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Representations of a symplectic type subalgebra of W_∞^N

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In this paper we classify the irreducible quasifinite highest weight modules over the symplectic type Lie subalgebra of the Lie algebra of all regular differential operators on circle that kill constants. We also realize them in terms of the representations theory of the complex Lie algebra $g\ell_\infty^{[m]}$ of infinite matrices with a finite number of non-zero diagonals with entries in the algebra of truncated polynomials and the corresponding subalgebras of type C. © *2011 American Institute of Physics*. [doi:10.1063/1.3596180]

I. INTRODUCTION

The study of representations of the Lie algebra $W_{1+\infty}$(the universal central extension of the Lie algebra of differential operators on the circle) was initiated in Ref. 7, where a characterization of its irreducible quasifinite highest weight representations was given. These modules were constructed in terms of irreducible highest weight representations of the Lie algebra of infinite matrices and the unitary ones were described. This analysis was continued in the framework of vertex algebra theory for the $W_{1+\infty}$ algebra[4,8] and for its matrix version.[3] The case of orthogonal subalgebras of $W_{1+\infty}$ was studied in Ref. 9. The symplectic subalgebra of $W_{1+\infty}$ was considered in Ref. 1 in relation to number theory.

In Ref. 6 a similar study was carried out for the Lie subalgebras $W_{\infty,p}$ of $W_{1+\infty}$, where $W_{\infty,p}$ ($p \in \mathbb{C}[x]$) is the central extension of the Lie algebra $\mathcal{D}p(t\partial_t)$ of differential operators on the circle that are a multiple of $p(t\partial_t)$. The most important of these subalgebras is $W_\infty = W_{\infty,x}$ that is obtained by taking $p(x)$. In this paper, Kac and Liberati also give some general results on the characterization of quasifinite representations of any $\mathbb{Z}$-graded Lie algebra. In the present paper we classify all irreducible quasifinite highest weight modules of the symplectic type $\mathcal{D}_{x,\theta}^N$, subalgebra of the Lie algebra of all regular differential operators on circle that kill constants, given by the minus fixed points of the anti-involution θ related to those introduced in Ref. 10.

The paper is organized as follows: In Sec. II we present some standard facts of representations of $\widehat{g\ell}_\infty$ and the subalgebra of type C. In Sec. III we introduce the subalgebra $\mathcal{D}_{x,\theta}^N$, and we study its structure of parabolic subalgebras. In Sec. IV a characterization of quasifinite highest weight modules of $\widehat{\mathcal{D}_{x,\theta}^N}$ is given. In Sec. V we establish the interplay between $\mathcal{D}_{x,\theta}^N\widehat{g\ell}_\infty$ and its subalgebra of type C. Finally, in Sec. VI we give the realization of quasifinite highest weight modules of $\widehat{\mathcal{D}_{x,\theta}^N}$.

II. LIE ALGEBRAS $\widehat{g\ell}_\infty^{[m]}$ AND $c_\infty^{[m]}$

Denote by $R_m = \mathbb{C}[u]/(u^{m+1})$, the quotient algebra of the polynomial algebra $\mathbb{C}[u]$ by the ideal generated by $u^{m+1}(m \in \mathbb{Z}_{\geqslant 0})$. Let $\mathbf{1}$ be the identity element in R_m. Denote by $g\ell_\infty^{[m]}$ the complex Lie algebra of all infinite matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with only finitely many nonzero diagonals with entries in R_m. Denote by $E_{i,j}$ the infinite matrix with 1 at (i, j)-entry and 0 elsewhere. There is a natural

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automorphism ν of $g\ell_\infty^{[m]}$ given by

$$\nu(E_{i,j}) = E_{i+1,j+1}. \tag{2.1}$$

Let the weight of $E_{i,j}$ be $j-i$. This defines the *principal* $\mathbb{Z}$-gradation $g\ell_\infty^{[m]} = \oplus_{j\in\mathbb{Z}} \left(g\ell_\infty^{[m]}\right)_j$. Denote by $\widehat{g\ell}_\infty^{[m]} = g\ell_\infty^{[m]} \oplus R_m$ the central extension of $g\ell_\infty^{[m]}$ given by the following two-cocycle with values in R_m:

$$C(A, B) = \mathrm{Tr}([J, A]\, B), \tag{2.2}$$

where $J = \sum_{i\le 0} E_{i,i}$. The $\mathbb{Z}$-gradation of the Lie algebra $g\ell_\infty^{[m]}$ extends to $\widehat{g\ell}_\infty^{[m]}$ by putting the weight of R_m to be 0. In particular we have the *triangular decomposition*

$$\widehat{g\ell}_\infty^{[m]} = \left(\widehat{g\ell}_\infty^{[m]}\right)_- \oplus \left(\widehat{g\ell}_\infty^{[m]}\right)_0 \oplus \left(\widehat{g\ell}_\infty^{[m]}\right)_+, \tag{2.3}$$

where

$$\left(\widehat{g\ell}_\infty^{[m]}\right)_\pm = \bigoplus_{j\in\mathbb{N}} \left(\widehat{g\ell}_\infty^{[m]}\right)_{\pm j} \qquad \text{and} \qquad \left(\widehat{g\ell}_\infty^{[m]}\right)_0 = \left(g\ell_\infty^{[m]}\right)_0 \oplus R_m.$$

Given $\lambda \in \left(\widehat{g\ell}_\infty^{[m]}\right)_0^*$, we let,

$$\begin{aligned} c_i &= \lambda(u^i), \\ {}^a\lambda_j^{(i)} &= \lambda(u^i\, E_{j,j}), \\ {}^a h_j^{(i)} &= {}^a\lambda_j^{(i)} - {}^a\lambda_{j+1}^{(i)} + \delta_{j,0}\, c_i, \end{aligned} \tag{2.4}$$

where $j\in\mathbb{Z}$ and $i = 0, \dots, m$. Let $L(\widehat{g\ell}_\infty^{[m]}, \lambda)$ be the irreducible highest weight $\widehat{g\ell}_\infty^{[m]}$-module with highest weight λ. The ${}^a\lambda_j^{(i)}$ are called the *labels* and c_i are the *central charges* of $L(\widehat{g\ell}_\infty^{[m]}, \lambda)$.

Consider the vector space $R_m[t, t^{-1}]$, and take the R_m-basis $v_i = t^{-i}$, $i\in\mathbb{Z}$. Now consider the following $\mathbb{C}$-bilinear form on $R_m[t, t^{-1}]$:

$$C(u^m v_i, u^n v_j) = u^m(-u^n)\delta_{i,1-j}\,. \tag{2.5}$$

Denote by $\bar{c}_\infty^{[m]}$ the Lie subalgebra of $g\ell_\infty^{[m]}$, which preserves the bilinear form $C(\ ,\ )$. We have

$$\bar{c}_\infty^{[m]} = \{(a_{ij}(u))_{i,j\in\mathbb{Z}} \in g\ell_\infty^{[m]} \mid a_{ij}(u) = (-1)^{i+j+1} a_{1-j,1-i}(-u)\}\,.$$

Denote by $c_\infty^{[m]} = \bar{c}_\infty^{[m]} \oplus R_m$ the central extension of $\bar{c}_\infty^{[m]}$ given by the restriction of the two-cocycle (2.2), defined in $g\ell_\infty^{[m]}$. This subalgebra inherits from $\widehat{g\ell}_\infty^{[m]}$ the principal $\mathbb{Z}$-gradation and the triangular decomposition, (see Refs. 5 and 9 for notation)

$$c_\infty^{[m]} = \oplus_{j\in\mathbb{Z}}(c_\infty^{[m]})_j \qquad c_\infty^{[m]} = (c_\infty^{[m]})_+ \oplus (c_\infty^{[m]})_0 \oplus (c_\infty^{[m]})_-\,.$$

In particular when $m = 0$, we have the usual Lie subalgebra of $g\ell_\infty$, denoted by c_∞.

Given $\lambda \in (c_\infty^{[m]})_0^*$, denote by $L(c_\infty^{[m]}; \lambda)$ the irreducible highest weight module over $c_\infty^{[m]}$ with highest weight λ. For each $\lambda \in (c_\infty^{[m]})_0^*$, we let

$$\begin{aligned} c_i &= \lambda(u^i), \\ {}^c\lambda_j^{(i)} &= \lambda(u^i\, E_{j,j} - (-u)^i\, E_{1-j,1-j}), \\ {}^c h_j^{(i)} &= {}^c\lambda_j^{(i)} - {}^c\lambda_{1+j}^{(i)}, \\ {}^c h_0^{(i)} &= {}^c\lambda_1^{(i)} + c_i \quad (i \text{ even}), \end{aligned} \tag{2.6}$$

where $j\in\mathbb{N}$ and $i = 0, \dots, m$. The ${}^c\lambda_j^{(i)}$ are called the *labels* and c_i are the ¨*central charges* of $L(c_\infty^{[m]}, \lambda)$.

III. THE SUBALGEBRA $\widehat{\mathcal{D}^N_{\theta,x}}$ AND ITS STRUCTURE OF PARABOLIC SUBALGEBRAS

Let N be a positive integer. Denote by $\mathcal{D}^N_{as}$ the associative algebra of all regular matrix differential operators on $\mathbb{C}^\times$ of the form

$$E = e_k(t)\partial_t^k + e_{k-1}(t)\partial_t^{k-1} + \cdots + e_0(t),$$

where

$$e_i(t) \in \mathrm{Mat}_N\mathbb{C}[t, t^{-1}]$$

and denote by $\mathcal{D}^N$ the corresponding Lie algebra. Here and further we denote by $\mathrm{Mat}_N R$ the associative algebra of all $N \times N$ matrices over an algebra R.

Set $D = t\partial_t$. The elements $t^k D^m e_{i,j}$ ($k \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geqslant 0}$, $i, j \in \{1 \dots N\}$) form a basis of $\mathcal{D}^N$. Here and further $e_{i,j}$ is the standard basis of $\mathrm{Mat}_N\mathbb{C}$.

We have the following two-cocycle on $\mathcal{D}^N$, (cf. Ref. 3)

$$\psi(t^r f(D)A, t^s g(D)B) = \begin{cases} \mathrm{tr}(AB) \displaystyle\sum_{-r\le m\le -1} f(m)g(m+r) & \text{if} \quad r = -s > 0 \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

where $r, s \in \mathbb{Z}$, $f, g \in \mathbb{C}[w]$, $A, B \in \mathrm{Mat}_N\mathbb{C}$ and tr is the usual trace.

Let

$$\widehat{\mathcal{D}^N} = \mathcal{D}^N \bigoplus \mathbb{C}C,$$

denote the central extension of $\mathcal{D}^N$ by a one-dimensional center $\mathbb{C}C$ corresponding to the two-cocycle ψ. The bracket in $\widehat{\mathcal{D}^N}$ is given by

$$[t^r f(D)A, t^s g(D)B] = t^{r+s}\Big(f(D+s)g(D)AB - f(D)g(D+r)BA\Big) + \psi\big(t^r f(D)A, t^s g(D)B\big)\, C. \tag{3.2}$$

Define the *weight* wt on $\widehat{\mathcal{D}^N}$ by

$$\mathrm{wt}\, t^k f(D)\, e_{i,j} = kN + i - j, \quad \mathrm{wt}\, C = 0. \tag{3.3}$$

This gives us the *principal* $\mathbb{Z}$*-gradation of* $\widehat{\mathcal{D}^N}$:

$$\widehat{\mathcal{D}^N} = \bigoplus_{j\in\mathbb{Z}} (\widehat{\mathcal{D}^N})_j. \tag{3.4}$$

Consider the following Lie subalgebra of $\mathcal{D}^N$

$$\mathcal{D}^N_x = \mathcal{D}^N DI.$$

Denote by W^N_∞ the central extension of $\mathcal{D}^N_x$ by $\mathbb{C}C$ corresponding to the restriction of the two-cocycle ψ.

Thus, W^N_∞ inherits the principal gradation of $\widehat{\mathcal{D}^N}$, namely,

$$W^N_\infty = \bigoplus_{j\in\mathbb{Z}} (W^N_\infty)_j.$$

Consider

$$\widetilde{\theta}(t^k f(D)De_{i,j}) = -t^k f(-D-k)De_{j,i},$$

the anti-involution corresponding to those that defines the symplectic type conformal subalgebra in $gc_{N,x}$ (cf. Ref. 10 p. 56). This anti-involution does not preserve the principal gradation of $\mathcal{D}^N$. However, it is conjugated by the automorphism $\sigma(t^k f(D)De_{i,j}) = t^k f(D)De_{i,N+1-j}$, to the following anti-involution

$$\theta(t^k f(D)De_{i,j}) = -t^k f(-D-k)De_{N+1-j,N+1-i}, \tag{3.5}$$

where $k \in \mathbb{Z}$. Observe that this coincides with Bloch's anti-involution for N=1 (cf. Ref. 1).

We denote by $\mathcal{D}_{\theta,x}^N$ the Lie subalgebra of $\mathcal{D}_x^N$ consisting of minus θ-fixed points.

Here and further we denote $D_k = D + \frac{k}{2}$. A set of generators of this subalgebra is

$$\{t^k \left(f(D_k)De_{i,N+1-j} + f(-D_k)De_{j,N+1-i} \right) : k \in \mathbb{Z},\ f \in \mathbb{C}[x], \quad 1 \leqslant i < j \leqslant N\}$$

together with the generators in the opposite diagonal

$$\{t^k f(D_k)De_{i,N+1-i} : k \in \mathbb{Z}, \quad f \in \mathbb{C}[x] \text{ even}, \quad 1 \leqslant i \leqslant N\}.$$

We denote again by ψ the restriction of the two-cocycle in (3.1) to $\mathcal{D}_{\theta,x}^N$. Denote by $\widehat{\mathcal{D}_{\theta,x}^N}$ the central extension of $\mathcal{D}_{\theta,x}^N$ by the one dimensional center $\mathbb{C}C$ corresponding to the two-cocycle above.Thus, $\widehat{\mathcal{D}_{\theta,x}^N}$ inherits the principal gradation of W_∞^N, namely,

$$\widehat{\mathcal{D}_{\theta,x}^N} = \bigoplus_{j \in \mathbb{Z}} (\widehat{\mathcal{D}_{\theta,x}^N})_j .$$

Recall that a *parabolic subalgebra* $\mathcal{P}$ of $\widehat{\mathcal{D}_{\theta,x}^N}$ is a subalgebra $\mathcal{P} = \bigoplus_{j \in \mathbb{Z}} \mathcal{P}_j$, where $\mathcal{P}_j = (\widehat{\mathcal{D}_{\theta,x}^N})_j$ if $j \geqslant 0$, and $\mathcal{P}_j \neq 0$ for some $j < 0$.

Given $a \in (\widehat{\mathcal{D}_{\theta,x}^N})_{-1}$, with $a \neq 0$, define $\mathcal{P}^a = \bigoplus_{j \in \mathbb{Z}} \mathcal{P}_j^a$, where $\mathcal{P}_j^a = (\widehat{\mathcal{D}_{\theta,x}^N})_j$ if $j \geqslant 0$, and recursively

$$\mathcal{P}_{-1}^a = \sum [\ldots [[a, (\widehat{\mathcal{D}_{\theta,x}^N})_0], (\widehat{\mathcal{D}_{\theta,x}^N})_0], \ldots] \quad \text{and} \quad \mathcal{P}_{-j-1}^a = [\mathcal{P}_{-1}^a, \mathcal{P}_{-j}^a],$$

for $j > 0$.

Here and further, we denote by

$$\delta_{n,\text{even}} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

for any $n \in \mathbb{N}$. Similarly for $\delta_{n,\text{odd}}$. Also we denote by $[s]$ the closest integer to s, which is not larger than s.

Remark 3.1:

(a) We have that the following properties are satisfied by $\widehat{\mathcal{D}_{\theta,x}^N}$:

(P_1) $(\widehat{\mathcal{D}_{\theta,x}^N})_0$ is commutative,

(P_2) if $a \in (\widehat{\mathcal{D}_{\theta,x}^N})_{-j}$ $(j > 0)$ and $[a, (\widehat{\mathcal{D}_{\theta,x}^N})_1] = 0$, then $a = 0$.

Observe that (P_1) is immediate from the definition of $(\widehat{\mathcal{D}_{\theta,x}^N})_0$. (P_2) follows by computing the bracket

$$0 = [a\ ,\ De_{q+1,q} + De_{N+1-q,N-q}],$$

with $a \in (\widehat{\mathcal{D}}_{\theta,x})_{-j}$; $j, q \in \mathbb{N}, 1 \leq q \leq \left[\frac{N}{2}\right] - \delta_{N,\text{even}}$, under the following considerations:

• If $j = kN$, $k \in \mathbb{Z}_{>0}$, let q be such that $1 \leqslant q \leqslant \left[\frac{N}{2}\right] - \delta_{N,\text{even}}$;

• If $j = kN + r$, $k \in \mathbb{Z}_{>0}$, with $1 \leq r \leq N-1$ and suppose that $1 \leq r < \left[\frac{N}{2}\right] - \delta_{N,\text{even}}$ then q is chosen in

$$\left\{2 \leq q \leq \frac{N-r+1}{2}\right\} \bigcup \left\{\left[\frac{r-1}{2}\right] - \delta_{N,\text{even}} \leq q \leq r\right\};$$

and if $\left[\frac{N}{2}\right] - \delta_{N,\text{even}} \leq r \leq N-1$, q is running in

$$\left\{2 \leq q \leq \frac{N-r+1}{2}\right\} \bigcup \left\{1 \leq q \leq \left[\frac{r-1}{2}\right] + \delta_{N,\text{even}}\right\} \bigcup \{q = N - r + 1\}.$$

(b) By Lemmas 2.1 and 2.2 in Ref. 6 we have that for any parabolic subalgebra $\mathcal{P}$ of $\widehat{\mathcal{D}_{\theta,x}^N}$, $\mathcal{P}_{-k} \neq 0$, implies $\mathcal{P}_{-k+1} \neq 0$ and $\mathcal{P}^a$ is the minimal subalgebra containing a.

Following Ref. 6 we call a parabolic subalgebra $\mathcal{P}$*non-degenerate* if $\mathcal{P}_{-j}$ has finite codimension in $(\widehat{\mathcal{D}_{\theta,x}^{N}})_{-j}$, for all $j > 0$, and an element $a \in (\widehat{\mathcal{D}_{\theta,x}^{N}})_{-1}$ *non-degenerate* if $\mathcal{P}^a$ is non-degenerate.

Let $\mathcal{P}$ be a parabolic subalgebra of $\widehat{\mathcal{D}_{\theta,x}^{N}}$. Using (3.3) and observing that for each positive integer j there exists a positive integer k such that $j = kN + r = (k+1)N - (N-r)$ with $0 \leqslant r \leqslant N-1$, we can describe $\mathcal{P}_{-j}$ as follows:

$$\mathcal{P}_{-j} = \Big\{t^{-k}\Big(f_i(D_{-k})\,e_{i,\,i+r} + f_i(-D_{-k})\,e_{N+1-r-i,\,N+1-i}\Big) \,:\, f_i \in I^i_{-j} \text{ and } 1 \le i \le \Big[\frac{N+1-r}{2}\Big]\Big\}$$
$$\bigcup \Big\{t^{-(k+1)}\Big(g_i(D_{-(k+1)})\,e_{i,\,i-N+r} + g_i(-D_{-(k+1)})\,e_{2N+1-i-r,\,N+1-i}\Big) \,:\, g_i \in L^i_{-j} \text{ and } N-r+1 \le i \le \Big[\frac{2N-r+1}{2}\Big]\Big\},$$

where I^i_{-j} and L^i_{j} are subspaces of $w\mathbb{C}[w]$. Let us take i such that $1 \le i \le \left[\frac{N+1-r}{2}\right]$, $f_i(w) \in I^i_{-j}$, and $g_i(w) \in \mathbb{C}[w]$. Computing the bracket

$$\Big[t^{-k}\Big(f_i(D_k)e_{i,\,i+r} + f_i(-D_{-k})e_{N+1-i-r,\,N+1-i}\Big),\; g_i(D_0)De_{i,\,i} + g_i(-D_0)De_{N+1-i,\,N+1-i}\Big]$$

for $j = Nk$ with $N \ge 2$, we see that I^i_{-j} satisfies

$$A^i_j I^i_{-j} \subseteq I^i_{-j}, \tag{3.7}$$

where $A^i_j = \{g_i(w)w - g_i(w-k)(w-k) : g_i(w) \in \mathbb{C}[w]\}$. For $j = Nk + r$ with $N > 1$, $r \neq 0$, as above, we see that I^i_{-j} satisfies (3.7) for $A^i_j = \{g_i(w-k)(w+k) : g_i(w) \in \mathbb{C}[w]\}$.

Now take l such that $N - r + 1 \le l \le \left[\frac{2N+1-r}{2}\right]$, $g_l(w) \in \mathbb{C}[w]$, and $f_l(w) \in L^l_{-j}$. Computing the bracket

$$\Big[t^{-(k+1)}\Big(f_l(D_{(k+1)})e_{l,\,l-N+r} + f_l(-D_{-(k+1)})e_{2N+1-l-r,\,N+1-l}\Big),\; g_l(D_0)De_{l,\,l} + g_l(-D_0)De_{N+1-l,\,N+1-l}\Big]$$

for $j = Nk + r$ with $N > 1$, $r \neq 0$ we see that L^l_{-j} satisfies

$$A^l_j L^l_{-j} \subseteq L^l_{-j}, \tag{3.8}$$

where $A^l_j = \{g_l(w-(k+1)(w-(k+1)) : g_l(w) \in \mathbb{C}[w]\}$.

Thus we have proved the following result:

Lemma 3.2:

(a) *I^i_{-j} and L^l_{-j} are ideals for all $j \in \mathbb{N}$ where $j = kN + r$ with $0 \leqslant r \leqslant N-1$ and $1 \le i \le \left[\frac{N+1-r}{2}\right]$, $1 \le l \le \left[\frac{2N+1-r}{2}\right]$.*

(b) *If $I^i_{-j} \neq 0$, and $L^l_{-j} \neq 0$ then they have finite codimension in $\mathbb{C}[w]$.*

Proof: The proof is analogous to that of Lemma 3.5 in Ref. 6. □

Now we have the following important proposition:

Proposition 3.3:

(a) *Any non-zero element* $d \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-1}$

$$d = \sum_{i=1}^{[\frac{N}{2}]-\delta_{N,even}} f_i(D_0)De_{i,\,i+1} + f_i(-D_0)De_{N-i,\,N+1-i}$$

$$+\ \delta_{N,\,even}\, f(D_0)De_{\frac{N}{2},\,\frac{N}{2}+1} + t^{-1}h(D_{-1})De_{N,1} \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-1},$$

where $f_i(w)$, $f(w)$, *and* $h(w)$ *are non-zero polynomials with* f *and* h *even, is non-degenerate.*

(b) *Let* $d \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-1}$ *as in (a). Then*

$$\begin{aligned}(\widehat{\mathcal{D}^N_{\theta,x}})^d_0 &:= [(\widehat{\mathcal{D}^N_{\theta,x}})_1, d]\\ &= span\Big\{g_k\,(D)\,D\,(D)^l\,\left(e_{k+1,k+1} - e_{k,k}\right) +\\ &\qquad g_k\,(-D)\,D\,(-D)^l\left(e_{N-k,N-k} - e_{N+1-k,N+1-k}\right):\\ &\qquad\qquad k = 1,\ldots,\left[\frac{N}{2}\right] - \delta_{N,even}\ and\ l \in \mathbb{Z}_{\geqslant 0}\Big\}\\ &\bigcup \delta_{N,even}\Big\{g(D)(D)^r D\,\left(e_{\frac{N}{2},\frac{N}{2}} - e_{\frac{N}{2}+1,\frac{N}{2}+1}\right)\\ &\qquad\qquad r \in \mathbb{Z}_{\geqslant 0},\ \ even\ and\ g \in \mathbb{C}[w]\ is\ odd\Big\}\\ &\bigcup\Big\{h(D_{-1})D(D^l + (-D-1)^l)\,e_{1,1} + h(-D_{-1})D((D-1)^l\\ &\qquad + (-D)^l)\,e_{N,N}\ : l \in \mathbb{Z}_{\geqslant 0}, even\ integer\ and\ h \in \mathbb{C}[w]\ is\ even\Big\},\end{aligned}$$

where $g_k(D) = f_k(D)D + f_{N-k}(-D)D$ *with* $k = 1,\ldots,\left[\frac{N}{2}\right] - \delta_{N,even}$ *and* $g(D) = f(D)D$.

Proof: Let $d \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-1}$, as in (a), since each $f_i(w)$, $f(w)$, and $h(w)$ is a non-zero polynomial L^N_{-j} and $I^i_{-j} \neq 0$ for $1 \leq i \leq N-1$ and for all $j \geq 1$. So, by lemma 3.2 (b), part (a) follows. Finally, part (b) follows by computing the commutator $[a, d]$ with $a = (D_0)^l D\, e_{k+1,k} + (-D_0)^l D\, e_{N+1-k,N-k}$ with $k = 1\ldots[\frac{N}{2}] - \delta_{N,\mathrm{even}}$;$a = \delta_{N,\mathrm{even}}\,(D_0)^r De_{\frac{N}{2}+1,\,\frac{N}{2}}$ and $a = t(D_1)^m D\, e_{1,N}$ with $l,\ r,\ m \in \mathbb{Z}_{\geq 0}$ and $r,\ m$ even integers. □

Remark 3.4: In Ref. 11 Proposition 3.3 (a) and (b) are not correct. There exists parabolic subalgebra $\mathfrak{p}^d$ of $\widehat{D^N_o}$ for $N > 1$, such that $\mathfrak{p}^d_j \neq 0$ for some $j < 0$, but $\mathfrak{p}^d_j = 0$ for $j \ll 0$, $d \in (\widehat{D^N_o})_{-1}$ (cf. Ref. 3. Remark 2.2). For example

$$d = f(D_0)De_{1,\,2} + f(-D_0)De_{N-1,\,N}, \qquad f \in \mathbb{C}[w].$$

It should be restated as Proposition 3.3 (a) above with the corresponding adjustment. The remaining results in Ref. 11 are valid since we always considered non-degenerate elements for the proof of the main theorem.Summarizing, we have that the following properties are satisfied by $\widehat{\mathcal{D}^N_{\theta,x}}$:

(P_1) $(\widehat{\mathcal{D}^N_{\theta,x}})_0$ is commutative,

(P_2) if $a \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-j}$ ($j > 0$) and $[a, (\widehat{\mathcal{D}^N_{\theta,x}})_1] = 0$, then $a = 0$,

(P_3) if $\mathcal{P}$ is non-degenerate parabolic subalgebra of $\widehat{\mathcal{D}^N_{\theta,x}}$, then there exists a non-degenerate element a such that $P^a \subseteq \mathcal{P}$.

Observe that (P_3) follows from Proposition 3.3 (a) and the fact that $\mathcal{P}$ is non-degenerate.

IV. CHARACTERIZATION OF QUASIFINITE HIGHEST WEIGHT MODULES OF $\widehat{\mathcal{D}^N_{\theta,x}}$

Now, we begin our study of quasifinite representations over $\widehat{\mathcal{D}^N_{\theta,x}}$. Consider $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ any $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$, and let $\mathfrak{g}_+ = \oplus_{j>0}\mathfrak{g}_j$. A $\mathfrak{g}$-module V is called $\mathbb{Z}$-*graded* if $V = \oplus_{j\in\mathbb{Z}} V_j$ and $\mathfrak{g}_i V_j \subset V_{i+j}$. A $\mathbb{Z}$ -graded $\mathfrak{g}$-module V is called *quasifinite* if $\dim V_j < \infty$ for all j.

Given $\lambda \in \mathfrak{g}_0^*$, a *highest weight module* with weight λ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module $V(\mathfrak{g}, \lambda)$ generated by a highest weight vector $v_\lambda \in V(\mathfrak{g}, \lambda)_0$ which satisfies

$$h v_\lambda = \lambda(h) v_\lambda \qquad (h \in \mathfrak{g}_0), \qquad \mathfrak{g}_+ v_\lambda = 0. \tag{4.1}$$

A non-zero vector $v \in V(\mathfrak{g}, \lambda)$ is called *singular* if $\mathfrak{g}_+ v = 0$.

The *Verma module* over $\mathfrak{g}$ is defined as usual:

$$M(\mathfrak{g}, \lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0\oplus\mathfrak{g}_+)} \mathbb{C}_\lambda, \tag{4.2}$$

where $\mathbb{C}_\lambda$ is the one-dimensional $(\mathfrak{g}_0 \oplus \mathfrak{g}_+)$-module given by $h \mapsto \lambda(h)$ if $h \in \mathfrak{g}_0$, $\mathfrak{g}_+ \mapsto 0$, and the action of $\mathfrak{g}$ is induced by the left multiplication in $\mathcal{U}(\mathfrak{g})$. Here and further $\mathcal{U}(\mathfrak{g})$ stands for the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. Any highest weight module $V(\mathfrak{g}, \lambda)$ is a quotient module of $M(\mathfrak{g}, \lambda)$. The irreducible module $L(\mathfrak{g}, \lambda)$ is the quotient of $M(\mathfrak{g}, \lambda)$ by the maximal proper graded submodule.

Consider a parabolic subalgebra $\mathcal{P} = \oplus_{j\in\mathbb{Z}}\mathcal{P}_j$ of $\mathfrak{g}$ and let $\lambda \in \mathfrak{g}_0^*$ be such that $\lambda|_{\mathfrak{g}_0\cap[\mathcal{P},\mathcal{P}]} = 0$. Then the $(\mathfrak{g}_0 \oplus \mathfrak{g}_+)$-module $\mathbb{C}_\lambda$ extends to a $\mathcal{P}$-module by letting $\mathcal{P}_j$ act as 0 for $j < 0$, and we may construct the highest weight module

$$M(\mathfrak{g}, \mathcal{P}, \lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathcal{P})} \mathbb{C}_\lambda,$$

called the *generalized Verma module*. Clearly all these highest weight modules are graded.

From now on we will consider $\mathfrak{g} = \widehat{\mathcal{D}^N_{\theta,x}}$ and $\lambda \in (\widehat{\mathcal{D}^N_{\theta,x}})_0^*$. By Theorem 2.5 in Ref. 6 we have,

Theorem 4.1: Since $\widehat{\mathcal{D}^N_{\theta,x}}$ satisfies (P_1), (P_2), and (P_3) the following conditions on $\lambda \in (\widehat{\mathcal{D}^N_{\theta,x}})^*$ are equivalent:

(a) *$M(\widehat{\mathcal{D}^N_{\theta,x}}; \lambda)$ contains a singular vector $a v_\lambda \in M(\widehat{\mathcal{D}^N_{\theta,x}}; \lambda)_{-1}$, where $a \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-1}$ is non-degenerate ;*

(b) *There exists a non-degenerate element $a \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-1}$, such that $\lambda([(\widehat{\mathcal{D}^N_{\theta,x}})_1, a]) = 0$;*

(c) *$L(\widehat{\mathcal{D}^N_{\theta,x}}; \lambda)$ is quasifinite;*

(d) *There exists a non-degenerate element $a \in (\widehat{\mathcal{D}^N_{\theta,x}})_{-1}$, such that $L(\widehat{\mathcal{D}^N_{\theta,x}}; \lambda)$ is the irreducible quotient of a generalized Verma module $M(\widehat{\mathcal{D}^N_{\theta,x}}; \mathcal{P}^a, \lambda)$.*

We shall write $M(\lambda)$ and $L(\lambda)$ in place of $M(\widehat{\mathcal{D}^N_{\theta,x}}, \lambda)$ and $L(\widehat{\mathcal{D}^N_{\theta,x}}, \lambda)$ if no ambiguity arises.

A functional $\lambda \in (\widehat{\mathcal{D}^N_{\theta,x}})_0^*$ is described by its *labels*

$$\Delta_{i,l} = -\lambda\Big((D_0)^l D\, e_{i,i} + (-D_0)^l D e_{N+1-i,N+1-i}\Big)$$

with $l \in \mathbb{Z}_{\geqslant 0}$, $i = 1 \ldots \left[\frac{N}{2}\right] + \delta_{N,\text{odd}}$ and the *central charge* $c = \lambda(C)$. We shall consider the generating series

$$\Delta_i(x) = \sum_{l\geqslant 0} \frac{x^l}{l!}\Delta_{i,l} \qquad i = 1 \ldots \left[\frac{N}{2}\right] + \delta_{N,\text{odd}}. \tag{4.3}$$

Recall that a *quasipolynomial* is a linear combination of functions of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial and $\alpha \in \mathbb{C}$, and an *even quasipolynomial* (*odd quasipolynomial*) is a quasipolynomial that is a solution of a non-trivial linear differential equation with constant coefficients $p(\partial_t) = 0$, where $p(x)$ is an even polynomial (respectively, odd polynomial).

One has the following characterization of quasifinite highest weight modules over $\widehat{\mathcal{D}^N_{\theta,x}}$.

Theorem 4.2: *A $\widehat{\mathcal{D}^N_{\theta,x}}$-module $L(\lambda)$ is quasifinite if and only if*

$$G_1(x) = \left(\frac{d}{dx} - \frac{1}{2}\right) e^{-\frac{x}{2}} \Delta_1(x) - \left(\frac{d}{dx} + \frac{1}{2}\right) e^{\frac{x}{2}} \Delta_1(-x)$$

with $G_1(x)$ an even quasipolynomial,

$$\Delta_k(x) - \Delta_{k+1}(x) = F_k(x)$$

for $k = 1, \ldots, [\frac{N}{2}] - \delta_{N,even}$, where each $F_k(x)$ is a quasipolynomial, and

$$F_{\frac{N}{2}}(x) = \delta_{N,\mathrm{even}} \left(\frac{\Delta_{\frac{N}{2}}(x) - \Delta_{\frac{N}{2}}(-x)}{2} \right),$$

where $F_{\frac{N}{2}}(x)$ is an odd quasipolynomial.

Proof: From Proposition 3.3 (c) and Theorem 4.1 part (b), we have that $L(\lambda)$ is quasifinite if only if there exists (monic) polynomials $g_k(x)$, $g(x)$, and $h(x)$ with $g(x)$ odd polynomial, $h(x)$ even polynomial and $k = 1 \cdots [\frac{N}{2}] - \delta_{N,\mathrm{even}}$, such that

$$\lambda \big(\cosh(D - 1/2)h(D - 1/2)(D - 1)De_{1,1} - \cosh(D + 1/2)h(D + 1/2)(D + 1)De_{N,N}\big) = 0 \tag{4.4}$$

$$\lambda \quad \big(g_k(D)De^{xD} \left[e_{k+1,k+1} - e_{k,k}\right] + g_k(-D)De^{-xD} \left[e_{N-k,N-k} - e_{N+1-k,N+1-k}\right]\big) = 0 \tag{4.5}$$

with $k = 1, \ldots, \left[\frac{N}{2}\right] - \delta_{N,\mathrm{even}}$, and

$$\delta_{N,\mathrm{even}}\, \lambda \left(g(D)D\cosh(xD)\left[e_{\frac{N}{2},\frac{N}{2}} - e_{\frac{N}{2}+1,\frac{N}{2}+1}\right]\right) = 0. \tag{4.6}$$

Using (4.3) together with the identities

$$f(D)e^{xD} = f\left(\frac{d}{dx}\right)\left(e^{xD}\right), \quad p(D)e^{x(D+1)} = e^x p(D)e^{xD} = e^x p\left(\frac{d}{dx}\right)e^{xD},$$

$$e^x p\left(\frac{d}{dx}\right) f(x) = p\left(\frac{d}{dx} - 1\right) e^x f(x)$$

and the fact that g is an odd polynomial and h is an even polynomial, conditions (4.4)–(4.6) can be rewritten as follows:

$$
\begin{aligned}
0 &= \lambda\ \big(\cosh(D-1/2)h(D-1/2)(D-1)De_{1,1} - \\
&\qquad \cosh(D+1/2)h(D+1/2)(D+1)De_{N,N}\big) \\
&= \frac{1}{2}\,\lambda\left(\left[h\left(\frac{d}{dx}\right)\left(\frac{d^2}{dx^2}-\frac{1}{4}\right)e^{x(D-1/2)}+\right.\right. \\
&\qquad \left. + h\left(-\frac{d}{dx}\right)\left(-\frac{d}{dx}-\frac{1}{2}\right)\left(-\frac{d}{dx}+\frac{1}{2}\right)e^{-x(D-1/2)}\right]e_{1,1} \\
&\quad -\left[h\left(\frac{d}{dx}\right)\left(\frac{d}{dx}-\frac{1}{2}\right)\left(\frac{d}{dx}+\frac{1}{2}\right)e^{x(D+1/2)} - \right. \\
&\qquad \left.\left. - h\left(-\frac{d}{dx}\right)\left(-\frac{d}{dx}-\frac{1}{2}\right)\left(-\frac{d}{dx}+\frac{1}{2}\right)e^{-x(D+1/2)}\right]e_{N,N}\right) \\
&= \frac{1}{2}\,h\left(\frac{d}{dx}\right)\ \left(\frac{d^2}{dx^2}-\frac{1}{4}\right)\ \lambda\ \big(\big(e^{x(D-1/2)}+e^{-x(D-1/2)}\big)\,e_{1,1} - \\
&\qquad \big(e^{x(D+1/2)}+e^{-x(D+1/2)}\big)\,e_{N,N}\big) \\
&= \frac{1}{2}\,h\left(\frac{d}{dx}\right)\ \left(\frac{d^2}{dx^2}-\frac{1}{4}\right)\ \lambda\ \big(e^{-x/2}(e^{xD}e_{1,1}-e^{-xD}e_{N,N}) \\
&\qquad -e^{x/2}(e^{xD}e_{N,N}-e^{-xD}e_{1,1})\big) \\
&= \frac{1}{2}\,h\left(\frac{d}{dx}\right)\ \ \lambda\ \left(\left(\frac{d}{dx}-\frac{1}{2}\right)e^{-x/2}(De^{xD}e_{1,1}+De^{-xD}e_{N,N})\right. \\
&\qquad \left. -\left(\frac{d}{dx}+\frac{1}{2}\right)e^{x/2}(De^{xD}e_{N,N}+De^{-xD}e_{1,1})\right) \\
&= -\frac{1}{2}h\left(\frac{d}{dx}\right)\left(\left(\frac{d}{dx}-\frac{1}{2}\right)\mathrm{e}^{-\frac{x}{2}}\Delta_1(x)-\left(\frac{d}{dx}+\frac{1}{2}\right)\mathrm{e}^{\frac{x}{2}}\Delta_1(-x)\right)
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
0 &= \lambda\left(g_k(D)D\mathrm{e}^{xD}\ \left[e_{k+1,k+1}-e_{k,k}\right]\right. \\
&\qquad \left. + g_k(-D)D\mathrm{e}^{-xD}\ \left[e_{N-k,N-k}-e_{N+1-k,N+1-k}\right]\right) \\
&= \lambda\left(g_k\left(\frac{d}{dx}\right)D\mathrm{e}^{xD}\ \left[e_{k+1,k+1}-e_{k,k}\right]\right. \\
&\qquad \left. + g_k\left(\frac{d}{dx}\right)D\mathrm{e}^{-xD}\ \left[e_{N-k,N-k}-e_{N+1-k,N+1-k}\right]\right) \\
&= \lambda\left(g_k\left(\frac{d}{dx}\right)\left[D\mathrm{e}^{xD}e_{k+1,k+1}+D\mathrm{e}^{-xD}e_{N-k,N-k}\right]\right. \\
&\qquad \left. - g_k\left(\frac{d}{dx}\right)\left[D\mathrm{e}^{xD}e_{k,k}+D\mathrm{e}^{-xD}e_{N+1-k,N+1-k}\right]\right) \\
&= -g_k\left(\frac{d}{dx}\right)(\Delta_{k+1}(x)-\Delta_k(x))\,.
\end{aligned}
\tag{4.8}
$$

and,

$$
\begin{aligned}
0 &= \lambda \left(g(D) D \cosh(xD) \left[e_{\frac{N}{2},\frac{N}{2}} - e_{\frac{N}{2}+1,\frac{N}{2}+1} \right] \right) \\
&= \lambda \left(g\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right) \cosh(xD) \left[e_{\frac{N}{2},\frac{N}{2}} - e_{\frac{N}{2}+1,\frac{N}{2}+1} \right] \right) \\
&= \frac{1}{2}\lambda \left(g\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right) \left(\mathrm{e}^{xD} e_{\frac{N}{2},\frac{N}{2}} - \mathrm{e}^{-xD} e_{\frac{N}{2}+1,\frac{N}{2}+1} + \right.\right. \\
&\qquad \left.\left. + \,\mathrm{e}^{-xD} e_{\frac{N}{2},\frac{N}{2}} - \mathrm{e}^{-xD} e_{\frac{N}{2}+1,\frac{N}{2}+1} \right)\right) \\
&= \frac{1}{2} g\left(\frac{d}{dx}\right) \lambda \left(D\mathrm{e}^{xD} e_{\frac{N}{2},\frac{N}{2}} + D\mathrm{e}^{-xD} e_{\frac{N}{2}+1,\frac{N}{2}+1} \right. \\
&\qquad \left. - D\mathrm{e}^{-xD} e_{\frac{N}{2},\frac{N}{2}} - D\mathrm{e}^{xD} e_{\frac{N}{2}+1,\frac{N}{2}+1} \right) \\
&= -g\left(\frac{d}{dx}\right) \frac{\Delta_{\frac{N}{2}}(x) - \Delta_{\frac{N}{2}}(-x)}{2}.
\end{aligned} \tag{4.9}
$$

Thus, $L(\lambda)$ is quasifinite if and only if there exist polynomials $g_k(x)$, $g(x)$, and $h(x)$ with $g(x)$ odd polynomial, $h(x)$ even polynomial and $k = 1 \cdots [\frac{N}{2}] - \delta_{N,\mathrm{even}}$, such that

$$
0 = h\left(\frac{d}{dx}\right) \left(\left(\frac{d}{dx} - \frac{1}{2} \right) \mathrm{e}^{-\frac{x}{2}} \Delta_1(x) - \left(\frac{d}{dx} + \frac{1}{2} \right) \mathrm{e}^{\frac{x}{2}} \Delta_1(-x) \right), \tag{4.10}
$$

$$
0 = g_k\left(\frac{d}{dx}\right) (\Delta_{k+1}(x) - \Delta_k(x)), \tag{4.11}
$$

$$
0 = g\left(\frac{d}{dx}\right) \frac{\Delta_{\frac{N}{2}}(x) - \Delta_{\frac{N}{2}}(-x)}{2}. \tag{4.12}
$$

Therefore, $G_1(x) = \left(\frac{d}{dx} - \frac{1}{2}\right) \mathrm{e}^{-\frac{x}{2}} \Delta_1(x) - \left(\frac{d}{dx} + \frac{1}{2}\right) \mathrm{e}^{\frac{x}{2}} \Delta_1(-x)$ is an even quasipolynomial, $F_k(x) = \Delta_{k+1}(x) - \Delta_k(x)$ are quasipolynomials, and $F_{\frac{N}{2}} = \left(\Delta_{\frac{N}{2}}(x) - \Delta_{\frac{N}{2}}(-x) \right)/2$ is an odd quasipolynomial. □

Remark 4.3: It is easy to see that this result coincides with the case $N = 1$ developed in Ref. 2.

V. INTERPLAY BETWEEN $\widehat{\mathcal{D}^N_{\theta,x}}$ AND THE INFINITE RANK CLASSICAL LIE ALGEBRAS OF TYPE *A* AND *C*

Let $\mathcal{O}$ denotes the algebra of all holomorphic functions on $\mathbb{C}$ with the topology of uniform convergence on compact sets. We consider the vector space $(\mathcal{D}_x^N)^{\mathcal{O}}$ spanned by the differential operators (of infinite order) of the form $t^k f(D) D\, e_{i,j}$, where $f \in \mathcal{O}$. The bracket in $\mathcal{D}_x^N$ extends to $(\mathcal{D}_x^N)^{\mathcal{O}}$. The principal gradation extends as well $(\mathcal{D}_x^N)^{\mathcal{O}} = \bigoplus_{l\in\mathbb{Z}} (\mathcal{D}_x^N)_l^{\mathcal{O}}$, where $t^k f(D) D e_{i,j} \in (\mathcal{D}_x^N)_l^{\mathcal{O}}$ if $l = kN + i - j$ and $f \in \mathcal{O}$. Similarly, we obtain the gradation for the central extension $\widehat{(\mathcal{D}_x^N)}^{\mathcal{O}}$. In the same way, we define a completion $(\mathcal{D}_{\theta,x}^N)^{\mathcal{O}}$ of $\mathcal{D}_{\theta,x}^N$ consisting of all differential operators of the form

$$
\{ t^k \left(f(D_k) D e_{i,j} + f(-D_k) D e_{N+1-j,N+1-i} \right) : k \in \mathbb{Z}, \quad 1 \leqslant i < j \leqslant N, \ f \in \mathcal{O} \},
$$

and on the opposite diagonal,

$$
\{ t^k f(D_k) D e_{i,N+1-i} \ : k \in \mathbb{Z}, \quad 1 \leqslant i \leqslant N, \ f \in \mathcal{O} \text{ even } \}.
$$

Then the two-cocycle ψ on $\mathcal{D}^N_{\theta,x}$ extends to a two-cocycle on $(\mathcal{D}^N_{\theta,x})^{\mathcal{O}}$. Let $(\widehat{\mathcal{D}^N_{\theta,x}})^{\mathcal{O}} = (\mathcal{D}^N_{\theta,x})^{\mathcal{O}} \bigoplus \mathbb{C}C$ be the corresponding central extension. In this case one obtains the gradation $(\mathcal{D}^N_{\theta,x})^{\mathcal{O}} = \bigoplus_{k\in\mathbb{Z}} (\widehat{\mathcal{D}^N_{\theta,x}})^{\mathcal{O}}_k$ by restriction of the gradation of $\widehat{(\mathcal{D}^N_x)}^{\mathcal{O}}$.

Given $s \in \mathbb{C}$, we have, (cf. (3.2) in Ref. 3), the embedding $\varphi_s : \mathcal{D}^N_x \to g\ell^{[m]}_\infty$ (respectively, $\varphi_s : (\mathcal{D}^N_x)^{\mathcal{O}} \to g\ell^{[m]}_\infty$) given by

$$\varphi^{[m]}_s \left(t^k f_i(D) D\, e_{i,\,j}\right) = \sum_{l\in\mathbb{Z}} f_i\,(-l+s+u)\,(-l+s+u) E_{(l-k)N-i+1,\,lN-j+1}\,,$$

which is an homomorphism of Lie algebras. Restricting these homomorphisms of Lie algebras to $\mathcal{D}^N_{\theta,x}$, we obtain a family of homomorphism of Lie algebras $\varphi_s : \mathcal{D}^N_{\theta,x} \to g\ell^{[m]}_\infty$ (respectively, to $(\mathcal{D}^N_{\theta,x})^{\mathcal{O}}, \varphi_s : (\mathcal{D}^N_{\theta,x})^{\mathcal{O}} \to g\ell^{[m]}_\infty$), namely,

$$\begin{aligned}
&\varphi^{[m]}_s\Big(t^k\big(f_i(D_k) D\, e_{i,\,j} + f_i(-D_k) D\, e_{N+1-j,\,N+1-i}\big)\Big) = \\
&= \sum_{l\in\mathbb{Z}} \Bigg[f_i \left(-l + \frac{k}{2} + s + u\right) (-l+s+u) E_{(l-k)N-i+1,\,lN-j+1} \\
&\qquad\qquad + f_i \left(l - \frac{k}{2} - s - u\right) (-l+s+u)\, E_{(l-k-1)N+j,\,(l-1)N+i} \Bigg] \\
&= \sum_{r=0}^{m} \sum_{l\in\mathbb{Z}} \Bigg[f^{(r)}_i \left(-l + \frac{k}{2} + s\right) \frac{(-l+s)u^r + u^{r+1}}{r!}\, E_{(l-k)N-i+1,\,lN-j+1} \\
&\qquad + (-1)^r\, f^{(r)}_i \left(l - \frac{k}{2} - s\right) \frac{(-l+s)u^r + u^{r+1}}{r!}\, E_{(l-k-1)N+j,\,(l-1)N+i} \Bigg], \qquad (5.1)
\end{aligned}$$

where $1 \leqslant i < j \leqslant N$ and $f^{(r)}$ denote the rth derivative of, f and similarly, in the other set of generators

$$\begin{aligned}
&\varphi^{[m]}_s \left(t^k f_i(D_k) D e_{i,N+1-i}\right) = \\
&\quad = \sum_{l\in\mathbb{Z}} f_i \left(-l + \frac{k}{2} + s + u\right) (-l+s+u) E_{(l-k)N-i+1,\,(l-1)N+i} \\
&\quad = \sum_{r=0}^{m} \sum_{l\in\mathbb{Z}} f^{(r)}_i \left(-l + \frac{k}{2} + s\right) \frac{(-l+s)u^r + u^{r+1}}{r!}\, E_{(l-k)N-i+1,\,(l-1)N+i}\,, \qquad (5.2)
\end{aligned}$$

where as above, $1 \leqslant i \leqslant N$, f is even and $f^{(r)}$ denote the rth derivative of f. For each $s \in \mathbb{C}$ and $k \in \mathbb{Z}$, set

$$\begin{aligned}
I^{[m]}_{s,k} = \Bigg\{ f \in \mathcal{O} :\ & f^{(r)} \left(n + \frac{k}{2} + s\right) = 0 \quad \text{and} \\
& f^{(r)} \left(-n - \frac{k}{2} - s\right) = 0, \quad \text{for all } n \in \mathbb{Z}, \quad \text{and all } r = 0, \ldots, m \Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{I}^{[m]}_{s,k} = \big\{ f \in \mathcal{O} :\ & f \text{ is even and} \\
& f^{(r)} \left(n + \frac{k}{2} + s\right) = 0, \quad \text{for all } n \in \mathbb{Z}, \quad \text{and all } r = 0, \ldots, m \big\}.
\end{aligned}$$

Let

$$J_s^{[m]} = \bigoplus_{k\in\mathbb{Z}} \Big\{ t^k \big(f(D_k) D\, e_{i,j} + f(-D_k) D\, e_{N+1-j,N+1-i} \big) : \ f \in I_{s,k}^{[m]} \text{ and } 1 \leqslant i < j \leqslant N \Big\} \bigoplus \bigoplus_{k\in\mathbb{Z}} \Big\{ t^k f(D_k) D\, e_{i,N+1-i} : \ f \in \widetilde{I}_{s,k}^{[m]} \Big\}.$$

We clearly have

$$\ker \varphi_s^{[m]} = J_s^{[m]}. \tag{5.3}$$

Fix $\vec{s} = (s_1, \ldots, s_M) \in \mathbb{C}^M$, such that $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$ and $s_i + s_j \notin \mathbb{Z}$ for all i, j. Also fix $\vec{m} = (m_1, \ldots, m_M) \in \mathbb{Z}_{\geqslant 0}^M$. Let $g\ell_\infty^{[\vec{m}]} = \oplus_{i=1}^M g\ell_\infty^{[m_i]}$ and consider the homomorphism

$$\varphi_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^M \varphi_{s_i}^{[m_i]} \ : \ \big(\mathcal{D}_{\theta,x}^N\big)^{\mathcal{O}} \longrightarrow g\ell_\infty^{[\vec{m}]}.$$

Proposition 5.1: Given $\vec{s}$ and $\vec{m}$ as above we have the following exact sequence of Lie algebras:

$$0 \longrightarrow J_{\vec{s}}^{[\vec{m}]} \longrightarrow \big(\mathcal{D}_{\theta,x}^N\big)^{\mathcal{O}} \stackrel{\varphi_{\vec{s}}^{[\vec{m}]}}{\longrightarrow} g\ell_\infty^{[\vec{m}]} \longrightarrow 0,$$

where $J_{\vec{s}}^{[\vec{m}]} = \bigcap_{i=1}^M J_{s_i}^{[m_i]}$.

Proof: For simplicity we prove this in the case $M = 1$. By the assumptions above we have that $\vec{m} = m \in \mathbb{Z}_{\geqslant 0}$ and $\vec{s} = s \notin \mathbb{Z}/2$. The general case is similar. It is clear that $\ker \varphi_s^{[m]} = J_s^{[m]}$. For the surjectivity we recall the following well known fact: for every discrete sequence of points in $\mathbb{C}$ and a non-negative integer t there exists $f(w) \in \mathcal{O}$ having the prescribed values of its first t derivatives at these points. Since $s \notin \mathbb{Z}/2$ the sequences $\{-l + \frac{k}{2} + s\}_{l\in\mathbb{Z}}$ and $\{l - \frac{k}{2} - s\}_{l\in\mathbb{Z}}$ are disjoint, then the Proposition follows. □

Now we want to extend the homomorphism $\varphi_s : \mathcal{D}_{\theta,x}^N \to g\ell_\infty^{[m]}$ (respectively, $\varphi_s : (\mathcal{D}_{\theta,x}^N)^{\mathcal{O}} \to g\ell_\infty^{[m]}$) to a homomorphism between the central extensions of the corresponding Lie algebras. Observe that these homomorphisms preserve the principal gradation.

Introduce the following functions, (cf. Ref. 9)

$$\eta_j(x,\mu) = \left(\frac{e^{\mu x} + (-1)^j e^{-\mu x}}{2} \right) \frac{x^j}{j!} \quad (j \in \mathbb{Z}_+, \mu \in \mathbb{C}). \tag{5.4}$$

The functions $\eta_j(x,\mu)$ satisfy

$$\eta_j(-x,\mu) = \eta_j(x,\mu), \quad \eta_j(x,-\mu) = (-1)^j \eta_j(x,\mu), \quad \eta_0(x,\mu) = \cosh(\mu x).$$

We have the following.

Proposition 5.2: The homomorphism $\varphi_s^{[m]}$ lifts to a Lie algebra homomorphism $\widehat{\varphi}_s^{[m]}$ of the corresponding central extensions as follows:

$$\widehat{\varphi}_s^{[m]}|_{(\widehat{\mathcal{D}_{\theta,x}^N})_j} = \varphi_s^{[m]}|_{(\mathcal{D}_{\theta,x}^N)_j} \qquad \textit{if } j \neq 0, \tag{5.5}$$

$$\widehat{\varphi}_s^{[m]}\left(e^{xD}D\,e_{i,i}+e^{xD}D\,e_{N+1-i,N+1-i}\right)$$
$$=\varphi_s^{[m]}\left(e^{xD}D\,e_{i,i}+e^{-xD}D\,e_{N+1-i,N+1-i}\right)-\frac{d}{dx}\left(\left(\frac{\cosh\left(\left(s-\frac{1}{2}\right)x\right)-\cosh\left(\frac{x}{2}\right)}{\sinh(\frac{x}{2})}\right)\right.$$
$$\left.+\sum_{1\leqslant j\leqslant m,}\frac{\eta_j(x,s)}{\sinh(\frac{x}{2})}u^j\right), \tag{5.6}$$

and

$$\widehat{\varphi}_s^{[m]}(C)=\mathbf{1}. \tag{5.7}$$

Proof: Straightforward using formulas (6) in Ref. 3 and the explicit formulas for $\varphi_s^{[m]}$ given in (5.1) and (5.2). □

The homomorphism $\varphi_s^{[m]}$ is defined for any $s\in\mathbb{C}$. However, for $s\in\mathbb{Z}/2$, it is no longer surjective. These cases are described by the following Propositions.

Proposition 5.3: For $s=0$ and $s=\frac{1}{2}$, we have the following exact sequence of Lie algebras:

$$0\longrightarrow J_s^{[m]}\longrightarrow(\mathcal{D}_{\theta,x}^N)^{\mathcal{O}}\stackrel{\varphi_s^{[m]}}{\longrightarrow}C\longrightarrow 0,$$

where $C\simeq\bar{c}_\infty^{[m]}$.

Proof: First consider $s=1/2$. The homomorphism $\varphi_s^{[m]}:\mathcal{D}_x^N\longrightarrow g\ell_\infty^{[m]}$ introduced in Sec. VI in Ref. 3 is surjective. Recall that we defined in $\mathcal{D}_x^N$ the anti-involution θ given in (3.5). It is easy to see that it transfers, via the $\varphi_s^{[m]}$, to an anti-involution $\omega:g\ell_\infty^{[m]}\longrightarrow g\ell_\infty^{[m]}$ as follows:

$$\omega(u^k-(1/2+\widetilde{m})u^{k-1})E_{i,j})=$$
$$((-u)^k-(1/2+n)(-u)^{k-1})E_{1-j,1-i})\text{ for }k\geq 1, \tag{5.8}$$

where $i=nN+q+q$; $j=\tilde{m}N+q$ with $1\leq q,\widetilde{q}\leq N$.

Therefore, the Lie algebra of $-\theta$-fixed points in $\mathcal{D}_x^N$, namely, $\mathcal{D}_{\theta,x}^N$, maps surjectively to the Lie algebra of $-\omega$ fixed points in $g\ell_\infty^{[m]}$. Then it is enough to show that ω is conjugated by an automorphism T of $g\ell_\infty^{[m]}$ to the anti-involution defining $\bar{c}_\infty^{[m]}$.

For this define

$$\begin{aligned}
T(u^m\ E_{i,i+1})&=(\widetilde{m}+1/2)\ u^m\ E_{i,i+1},\\
T(u^l\ E_{i,i+1})&=(u^{l+1}-(\widetilde{m}+1/2)\ u^l)\ E_{i,i+1}\quad\text{for}\quad 0\leq l\leq m-1,\\
T(u^m\ E_{i+1,i})&=\frac{-1}{(n-1/2)}\ (-u)^m\ E_{i+1,i},\\
T(u^l\ E_{i+1,i})&=\frac{1}{u-(n-1/2)}(-u)^l\ E_{i+1,i}\quad\text{for}\quad 0\leq l\leq m-1,
\end{aligned} \tag{5.9}$$

where $i+1=\widetilde{m}N+q$, $i=nN+\bar{q}$ with $1\leq q,\bar{q}\leq N$. It is a straightforward verification that this extends to an automorphism of the associative algebra $g\ell_\infty^{[m]}$ that conjugates ω to the anti-involution defining $\bar{c}_\infty$.

Now, consider the case $s=0$. In this case, The homomorphism $\varphi_0^{[m]}:\mathcal{D}_x^N\longrightarrow g\ell_\infty^{[m]}$ introduced in Sec. VI in Ref. 3 is no longer surjective. However, it is surjective if we restrict $\varphi_0^{[m]}:\mathcal{D}_x^N\longrightarrow\mathfrak{g}^{[m]}$, where $\mathfrak{g}^{[m]}$ is the subalgebra of $g\ell_\infty^{[m]}$ generated by $\{E_{sN-i+1,sN-j+1}: i\neq 1,\ldots N$ and $j\neq 1,\ldots,N\}$ with entries in R_m. We will call such homomorphism also $\varphi_0^{[m]}$. Now, as above the anti-involution θ in (3.5) transfers to $\mathfrak{g}^{[m]}$ as follows:

$$\omega_0((u^k-(\widetilde{m}+1)u^{k-1})E_{ij})=((-u)^k-(n+1)(-u)^{k-1})E_{-N+1-j,-N+1-i}, \tag{5.10}$$

for $k \geq 1$, where $i = nN + q$; $j = \widetilde{m}N + \bar{q}$ with $1 \leq q, \bar{q} \leq N$. As above, it is enough to show that ω_0 is conjugated by an isomorphism $T : \mathfrak{g}^{[m]} \longrightarrow g\ell_\infty^{[m]}$ to the anti-involution defining $\bar{c}_\infty^{[m]}$.

One should take $T = \pi \circ T'$, where π is the natural projection of $\mathfrak{g}^{[m]}$ onto $g\ell_\infty^{[m]}$ and T' is the automorphism of $\mathfrak{g}^{[m]}$ defined by

$$\begin{aligned} T'(u^m\ E_{i,i+1}) &= -(\widetilde{m}+1)\,(-u)^m\ E_{i,i+1}, \\ T'(u^l\ E_{i,i+1}) &= (-1)^l(u^{l+1} - (\widetilde{m}+1)\,u^l)\ E_{i,i+1} \quad \text{for} \quad 0 \leq l \leq m-1, \\ T'(u^m\ E_{i+1,i}) &= \frac{-1}{(n+1)}\,(-u)^m\ E_{i+1,i}, \\ T'(u^l\ E_{i+1,i}) &= \frac{1}{u-(n+1)}(-u)^l\ E_{i+1,i} \quad \text{for} \quad 0 \leq l \leq m-1, \end{aligned} \tag{5.11}$$

where $i + 1 = \widetilde{m}N + q$; $i = nN + \bar{q}$ with $1 \leq q \leq N$. Finishing the proof. □

Remark 5.4:

(a) For $s = 0$ and $s = 1/2$, in view of the proposition above, by an abuse of notation we will denote again $\varphi_s^{[m]}$ the surjective homomorphism $\mathcal{D}_{\theta,x}$ onto $\bar{c}_\infty^{[m]}$ given by the old $\varphi_s^{[m]}$ composed with the isomorphism $C \simeq \bar{c}_\infty^{[m]}$.

(b) For $s \in \mathbb{Z}/2$ the image of $\mathcal{D}_{\theta,x}^N$ under the homomorphism $\varphi_s^{[m]}$ is $\nu^{\widetilde{s}}(\bar{c}_\infty^{[m]})$, where ν was defined in (2.1) and $\widetilde{s} = s$ if $s \in \mathbb{Z}$ and $\widetilde{s} = s - 1/2$ if $s \in \mathbb{Z} + 1/2$. Therefore, we will only consider $s = 0,\ 1/2$ throughout the paper.

Given $\vec{m} = (m_1, \ldots m_M) \in \mathbb{Z}_{\geqslant 0}^M$ and $\vec{s} = (s_1, \ldots, s_M)$ such that, $s_i \in \mathbb{Z}$ implies $s_i = 0$; $s_i \in \mathbb{Z} + 1/2$ implies $s_i = 1/2$ and $s_i \neq \pm s_j \mod \mathbb{Z}$ for $i \neq j$, and combining Propositions 5.1 and 5.3, we obtain a homomorphism of Lie algebras

$$\widehat{\varphi}_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^{M} \varphi_{s_i}^{[m_i]} : \widehat{\mathcal{D}_o^N} \longrightarrow \mathfrak{g}^{[\vec{m}]} := \oplus_{i=1}^M \mathfrak{g}^{[m_i]}, \tag{5.12}$$

where

$$\mathfrak{g}^{[m]} = \begin{cases} \widehat{g\ell}_\infty^{[m]} & \text{if} \quad s \notin \mathbb{Z}/2, \\ c_\infty^{[m]} & \text{if} \quad s = 0 \quad \text{or} \quad s = 1/2. \end{cases} \tag{5.13}$$

We can prove the following Proposition in the same way as Proposition (5.1).

Proposition 5.5: The homomorphism $\widehat{\varphi}_{\vec{s}}^{[\vec{m}]}$ extends to a surjective homomorphism of Lie algebras, which is denoted again by $\widehat{\varphi}_{\vec{s}}^{[\vec{m}]}$

$$\widehat{\varphi}_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^{M} \widehat{\varphi}_{s_i}^{[m_i]} : (\widehat{\mathcal{D}_{\theta,x}^N})^{\mathcal{O}} \longrightarrow \mathfrak{g}^{[\vec{m}]}.$$

VI. REALIZATION OF QUASIFINITE HIGHEST WEIGHT MODULES OF $\widehat{\mathcal{D}_{\theta,x}^N}$

Let $\mathfrak{g}^{[m]}$ as (5.13). The proof of the following proposition is standard. We will use the notation introduced in Sec. II.

Proposition 6.1: The $\mathfrak{g}^{[m]}$-module $L(\mathfrak{g}^{[m]},\ \lambda)$ is quasifinite if and only if all but finitely many of the $\,^*h_k^{(i)}$ are zero, where $*$ represents a, or c depending on whether $\mathfrak{g}^{[m]}$ is $\widehat{g\ell}_\infty^{[m]}$, or $c_\infty^{[m]}$, (cf. Ref. 5).

Given $\vec{m} = (m_1, \ldots m_M) \in \mathbb{Z}_{\geqslant 0}^M$, take a quasifinite $\lambda_i \in (\mathfrak{g}^{[m_i]})_0^*$ for each $i = 1, \ldots, M$ and let $L(\mathfrak{g}^{[m_i]},\ \lambda_i)$ be the corresponding irreducible $\mathfrak{g}^{[m_i]}$-module. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_M)$. Then the tensor product

$$L\left(\mathfrak{g}^{[\vec{m}]}, \vec{\lambda}\right) = \bigotimes_{i=1}^{M} L\left(\mathfrak{g}^{[m_i]}, \ \lambda_i\right) \tag{6.1}$$

is an ir-reducible $\mathfrak{g}^{[\vec{m}]}$-module, with $\mathfrak{g}^{[\vec{m}]} = \bigoplus_{i=1}^{M} \mathfrak{g}^{[m_i]}$. The module $L(\mathfrak{g}^{[\vec{m}]}, \vec{\lambda})$ can be regarded as a $\widehat{\mathcal{D}^N_{\theta,x}}$- module via the homomorphism $\varphi_{\vec{s}}^{[\vec{m}]}$ and will be denoted by $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$. We shall need the following Proposition:

Proposition 6.2: Let V be a quasifinite $\widehat{\mathcal{D}^N_{\theta,x}}$-module. Then the action of $\widehat{\mathcal{D}^N_{\theta,x}}$ on V naturally extends to the action of $(\widehat{\mathcal{D}^N_{\theta,x}})_k^{\mathcal{O}}$ on V for any $k \neq 0$.

Proof: Replace $B = ad\, D^2 - k^2$ by

$$B = ad\left[\left(D + \frac{k+1}{2}\right) E_{i,i} + \left(D + \frac{k+1}{2} - k\right) E_{j,j}\right] + ad\left[\left(-D - \frac{k+1}{2}\right) E_{N+1-j\,,N+1-j} + \left(-D - \frac{k+1}{2} + k\right) E_{N+1-i\,,N+1-i}\right],$$

in the proof of Proposition 4.3 in Ref. 7. The rest of proof remains the same. □

Theorem 6.3: *Let V be a quasifinite $\mathfrak{g}^{[\vec{m}]}$-module, which is regarded as a $\widehat{\mathcal{D}^N_{\theta,x}}$-module via the homomorphism $\varphi_{\vec{s}}^{[\vec{m}]}$. Then any $\widehat{\mathcal{D}^N_{\theta,x}}$-submodule of V is also a $\mathfrak{g}^{[\vec{m}]}$-submodule. In particular, the $\widehat{\mathcal{D}^N_{\theta,x}}$-module $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$ are irreducible if $\vec{s} = (s_1, \ldots, s_M)$ is such that, $s_i \in \mathbb{Z}$ implies $s_i = 0$; $s_i \in \mathbb{Z} + 1/2$ implies $s_i = 1/2$ and $s_i \neq \pm s_j \mod \mathbb{Z}$ for $i \neq j$.*

Proof: Let U be a $\widehat{\mathcal{D}^N_{\theta,x}}$-submodule of V. U is a quasifinite $\widehat{\mathcal{D}^N_{\theta,x}}$-module as well, hence by Proposition 6.2, it can be extended to $(\widehat{\mathcal{D}^N_{\theta,x}})_k^{\mathcal{O}}$ for any $k \neq 0$. By Proposition 5.5, the map $\varphi_{\vec{s}}^{[\vec{m}]}$: $(\widehat{\mathcal{D}^N_{\theta,x}})_k^{\mathcal{O}} \longrightarrow (\mathfrak{g}^{[\vec{m}]})_k$ is surjective for any $k \neq 0$. Thus U is invariant with respect to all members of the principal gradation of $(\mathfrak{g}^{[\vec{m}]})_k$ with $k \neq 0$. Since $\mathfrak{g}^{[\vec{m}]}$ coincides with its derived algebra, this proves the theorem. □

Now, we will show that in fact all the quasifinite $\widehat{\mathcal{D}^N_{\theta,x}}$-modules can be realized as some $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$, for $\vec{m} \in \mathbb{Z}^M_{\geq 0}$ and $\vec{s} \in \mathbb{C}^M$ such that $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$ and $s_i + s_j \notin \mathbb{Z}$ for all i, j. For simplicity we will consider the case $M = 1$. But first we will calculate the generating series $\Delta_{m,s,\lambda,i}$ of the highest weight and central charge c of the $\widehat{\mathcal{D}^N_{\theta,x}}$-module $L_s^{[m]}(\lambda)$.

Let $s \notin \mathbb{Z}/2$. Using formula (5.6), the fact that

$$\lambda(\exp(xD)De_{i,i} + \exp(-xD)De_{N+1-i,N+1-i}), \tag{6.2}$$

with $i = 1, \ldots, [\frac{N}{2}] + \delta_{N,\mathrm{odd}}$ and (2.4) we have that

$$\Delta_{m,\,s,\,\lambda,\,i}(x) = \frac{d}{dx}\sum_{l\in\mathbb{Z}}\sum_{r=0}^{m}\left(\frac{{}^a\lambda^{(r)}_{(l-1)N+i}}{r!}(-x)^r e^{x(l-s)} - \frac{{}^a\lambda^{(r)}_{lN-i+1}}{r!}x^r e^{-x(l-s)}\right) + \frac{d}{dx}\sum_{r=0}^{m}\frac{\eta_r(x, s-\frac{1}{2})c_r}{\sinh\frac{x}{2}} - \frac{d}{dx}\left(\frac{\cosh\left(\frac{x}{2}\right)c_0}{\sinh\frac{x}{2}}\right). \tag{6.3}$$

Introduce the polynomials

$$ {}^{a}g_t(x) = \sum_{r=0}^{m} {}^{a}h_t^{(r)} \frac{x^r}{r!}. $$

Then

$$ \Delta_{m,s,\lambda,i}(x) - \Delta_{m,s,\lambda,i+1}(x) = \frac{d}{dx} \sum_{l\in\mathbb{Z}} \Big(e^{x\left(l-s-\frac{1}{2}\right)}\, {}^{a}g_{(l-1)N+i}(-x) + e^{-x\left(l-s+\frac{1}{2}\right)}\, {}^{a}g_{lN-i}(x) \Big), \tag{6.4} $$

$$ \left(\frac{d}{dx}+\frac{1}{2}\right) e^{\frac{-x}{2}} \Delta_{m,s,\lambda,1}(x) - \left(\frac{d}{dx}-\frac{1}{2}\right) e^{\frac{x}{2}} \Delta_{m,s,\lambda,1}(-x) = -\left(\frac{d^2}{dx^2}-\frac{1}{4}\right) \sum_{l\in\mathbb{Z}} \sum_{r=0}^{m} \left({}^{a}h_{(l-1)N}^{(r)} + \delta_{l,1} c_r \right) \eta_r\left(x, -l+s+\frac{1}{2}\right) - \cosh\left(\frac{x}{2}\right) c_0, \tag{6.5} $$

and if N is even we also have

$$ \Delta_{m,s,\lambda,\frac{N}{2}}(x) - \Delta_{m,s,\lambda,\frac{N}{2}}(-x) = \frac{d}{dx} \sum_{l\in\mathbb{Z}} \sum_{r=0}^{m} {}^{a}h_{(l-\frac{1}{2})N}^{(r)}\, \eta_r\,(x, s-l)\,. \tag{6.6} $$

Now consider $s = 1/2$. Recall that by Remark 5.4 (a), in this case we have that the embedding $\widehat{\varphi}_s^{[m]} : \widehat{\mathcal{D}_{\theta,x}^N} \longrightarrow c_\infty^{[m]}$ is actually the embedding given by (5.5)–(5.7) composed with T^{-1}, where T was introduced in the proof of Proposition 5.3. Using this, (6.2) and (2.6) we have that

$$ \Delta_{m,\frac{1}{2},\lambda,i}(x) = \frac{d}{dx} \sum_{l\geq 1} \sum_{r=0}^{m} \left(\frac{{}^{c}\lambda_{(l-1)N+i}^{(r)}}{r!} (-x)^r e^{\left(l-\frac{1}{2}\right)x} - \frac{{}^{c}\lambda_{lN-i+1}^{(r)}}{r!} x^r e^{\left(-l+\frac{1}{2}\right)x} \right) + \frac{d}{dx} \sum_{r=0}^{m} \frac{\eta_r\,(x,0)\, c_r}{\sinh \frac{x}{2}} - \frac{d}{dx} \left(\frac{\cosh\left(\frac{x}{2}\right) c_0}{\sinh \frac{x}{2}} \right). \tag{6.7} $$

Introduce the polynomials

$$ {}^{c}g_t(x) = \sum_{r=0}^{m} {}^{c}h_t^{(r)} \frac{x^r}{r!}. $$

Then

$$ \Delta_{m,\frac{1}{2},\lambda,i}(x) - \Delta_{m,\frac{1}{2},\lambda,i+1}(x) = \frac{d}{dx} \sum_{l\geq 1} \Big(e^{\left(l-\frac{1}{2}\right)x}\, {}^{c}g_{(l-1)N+i}(-x) + e^{\left(-l+\frac{1}{2}\right)x}\, {}^{c}g_{lN-i}(x) \Big), \tag{6.8} $$

$$ \left(\frac{d}{dx}+\frac{1}{2}\right) e^{\frac{-x}{2}} \Delta_{m,\frac{1}{2},\lambda,1}(x) - \left(\frac{d}{dx}-\frac{1}{2}\right) e^{\frac{x}{2}} \Delta_{m,\frac{1}{2},\lambda,1}(-x) = -\left(\frac{d^2}{dx^2}-\frac{1}{4}\right) \sum_{l\geq 1} \sum_{r=0}^{m} \left({}^{c}h_{(l-1)N}^{(r)} + \delta_{l,1} c_r \right) \eta_r\,(x, 1-l) - \cosh\left(\frac{x}{2}\right) c_0, \tag{6.9} $$

and if N is even we also have

$$\Delta_{m,\,\frac{1}{2},\,\lambda,\,\frac{N}{2}}(x) - \Delta_{m,\,\frac{1}{2},\,\lambda,\,\frac{N}{2}}(-x) = \frac{d}{dx}\sum_{l\geq 1}\sum_{r=0}^{m} {}^{c}h^{(r)}_{(l-\frac{1}{2})N}\,\eta_r\left(x, \frac{1}{2}-l\right). \tag{6.10}$$

Finally, consider $s = 0$. Recall that by Remark 5.4 (a), in this case we have that the embedding $\widehat{\varphi}_0^{[m]} : \widehat{\mathcal{D}^N_{\theta,x}} \longrightarrow c_\infty^{[m]}$ is actually the embedding given by (5.5)–(5.7) composed with T^{-1}, where T was introduced in the proof of Proposition 5.3. Using this, (6.2) and (2.6) we have that

$$\Delta_{m,\,0,\,\lambda,\,i}(x) = \frac{d}{dx}\sum_{l\geq 1}\sum_{r=0}^{m}\left(\frac{{}^{c}\lambda^{(r)}_{(l-1)N+i}}{r!}x^r e^{lx} - \frac{{}^{c}\lambda^{(r)}_{lN-i+1}}{r!}(-x)^r e^{-lx}\right)$$
$$+\frac{d}{dx}\sum_{r=0}^{m}\frac{\eta_r\left(x, \frac{1}{2}\right)c_r}{\sinh\frac{x}{2}} - \frac{d}{dx}\left(\frac{\cosh\left(\frac{x}{2}\right)c_0}{\sinh\frac{x}{2}}\right). \tag{6.11}$$

Introduce the polynomials

$${}^{c}g_t(x) = \sum_{r=0}^{m} {}^{c}h^{(r)}_t \frac{x^r}{r!}.$$

Then

$$\Delta_{m,\,0,\,\lambda,\,i}(x) - \Delta_{m,\,0,\,\lambda,\,i+1}(x) = \frac{d}{dx}\sum_{l\geq 1}\left(e^{lx}\;{}^{c}g_{(l-1)N+i}(x)\right.$$
$$\left.+\; e^{-lx}\;{}^{c}g_{lN-i}(-x)\right), \tag{6.12}$$

$$\left(\frac{d}{dx}+\frac{1}{2}\right)e^{\frac{-x}{2}}\Delta_{m,\,0,\,\lambda,\,1}(x) - \left(\frac{d}{dx}-\frac{1}{2}\right)e^{\frac{x}{2}}\Delta_{m,\,0,\,\lambda,\,1}(-x) =$$
$$-\left(\frac{d^2}{dx^2}-\frac{1}{4}\right)\sum_{l\geq 1}\sum_{r=0}^{m}\left({}^{c}h^{(r)}_{(l-1)N}+\delta_{l,1}c_r\right)\eta_r\left(x, l-\frac{1}{2}\right)$$
$$-\cosh\left(\frac{x}{2}\right)c_0, \tag{6.13}$$

and if N is even we also have

$$\Delta_{m,\,0,\,\lambda,\,\frac{N}{2}}(x) - \Delta_{m,\,0,\,\lambda,\,\frac{N}{2}}(-x) = \frac{d}{dx}\sum_{l\geq 1}\sum_{r=0}^{m} {}^{c}h^{(r)}_{(l-\frac{1}{2})N}\,\eta_r(x, l). \tag{6.14}$$

Now we can realize the irreducible quasifinite high weigh $\widehat{\mathcal{D}^N_{\theta,x}}$−module. Take an irreducible quasifinite weight $\widehat{\mathcal{D}^N_{\theta,x}}$−module V with central charge c and generating series $\Delta_i(x)$ such that

$$\left(\frac{d^2}{dx^2}-\frac{1}{4}\right)G_1(x) = \left(\frac{d}{dx}-\frac{1}{2}\right)\mathrm{e}^{-\frac{x}{2}}\Delta_1(x) + \left(\frac{d}{dx}-\frac{1}{2}\right)\mathrm{e}^{\frac{x}{2}}\Delta_1(-x)$$

is an even quasipolynomial,

$$F_i(x) = \Delta_i(x) - \Delta_{i+1}(x)$$

for $i = 1, \ldots, [\frac{N}{2}] - \delta_{N,\mathrm{even}}$, are quasipolynomials, and when N is even we also have that

$$F_{\frac{N}{2}}(x) = \frac{\Delta_{\frac{N}{2}}(x) - \Delta_{\frac{N}{2}}(-x)}{2}$$

is an odd quasipolynomial. We may write

$$F_i(x) = \sum_{s\in\mathbb{C}} p_{i,s}(x)e^{sx} \tag{6.15}$$

with $p_{i,s}(x)$ polynomials,

$$G_1(x) = \sum_{s\in\mathbb{C}} \sum_{j=0}^{m_s} a_{s,j}\, \eta_j(x,s) \tag{6.16}$$

and

$$F_{\frac{N}{2}}(x) = \sum_{s\in\mathbb{C}} \sum_{j=0}^{\widetilde{m}_s} b_{s,j}\, \eta_j(x,s), \tag{6.17}$$

where $a_{s,j},\ b_{s,j} \in \mathbb{C}$ and $a_{s,j} \neq 0,\ b_{s,j} \neq 0$ for only finitely many $s \in \mathbb{C}$.

Remark 6.4: Since, by definition of η_r, we have that $\eta_r(x,-s) = (-1)^r \eta_r(x,s)$, to avoid ambiguities in the expression of $G_1(x)$ and $F_{\frac{N}{2}}$ above, we will choose the parameter s following these rules: when $s \in \mathbb{Z}$, we require $s \leq 0$; when $s \in \frac{1}{2} + \mathbb{Z}$,we require $s \leq \frac{1}{2}$; when $s \notin \mathbb{Z}/2$, we require that $\operatorname{Im} s > 0$ if $\operatorname{Im} s \neq 0$, or $s - [s] < \frac{1}{2}$ if $s \in \mathbb{R}$, where $\operatorname{Im} s$ denotes the imaginary part of s and again $[s]$ denote the closest integer to s, which is not larger than s. Decompose the set $\{s \in \mathbb{C} : a_{s,j} \neq 0 \text{ for some } j\} \bigcup \{s \in \mathbb{C} : b_{s,j} \neq 0 \text{ for some } j\} \bigcup \{s \in \mathbb{C} : p_{i,s}(x) \neq 0\}$ into a disjoint union of classes under the equivalence condition: $s \sim \dot{s}$ if and only if $s = \pm\dot{s}$ (mod$\mathbb{Z}$). Pick a representative s in an equivalence class S such that $s = 0$ if the equivalence class is in $\mathbb{Z}$, and $s = \frac{1}{2}$ if the equivalence class is in $\mathbb{Z} + \frac{1}{2}$. Let $S = \{s, s - k_1, s - k_2, \ldots\}$ be such an equivalence class and take $m = \max_{s\in S}\{m_s, \widetilde{m}_s, \deg p_{i,s}\}$. Put $k_0 = 0$. It is easy to see that if $s = 0$ or $s = \frac{1}{2}$, then $k_i \in \mathbb{N}$.

We will associate with each S the $\mathfrak{g}^{[m]}-$module $L_S^{[m]}(\lambda_S)$ in the following way.

- If $s \notin \mathbb{Z}/2$, set

$$^{a}h^{(r)}_{(k_j-1)N} + \delta_{1,k_j} c_r = \frac{1}{4}\, a_{-k_j+s+\frac{1}{2},r}, \quad c_0 = -\,a_{\frac{1}{2},0}, \tag{6.18}$$

$$^{a}h^{(r)}_{k_jN-i} = \left(\frac{d}{dx}\right)^r p_{i,-k_j+s}(0), \tag{6.19}$$

$$^{a}h^{(r)}_{(k_j-1)N+i} = (-1)^r \left(\frac{d}{dx}\right)^r p_{i,k_j-s}(0), \tag{6.20}$$

and if N is even

$$^{a}h^{(r)}_{(k_j-\frac{1}{2})N} = 2b_{-k_j+s,r} \tag{6.21}$$

for $r = 0\cdots m$ and $j = 0, 1, 2, \ldots$.

We associate with S the $\widehat{\mathfrak{gl}}^{[m]}_\infty-$module $L_S^{[m]}(\lambda_S)$ with central charges

$$c_r = \sum_i \sum_{k_j} \left({}^{a}h^{(r)}_{(k_j-1)N+i} + {}^{a}h^{(r)}_{k_jN-i} \right) + \sum_{k_j} \left({}^{a}h^{(r)}_{(k_j-1)N} + \delta_{N,\text{even}}\, {}^{a}h^{(r)}_{(k_j-\frac{1}{2})N} \right) \tag{6.22}$$

and labels

$$\lambda^{(r)}_j = \sum_{(k_j-1)N+i\geq j} {}^{a}\widetilde{h}^{(r)}_{(k_j-1)N+i} + \sum_{k_jN-i\geq j} {}^{a}\widetilde{h}^{(r)}_{k_jN-i}$$

$$+\ \delta_{N,\text{even}} \sum_{(k_j-\frac{1}{2})N\geq j} {}^{a}\widetilde{h}^{(r)}_{(k_j-\frac{1}{2})N} + \sum_{(k_j-1)N\geq j} {}^{a}\widetilde{h}^{(r)}_{(k_j-1)N}, \tag{6.23}$$

where ${}^{a}\widetilde{h}^{(r)}_k = {}^{a}h^{(r)}_k - c_r\delta_{k,0}$.

- If $s = 0$, set

$$^{c}h^{(r)}_{k_jN} - \delta_{k_j,0} c_r = a_{k_j+\frac{1}{2},r}, \quad c_0 = -\,a_{\frac{1}{2},0}, \tag{6.24}$$

$$^{c}h^{(r)}_{k_jN+i} = \left(\frac{d}{dx}\right)^r p_{i\,,k_j+1}, \tag{6.25}$$

$$^{c}h^{(r)}_{(k_j+1)N-i} = (-1)^r \left(\frac{d}{dx}\right)^r p_{i\,,-k_j-1}(0), \tag{6.26}$$

and if N is even

$$^{d}h^{(r)}_{(k_j+\frac{1}{2})N} = 2b_{k_j+1,\,r} \tag{6.27}$$

for $r = 0 \cdots m$ and $j = 0, 1, 2 \ldots.$

We associate with S the $c_\infty^{[m]}-$module $L_S^{[m]}(\lambda_S)$ with central charges

$$c_r = \sum_i \sum_{k_j} \left({}^{c}h^{(r)}_{k_jN+i} + {}^{c}h^{(r)}_{(k_j+1)N-i}\right) + \sum_{k_j} \left(\delta_{N,\text{even}}\, {}^{c}h^{(r)}_{(k_j+\frac{1}{2})N} + {}^{c}h^{(r)}_{k_jN}\right) \tag{6.28}$$

and labels

$$\lambda_j^{(r)} = \sum_{k_jN+i\geq j} {}^{c}h^{(r)}_{k_jN+i} + \sum_{(k_j+1)N-i\geq j} {}^{c}h^{(r)}_{(k_j+1)N-i} + \sum_{k_jN\geq j} {}^{c}h^{(r)}_{k_jN}$$

$$+\ \delta_{N,\text{even}} \sum_{(k_j+\frac{1}{2})N\geq j} {}^{c}h^{(r)}_{(k_j+\frac{1}{2})N}. \tag{6.29}$$

- If $s = \frac{1}{2}$,

$$^{c}h^{(r)}_{k_jN} + \delta_{0,k_j} c_r = a_{k_j,\,r}, \quad c_0 = -a_{-\frac{1}{2},0}, \tag{6.30}$$

$$^{c}h^{(r)}_{k_jN+i} = (-1)^r \left(\frac{d}{dx}\right)^r p_{i,\,k_j+\frac{1}{2}}(0), \tag{6.31}$$

$$^{c}h^{(r)}_{(k_j+1)N-i} = \left(\frac{d}{dx}\right)^r p_{i,\,-k_j-\frac{1}{2}}(0), \tag{6.32}$$

and if N is even

$$^{c}h^{(r)}_{(k_j+\frac{1}{2})N} = 2b_{-k_j+\frac{1}{2},\,r} \tag{6.33}$$

for $r = 0 \cdots m$ and $j = 0, 1, \ldots.$

We associate with S the $c_\infty^{[m]}-$module $L_S^{[m]}(\lambda_S)$ with central charges

$$c_r = \sum_i \sum_{k_j} \left({}^{c}h^{(r)}_{k_jN+i} + {}^{c}h^{(r)}_{(k_j+1)N-i}\right) + \sum_{k_j} \left(\delta_{N,\text{even}}\, {}^{c}h^{(r)}_{(k_j+\frac{1}{2})N} + {}^{c}h^{(r)}_{k_jN}\right) \tag{6.34}$$

and labels

$$\lambda_j^{(r)} = \sum_{k_jN+i\geq j} {}^{c}h^{(r)}_{k_jN+i} + \sum_{(k_j+1)N-i\geq j} {}^{c}h^{(r)}_{(k_j+1)N-i} + \sum_{k_jN\geq j} {}^{c}h^{(r)}_{k_jN}$$

$$+\ \delta_{N,\text{even}} \sum_{(k_j+\frac{1}{2})N\geq j} {}^{c}h^{(r)}_{(k_j+\frac{1}{2})N}. \tag{6.35}$$

Denote by $\{s_1, s_2, \ldots, s_M\}$ a set of representatives of equivalence classes in the set $\{s \in \mathbb{C} : a_{s,j} \neq 0 \text{ for some } j\} \bigcup \{s \in \mathbb{C} : b_{s,j} \neq 0 \text{ for some } j\} \bigcup \{s \in \mathbb{C} : p_{i,s}(x) \neq 0\}$. By theorem 6.3, the $\widehat{\mathcal{D}^N_{\theta,x}}$-module $L^{[\vec{m}]}_{\vec{s}}(\vec{\lambda})$ is irreducible for $\vec{s} = (s_1, s_2, \ldots, s_M)$ such that $s_i \in \mathbb{Z}$ implies $s_i = 0$, $s_i \in \mathbb{Z} + \frac{1}{2}$

implies $s_i = \frac{1}{2}$, and $s_i \neq \pm s_j$ mod$\mathbb{Z}$ for $i \neq j$. We have

$$\Delta_{\vec{m},\vec{s},\vec{\lambda}}(x) = \sum_i \Delta_{m_i,s_i,\lambda_i}(x) \quad \text{and} \quad c = \sum_i c_0(i).$$

By theorem 4.2 and summarizing the above we have proved the following:

Theorem 6.5: *Let V be an irreducible quasifinite highest weight $\widehat{\mathcal{D}^N_{\theta,x}}$-module with central charge c and generating series $\Delta_i(x)$ such that*

$$\left(\frac{d^2}{dx^2} - \frac{1}{4}\right) G_1(x) = \left(\frac{d}{dx} - \frac{1}{2}\right) e^{-\frac{x}{2}} \Delta_1(x) - \left(\frac{d}{dx} - \frac{1}{2}\right) e^{\frac{x}{2}} \Delta_1(-x)$$

is an even quasipolynomial,

$$F_k(x) = \Delta_k(x) - \Delta_{k+1}(x)$$

for $k = 1, \ldots, [\frac{N}{2}] - \delta_{N,even}$, are quasipolynomials, and if N is even

$$F_{\frac{N}{2}}(x) = \frac{\Delta_{\frac{N}{2}}(x) - \Delta_{\frac{N}{2}}(-x)}{2}$$

is an odd quasipolynomial. Then V is isomorphic to the tensor product of all the modules $L_s^{[m]}(\lambda_S)$ for different equivalence classes S.

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