Noninvertible symmetries in type IIB supergravity

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In this work, we uncover a collection of noninvertible topological operators linked to the 0-, 2-, 4-, and 6-form symmetries related to the type IIB superstring effective theory. By pinpointing the $\mathrm{SL}(2,\mathbb{Z})$-covariant conserved currents corresponding to these symmetries, we first derive a set of $\mathrm{SL}(2,\mathbb{Z})$-invariant invertible topological operators that encapsulate the integer Bogomol'nyi-Prasad-Sommerfield charges inherent to the theory. Moving forward, by incorporating fractional charges while maintaining gauge invariance, we introduce the noninvertible topological operators for each generalized symmetry and, in particular, for the $\mathrm{SL}(2,\mathbb{Z})$ 0-form symmetry. Identifying them as a novel kind of symmetries reminiscent of fractional quantum-Hall-effect-like noninvertible operators, we study their action on charged objects and their associated topological quantum field theories obtained via half (higher) gauging.

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I. INTRODUCTION

In recent years, the discovery of higher-form symmetries [1,2], higher-group symmetries [3–7], and noninvertible symmetries [8–18] has opened new avenues in quantum field theories (QFTs) (see [19–23] for some reviews). These aspects unveil new layers of complexity and richness into the symmetry structures of physical theories, shifting the paradigm of how we understand these systems.

Numerous works have significantly advanced the understanding of higher-form and noninvertible symmetries in string theory and supergravity [24–33]. See [34,35] for gauge noninvertible symmetries from a world sheet perspective.

In the context of supergravity, despite the fact that every global symmetry is expected to be broken or gauged in string theory, we aim to understand whether new phenomena associated with noninvertible symmetries can effectively emerge at the intermediate scale between the continuous IR and discrete UV symmetries. On the other hand, dealing with both Page currents [36] (see also [37,38]) and higher-group structures entails an obstacle for assembling topological operators associated with these symmetries. This eventually translates to a formulation of BF theories in terms of the supergravity field strengths (see [29] for example).

In this paper we elaborate on this to understand the landscape of global higher-form and noninvertible symmetries in a particular supergravity theory. We want to scrutinize the symmetry structure of the bosonic sector of type IIB superstring effective theory and highlight its significance and implications, where the $\mathrm{SL}(2,\mathbb{R})$ 0-form symmetry serves a crucial function. More in detail, we would like to understand the role of the non-Abelian 0-form symmetry in the construction of the (non)invertible topological operators of the theory and their actions.

Some works involving type IIB backgrounds in the literature amount to the obtaining of topological quantum field theories (TQFTs) [39], the analysis of topological T duality [40], topological duality defects [41], and branes [42,43].

The paper is organized as follows. In Sec. II, we introduce the manifestly invariant $\mathrm{SL}(2,\mathbb{R})$ type IIB superstring effective theory, together with its global and gauge symmetries. In Sec. III, we obtain the invertible topological operators associated with the unbroken global higher-form symmetries of the theory and their action on charged operators. Section IV contains a set of noninvertible

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topological operators for each global symmetry of the theory, together with their genesis from a half higher gauging and associated TQFTs. This, in turn, allows us to calculate their action on charged operators. Finally, our conclusions are presented in Sec. V.

II. SYMMETRIES OF TYPE IIB SUPERGRAVITY

In its manifestly $\mathrm{SL}(2,\mathbb{R})$-invariant formulation [44], the bosonic type IIB supergravity pseudoaction reads

$$S=\int_{\Sigma_{10}}\left[-R\star 1-\frac{1}{4}\mathrm{d}M^{ij}\wedge\star\,\mathrm{d}M_{ij}+\frac{1}{2}M^{ij}H_i\wedge\star\,H_j+\frac{1}{4}F\wedge\star\,F-\frac{1}{4}\epsilon^{ij}D\wedge H_i\wedge H_j\right], \quad (1)$$

where B_i is an $\mathrm{SL}(2,\mathbb{R})$ doublet of 2-forms, D is a 4-form singlet, and M is an $\mathrm{SL}(2,\mathbb{R})$ symmetric matrix that depends on the scalars of the theory. This pseudoaction must be complemented by the self-duality relation $F=\star F$, and

$$H_i\equiv \mathrm{d}B_i,\qquad F\equiv \mathrm{d}D-\frac{1}{2}\epsilon^{ij}B_i\wedge H_j \quad (2)$$

are the field strengths that are invariant under the following gauge transformations:

$$B_i\to B_i+\mathrm{d}\lambda_i^{(1)},\quad D\to D+\mathrm{d}\lambda^{(3)}+\frac{1}{2}\epsilon^{ij}\mathrm{d}\lambda_i^{(1)}\wedge B_j. \quad (3)$$

When Σ_{10} has a nontrivial topology that permits the existence of closed 2-forms $\Lambda_i^{(2)}$, 4-forms $\Lambda^{(4)}$, or both, the equations of motion (EOM) are invariant under the transformations

$$B_i\to B_i+\Lambda_i^{(2)},\qquad D\to D+\Lambda^{(4)}+\frac{1}{2}\epsilon^{ij}\Lambda_i^{(2)}\wedge B_j. \quad (4)$$

Thus, while the EOM exhibit a classical $\mathrm{U}(1)^{(2)}\times\mathrm{U}(1)^{(2)}$ 2-form symmetry and a $\mathrm{U}(1)^{(4)}$ 4-form symmetry, the Chern-Simons term in the action breaks both of them for generic topologies (see also [29,45]). Moreover, the Bianchi identities $\mathrm{d}H_i=0$ yield a $\mathrm{U}(1)^{(6)}\times\mathrm{U}(1)^{(6)}$ 6-form symmetry.

In contrast, this pseudoaction is manifestly invariant under the following $\mathrm{SL}(2,\mathbb{R})$ 0-form symmetry transformations [46]:

$$M'_{ij}=\Omega^k{}_i\Omega^l{}_jM_{kl},\qquad B'_i=\Omega^j{}_iB_j, \quad (5)$$

where $\Omega^i{}_j\in\mathrm{SL}(2,\mathbb{R})$. This $\mathrm{SL}(2,\mathbb{R})$ symmetry acts as an outer automorphism on the U(1) factors of both 2- and 6-form symmetries.

Finally, the transformations (3) allow us to identify a $(\mathrm{U}(1)^{(2)}\times\mathrm{U}(1)^{(2)})\times_\kappa\mathrm{U}(1)^{(4)}$ 4-group classical symmetry [5], which is fully characterized by the invariant of the $\mathrm{SL}(2,\mathbb{R})$ outer automorphism, $\kappa\equiv\epsilon^{ij}$.

III. INVERTIBLE CHARGED OPERATORS AND ACTIONS

The type IIB pseudoaction is invariant only under the 0- and 6-form symmetries. In this section we are going to obtain the invertible topological operators using their respective conserved currents.

The $\mathrm{SL}(2,\mathbb{R})$ codimension-1 conserved Noether current associated with the 0-form symmetry satisfies $\mathrm{d}\star j_a=0$, where the index a transforms in the adjoint representation of $\mathrm{SL}(2,\mathbb{R})$ and the current $\star j_a$ is given by

$$\star j_a=4(t_a)^j{}_i\left[M^{ik}\star\mathrm{d}M_{kj}+B_j\wedge M^{ik}\star H_k+\frac{1}{4}\epsilon^{ik}B_j\wedge(B_k\wedge\star\,F-2D\wedge H_k)\right], \quad (6)$$

where $(t_a)^j{}_i$ are the generators of the $\mathrm{SL}(2,\mathbb{R})$ group.

Regarding the $\mathrm{U}(1)^{(6)}\times\mathrm{U}(1)^{(6)}$ symmetry, the conserved current $\star\, j_i$ arising from the Bianchi identities of the 2-forms B_i is [47,48]

$$\mathrm{d}\star j_i=0,\qquad \star\, j_i=H_i. \quad (7)$$

Thus, we construct the invertible topological operators $U(\Sigma_{\hat p})$ associated with each of the two currents as follows:

$$U(\Sigma_9)=\exp\int_{\Sigma_9}\star j^{(1)},\quad U(\Sigma_3)=\exp\int_{\Sigma_3}\star j^{(7)}, \quad (8)$$

where $\Sigma_{\hat p}$ represents a $\hat p$-dimensional closed manifold, the currents are

$$\star j^{(1)}\equiv q^a\star j_a,\qquad \star j^{(7)}\equiv 2\pi\mathrm{i}\tilde q^i\star j_i, \quad (9)$$

and $\{q^a,\tilde q^i\}$ are the parameters of the symmetry transforming in the adjoint and fundamental $\mathrm{SL}(2,\mathbb{Z})$ representations, respectively. At this moment, the operators are not gauge invariant, as the charges are still real numbers.

To evaluate the action of these operators, we introduce the charged objects $\mathcal{O}_{(2)}$, $\mathcal{O}_{(4)}$, and $\mathcal{O}_{(6)}$ which, respectively, carry charges $\{Q^i,Q,\tilde Q_i\}\in\mathbb{Z}$ transforming, respectively, in the fundamental, singlet, and antifundamental representations of $\mathrm{SL}(2,\mathbb{Z})$ and that correspond to F1/D1, D3, and NS5/D5 charges [37]. We will denote the insertion of these operators along a generic submanifold Σ_p as $\mathcal{O}_{(p)}\equiv\mathcal{O}(\Sigma_p)$ [49].

Then, the actions of $U(\Sigma_{\hat{p}})$ on these operators are

$$U(\Sigma_{\hat{p}}):\ \mathcal{O}(M_p) \mapsto \mathcal{O}(M_p).$$

Let us emphasize that the trivial action of the 0-form operator $U(\Sigma_9)$ is a consequence of the invariance of every $\mathcal{O}(M_p)$ under $\mathrm{SL}(2,\mathbb{Z})$. Precisely, its action on operators $\mathcal{O}_i(M_p)$ and $\mathcal{O}_a(M_p)$ transforming, respectively, under the fundamental and the adjoint representations is

$$U(\Sigma_9):\ \mathcal{O}_i(M_p) \mapsto \Omega^j{}_i \mathcal{O}_j(M_p),\qquad \mathcal{O}_a(M_p) \mapsto \Omega^b{}_a \mathcal{O}_b(M_p), \tag{10}$$

where $\Omega^j{}_i \equiv \exp C^j{}_i$, $\Omega^b{}_a \equiv \exp(q^c f^b{}_{ca})$ and $C^j{}_i \equiv q^a (t_a)^j{}_i$.

The gauge invariance of the operator $U(\Sigma_9)$ requires the components of C to be integers. Because $\Omega \in \mathrm{SL}(2,\mathbb{Z})$, this further constrains C to satisfy $\det C = 0$. These two conditions in turn imply that Ω is not the most general $\mathrm{SL}(2,\mathbb{Z})$ element. Instead it amounts to the $\mathrm{SL}(2,\mathbb{Z})$ monodromies of the D7-brane [51], which precisely leaves C invariant.

It would be interesting to explore whether there exists a richer structure in the 0-form sector, reflecting the non-Abelian nature of $\mathrm{SL}(2,\mathbb{R})$, which requires a more careful analysis of the charges q's as representations.

IV. NONINVERTIBILITY AND ACTION OF OPERATORS

In this section we distinguish the different situations of the higher-form symmetries at the level of the action.

For nontrivial topologies, the $\mathrm{U}(1)^{(2)} \times \mathrm{U}(1)^{(2)}$ and the $\mathrm{U}(1)^{(4)}$ symmetries are not symmetries of the pseudoaction due to the Chern-Simons terms. However, we are going to show that the theory still hosts a subgroup of them as 2- and 4-form noninvertible symmetries.

A different situation occurs with the 0- and 6-form symmetries. Being symmetries of the action, we can promote the integer parameters of their associated topological operators to rational while still preserving gauge invariance. Because of its simple structure, $\mathrm{U}(1)^{(6)} \times \mathrm{U}(1)^{(6)}$ does not require any further treatment for this. However, in full analogy with the fractional quantum Hall effect (FQHE), the $\mathrm{SL}(2,\mathbb{Z})$ symmetry will necessarily become noninvertible when considering fractionary parameters.

Let us first introduce the classical conserved global currents associated with the 2- and 4-form symmetries. These, in turn, will be improved to build gauge-invariant topological operators with fractionary parameters.

The conserved current of the $\mathrm{U}(1)^{(2)} \times \mathrm{U}(1)^{(2)}$ symmetry associated with $\Lambda_i^{(2)}$ is

$$\mathrm{d}(M^{ij} \star H_j) + \frac{1}{2}\epsilon^{ij} H_j \wedge (F + \star F) = 0, \tag{11}$$

where, upon assuming $F = \star F$, it can be rewritten as

$$\mathrm{d} \star j^i = 0,\qquad \star j^i = 3\left(M^{ij} \star H_j + \frac{1}{3}\epsilon^{ij} B_j \wedge (F + 2\mathrm{d}D)\right). \tag{12}$$

Regarding the 4-form symmetry $\mathrm{U}(1)^{(4)}$, the conserved current associated with $\Lambda^{(4)}$ is

$$\mathrm{d} \star j = 0, \qquad \star j = 2\left(\star F + \frac{1}{2}\epsilon^{ij} B_i \wedge H_j\right). \tag{13}$$

A. Noninvertible operators

In this section we build a set of noninvertible operators associated with the generalized currents discussed above. To do so, we relax the integer condition over the parameters to rational numbers, $q \to q/N$ with $N \in \mathbb{Z}$, while preserving the gauge invariance of the operators. This is achieved by introducing a set of auxiliary fields which will be the degrees of freedom (d.o.f.) of the TQFTs associated with each noninvertible operator. In Fig. 1 the arrows denote the auxiliary fields entering each noninvertible operator $\mathcal{D}(\Sigma_p)$. The role of these fields is twofold: while they allow us to restore the gauge invariance, the integration over these new d.o.f. makes the resulting operator noninvertible. In analogy to [17,18], our result consists of a set of FQHE operators for the symmetries of type IIB supergravity [52].

Let us first consider $U(\Sigma_3)$, the topological operator of the $\mathrm{U}(1)^{(6)} \times \mathrm{U}(1)^{(6)}$ symmetry. Being gauge invariant even when q^i is a continuous parameter, it does not require any treatment. Thus, there is no noninvertible counterpart associated and the symmetry becomes $\mathbb{Z}_N^{(6)} \times \mathbb{Z}_N^{(6)}$.

Let us promote the parameters that multiply the 0-, 2-, and 4-form currents to rational values,

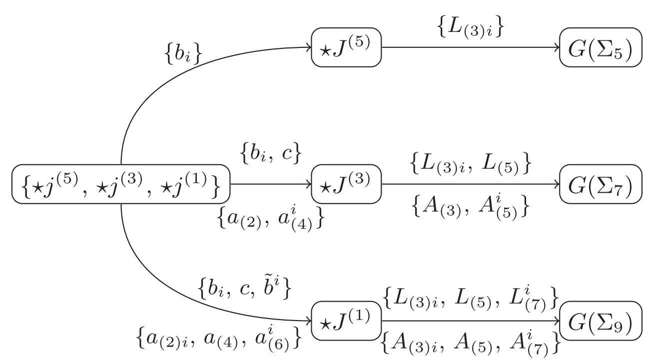

FIG. 1. Left: higher-form currents of type IIB supergravity, where $\star j^{(7)}$ is omitted due to its trivial nature. Center: noninvertible $\star J^{(p)}$ currents and the associated auxiliary fields in the arrows. Right: half higher gauging terms. Above and below each arrow, we show the gauge fields that induce gauge transformations on the auxiliary fields above and below the previous arrow, respectively.

$$q \to \frac{q}{N}, \qquad q^i \to \frac{q^i}{N}, \qquad q^a \to \frac{q^a}{N}. \tag{14}$$

This would break the gauge invariance of some of their associated topological operators. However, upon introducing the auxiliary fields to restore the gauge invariance, they will automatically mutate to noninvertible operators.

Regarding the vector parameters $q^i \in \mathbb{Z}$ and $q^a \in \mathbb{Z}$, it is sufficient considering the same integer N for every component. Dividing each of its components by a different integer yields an equivalent result: $q^I/N^I = q'^I/N$, for $I = \{i, a\}$ [55].

Let us study then the gauge-invariant operator associated with the classical $\mathrm{U}(1)^{(4)}$ that is compatible with $q \to \frac{1}{N} q$. The Chern-Simons structure of $\star j^{(5)}$ in (13) allows for the same treatment as to the original FQHE case [17,18]. In our case, the noninvertible operator results

$$\mathcal{D}(\Sigma_5) = \int [\mathrm{D}b_i] \exp\left[2\pi \mathrm{i} \int_{\Sigma_5} \star J^{(5)}\right],$$
$$\star J^{(5)} \equiv q\left(\frac{2}{N} \star F - \epsilon^{ij}(2H_i - N h_i) \wedge b_j\right), \tag{15}$$

where b_i is an $\mathrm{SL}(2, \mathbb{Z})$ doublet of 2-forms and $h_i \equiv \mathrm{d}b_i$.

We analyze now the 2-form symmetry topological operator, which is associated with $\mathrm{U}(1)^{(2)} \times \mathrm{U}(1)^{(2)}$, with a fractional parameter $q^i \to \frac{1}{N} q^i$. In order to write a noninvertible one, it is crucial to observe that the last two terms of $\star j^i$ in (12), which is the nongauge-invariant sector, have been expressed as the sum of two BF theories. This part can be extended to a gauge-invariant one as follows:

$$\epsilon^{ij} B_j \wedge (F + 2\mathrm{d}D) \to I^{(3)i} \equiv \epsilon^{ij}(b_j \wedge F - 2c \wedge H_j), \tag{16}$$

where b_i and c are a doublet of 2-forms and a 4-form auxiliary field, respectively, whose gauge transformations are similar to the ones in (3),

$$b_i \to b_i + \mathrm{d}\lambda_i^{(1)}, \qquad c \to c + \mathrm{d}\lambda^{(3)} + \frac{1}{2}\epsilon^{ij}\mathrm{d}\lambda_i^{(1)} \wedge B_j.$$

In addition, $f \equiv dc - \frac{1}{2}\epsilon^{kl} b_k \wedge H_l$ is a gauge-invariant tensor mimicking the field strength F.

The second and final task consists of imposing the gauge-invariant constraints

$$H_i - N h_i = 0, \qquad F - N f = 0, \tag{17}$$

whose solutions, up to gauge transformations, are

$$\{b_i, c\} = \frac{1}{N}\{B_i, D\}. \tag{18}$$

Let us note that, while this equation holds locally, a more careful analysis is necessary when considering a nontrivial spacetime topology, for which the gauge fields are not defined as differential forms.

Thus, upon integrating out the auxiliary fields, we recover the $\mathbb{Z}_N^{(2)} \times \mathbb{Z}_N^{(2)}$ current $\star j^{(3)}/N$. To impose the constraints (17), we need an extra pair of auxiliary fields, $\{a_{(2)}, a^i_{(4)}\}$ which, being invariant under gauge transformations, can be understood as Lagrange multipliers. Then the operator will contain the gauge-invariant pieces,

$$2(F - Nf) \wedge a_{(2)} + (H_i - N h_i) \wedge a^i_{(4)}, \tag{19}$$

which trivially establish the solution (18). Thus, combining (16) and (19), the noninvertible operator is assembled as follows:

$$\mathcal{D}(\Sigma_7) = \int \mathrm{D}[b_i, c, a_{(2)}, a^i_{(4)}] \exp\left[2\pi \mathrm{i} \int_{\Sigma_7} \star J^{(3)}\right],$$
$$\star J^{(3)} \equiv \frac{3}{N} q_i M^{ij} \star H_j + q_i \epsilon^{ij}(b_j \wedge F - 2c \wedge H_j)$$
$$+ 2(F - Nf) \wedge a_{(2)} + (H_i - N h_i) \wedge a^i_{(4)}. \tag{20}$$

Being gauge invariant, a nongauge-invariant operator with $\mathbb{Z}_N$ parameter is exactly recovered in two ways: (i) either upon integration of the fields $\{a_{(2)}, a^i_{(4)}\}$, which straightforwardly implies (18), or (ii) by integrating out the fields $\{b_i, c\}$. This property arises from the fact that the auxiliary fields are linearly present in each term of (20). We emphasize that, due to the linear dependence on the auxiliary fields, these integrals can be explicitly done.

Let us finally consider a noninvertible operator for the $\mathrm{SL}(2, \mathbb{Z})$ 0-form symmetry of the theory. We first note that, although $\star j_a$ accounts for a term proportional to $B_j \wedge M^{ik} \star H_k$, which does not admit an integration by parts, it is potentially related to the EOM of B_i, Eq. (12). Precisely, this EOM authorizes the introduction of a doublet of dual 7-forms $\tilde{H}^i$, which is defined as big

$$\tilde{H}^i = M^{ij} \star H_j, \tag{21}$$

and can be locally written as

$$\tilde{H}^i = \mathrm{d}\tilde{B}^i - \frac{1}{3}\epsilon^{ij}(B_j \wedge F - 2D \wedge H_j), \tag{22}$$

where $\tilde{B}^i$ is an $\mathrm{SL}(2, \mathbb{Z})$ 6-form doublet whose gauge transformation is $\tilde{B}^i \to \tilde{B}^i + \tilde{\Delta}^i$ with

$$\tilde{\Delta}^i \equiv \mathrm{d}\tilde{\lambda}^i_{(5)} + \frac{2}{3}\epsilon^{ij}\mathrm{d}\lambda^{(3)} \wedge B_j$$
$$- \frac{1}{3}\epsilon^{ij}\left(\mathrm{d}\lambda_j^{(1)} \wedge D - \frac{1}{2}\epsilon^{kl} B_j \wedge \mathrm{d}\lambda_k^{(1)} \wedge B_l\right). \tag{23}$$

Based on the $\mathrm{SL}(2,\mathbb{R})$-invariant democratic formulation of type IIB supergravity [56], we introduce the auxiliary fields to construct the 0-form noninvertible operator.

In particular, plugging $\tilde{H}^i$ into (6), $\star j_a$ is rewritten as a composite of two BF theories,

$$\star j_a = 4(t_a)^j{}_i \left[M^{ik} \star \mathrm{d}M_{kj} + \frac{1}{4} B_j \wedge (\tilde{H}^i + 3\mathrm{d}\tilde{B}^i) \right]. \quad (24)$$

Thus, mimicking the form of $\tilde{H}^i$, we define the field strength $\tilde{h}^i$ for the 6-form auxiliary fields $\tilde{b}^i$ as

$$\tilde{h}^i = \mathrm{d}\tilde{b}^i - \frac{1}{3} I^{(3)i}. \quad (25)$$

In analogy with $\tilde{B}^i$, for this field strength to be gauge invariant $\tilde{b}^i$ transforms as $\tilde{b}^i \to \tilde{b}^i + \tilde{\Delta}^i$. Therefore, the noninvariant part of (24) is promoted to the following gauge-invariant BF terms:

$$\begin{aligned} &(t_a)^j{}_i B_j \wedge (\tilde{H}^i + 3\mathrm{d}\tilde{B}^i) \\ &\quad \to I_a^{(4)} \equiv (t_a)^j{}_i (b_j \wedge M^{ik} \star H_k + 3B_j \wedge \mathrm{d}\tilde{b}^i), \end{aligned} \quad (26)$$

where we have traded the dual 7-form $\tilde{H}^i$ using (21).

Finally, to recover the original current, we have to supplement the operator by the appropriate terms that realize the Eq. (17), together with the new gauge-invariant condition

$$M^{ij} \star H_j - N\tilde{h}^i = 0. \quad (27)$$

This is accomplished by considering the gauge-invariant Lagrange-multiplier-like fields $\{a_{(2)i}, a_{(4)}, a^i_{(6)}\}$, which are, respectively, imposing the constraints (17) and (27).

Bringing together these requirements, we assemble the noninvertible $\mathrm{SL}(2,\mathbb{Z})$ operator

$$\begin{aligned} \mathcal{D}(\Sigma_9) &= \int \mathrm{D}[b_i, c, b^i, a_{(2)i}, a_{(4)}, a^i_{(6)}] \exp\left[\int_{\Sigma_9} \star J^{(1)}\right], \\ \star J^{(1)} &\equiv \frac{4}{N} C^j{}_i M^{ik} \star \mathrm{d}M_{kj} + C^j{}_i (b_j \wedge \tilde{H}^i + 3B_j \wedge \mathrm{d}\tilde{b}^i) \\ &\quad + 3(\tilde{H}^i - N\tilde{h}^i) \wedge a_{(2)i} + 2(F - Nf) \wedge a_{(4)} \\ &\quad + (H_i - Nh_i) \wedge a^i_{(6)}, \end{aligned} \quad (28)$$

where we have used (21).

Similar to the previous cases, every auxiliary field appears linearly in $\star J^{(1)}$. This feature enables us to straightforwardly check that either integrating out first the fields $\{a_{(2)i}, a_{(4)}, a^i_{(6)}\}$, or the set $\{b_i, c, \tilde{b}^i\}$, the invertible operator $U(\Sigma_9)$ with parameter q^a/N is recovered.

B. Half (higher) gaugings and TQFTs

While half gauging a symmetry induces a codimension-1 topological interface between two different QFTs implementing a noninvertible 0-form symmetry [10,11,13], for a higher-form symmetry the notion of higher gauging is introduced [13,29,57–60]. In this case, p gauging of a q-form global symmetry consists of inserting a system of q-form symmetry defects along a codimension-p manifold inside the bulk spacetime. The isomorphism between the ungauged and gauged QFTs is ensured by the self-duality condition, which implies the relation $q = \frac{1}{2}(d + p - 2)$ [45,61].

Additionally, d-dimensional theories can also be self-dual under p gauging a discrete q-form $\times$ $(d + p - q - 2)$-form symmetry $(0 \leq q \leq d - 1)$ [45,62].

On the other hand, the obtaining of the half (higher) gauging structure provides more rigorous evidence of the topological nature of the noninvertible operators [17,20,63].

Apart from introducing the above auxiliary fields, this method requires some additional gauge fields to realize the gauging [29,45]. The arrows in Fig. 1 show the set of necessary fields for every higher-form symmetry. Let us stress that, for these gauge fields, which are just defined inside the region of the gauging, we impose Dirichlet boundary conditions $A|_{\Sigma_p} = 0$, and $L|_{\Sigma_p} = 0$, where $\Sigma_p = \partial\Sigma_{p+1}$ and corresponds to the closed manifold where the noninvertible operator is inserted.

We also stress that the gauge fields entering the gauging procedure might be understood as the d.o.f. of the TQFTs associated with each noninvertible operator.

Let us first discuss the operator $\mathcal{D}(\Sigma_5)$, which is obtained by four gauging the $\mathrm{U}(1)^{(6)} \times \mathrm{U}(1)^{(6)}$ symmetry to $\mathbb{Z}_N^{(6)} \times \mathbb{Z}_N^{(6)}$ as follows:

$$\begin{aligned} \mathcal{D}(\Sigma_5) &= \int \mathrm{D}[b_i, A_{(3)i}] \exp\left[2\pi \mathrm{i} G(\Sigma_5)\right], \\ G(\Sigma_5) &\equiv \int_{\Sigma_5} \frac{2q}{N} \star F + \int_{\Sigma_6} \frac{q}{N} \epsilon^{ij} H_i \wedge H_j \\ &\quad - 2\epsilon^{ij} A_{(3)i} \wedge (H_j - Nh_j) \\ &\quad + N\epsilon^{ij} A_{(3)i} \wedge A_{(3)j}, \end{aligned} \quad (29)$$

with $\Sigma_5 = \partial\Sigma_6$. Here, the gauge transformation $A_{(3)i} \to A_{(3)i} + \mathrm{d}\Xi_i^{(2)}$ induces the transformation $b_i \to b_i + \Xi_i^{(2)}$. In this case, $p = 4$ and $q = 6$, so the above self-duality condition is fulfilled.

The operator $\mathcal{D}(\Sigma_7)$ associated with $\mathrm{U}(1)^{(2)} \times \mathrm{U}(1)^{(2)}$ is written in terms of the half higher gauging $G(\Sigma_7)$ as

$$\mathcal{D}(\Sigma_7) = \int \mathrm{D}[\Phi_7] \exp\left[2\pi \mathrm{i} G(\Sigma_7)\right], \quad (30)$$

where $\mathrm{D}[\Phi_7] \equiv \mathrm{D}[b_i, c, a_{(2)}, a^i_{(4)}, L_{(3)i}, L_{(5)}, A_{(3)}, A^i_{(5)}]$, $\Sigma_7 = \partial\Sigma_8$ is a closed manifold, and $G(\Sigma_7)$ is

$$\begin{aligned} G(\Sigma_7) = & \int_{\Sigma_7} \frac{3q_i}{N} M^{ij} \star H_j + \int_{\Sigma_8} \frac{3q_i}{N} \epsilon^{ij} H_j \wedge F \\ & + (H_j - N h_j) \wedge A^j_{(5)} \\ & + L_{(3)j} \wedge (-q_i \epsilon^{ij} F - N da^j_{(4)} - N \epsilon^{ji} H_i \wedge a_{(2)}) \\ & - N L_{(3)j} \wedge A^j_{(5)} + 2(F - Nf) \wedge A_{(3)} \\ & + 2 L_{(5)} \wedge (q_i \epsilon^{ij} H_j - N da_{(2)}) - 2N L_{(5)} \wedge A_{(3)}. \end{aligned} \quad (31)$$

Gauge invariance under transformations of the new fields $\{L_{(3)i}, L_{(5)}, A_{(3)}, A^i_{(5)}\}$ yields nontrivial gauge transformations of $\{b_i, c, a_{(2)}, a^i_{(4)}\}$. As $p = 2$, $q = 4$, the self-duality amounts to a discrete $\mathbb{Z}^{(4)}_N \times \mathbb{Z}^{(6)}_N$ symmetry [45].

Finally, we consider the codimension-1 defect $\mathcal{D}(\Sigma_9)$ associated with the $\mathrm{SL}(2,\mathbb{R})$ 0-form symmetry. Here the operator is written in terms of $G(\Sigma_9)$ as

$$\mathcal{D}(\Sigma_9) = \int \mathrm{D}[\Phi_9] \exp[G(\Sigma_9)], \quad (32)$$

where $\mathrm{D}[\Phi_9] \equiv \mathrm{D}[b_i, c, \tilde{b}^i, a^{(2)}_i, a_{(4)}, a^i_{(6)}, L^{(3)}_i, L^{(5)}, L^i_{(7)}, A^{(3)}_i, A^{(5)}, A^i_{(7)}]$, $\Sigma_9 = \partial\Sigma_{10}$ is closed, and $G(\Sigma_9)$ is

$$\begin{aligned} G(\Sigma_9) = & \int_{\Sigma_9} \frac{4}{N} C^j{}_i M^{ik} \star \mathrm{d}M_{kj} + \int_{\Sigma_{10}} \frac{1}{N} C^j{}_i (H_i \wedge \tilde{H}^i - \epsilon^{ik} B_j \wedge H_k \wedge F) \\ & - N L^{(3)}_i \wedge \left(\frac{C^i{}_j}{N} \tilde{H}^j + \mathrm{d}a^i_{(6)} - \epsilon^{ij} a^{(2)}_j \wedge F - \epsilon^{ij} H_j \wedge a_{(4)} \right) + (H_i - N h_i) \wedge A^i_{(7)} - N L^{(3)}_i \wedge A^i_{(7)} \\ & + 2N L_{(5)} \wedge (-N \mathrm{d}a_{(4)} + \epsilon^{ij} a^{(2)}_i \wedge H_j) + 2(F - Nf) \wedge A_{(5)} - 2N L_{(5)} \wedge A_{(5)} \\ & + 3 L^{(7)}_i \wedge (-N \mathrm{d}a^{(2)}_i + C^j{}_i H_j) + 3(\tilde{H}^i - N \tilde{h}^i) \wedge A^{(3)}_i - 3N L^i_{(7)} \wedge A^{(3)}_i. \end{aligned} \quad (33)$$

Interestingly ($p = 0$, $q = 2$), the 2- and 6-form symmetries are democratically gauged to $(\mathbb{Z}^{(2)}_N \times \mathbb{Z}^{(2)}_N) \times (\mathbb{Z}^{(6)}_N \times \mathbb{Z}^{(6)}_N)$. However, as expected from its invariance under $\mathrm{SL}(2,\mathbb{R})$, the 4-form symmetry is fictitiously gauged. This is also confirmed in the next section.

Thus, the discovery of these noninvertible operators might imply the existence of associated TQFTs that are described by the actions $G(\Sigma_p)$ in (29), (31), and (33).

C. Action of operators

The half higher gauging method has been proven to be useful to calculate the action of the noninvertible operators on charged objects [9,17,45]. In this subsection we are going to calculate the action of the symmetry operators on branes by reading off how the magnetic current couples to the auxiliary fields in the half (higher) gauging construction. Alternative approaches for this calculation can be found in [53].

Starting with $\mathcal{D}(\Sigma_3) \equiv U(\Sigma_3)$ with $\tilde{q}^i \to \tilde{q}^i/N$, its nontrivial and invertible action on the aforementioned charged operators reduces to

$$\mathcal{D}(\Sigma_3){:}\ \mathcal{O}(M_6) \mapsto e^{2\pi \mathrm{i} \frac{\tilde{q}^i \tilde{Q}_i}{N}} \mathcal{O}(M_6). \quad (34)$$

The action of the 4-form symmetry operator $\mathcal{D}(\Sigma_5)$ is

$$\begin{aligned} \mathcal{D}(\Sigma_5){:}\ & \mathcal{O}(M_2) \mapsto \mathcal{O}(M_2), \\ & \mathcal{O}(M_4) \mapsto e^{2\pi \mathrm{i} \frac{qQ}{N}} \mathcal{O}(M_4), \\ & \mathcal{O}(M_6) \mapsto \mathcal{O}(M_6) e^{-2\pi \mathrm{i} \tilde{Q}_i \epsilon^{ij} \int_{\mathcal{M}_3} 2A_{(3)j}}, \end{aligned} \quad (35)$$

where $\mathcal{M}_3 \subset M_6 \cap \Sigma_5$ is a compact manifold such that $\oint_{\partial\mathcal{M}_3} \Xi_{(2)i} \in 2\pi\mathbb{Z}_i$. Using the EOM of $A_{(3)i}$ we obtain

$$\mathcal{D}(\Sigma_5)\mathcal{O}(M_6) = \mathcal{O}(M_6) e^{-2\pi \mathrm{i} \epsilon^{ij} \frac{2\tilde{Q}_i}{N} \int_{\mathcal{M}_3} H_j}. \quad (36)$$

Second, the action of $\mathcal{D}(\Sigma_7)$ amounts to

$$\begin{aligned} \mathcal{D}(\Sigma_7){:}\ & \mathcal{O}(M_2) \mapsto e^{2\pi \mathrm{i} \frac{3q_i Q^i}{N}} \mathcal{O}(M_2), \\ & \mathcal{O}(M_4) \mapsto \mathcal{O}(M_4) e^{2\pi \mathrm{i} Q \int_{\mathcal{M}_3} 2A_{(3)} + q_i \epsilon^{ij} L_{(3)j}}, \\ & \mathcal{O}(M_6) \mapsto \mathcal{O}(M_6) e^{2\pi \mathrm{i} \tilde{Q}_i \int_{\mathcal{M}_5} A^i_{(5)} + 2\epsilon^{ij} q_j L_{(5)}}, \end{aligned}$$

where $\mathcal{M}_3 \subset M_4 \cap \Sigma_7$ and $\mathcal{M}_5 \subset M_6 \cap \Sigma_7$ are submanifolds such that the integration of the gauge parameters of $\{L_{(3)i}, A_{(3)}\}$ and $\{L_{(5)}, A^i_{(5)}\}$ are, respectively, quantized over $\partial\mathcal{M}_3$ and $\partial\mathcal{M}_5$. Subsequently, if we evaluate the noninvertible phases on shell, we obtain

$$\mathcal{D}(\Sigma_7)\mathcal{O}(M_4) = \mathcal{O}(M_4) e^{2\pi \mathrm{i} Q \frac{5 q_i \epsilon^{ij}}{N} \int_{\mathcal{M}_3} H_j}, \quad (37)$$

$$\mathcal{D}(\Sigma_7)\mathcal{O}(M_6) = \mathcal{O}(M_6) e^{2\pi \mathrm{i} \tilde{Q}_i \frac{5 \epsilon^{ij} q_j}{N} \int_{\mathcal{M}_5} F}. \quad (38)$$

Finally, the action of the $\mathrm{SL}(2,\mathbb{R})$ noninvertible operator nontrivially acts over all the charged objects,

$$\begin{aligned}\mathcal{D}(\Sigma_9)\colon\ \mathcal{O}(M_2) &\mapsto \mathcal{O}(M_2)e^{-Q^i\int_{\mathcal{M}_3}L_i^{(3)}+C^j{}_iA_j^{(3)}},\\ \mathcal{O}(M_4) &\mapsto \mathcal{O}(M_4)e^{-Q\int_{\mathcal{M}_5}L_{(5)}},\\ \mathcal{O}(M_6) &\mapsto \mathcal{O}(M_6)e^{\tilde{Q}_i\int_{\mathcal{M}_7}3C^i{}_jA^j_{(7)}-L^i_{(7)}},\end{aligned}$$

such that $\partial\mathcal{M}_3 = M_2 \subset \Sigma_9$, $\partial\mathcal{M}_5 = M_4 \subset \Sigma_9$, and $\partial\mathcal{M}_7 = M_6 \subset \Sigma_9$. Using the EOM of the gauge fields Ls and As, these noninvertible phases result

$$\mathcal{D}(\Sigma_9)\mathcal{O}(M_2) = \mathcal{O}(M_2)e^{Q^i\frac{4}{N}C^j{}_i\int_{\mathcal{M}_3}H_j}, \tag{39}$$

$$\mathcal{D}(\Sigma_9)\mathcal{O}(M_4) = \mathcal{O}(M_4), \tag{40}$$

$$\mathcal{D}(\Sigma_9)\mathcal{O}(M_6) = \mathcal{O}(M_6)e^{-\tilde{Q}_i\frac{4}{N}C^i{}_j\int_{\mathcal{M}_7}M^{jk}\star H_k}. \tag{41}$$

V. CONCLUSIONS

In this work we have studied the higher-form symmetries of type IIB supergravity and found the invertible topological operators associated with them. Interestingly, we have obtained an invertible operator that carries out $\mathrm{SL}(2,\mathbb{Z})$ transformations on operators or objects charged under this symmetry. We have provided a mechanism to systematically obtain a set of noninvertible operators inspired by BF theories. Then, led by the half higher gauging construction, we have obtained their action on charged objects. Being this a first step, a more detailed analysis of these operators [64] could shed some light on their uniqueness and potential applications.

Likewise, understanding the half higher gauging as an effective description of some phenomena occurring at intermediate energy scales in supergravity (with generalized symmetries in gravity [65]) could be interesting.

Finally, we would like to understand whether there exists a relation between the operators with continuous symmetries and the fluxbranes discussed in [43]. It would be interesting to compare the TQFTs and their associated d.o.f.

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[48] Different from [29], the $\mathrm{SL}(2,\mathbb{R})$-covariant formulation enables us to have a doublet of invertible operators.
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