Linear Algebra and its Applications 439 (2013) 1286–1293

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

A-partial isometries and generalized inverses

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ARTICLE INFO

Article history:
Received 15 January 2013
Accepted 6 April 2013
Available online 8 May 2013

Submitted by P. Šemrl

AMS classification:
46C05
47A05

Keywords:
Generalized inverses
Partial isometries
Semi-inner products
Compatibility

ABSTRACT

In this work we study the relationship between A-partial isometries and generalized inverses.

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This paper is devoted to the study of the relationship between generalized inverses and A-partial isometries. By A-partial isometries we mean bounded linear operators defined on a Hilbert space that behave (in some sense) as partial isometries when the semi-inner product induced by a positive (semidefinite) operator is considered. This concept was first introduced in [1] and studied also in [3,13] where several results concerning projections and metrical properties were developed. However, properties regarding the relationship between A-partial isometries and generalized inverses have not been fully developed yet. The study of this relationship is motivated by recent publications about classical partial isometries and generalized inverses (see [5,10–12] and the references given there). In fact, classical partial isometries can be defined as those operators whose adjoint coincides with their Moore–Penrose generalized inverse. In this article we shall explore, among other things, if a similar equivalence holds for A-partial isometries. Nevertheless, since the metrical structure induced by a positive operator is weaker than that of a Hilbert space, additional difficulties arise. For instance, the existence of an adjoint operator with respect to the semi-inner product is not guaranteed. Moreover,

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http://dx.doi.org/10.1016/j.laa.2013.04.011

not all A-partial isometries admit generalized inverses. These facts show that special considerations must be taken into account while working with A-partial isometries. Additional conditions, such as compatibility or angle conditions, are required in order to get results which are extensions of those which are valid for classical partial isometries.

Another goal of this paper is to study similarity in the context of A-partial isometries. Many articles deal with the problem of characterizing those operators which are similar to partial isometries (see [5,11,12]). Again, generalized inverses play a fundamental role in many of these characterizations. Therefore, we study some of these results for A-partial isometries. Once more, extra hypotheses are needed in order to obtain similar results in this context.

The contents of the paper are the following. In Section 1 we set up notation and terminology. Furthermore, we collect some facts about A-partial isometries. Section 2 contains the main results of the paper concerning the relationship between A-partial isometries and generalized inverses. Finally, in Section 3 we explore similarity to A-partial isometries.

1. Preliminaries

Throughout this manuscript, $\mathcal{H}$ denotes a Hilbert space, $L(\mathcal{H})$ is the algebra of bounded linear operators defined on $\mathcal{H}$ and $L(\mathcal{H})^+$ is the cone of all positive (semidefinite) operators of $L(\mathcal{H})$, i.e., $L(\mathcal{H})^+ = \{A \in L(\mathcal{H}) : \langle A\xi, \xi\rangle \geqslant 0\ \forall \xi \in \mathcal{H}\}$. For every $T \in L(\mathcal{H})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^*. In the sequel we denote by $\mathcal{S} \dot{+} \mathcal{T}$ the direct sum of the closed subspaces $\mathcal{S}$ and $\mathcal{T}$. In particular, if $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ then $Q_{\mathcal{S}//\mathcal{T}}$ stands for the unique projection with $R(Q) = \mathcal{S}$ and $N(Q) = \mathcal{T}$.

Given $A \in L(\mathcal{H})^+$, we consider the semi-inner product $\langle\ ,\ \rangle_A$ defined by

$$\langle \xi, \eta\rangle_A := \langle A\xi, \eta\rangle\ \forall \xi, \eta \in \mathcal{H}.$$

Naturally, this semi-inner product induces a seminorm, $\|\cdot\|_A$, defined by $\|\xi\|_A = \|A^{1/2}\xi\|$. Besides, this structure induces an adjoint operation. However, this operation is defined for no all bounded linear operator on $\mathcal{H}$, unless A is invertible. Given $T \in L(\mathcal{H})$ we say that $W \in L(\mathcal{H})$ is an A-adjoint of T if

$$\langle T\xi, \eta\rangle_A = \langle \xi, W\eta\rangle_A,$$

for every $\xi, \eta \in \mathcal{H}$ or, equivalently, if $AW = T^*A$. Since the equation $AX = T^*A$ is not solvable for every $T \in L(\mathcal{H})$, then not every $T \in L(\mathcal{H})$ admits an A-adjoint operator. Thus, we shall denote by

$$L_A(\mathcal{H}) := \{T \in L(\mathcal{H}) : T \text{ admits A-adjoint}\}.$$

In particular, given $T \in L_A(\mathcal{H})$ and $\mathcal{M}$ a closed subspace of $\mathcal{H}$ such that $\mathcal{M} \dot{+} N(A) = \mathcal{H}$ we shall denote by $T^{\sharp_{\mathcal{M}}}$ the reduced solution for $\mathcal{M}$ of

$$AX = T^*A. \tag{1.1}$$

This means that $AT^{\sharp_{\mathcal{M}}} = T^*A$ and $R(T^{\sharp_{\mathcal{M}}}) \subseteq \mathcal{M}$. In such case, we shall say that $T^{\sharp_{\mathcal{M}}}$ is a reduced A-adjoint of T. In addition, when no confusion arises, we shall simply write $T^\sharp$ instead of $T^{\sharp_{\mathcal{M}}}$. That is, $T^\sharp$ is a reduced A-adjoint of T. Note that fixed the subspace $\mathcal{M}$, there exists a unique solution of (1.1) with range included in $\mathcal{M}$. Moreover, it is easy to prove that $N(T^{\sharp_{\mathcal{M}}}) = N(T^*A)$. In addition, observe that, in general, $(T^\sharp)^\sharp \neq T$. The reader is referred to [4] for a treatment of reduced solutions.

If $AT = T^*A$ then we say that T is A-selfadjoint. Observe that if T is A-selfadjoint then it does not mean, in general, that $T = T^\sharp$. In fact, $T = T^\sharp$ if and only if T is A-selfadjoint and $R(T) \subseteq \mathcal{M}$.

On the other hand, if $T \in L_A(\mathcal{H})$ then there exits a constant $c > 0$ such that $\|T\xi\|_A \leqslant c\|\xi\|_A$ for every $\xi \in \mathcal{H}$ (see [3, Proposition 1.2]). Then, for every $T \in L_A(\mathcal{H})$ we can define the next seminorm of T

$$\|T\|_A := \sup_{\xi \notin N(A)} \frac{\|T\xi\|_A}{\|\xi\|_A}.$$

In the next lemma we collect some properties of $T^\sharp$ and its relationship with the seminorm $\|.\|_A$. For the proof see [1,2].

Lemma 1.1. *Let $T, W \in L_A(\mathcal{H})$. The following properties hold:*

(1) $TW \in L_A(\mathcal{H})$ and $(TW)^{\sharp_\mathcal{M}} = W^{\sharp_\mathcal{M}} T^{\sharp_\mathcal{M}}$.
(2) $TT^\sharp$ and $T^\sharp T$ are A-selfadjoint operators.
(3) $\|T\|_A^2 = \|T^\sharp\|_A^2 = \|T^\sharp T\|_A$.

Given $T \in L(\mathcal{H})$ with closed range we shall say that $T' \in L(\mathcal{H})$ is a generalized inverse of T if $TT'T = T$. It is well-known that T has closed range if and only if T has a generalized inverse. We desire to study the relationship between A-partial isometries and generalized inverses. This leads to consider weighted generalized inverses. Given $A \in L(\mathcal{H})^+$ we shall say that $T' \in L(\mathcal{H})$ is an A-generalized inverse of T if

$$TT'T = T, T'TT' = T', ATT' = (TT')^*A, AT'T = (T'T)^*A. \tag{1.2}$$

Note that if $A = id$, then T' coincides with the Moore-Pensore generalized inverse of T, $T^\dagger$. The existence of A-generalized inverse is no longer guaranteed for all closed range operator, unless A is invertible. The following result due to Corach et al. characterizes the existence of A-generalized inverses in terms of compatibility. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$ we say that the pair $(A, \mathcal{S})$ is compatible if $\mathcal{P}(A, \mathcal{S}) := \{Q \in L(\mathcal{H}) : Q^2 = Q, R(Q) = \mathcal{S}\ AQ = Q^*A\}$ is not empty. Equivalently, a pair $(A, \mathcal{S})$ is compatible if and only if $\mathcal{S} + \mathcal{S}^{\perp_A} = \mathcal{H}$ where $\mathcal{S}^{\perp_A} = \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle_A = 0\ \forall \eta \in \mathcal{S}\}$. Given a compatible pair $(A, \mathcal{S})$, $\mathcal{P}(A, \mathcal{S})$ has one or infinitely many elements. Indeed, $\mathcal{P}(A, \mathcal{S})$ has a unique element if and only if $\mathcal{S} \cap N(A) = \{0\}$. For a complete treatment on compatibility we recommend [7] and references therein.

Theorem 1.2 [6, Theorem 3.1]**.** *Given $T \in L(\mathcal{H})$ with closed range, T admits an A-generalized inverse if and only if the pairs $(A, R(T))$ and $(A, N(T))$ are compatible.*

Let us now introduce some definitions.

Definition 1.3. Let $T \in L(\mathcal{H})$ we say that:

(1) T is an A-contraction if $\|T\xi\|_A \leqslant \|\xi\|_A$, for all $\xi \in \mathcal{H}$.
(2) T is an A-isometry if $\|T\xi\|_A = \|\xi\|_A$, for all $\xi \in \mathcal{H}$.
(3) T is an A-partial isometry if $\|T\xi\|_A = \|\xi\|_A$ for all $\xi \in N(AT)^{\perp_A}$.

It is straightforward that T is an A-contraction (resp. A-isometry) if and only if $T^*AT \leqslant A$ (resp. $T^*AT = A$). Moreover, $T \in L_A(\mathcal{H})$ is an A-contraction if and only if $\|T\|_A \leqslant 1$. For a deep treatment about A-contractions the reader is referred to [14] and references therein.

In the next results we collect some properties of A-partial isometries. The reader is referred to [1] for their proofs.

Lemma 1.4. *Let $T \in L_A(\mathcal{H})$. Then $\overline{R(T^\sharp T)} + N(A) = N(AT)^{\perp_A}$ for every reduced solution of (1.1), $T^\sharp$.*

Proposition 1.5. *Let $T \in L_A(\mathcal{H})$. Hence, the following statements are equivalent:*

(1) T is an A-partial isometry.
(2) $\|T\xi\|_A = \|\xi\|_A$ for every $\xi \in \overline{R(T^\sharp T)}$.
(1) If furthermore the pair $(A, \overline{R(T^\sharp T)})$ is compatible then these conditions are equivalent to
(3) $T^\sharp T$ is a projection.

In [1,2], given $T \in L_A(\mathcal{H})$ then $T^\sharp$ denotes the A-adjoint of T with $R(T) \subseteq \overline{R(A)}$. Therefore, the proofs of Lemmas 1.1 and 1.4 and Proposition 1.5 given in [1,2], are done under this notation. However, it is simple to check that the same proofs are valid if any reduced A-adjoint of T is considered.

In the next proposition we prove that the condition of $T^\sharp T$ being a projection is independent of the reduced solution $T^\sharp$.

Proposition 1.6. *Let $T \in L_A(\mathcal{H})$. The following conditions are equivalent:*

(1) $T^\sharp T$ is a projection for some reduced A-adjoint of T, $T^\sharp$.
(2) $T^\sharp T$ is a projection for every reduced A-adjoint of T, $T^\sharp$.

Proof. Let $T^{\sharp_1}$, $T^{\sharp_2}$ be the reduced solutions of (1.1) for $\mathcal{M}_1$, $\mathcal{M}_2$ respectively. Then, $T^{\sharp_2} = T^{\sharp_1} + B$ for some $B \in L(\mathcal{H})$ such that $R(B) \subseteq N(A)$. Thus, if $T^{\sharp_1}T$ is a projection then

$$\begin{aligned} A(T^{\sharp_2}T)^2 &= A(T^{\sharp_1}T + T^{\sharp_1}TBT + BTT^{\sharp_1}T + BTBT) \\ &= A(T^{\sharp_1}T + T^{\sharp_1}TBT) \\ &= AT^{\sharp_1}T + (T^{\sharp_1}T)^*ABT \\ &= AT^{\sharp_1}T \\ &= AT^{\sharp_2}T \end{aligned}$$

So, $(T^{\sharp_2}T)^2$ and $T^{\sharp_2}T$ are reduced solutions of the same equation and $R((T^{\sharp_2}T)^2), R(T^{\sharp_2}T) \subseteq \mathcal{M}_2$. Hence, by the uniqueness of the reduced solution we get that $(T^{\sharp_2}T)^2 = T^{\sharp_2}T$. $\square$

2. *A*-partial isometries and generalized inverses

Partial isometries are usually defined as those operators that behave as an isometry onto the orthogonal complement of their kernels. Considering this definition, it is clear that the concept of A-partial isometry extends the one of partial isometry. In fact, a partial isometry is an A—partial isometry taking $A = id$. However, partial isometries can also be defined in the following equivalent ways: T is a partial isometry if and only if

(i) T^* is generalized inverse of T, i.e., $T = TT^*T$
or, equivalently,
(ii) $T^* = T^\dagger$, where $T^\dagger$ denotes the Moore-Penrose generalized inverse of T.

Therefore, the aim of this section is to study if similar equivalent conditions hold for A-partial isometries. More precisely, regarding conditions i) and ii) the question that naturally arises is if the following statements are equivalent:

(a) $T \in L_A(\mathcal{H})$ is an A-partial isometry;
(b) $TT^\sharp T = T$;
(c) $T^\sharp$ is an A-generalized inverse of T.

The first part of this section is devoted to answer this question. As we shall see extra conditions are required in order to obtain the equivalence. For instance, not all A-partial isometries admit a generalized inverse. This is shown in the next example:

Example 2.1. Let $A \in L(\mathcal{H})^+$ and consider the matrix representation given by the decomposition $\mathcal{H} = \overline{R(A)} \oplus N(A)$. Then $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Now, define $T = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ with $d \in L(N(A))$ with not closed range. Hence, T has not closed range (so, it has not a generalized inverse) and it is easy to check that $T^*AT = A$, i.e., T is an A-isometry and, in particular, T is an A-partial isometry.

In what follows, we consider T with closed range. Under this assumption, we begin by showing that (b) and (c) are equivalent.

Proposition 2.2. *Let $T \in L_A(\mathcal{H})$. The following assertions are equivalent:*

(1) $TT^\sharp T = T$.
(2) $T^\sharp$ is an A-generalized inverse of T.

Proof. Let $TT^\sharp T = T$. Then, as $AT^\sharp = T^*A$, we have $AT^\sharp = T^*(T^\sharp)^*T^*A = T^*(T^\sharp)^*AT^\sharp = T^*ATT^\sharp = AT^\sharp TT^\sharp$. Hence, $T^\sharp$ and $T^\sharp TT^\sharp$ are both reduced solutions of the same equation with $R(T^\sharp), R(T^\sharp TT^\sharp) \subseteq \mathcal{M}$. So, by the uniqueness of the reduced solution we get that $T^\sharp = T^\sharp TT^\sharp$. Finally, by Lemma 1.1, $TT^\sharp$ and $TT^\sharp$ are A-selfadjoint. So, $T^\sharp$ is an A-generalized inverse of T. The converse is trivial. $\square$

Now, let us study the relationship between statmentes (a) and (b):

Proposition 2.3. *Let $T \in L_A(\mathcal{H})$. If $TT^\sharp T = T$ then T is an A-partial isometry. The converse is false, in general.*

Proof. Clearly, if $TT^\sharp T = T$ then $T^\sharp T$ is a projection and, by Proposition 1.5, T is an A-partial isometry.

In order to see that, in general, the converse fails, let us consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ and $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. Then, $T^*AT = A$, i.e, T is an A-isometry and, in particular, an A-partial isometry. However, let us show that $T \neq TT^\sharp T$ for every reduced A-adjoint of T, $T^\sharp$. For this, note that $T^\sharp = Q_{\mathcal{M}//N(A)}A^\dagger T^*A$ where $A^\dagger$ denotes the Moore-Penrose generalized inverse of A. Hence,

$$TT^\sharp T = T(I - Q_{N(A)//\mathcal{M}})A^\dagger T^*AT = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - TQ_{N(A)//\mathcal{M}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, since $N(A) = span\{(0, 1)\}$ then $Q_{N(A)//\mathcal{M}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$ for some $\lambda \in \mathbb{C}$. So,

$$TT^\sharp T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1-\lambda & 0 \end{pmatrix} \neq T,$$

for every $\lambda \in \mathbb{C}$. $\square$

Last proposition implies that extra hypotheses are required in order to T being a A-partial isometry implies $TT^\sharp T = T$. This implication is studied in our next theorem. Before that, let us present a technical result.

Lemma 2.4. *Let $T \in L_A(\mathcal{H})$ with closed range such that $(A, R(T))$ is compatible. Then, $\mathcal{H} = R(T) + N(T^\sharp)$ and $R(T^\sharp T) = R(T^\sharp)$.*

Proof. Let the pair $(A, R(T))$ be compatible. Then $\mathcal{H} = R(T) + R(T)^{\perp_A}$. Now, $R(T)^{\perp_A} = N(T^*A) = N(T^\sharp)$. Thus, $\mathcal{H} = R(T) + N(T^\sharp)$ and so, $R(T^\sharp T) = R(T^\sharp)$. $\square$

Theorem 2.5. *Let $T \in L_A(\mathcal{H})$ with closed range. The following conditions are equivalent:*

(1) $TT^\sharp T = T$;
(2) *T is an A-partial isometry such that $(A, R(T))$ is compatible and* $\mathcal{H} = R(T^\sharp) + N(T)$;
(3) the next two conditions hold:
  (1) *$T^\sharp T$ is a projection (i.e., T is an A-partial isometry);*
  (2) *$\mathcal{P}(A, R(T))$ has a unique element.*

Proof. 1 $\Leftrightarrow$ 2 Suppose that $TT^\sharp T = T$. Then, by Proposition 2.3, T is an A-partial isometry. Moreover, by Proposition 2.2 and Theorem 1.2, the pair $(A, R(T))$ is compatible. In addition, from $TT^\sharp T = T$, we have that $T^\sharp T$ is a projection. So, $\mathcal{H} = R(T^\sharp T) + N(T^\sharp T)$. Now, from $TT^\sharp T = T$, we get that $N(T^\sharp T) = N(T)$ and , by the previous Lemma, $R(T^\sharp T) = R(T^\sharp)$. Therefore, $\mathcal{H} = R(T^\sharp) + N(T)$.

Conversely, if the pair $(A, R(T))$ is compatible then, by Lemma 2.4, $\mathcal{H} = R(T) + N(T^\sharp)$ and $R(T^\sharp T) = R(T^\sharp)$. Furthermore, as $N(T) \subseteq N(AT) = R(T^\sharp T)^{\perp_A}$ then $\mathcal{H} = \overline{R(T^\sharp T)} + \overline{R(T^\sharp T)}^{\perp_A}$, i.e., the pair $(A, \overline{R(T^\sharp T)})$ is compatible. Thus, if T is an A-partial isometry then $T^\sharp T$ is an A-selfadjoint projection. Hence, as $R(T^\sharp T) = R(T^\sharp)$, then $T^\sharp TT^\sharp = T^\sharp$. Moreover, from this last equality we also obtain that $TT^\sharp$ is a projection. Now, as $\mathcal{H} = R(T^\sharp) + N(T)$, then $R(TT^\sharp) = R(T)$. So, $TT^\sharp T = T$.

1 $\Leftrightarrow$ 3 Assume that $TT^\sharp T = T$. Then, item (a) holds trivially. Moreover, by Proposition 2.2 and Theorem 1.2, the pair $(A, R(T))$ is compatible, i.e., $\mathcal{P}(A, R(T))$ is not empty. In order to see that $\mathcal{P}(A, R(T))$ has a unique element, we must show that $R(T) \cap N(A) = \{0\}$. Now, since $TT^\sharp T = T$, then $TT^\sharp = Q_{R(T)//N(TT^\sharp)}$, i.e., $R(T) \cap N(TT^\sharp) = \{0\}$. On the other hand, $N(A) \subseteq N(T^*A) = N(T^\sharp) \subseteq N(TT^\sharp)$ and the assertion follows.

Conversely, suppose that items (a) and (b) hold. Then, $(A, R(T))$ is compatible and, by Lemma 2.4, $R(T^\sharp T) = R(T^\sharp)$. Thus, $T^\sharp T \in \mathcal{P}(A, R(T^\sharp))$ and so $T^\sharp TT^\sharp = T^\sharp$. From this, we obtain that $TT^\sharp$ is a projection. Now, since $\mathcal{H} = R(T^\sharp) + R(T^\sharp)^{\perp_A}$ (because of the compatibility of the pair $(A, R(T^\sharp))$) and $R(T) \cap N(A) = \{0\}$ we get that $R(T^\sharp)^{\perp_A} = N(AT) = N(T)$ and $R(TT^\sharp) = R(T)$. Therefore $TT^\sharp \in \mathcal{P}(A, R(T))$ and so $TT^\sharp T = T$. $\square$

Corollary 2.6. *Let $T \in L_A(\mathcal{H})$. The following statements are equivalent:*

(1) *$TT^\sharp T = T$ for some reduced A-adjoint of T, $T^\sharp$.*
(2) *$T\tilde{T}T = T$ for every A-adjoint of T, $\tilde{T}$.*

Proof. Let $TT^\sharp T = T$ for some reduced A-adjoint of T. Hence, by Theorem 2.5, $R(T) \cap N(A) = \{0\}$. Now, let $\tilde{T}$ be an A-adjoint of T (no necessarily reduced A—adjoint of T). Then, as $AT^\sharp = T^*A = A\tilde{T}$, we have that $\tilde{T} = T^\sharp + B$ for some $B \in L(\mathcal{H})$ such that $AB = 0$. Therefore, $T\tilde{T}T = TT^\sharp T + TBT = T + TBT$. Now, $ATBT = (T^\sharp)^*ABT = 0$, so $R(TBT) \subseteq R(T) \cap N(A) = \{0\}$, i.e., $TBT = 0$ and so $T\tilde{T}T = T$. The converse is trivial. $\square$

In [10] it is proven that a contraction is a partial isometry if and only if it admits a contractive generalized inverse. Our next goal is to get a similar result for A-partial isometries. With this purpose, we shall denote by $\cos_0(\mathcal{S}, \mathcal{T})$ the cosine of the Dixmier angle between the closed subspaces $\mathcal{S}$ and $\mathcal{T}$, i.e,

$$\cos_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \nu \rangle| : \xi \in \mathcal{S}, \nu \in \mathcal{T} \text{ and } \|\xi\| \leqslant 1, \|\nu\| \leqslant 1\}.$$

The next result will be useful in what follows.

Theorem 2.7. *Let $\mathcal{S}, \mathcal{T}$ be two closed subspaces of $\mathcal{H}$. Then, $\cos_0(\mathcal{S}, \mathcal{T}) < 1$ if and only if $\mathcal{S} \cap \mathcal{T} = \{0\}$ and $\mathcal{S} + \mathcal{T}$ is a closed subspace.*

For the proof of the previous theorem and a treatment on the theory of angles between subspaces the reader is referred to [8,9].

Proposition 2.8. *Let $T \in L_A(\mathcal{H})$ be an A-contraction. The following conditions are equivalent:*

(1) $TT^\sharp T = T$;
(2) there exists an A-generalized inverse of T, $S \in L_A(\mathcal{H})$, *such that* $\|S\|_A \leqslant 1$ *and* $\cos_0(R(S), N(A)) < 1$.

Proof. $1 \Rightarrow 2$. The result follows by taking $S = T^\sharp$, and applying Theorem 2.3. in [4].

$2 \Rightarrow 1$. First note that since T admits A-generalized inverse then the pair $(A, R(T))$ is compatible.

Secondly, the condition $\cos_0(R(S), N(A)) < 1$ implies that $R(S) \subseteq \mathcal{M}$ for some closed subspace of $\mathcal{H}$ such that $\mathcal{M} \dot{+} N(A) = \mathcal{H}$ (for instance, $\mathcal{M} = (R(S) + N(A))^\perp \dot{+} R(S)$) Now, from $TST = T$ we have that ST is a projection. Moreover, since $S, T \in L_A(\mathcal{H})$ are A-contractions then $\|ST\|_A \leqslant \|S\|_A\|T\|_A \leqslant 1$ and we get that ST is A-selfadjoint (see [2]). Furthermore, $R(ST) \subseteq R(S) \subseteq \mathcal{M}$, then $(ST)^{\sharp_\mathcal{M}} = ST$. Therefore, $T^{\sharp_\mathcal{M}} = (TST)^{\sharp_\mathcal{M}} = (ST)^{\sharp_\mathcal{M}} T^{\sharp_\mathcal{M}} = STT^{\sharp_\mathcal{M}}$, and so $R(T^{\sharp_\mathcal{M}}) \subseteq R(ST)$. It means that $\overline{R(T^{\sharp_\mathcal{M}}T)} \subseteq R(ST)$. As a consequence, for every $\xi \in \overline{R(T^{\sharp_\mathcal{M}}T)}$ we have

$$\|\xi\|_A = \|ST\xi\|_A \leqslant \|T\xi\|_A \leqslant \|\xi\|_A.$$

Hence, T is an A-partial isometry.

Finally, from $TST = T$ we have that $R(ST) + N(T) = \mathcal{H}$. Now, since $R(ST) = R(T^{\sharp_\mathcal{M}} S^{\sharp_\mathcal{M}}) \subseteq R(T^{\sharp_\mathcal{M}})$, we get $\mathcal{H} = R(T^{\sharp_\mathcal{M}}) + N(T)$, and the result follows by Theorem 2.5. $\square$

3. Similarity to *A*-partial isometries

The aim of this section is to study conditions for the similarity to A-partial isometries by means of generalized inverses. An interesting result in this direction, but for partial isometries, can be found in [5] where the authors prove that T is similar to a partial isometry if and only if T^* is similar (by a positive operator) to a generalized inverse of T. Our goal is to study this problem for A-partial isometries.

Before that, let us recall some concepts and introduce some notation. Two operators $T, W \in L(\mathcal{H})$ are similar if there exists an invertible operator $L \in L(\mathcal{H})$ such that $T = LWL^{-1}$, in such case, we denote $T \sim W$. If the operator L is also positive then we denote $T \sim_+ W$ and, if in addition, L is A-positive we write $T \sim_{+_A} W$. An operator L is A-positive if AL is positive. Moreover, it holds that L is A-positive if and only if L is $A^{1/2}$-positive. In this section, given $T \in L_A(\mathcal{H})$, $T^\sharp$ denotes the A-adjoint of T with $R(T^\sharp) \subseteq \overline{R(A)}$.

Theorem 3.1. *Let $T \in L_A(\mathcal{H})$ with closed range. Hence, the following conditions are equivalent:*

(1) $T^\sharp \sim_{+_A} T'$ *for some generalized inverse of* T, T';
(2) there exists $V \in L_A(\mathcal{H})$ *such that* $T \sim_{+_A} V$ *with* $VV^\sharp V = V$.

Proof. $1 \Rightarrow 2$. Let $T' = LT^\sharp L^{-1}$ with $L \in L(\mathcal{H})^+$ an A-positive operator. Then, $T = TT'T = TLT^\sharp L^{-1}T$ and so $L^{-1/2}TL^{1/2} = (L^{-1/2}TL^{1/2})(L^{1/2}T^\sharp L^{-1/2})(L^{-1/2}TL^{1/2})$. Define $V := L^{-1/2}TL^{1/2}$ and let us show that $V^\sharp = L^{1/2}T^\sharp L^{-1/2}$. For this, note first that $AL^{1/2}T^\sharp L^{-1/2} = L^{1/2}AT^\sharp L^{-1/2} = L^{1/2}T^*AL^{-1/2} = L^{1/2}T^*L^{-1/2}A = V^*A$. So, $L^{1/2}T^\sharp L^{-1/2}$ is an A-adjoint of V. Furthermore, $R(L^{1/2}T^\sharp L^{-1/2}) = R(L^{1/2}T^\sharp) \subseteq L^{1/2}\overline{R(A)} \subseteq \overline{R(L^{1/2}A)} = \overline{R(AL^{1/2})} \subseteq \overline{R(A)}$. Thus, $L^{1/2}T^\sharp L^{-1/2} = V^\sharp$ and the result is obtained.

$2 \Rightarrow 1$ Let $V \in L_A(\mathcal{H})$ such that $T \sim_{+_A} V$ with $VV^\sharp V = V$. Thus, $T = LVL^{-1}$ with $L \in L(\mathcal{H})^+$ such that $AL = LA \in L(\mathcal{H})^+$. Hence, $V^\sharp = LT^\sharp L^{-1}$. This last fact can be proved as above. So, as $V = VV^\sharp V$ we obtain that $L^{-1}TL = L^{-1}TLLT^\sharp L^{-1}L^{-1}TL$, i.e., $T = TLLT^\sharp(LL)^{-1}T$. So, the result follows considering $T' = LLT^\sharp(LL)^{-1}$. $\square$

Now, applying Theorems 2.5 and 3.1 we obtain the following result regarding A-partial isometries:

Corollary 3.2. *Let $T \in L_A(\mathcal{H})$ with closed range. Hence, if $T^\sharp \sim_{+_A} T'$ for some generalized inverse of T, T', then $T \sim V$ for some A-partial isometry V.*

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