbf{B}$. Since $\forall\mathbf{A}$ is totally ordered, then for each $z \in A$, we have that $\forall([\forall\mathbf{z}))$ is totally ordered. The property of being totally ordered is a first order property; thus, it is preserved under ultraproducts. It follows that $\forall\mathbf{B}$ is totally ordered. □

In every bounded ML-algebra, we can define the structure of an MMV-algebra. Then in Lemma 4.2, we prove that every subdirectly irreducible ML-algebra is isomorphic to a monadic implicational subreduct of an MMV-algebra.

Proposition 4.3. *Every ML-algebra is isomorphic to a monadic implicational subreduct of an MMV-algebra.*

Proof. Let $\mathbf{A} \in \mathcal{ML}$. Let $\varphi\colon \mathbf{A} \to \prod_{i\in I} \mathbf{A}_i$, where each $\mathbf{A}_i$ is subdirectly irreducible, be a subdirect representation of $\mathbf{A}$. From Lemma 4.2, we know that for each $i \in I$, there exists $\mathbf{B}_i \in \mathcal{MMV}$ such that $\forall\mathbf{B}_i$ is totally ordered and $\mathbf{A}_i$ is isomorphic to a monadic implicational subreduct of $\mathbf{B}_i$. Then $\mathbf{A}$ is isomorphic to a monadic implicational subreduct of $\prod_{i\in I} \mathbf{B}_i$. □

It is straightforward to see the next results.

Proposition 4.4. *If* $\mathcal{V}$ *is a variety of MMV-algebras, then* $\mathcal{S}^{\{\to,\forall,1\}}(\mathcal{V})$*, the class of all monadic implicational subreducts of* $\mathcal{V}$*, is a variety of ML-algebras.*

Corollary 4.5. *Let* $\mathbf{B}$ *be an MMV-algebra and* $\mathbf{A}$ *its* $\{\to,\forall,1\}$*-reduct. Then*

$$\mathcal{V}_{\mathcal{ML}}(\mathbf{A}) = \mathcal{S}^{\{\to,\forall,1\}}(\mathcal{V}_{\mathcal{MMV}}(\mathbf{B})).$$

Corollary 4.6. *The monadic implicational subreduct of the functional MMV-algebra* $[\mathbf{0},\mathbf{1}]^{\mathbb{N}}$ *generates the variety of monadic Łukasiewicz implication algebras. That is,* $\mathcal{ML} = \mathcal{V}(\langle[0,1]^{\mathbb{N}}; \to, \forall_\wedge, 1\rangle)$.

Proof. From the previous corollary, Proposition 4.3, and since the variety of MMV-algebra is generated by the MMV-algebra $[\mathbf{0},\mathbf{1}]^{\mathbb{N}}$ (see [7]), we have that

$$\begin{aligned}\mathcal{ML} &= \mathcal{S}^{\{\to,\forall,1\}}(\mathcal{MMV}) = \mathcal{S}^{\{\to,\forall,1\}}(\mathcal{V}(\langle[0,1]^{\mathbb{N}}; \oplus, \neg, \forall_\wedge, 0\rangle)) \\ &= \mathcal{V}(\langle[0,1]^{\mathbb{N}}; \to, \forall_\wedge, 1\rangle).\end{aligned}$$

□

From Lemma 3.11 and since the variety $\mathcal{MMV}$ is generated by its finite members [7] and $\mathcal{ML}$ is the class of all monadic implicational subreducts of $\mathcal{MMV}$, we obtain the following result.

Corollary 4.7. *The variety* $\mathcal{ML}$ *is generated by its finite members. More precisely,* $\mathcal{ML} = \mathcal{V}\left(\{\mathbf{L}_n^k : n, k \in \mathbb{N}\}\right)$.

5. The lattice of subvarieties

In this section, we completely describe the lattice of subvarieties of $\mathcal{ML}$. We also give an equational basis for each proper subvariety.

Let us consider the subvariety of MMV-algebras

$$\mathcal{K}_n = \mathcal{V}_{\mathcal{MMV}}(\{\mathbf{S}_1^{\mathbb{N}}, \mathbf{S}_2^{\mathbb{N}}, \dots, \mathbf{S}_n^{\mathbb{N}}\})$$

characterized in $\mathcal{MMV}$ by the identity (ϵ_n) (see Section 2). We denote by $\mathbf{L}_k^{\mathbb{N}}$ the monadic implicational reduct of the MMV-algebra $\mathbf{S}_k^{\mathbb{N}}$. From Proposition 4.4, we know that $\mathcal{S}^{\{\to,\forall,1\}}(\mathcal{K}_n)$ is a variety of ML-algebras. Moreover, $\mathcal{S}^{\{\to,\forall,1\}}(\mathcal{V}_{\mathcal{MMV}}\{\mathbf{S}_m^{\mathbb{N}} : 1 \leq m \leq n\}) = \mathcal{V}_{\mathcal{ML}}\left(\mathbf{L}_n^{\mathbb{N}}\right)$. Therefore, we have the following result.

Lemma 5.1. *For each positive integer* n*, the subvariety* $\mathcal{V}(\mathbf{L}_n^{\mathbb{N}})$ *is characterized by the identity* (ϵ_n).

Since $x \overset{n}{\to} y \leq x \overset{n+1}{\to} y$ is satisfied in every ML-algebra, the identity (ϵ_n) is equivalent to the identity $(\epsilon'_n)(x \overset{n+1}{\to} y) \to (x \overset{n}{\to} y) \approx 1$. The following relation between the subvarieties $\mathcal{V}(\mathbf{L}_n^{\mathbb{N}})$ is easily proved.

Corollary 5.2. *The subvarieties* $\mathcal{V}(\mathbf{L}_n^{\mathbb{N}})$ *form an* $\omega + 1$*-chain*

$$\mathcal{V}(\mathbf{L}_1^{\mathbb{N}}) \subsetneq \mathcal{V}(\mathbf{L}_2^{\mathbb{N}}) \subsetneq \cdots \subsetneq \mathcal{V}(\mathbf{L}_n^{\mathbb{N}}) \subsetneq \cdots \subsetneq \mathcal{V}([\mathbf{0},\mathbf{1}]^{\mathbb{N}}) = \mathcal{ML},$$

in the lattice of subvarieties of $\mathcal{ML}$.

Let us recall that the subvariety of MMV-algebras generated by the MMV-algebra $[\mathbf{0},\mathbf{1}]^k$ is characterized by the identity (α^k) (see Section 2). Since $x \vee y = (x \to y) \to y$, we have that (α^k) is an identity for the monadic Łukasiewicz implication algebras. As a consequence of this and Corollary 4.5, we have the following result.

Lemma 5.3. *Let* k *be a positive integer. The subvariety of ML-algebras generated by* $\langle [0,1]^k; \to, \forall_\wedge, 1\rangle$ *is characterized by the identity* (α^k)*. In addition,* $\mathcal{V}(\{\mathbf{L}_n^k : n \in \mathbb{N}\}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$.

Since $\mathcal{V}_{\mathcal{MMV}}([\mathbf{0},\mathbf{1}]^s) \subsetneq \mathcal{V}_{\mathcal{MMV}}([\mathbf{0},\mathbf{1}]^k)$ if and only if $s < k$, we obtain the following result.

Corollary 5.4. *There is an* $\omega + 1$*-chain in the lattice of subvarieties of* $\mathcal{ML}$ *given by*

$$\mathcal{V}([\mathbf{0},\mathbf{1}]) \subsetneq \mathcal{V}([\mathbf{0},\mathbf{1}]^2) \subsetneq \cdots \subsetneq \mathcal{V}([\mathbf{0},\mathbf{1}]^k) \subsetneq \cdots \subsetneq \mathcal{V}([\mathbf{0},\mathbf{1}]^{\mathbb{N}}) = \mathcal{ML}.$$

Let us consider a subdirectly irreducible monadic Łukasiewicz implication algebra $\mathbf{A}$ that satisfies (α^k) for some positive integer k. We know that $\mathbf{A}$ is a monadic implicational subreduct of an MMV-algebra $\mathbf{B}$, and from the construction of $\mathbf{B}$ (see Lemma 4.2), we have that $\mathbf{B}$ also satisfies (α^k). In addition, as $\forall\mathbf{B}$ is totally ordered, then the MMV-algebra $\mathbf{B}$ is isomorphic to a subalgebra of the functional MMV-algebra $\langle(\forall\mathbf{B})^k;\forall_\wedge\rangle$ (see Proposition 2.2). Let us see that $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^k$. The proof is similar to the proof of Proposition 2.2, but it has some changes.

Proposition 5.5. *Let n be a positive integer. If $\mathbf{A}$ is a monadic Łukasiewicz implication subalgebra of $\langle\mathbf{V}^n;\forall_\wedge\rangle$ such that $\forall_\wedge\mathbf{A}$ is totally ordered and $\mathbf{V}$ is a totally ordered Łukasiewicz implication algebra, then $\mathbf{A}$ is a subalgebra of $\langle(\forall_\wedge\mathbf{A})^n;\forall_\wedge\rangle$.*

Proof. For each $i \in \{1,\dots,n\}$, let us consider the epimorphism $\pi_i\restriction_A\colon \mathbf{A}\to\mathbf{V}$. We will show that for each i, $\pi_i\restriction_A(\forall_\wedge A) = \pi_i\restriction_A(A)$. Clearly, $\pi_i\restriction_A(\forall_\wedge A)\subseteq \pi_i\restriction_A(A)$. Let us prove that for every $b\in A$, there exists $c\in\forall_\wedge A$ such that $\pi_i(b)=\pi_i(c)$. To see this, we use an induction argument on n.

The case $n=1$ is trivial because $A=\forall_\wedge A$. Let us suppose that it is true for $n=k$. Let $A\subseteq V^{k+1}$ and $a=\langle a_1,a_2,\dots,a_k,a_{k+1}\rangle\in A$. Since $\mathbf{V}$ is a chain and $a_i\in V$, we can assume, without loss of generality, that we have $a_1\le a_2\le\cdots\le a_k\le a_{k+1}$. So, $\pi_1(a)=a_1=\pi_1(\forall_\wedge a)$. We define $\exists_\vee\colon A\to A$ by $\exists_\vee a=\forall_\wedge(a\to\forall_\wedge a)\to\forall_\wedge a$, for each $a\in A$. Then $\pi_{k+1}(a)=a_{k+1}=\pi_{k+1}(\exists_\vee a)$. Let us calculate $(a\to\forall_\wedge a)\vee(\exists_\vee a\to a)$. We have that $a\to\forall_\wedge a=\langle 1, a_2\to a_1,\dots,a_{k+1}\to a_1\rangle$ and also that $\exists_\vee a\to a=\langle a_{k+1}\to a_1, a_{k+1}\to a_2,\dots,a_{k+1}\to a_k,1\rangle$. Hence,

$$(a\to\forall_\wedge a)\vee(\exists_\vee a\to a)=\langle 1,(a_2\to a_1)\vee(a_{k+1}\to a_2),\dots,(a_k\to a_1)\vee(a_{k+1}\to a_k),1\rangle.$$

Let $\mathbf{B}$ be the subalgebra of $\mathbf{V}^{k+1}$ with $B=\{a\in V^{k+1} : a_1=a_{k+1}\}$. Thus, $\mathbf{B}\cong\mathbf{V}^k$ and $(a\to\forall_\wedge a)\vee(\exists_\vee a\to a)\in B$; in fact, $(a\to\forall_\wedge a)\vee(\exists_\vee a\to a)\in A\cap B$.

Let us consider i such that $1<i<k+1$. Then $\pi_i((a\to\forall_\wedge a)\vee(\exists_\vee a\to a)) = (a_i\to a_1)\vee(a_{k+1}\to a_i)$. Since $\mathbf{V}$ is a chain, two cases arise.

Case 1: $a_i\to a_1\ge a_{k+1}\to a_i$.
Then$((a\to\forall_\wedge a)\vee(\exists_\vee a\to a))\to\forall_\wedge a=\langle e_j\rangle_{1\le j\le k+1}$, where

$$e_j=\begin{cases} a_1 & \text{if } j=1 \text{ or } j=k+1,\\ ((a_j\to a_1)\vee(a_{k+1}\to a_j))\to a_1 & \text{if } j\notin\{1,i,k+1\},\\ (a_i\to a_1)\to a_1 & \text{if } j=i.\end{cases}$$

Then the i-component of $((a\to\forall_\wedge a)\vee(\exists_\vee a\to a))\to\forall_\wedge a$ is equal to $a_i\vee a_1=a_i$. In addition, $((a\to\forall_\wedge a)\vee(\exists_\vee a\to a))\to\forall_\wedge a\in B\cap A$, and by the induction hypothesis over $\mathbf{A}\cap\mathbf{B}\cong\mathbf{A}\cap\mathbf{V}^k$, there exists $c\in\forall_\wedge(A\cap B)\subseteq\forall_\wedge A$ such that $\pi_i(c)=a_i$.

Case 2: $a_i\to a_1\le a_{k+1}\to a_i$.
Then $\pi_i((a\to\forall_\wedge a)\vee(\exists_\vee a\to a))=a_{k+1}\to a_i$. Let us consider the MMV-algebra $[\forall_\wedge\mathbf{a})$, where $\neg_{\forall_\wedge a}x := x\to\forall_\wedge a$ and $x\odot_{\forall_\wedge a} y := \neg_{\forall_\wedge a}(x\to\neg_{\forall_\wedge a}y)$. Let us

note that $\forall_\wedge a \leq (a \to \forall_\wedge a) \vee (\exists_\vee a \to a)$ and $\forall_\wedge a \leq \exists_\vee a$. Consequently, we have that $((a \to \forall_\wedge a) \vee (\exists_\vee a \to a)) \odot_{\forall_\wedge a} \exists_\vee a \in [\forall_\wedge a)$. In addition,

$$\pi_1(((a \to \forall_\wedge a) \vee (\exists_\vee a \to a)) \odot_{\forall_\wedge a} \exists_\vee a) = (a_{k+1} \to a_1) \to a_1 = a_{k+1} \vee a_1$$
$$= a_{k+1} = \pi_{k+1}(((a \to \forall_\wedge a) \vee (\exists_\vee a \to a)) \odot_{\forall_\wedge a} \exists_\vee a).$$

Then $((a \to \forall_\wedge a) \vee (\exists_\vee a \to a)) \odot_{\forall_\wedge a} \exists_\vee a \in [\forall_\wedge a) \cap B \subseteq A \cap B$, and by the induction hypothesis, we have that there exists $d \in \forall_\wedge(A \cap B) \subseteq \forall_\wedge A$ such that

$$\begin{aligned}\pi_i(d) &= \pi_i(((a \to \forall_\wedge a) \vee (\exists_\vee a \to a)) \odot_{\forall_\wedge a} \exists_\vee a)\\ &= ((a_{k+1} \to a_i) \to (a_{k+1} \to a_1)) \to a_1\\ &= ((a_i \to a_{k+1}) \to (a_i \to a_1)) \to a_1 = (a_i \to a_1) \to a_1 = a_i.\end{aligned}$$ □

Corollary 5.6. *If* $\mathbf{A}$ *is a subdirectly irreducible Łukasiewicz implication algebra that satisfies* (α^k) *for some* k*, then* $\mathbf{A}$ *is isomorphic to a subalgebra of* $(\forall\mathbf{A})^k$*.*

The previous result motivates the following definition.

Definition 5.7. Let $\mathbf{A} \in \mathcal{ML}$. We define the *width of* $\mathbf{A}$, which is denoted by $\operatorname{width}\mathbf{A}$, as the least integer k such that (α^k) holds in $\mathbf{A}$. If k does not exist, then we say that the width of $\mathbf{A}$ is infinite and we write $\operatorname{width}\mathbf{A} = \omega$.

In the following theorem, we characterize the subvariety generated by a subdirectly irreducible algebra $\mathbf{A}$ by means of the order of $\forall\mathbf{A}$ and the width of $\mathbf{A}$.

Theorem 5.8. *Let* $\mathbf{A}$ *be a subdirectly irreducible monadic Łukasiewicz implication algebra.*

(1) *If* $\operatorname{ord}\forall\mathbf{A} = n < \omega$ *and* $\operatorname{width}\mathbf{A} = k < \omega$*, then* $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{L}_n^k)$*.*
(2) *If* $\operatorname{ord}\forall\mathbf{A} = n < \omega$ *and* $\operatorname{width}\mathbf{A} = \omega$*, then* $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{L}_n^{\mathbb{N}})$*.*
(3) *If* $\operatorname{ord}\forall\mathbf{A} = \omega$ *and* $\operatorname{width}\mathbf{A} = k < \omega$*, then* $\mathcal{V}(\mathbf{A}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$*.*
(4) *If* $\operatorname{ord}\forall\mathbf{A} = \omega$ *and* $\operatorname{width}\mathbf{A} = \omega$*, then* $\mathcal{V}(\mathbf{A}) = \mathcal{ML}$*.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible monadic Łukasiewicz implication algebra. By Corollary 3.9, we know that $\forall\mathbf{A}$ is a subdirectly irreducible Łukasiewicz implication algebra.

(1): If $\operatorname{ord}\forall\mathbf{A} = n < \omega$, then $\forall\mathbf{A}$ is isomorphic to $\mathbf{L}_n$. In particular, $\mathbf{A}$ is bounded. From Corollary 3.6, we have that $\mathbf{A}_0 = \langle A; \forall\rangle$ is an MMV-algebra. Since $\operatorname{width}\mathbf{A} = k < \omega$, then $\operatorname{width}\mathbf{A}_0 = k$ and $\mathbf{A}_0$ is isomorphic to $\mathbf{S}_n^k$. Thus, $\mathbf{A}$ is isomorphic to $\mathbf{L}_n^k$. This implies that $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{L}_n^k)$.

(2): Analogously to (1), $\mathbf{A}_0$ is an MMV-algebra and $\forall\mathbf{A}$ is isomorphic to $\mathbf{L}_n$. Since $\operatorname{width}\mathbf{A} = \omega$, then $\operatorname{width}\mathbf{A}_0 = \omega$, and we have that $\mathcal{V}_{\mathcal{MMV}}(\mathbf{A}_0) = \mathcal{V}_{\mathcal{MMV}}(\mathbf{S}_n^{\mathbb{N}})$ (see Section 2). Then from Corollary 4.5, we obtain that

$$\mathcal{V}_{\mathcal{ML}}(\mathbf{A}) = \mathcal{S}^{\{\to,\forall,1\}}(\mathcal{V}_{\mathcal{MMV}}(\mathbf{A}_0)) = \mathcal{S}^{\{\to,\forall,1\}}(\mathcal{V}_{\mathcal{MMV}}(\mathbf{S}_n^{\mathbb{N}})) = \mathcal{V}_{\mathcal{ML}}(\mathbf{L}_n^{\mathbb{N}}).$$

(3): Suppose that $\operatorname{ord} \forall \mathbf{A} = \omega$ and $\operatorname{width} \mathbf{A} = k < \omega$. From Lemma 5.3, we have that $\mathbf{A} \in \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$. Also, for all n, there exists $\forall a \in \forall A$ such that $\operatorname{ord}(\forall a) = n$. Since $[\forall a)$ is bounded, we can define in $[\forall a)$ an MMV-algebra structure. Then $\mathcal{V}_{\mathcal{MMV}}(\mathbf{S}_n^k) \subseteq \mathcal{V}_{\mathcal{MMV}}([\forall \mathbf{a}))$ for all n (see [8]). Then $\mathcal{V}(\mathbf{L}_n^k) \subseteq \mathcal{V}(\mathbf{A})$ for all n. Hence, from Lemma 5.3, we have that $\mathcal{V}([\mathbf{0},\mathbf{1}]^k) = \mathcal{V}(\mathbf{A})$.

(4): Suppose that $\operatorname{ord} \forall \mathbf{A} = \omega$ and $\operatorname{width} \mathbf{A} = \omega$. By a similar argument to that of (3), we can show that $\mathcal{V}(\mathbf{L}_n^k) \subseteq \mathcal{V}(\mathbf{A})$ for all n and k. Hence, from Proposition 4.7, $\mathcal{ML} = \mathcal{V}(\mathbf{A})$. □

Let us recall that an MMV-algebra $\mathbf{A} \in \mathcal{K}_n^k = \mathcal{V}(\{\mathbf{S}_1^k, \ldots, \mathbf{S}_n^k\})$ if and only if $\mathbf{A}$ satisfies (α^k) and (δ_n) (see Section 2). Then as an immediate consequence of Theorem 5.8 (1) and having into account that $\mathbf{L}_n^k$ is the monadic implicational reduct of the MMV-algebra $\mathbf{S}_n^k$, we have the following corollary.

Corollary 5.9. *Let n and k be positive integer. Then $\mathbf{A} \in \mathcal{V}(\mathbf{L}_n^k)$ if and only if (ϵ_n') and (α^k) hold in $\mathbf{A}$.*

In the next lemma, we give a single identity that characterizes the subvariety generated by $\mathbf{L}_n^k$. This identity will be needed later.

Lemma 5.10. *Let n and k be positive integers. Then the variety $\mathcal{V}(\mathbf{L}_n^k)$ is characterized by the following identity (β_n^k):*

$$\Big[\Big(\forall\big((x \overset{n+1}{\to} y) \to (x \overset{n}{\to} y)\big) \to \forall z\Big) \vee \Big(\forall\Big(\bigvee_{1 \le i < j \le k+1} (\forall(x_i \vee x_j) \to \bigvee_{s=1}^{k+1} \forall x_s)\Big) \to \forall z\Big)\Big] \to \forall z \approx 1.$$

Proof. Let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathcal{ML}$. If (ϵ_n') and (α^k) hold in $\mathbf{A}$, then it is straightforward to see that (β_n^k) holds, too. Reciprocally, let us suppose that (β_n^k) holds in $\mathbf{A}$, that there exists $a, b \in A$ such that $\epsilon_n'(a,b) = (a \overset{n+1}{\to} b) \to (a \overset{n}{\to} b) < 1$, and that there exist $a_1, \ldots, a_{k+1} \in A$ such that $\alpha^k(a_1, \ldots, a_{k+1}) = \bigvee_{1 \le i < j \le k+1} \big(\forall(a_i \vee a_j) \to \bigvee_{s=1}^{k+1} \forall a_s\big) < 1$. Since $\forall \mathbf{A}$ is a chain, we know that there is $c \in A$ such that $\forall c \le \forall(\epsilon_n'(a,b)) < 1$ and $\forall c \le \forall(\alpha^k(a_1, \ldots, a_{k+1})) < 1$. Then

$$\begin{aligned}\big[(\forall(\epsilon_n'(a,b)) \to \forall c) \vee \big(\forall(\alpha^k(a_1, \ldots, a_{k+1})) \to \forall c\big)\big] \to \forall c \\ = \forall(\epsilon_n'(a,b)) \wedge \forall\big(\alpha^k(a_1, \ldots, a_{k+1})\big) < 1,\end{aligned}$$

which is impossible since (β_n^k) holds in $\mathbf{A}$. Therefore, (ϵ_n') and (α^k) hold in $\mathbf{A}$. □

Since $\mathcal{ML}$ is congruence distributive, from Jónsson's results, we know that the lattice of subvarieties $\Lambda(\mathcal{ML})$ is also distributive. Next, we characterize the ordered set $\mathcal{J}(\Lambda(\mathcal{ML}))$ of join-irreducible elements of $\Lambda(\mathcal{ML})$ with the objective of determining $\Lambda(\mathcal{ML})$.

Let n, m, s, and t be positive integers. If $n \leq m$ and $s \leq t$, then $\mathbf{L}_n^s$ is a subalgebra of $\mathbf{L}_m^t$ and $\mathcal{V}(\mathbf{L}_n^s) \subseteq \mathcal{V}(\mathbf{L}_m^t)$. In addition, $\mathbf{L}_n^k$ is a subalgebra of the algebras $[\mathbf{0},\mathbf{1}]^k$ and $\mathbf{L}_n^{\mathbb{N}}$. Then, $\mathcal{V}(\mathbf{L}_n^k) \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ and $\mathcal{V}(\mathbf{L}_n^k) \subseteq \mathcal{V}(\mathbf{L}_n^{\mathbb{N}})$.

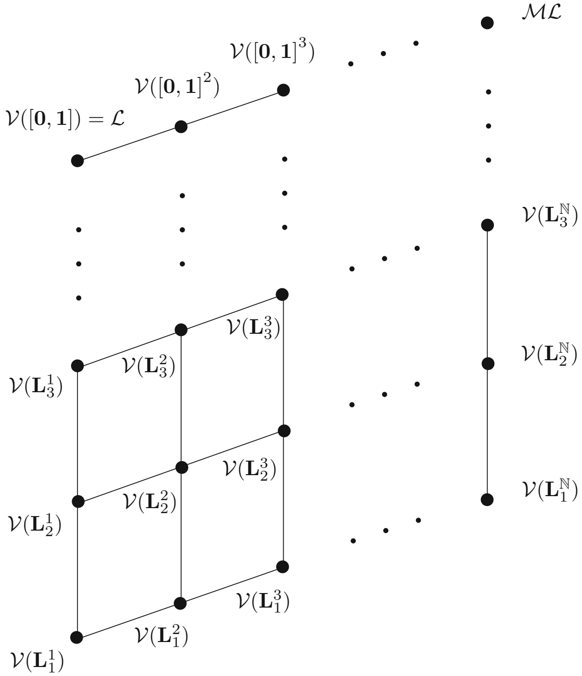

FIGURE 1. $\mathcal{J}(\Lambda(\mathcal{ML}))$

Theorem 5.11. *The set of join-irreducible subvarieties in* $\Lambda(\mathcal{ML})$ *is*

$$\mathcal{J}(\Lambda(\mathcal{ML})) = \left\{\mathcal{V}(\mathbf{L}_n^k) : n,k \in \mathbb{N}\right\} \cup \{\mathcal{V}([\mathbf{0},\mathbf{1}]^k) : k < \omega\} \cup \{\mathcal{V}(\mathbf{L}_n^{\mathbb{N}}) : n < \omega\} \cup \{\mathcal{V}([\mathbf{0},\mathbf{1}]^{\mathbb{N}})\}.$$

If $\mathcal{V}$ *is a proper non-trivial subvariety of ML-algebras, then* $\mathcal{V}$ *is a supremum of a finite number of subvarieties of the set*

$$\{\mathcal{V}(\mathbf{L}_n^k) : n,k \in \mathbb{N}\} \cup \{\mathcal{V}([\mathbf{0},\mathbf{1}]^k) : k < \omega\} \cup \{\mathcal{V}(\mathbf{L}_n^{\mathbb{N}}) : n < \omega\}.$$

Proof. Let $\mathcal{V}$ be a proper non-trivial subvariety of ML-algebras. We denote by $si(\mathcal{V})$ the family of subdirectly irreducible members of $\mathcal{V}$. Let us consider the subset $\mathfrak{A}$ of $(\mathbb{N} \cup \{\omega\}) \times (\mathbb{N} \cup \{\omega\})$ defined by the pairs (n,k) which satisfy that there exists $\mathbf{A} \in si(\mathcal{V})$ such that $\operatorname{ord} \forall \mathbf{A} = n$ and $\operatorname{width} \mathbf{A} = k$. Since $\mathcal{V}$ is non-trivial, then $\mathfrak{A} \neq \emptyset$. We define in $\mathfrak{A}$ the partial order $(n,s) \leq (m,t)$ if and only if $n \leq m$ and $s \leq t$. Since $\mathcal{V}$ is proper, from Theorem 5.8 (4), we know that there is not $\mathbf{A} \in si(\mathcal{V})$ such that $\operatorname{ord} \forall \mathbf{A} = \omega$ and $\operatorname{width} \mathbf{A} = \omega$. In addition, there are not $\mathbf{A}_i \in si(\mathcal{V})$ such that $(\operatorname{ord} \forall \mathbf{A}_i, \operatorname{width} \mathbf{A}_i)$ is a strictly increasing infinite sequence.

Let us prove that there exists a finite set $\mathfrak{B} = \{(n_i, k_i) : 1 \leq i \leq p\}$ of maximal elements in $\mathfrak{A}$ such that $\mathcal{V} = \bigvee_{i=1}^{p} \mathcal{V}(\mathbf{A}_i)$, where $\operatorname{ord} \forall \mathbf{A}_i = n_i$ and $\operatorname{width} \mathbf{A}_i = k_i$, for each i.

Let us suppose that there is in $\mathfrak{A}$ an element of the form (ω, k_1) or that there are elements in $\mathfrak{A}$ of the form (n_i, k_1) such that $\{n_i\}$ is a strictly increasing infinite sequence. In this case, there exists a maximal element of the form $m_1 = (\omega, k_1)$ with $k_1 \in \mathbb{N}$, and in addition, it is unique. Analogously, if there exists a maximal element of the form $m_2 = (n_2, \omega)$, then it is unique.

Let $\mathfrak{A}' = \mathfrak{A} - \{(n,k) : (n,k) \leq m_1 \text{ or } (n,k) \leq m_2\}$. If $\mathfrak{A}' = \emptyset$, then $\mathcal{V}$ has one of the following forms:

(a) $\mathcal{V} = \mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1})$,
(b) $\mathcal{V} = \mathcal{V}(\mathbf{L}_{n_2}^{\mathbb{N}})$,
(c) $\mathcal{V} = \mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1}) \vee \mathcal{V}(\mathbf{L}_{n_2}^{\mathbb{N}})$.

Let us suppose that $\mathfrak{A}' \neq \emptyset$. Then $\mathfrak{A}' \subseteq \mathbb{N} \times \mathbb{N}$ and it is finite. Thus, there exists in $\mathfrak{A}'$ a finite set of maximal elements (n_i, k_i) for $i \in I'$.

Let $I = I' \cup \{m_1, m_2\}$. It is clear that $\bigvee_{i \in I} \mathcal{V}(\mathbf{A}_i) \subseteq \mathcal{V}$. Let $\mathbf{A} \in si(\mathcal{V})$. Then $(\operatorname{ord} \forall \mathbf{A}, \operatorname{width} \mathbf{A}) \in \mathfrak{A}$. Then there exists (n_i, k_i) that is maximal and $(\operatorname{ord} \forall \mathbf{A}, \operatorname{width} \mathbf{A}) \leq (n_i, k_i)$. Thus, $\mathcal{V}(\mathbf{A}) \subseteq \mathcal{V}(\mathbf{A}_i)$. Therefore, $\mathcal{V} = \bigvee_{i \in I} \mathcal{V}(\mathbf{A}_i)$. □

From Lemma 5.10, Lemma 5.3, and Lemma 5.1, we have the identity that characterizes each join-irreducible subvariety in $\mathcal{ML}$. Finally, in the next theorem we obtain the identity for each proper subvariety of $\mathcal{ML}$.

Theorem 5.12. *Let $\{\mathcal{V}_i : 1 \leq i \leq s\}$ be a finite set of join-irreducible subvarieties in $\Lambda(\mathcal{ML})$ and let $\lambda_{\mathcal{V}_i}(x_{i1}, \ldots, x_{in_i}) \approx 1$ be the identity that characterizes $\mathcal{V}_i$, for each $i = 1, \ldots, s$. If $\mathcal{V} = \bigvee_{i=1}^{s} \mathcal{V}_i$ then the identity*

$$\lambda_{\mathcal{V}}(x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, x_{s1}, \ldots, x_{sn_s}) = \bigvee_{i=1}^{s} \forall (\lambda_{\mathcal{V}_i}(x_{i1}, \ldots, x_{in_i})) \approx 1,$$

characterizes the subvariety $\mathcal{V}$.

Proof. Let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathcal{ML}$. Let us suppose that $\mathbf{A} \in si(\mathcal{V})$. We know that $\mathbf{A} \in si(\mathcal{V}_i)$ for some $i = 1, \ldots, s$. Then $\lambda_{\mathcal{V}_i}(x_{i1}, \ldots, x_{in_i}) \approx 1$ holds in $\mathbf{A}$, and consequently, $\forall (\lambda_{\mathcal{V}_i}(x_{i1}, \ldots, x_{in_i})) \approx 1$ also holds in $\mathbf{A}$. Hence, $\mathbf{A}$ satisfies $\bigvee_{i=1}^{s} \forall (\lambda_{\mathcal{V}_i}(x_{i1}, \ldots, x_{in_i})) \approx 1$. Conversely, let us suppose that $\mathbf{A} \notin si(\mathcal{V})$. Then, $\mathbf{A} \notin si(\mathcal{V}_i)$ for any $i = 1, \ldots, s$. We choose elements $a_{i1}, \ldots, a_{in_i} \in A$ such that $\lambda_{\mathcal{V}_i}(a_{i1}, \ldots, a_{in_i}) < 1$. Then, $\forall (\lambda_{\mathcal{V}_i}(a_{i1}, \ldots, a_{in_i})) < 1$, for each i. Since $\forall \mathbf{A}$ is totally ordered, there exists $t \in \{1, \ldots, s\}$ such that

$$\bigvee_{i=1}^{s} \forall (\lambda_{\mathcal{V}_i}(a_{i1}, \ldots, a_{in_i})) = \forall (\lambda_{\mathcal{V}_t}(a_{t1}, \ldots, a_{tn_t})) < 1.$$

Thus, $\bigvee_{i=1}^{s} \forall (\lambda_{\mathcal{V}_i}(x_{i1}, \ldots, x_{in_i})) \approx 1$ does not hold in $\mathbf{A}$. □

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Cecilia R. Cimadamore

Departamento de Matemática, Universidad Nacional del Sur, Instituto de Matemática de Bahía Blanca (INMABB) (CONICET-UNS), Alem 1253. Bahía Blanca (8000), Argentina

e-mail: `crcima@criba.edu.ar`

J. Patricio Díaz Varela

Departamento de Matemática, Universidad Nacional del Sur, Instituto de Matemática de Bahía Blanca (INMABB) (CONICET-UNS), Alem 1253. Bahía Blanca (8000), Argentina

e-mail: `jpdiazvarela@gmail.com`