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LETTERS AND COMMENTS

Energy in one-dimensional linear waves

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Abstract

This work is based on propagation phenomena that conform to the classical wave equation. General expressions of power, the energy conservation equation in continuous media and densities of the kinetic and potential energies are presented. As an example, we study the waves in a string and focused attention on the case of standing waves. The treatment is applicable to introductory science textbooks.

1. Introduction

In a recently published letter [1], the energy content and energy transfer involved in one-dimensional waves applied to a string were studied. The author discusses the validity of the description of the potential energy in terms of the slope of the string for standing waves.

We believe that in the case of standing waves, the assumption made by Burko [1] that there is no flow of energy is incorrect. In a standing wave there is a flow of energy, which explains why the energy of a string element is not constant in time. At an antinode, at the instant of maximum displacement, the potential and kinetic energies of the string element are zero. When the string element is passing through the equilibrium position, the potential energy is zero but its kinetic energy is maximum. This string element performs simple harmonic motion as in the case of a spring–mass oscillator, but unlike the latter, each element of the string interacts with contiguous elements that make up the continuum, and therefore its energy does not behave in the same way as that of the oscillator. This apparent analogy may lead to misinterpretations [2].

In this work, we will study one-dimensional waves of small amplitudes propagating in a continuous elastic media. Particularly, we study the waves in a string, although the results may also apply in the cases of the propagation of sound and electromagnetic waves.

This work is based on propagation phenomena that conform to the classical wave equation. Section 2 is devoted to the study of the energy conservation equation in continuous media. In section 3, we analyse the densities of the string's kinetic and potential energies and energy flux to the case of stationary waves.

2. Classical wave equation and energy conservation

It is well known that in the small oscillation approximation, the classical wave equation takes the form

$$\frac{\partial^2\Phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0, \tag{2.1}$$

where $\Phi = \Phi(x, t)$ represents the displacement of the string element at the point x at time t measured from the equilibrium position. Moreover, it is assumed that the tension T of the string is uniform at the rest position. Outside this position, as the string oscillates, its length increases and, as a consequence, the value of the tension changes. However, if $\partial\Phi/\partial x \ll 1$, the increment of the string length is small and it can be assumed that the tension remains constant. Thus, T has the same value as when the string is in the rest position [3, 4]. In the case of waves in strings, c in (2.1) takes the form $c = \sqrt{T/\mu}$, where μ is the mass per unit length.

The local conservation energy equation is indeed a consequence of the wave equation (2.1) [5]. Indeed, we multiply equation (2.1) by $\partial\Phi/\partial t$, and after some algebra we arrive at

$$\frac{\partial}{\partial x}\left(-T\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\left(\frac{\partial\Phi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial\Phi}{\partial x}\right)^2\right) = 0, \tag{2.2}$$

that is, the local conservation energy equation, because it is of the form

$$\nabla\cdot\vec{J} + \frac{\partial u}{\partial t} = 0. \tag{2.3}$$

The energy flux in the x-direction

$$J_x = -T\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial t} \tag{2.4}$$

is the rate at which a segment of string is doing work on its neighbour to the right, and

$$u = \frac{1}{2}\mu\left(\frac{\partial\Phi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial\Phi}{\partial x}\right)^2 \tag{2.5}$$

is the local energy density.

Phenomenologically, equation (2.4) is indeed the energy flux as the product of the transverse tension (for small amplitudes) times the transverse velocity [6]. Clearly, the first term of (2.5) is the linear density of the kinetic energy u_c. As the wave energy is partly kinetic and partly potential, the second term of (2.5) is the linear density of the potential energy u_p.

Then, equation (2.3) states and shows the details of how energy is conserved. If the energy of a certain region decreases, it is because it flows through the region boundaries.

We can note that equations (2.4) and (2.5) are not a direct consequence of comparing (2.2) with (2.3). Indeed, if we make the transformations $u \to u + f(x)$ and $J_x \to J_x + g(t)$, where f and g are two arbitrary functions, equation (2.3) is still satisfying. However, if we are studying the propagation of a wave, all physical quantities involved in the wave must depend on both space and time. Thus, the functions f and g have no connection with the wave and we take as identically zero.

Integrating equation (2.3) over x, between arbitrary points (a, b) on the string, gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u\,\mathrm{d}x = \frac{\mathrm{d}E}{\mathrm{d}t} = T\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial t}\bigg|_a^b. \tag{2.6}$$

The wave energy in any region (a, b) of the string is not constant, except in two situations: first, when the points a and b are the fixed/free ends of a string of length $L = b - a$ and second, if the wave extends spatially over a region fully contained in the interval (a, b). In these conditions we can say that energy is the global or total energy of the wave.

3. Power and energy density of a stationary harmonic wave

In this section, we will calculate the power and energy density for a standing wave. In particular, we consider the oscillations of the fundamental mode of a string of length L that is fixed at both ends. The equation that describes this mode is

$$\Phi(x, t) = A \sin(kx) \sin(\omega t), \tag{3.1}$$

where $k = \pi/L$ and $\omega = \pi c/L$ are the wave number and angular frequency of the fundamental mode, respectively. Differentiating this expression with respect to position and time, and substituting it into equations (2.4) and (2.5), we obtain

$$\begin{aligned} u_c(x, t) &= \frac{\mu\omega^2 A^2}{2} \sin^2(kx) \cos^2(\omega t) \\ u_p(x, t) &= \frac{\mu\omega^2 A^2}{2} \cos^2(kx) \sin^2(\omega t) \\ J_x(x, t) &= -\frac{\mu\omega^2 A^2 c}{4} \sin(2kx) \sin(2\omega t). \end{aligned} \tag{3.2}$$

It is straightforward to show that (2.3) is satisfied by (3.2), but the alternative expression (2.4) of [1] for the potencial energy density does not, since it gives a time-independent total energy density. Hence, to the extent that (2.4) follows both as a first integral of the wave equation (2.1) and phenomenologically as remarked in the previous section, and assuming energy conservation, equation (4) of [1] cannot be valid locally.

In order to better analyse the results (3.2) in detail, we plot the displacement, the energy densities and power as a function of the coordinate x for different values of time, t. In the left panel of figure 1, we show the behaviour of displacement $\Phi(x, t)$ as a function of x for $t = 0$, $T/8$, $T/4$ and $3T/8$, where $T = 2\pi/\omega$, the period of the fundamental mode.

In the middle panel of figure 1, we show graphs of the kinetic energy density $u_c(x, t)$ (solid lines) and potential energy density $u_p(x, t)$ (dashed lines) for the same values of time. We can see clearly the different spatial and temporal distributions of energy densities. It is important to note that the kinetic energy density is large in the central region of the string and decreases as we go to extremes $x = 0$ and $x = L$, where the string is fixed. The potential energy density is always zero at $x = L/2$ because that segment of the string oscillates without deformation. The potential energy density grows as we approach the fixed ends and is a maximum at $x = 0$ and $x = L$. At these points, the deformation (elongation) of the string is also a maximum.

Finally, the right panel shows the power $J_x(x, t)$. When the power moves from the middle towards the ends, the kinetic energy density decreases in the center and the potential energy density increases at the ends. For the last two points in time the reverse situation occurs. The power at any point on the string periodically changes sign. Therefore, the temporal and spatial average values of it are zero.

This example clearly shows the applicability of the expressions obtained in the previous sections, based on the validity of the wave equation for small perturbations. In the opinion of the authors, these results do not present any contradiction; the total energy of the string is a constant but its distribution varies in space and time.

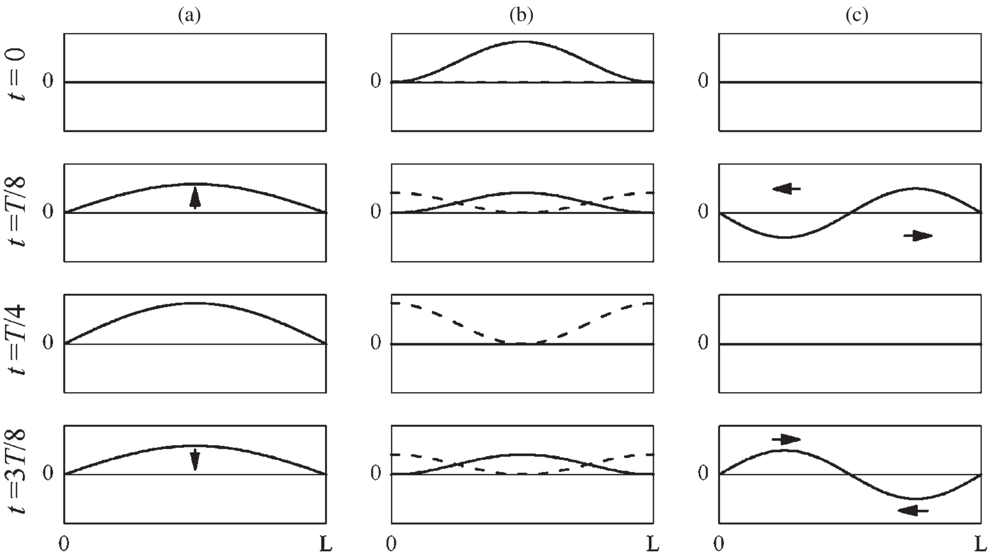

Figure 1. Displacement (a), kinetic and potential energy densities (b) and power (c) distributions along the string's length at different times. In (b) solid and dashed lines represent kinetic and potential energy densities, respectively.

4. Conclusions

Based on the validity of the classical wave equation, we have obtained general expressions for the power, kinetic and potential energy densities and the equation of energy conservation for continuum media. It was shown that these expressions can be applied to stationary solutions without contradiction. This deduction is suitable for teaching physics at an introductory level to students studying science and engineering.

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