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A NOTE ON THE COMPUTATION OF SPARSE RESULTANTS

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RÉSUMÉ. Le but de cette note est de discuter la relation entre les résidus toriques et les résultants creux et son application au calcul de ces derniers.

ABSTRACT. The purpose of this note is to discuss the relationship between toric residues and sparse resultants and its application to the computation of the latter.

1. Introduction. Resultants provide efficient ways for studying and solving polynomial systems. They have very classical origins [3, 11, 25, 37, 38, 41, 46] and are important in applications [1, 4, 28, 31, 36] as well as from a theoretical perspective [29, 32, 26].

This paper considers the sparse (or toric) resultant, which can exploit an *a priori* knowledge on the support of the equations [29]. There are several approaches to the problem of computing sparse resultants: Canny-Emiris algorithms based on mixed subdivision of the Minkowski sum of the Newton polytopes of the input polynomials [6, 7, 18], computations based on resultant complexes [29, 45, 48, 19, 20, 22, 33, 34], algorithms based on Bezoutians (also called Dixon-matrices) [5, 27, 13, 14], and of course the direct method of computing a Gröbner basis of the input polynomials with respect to an elimination order.

In this mostly expository note, we consider the global toric residue associated to a generic family of polynomials with given supports, of any polynomial with critical degree, to devise a different method to compute sparse resultants. The toric residue is a rational function of the coefficients of the given polynomials. Our work is inspired by the work of Jouanolou for the homogeneous case [32] and our previous work with Sturmfels [10], that describe the sparse resultant as the denominator of the global toric residue associated to a generic family of polynomials with given supports, of any polynomial with critical degree. But instead, we propose to use a particular polynomial with residue 1 to read the sparse resultant from the numerator of its normal form with respect to any Gröbner basis.

In Section 2 we recall the definitions and basic facts about sparse resultants and toric residues, and we relate them in Theorem 2.2. In Section 3 we prove the main result about elements with constant residue and resultants (Theorem 3.1) and state the

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resulting Algorithm 3.2. This algorithm works, in principle, if two conditions are satisfied. First of all, the codimension of a certain graded piece of a multi-homogeneous ideal must be one and then we must be able to exhibit an explicit element of known constant residue. In Section 4 we survey recent developments on these matters [17, 35]. The last section is devoted to examples.

The interest of our approach relating these two important invariants of polynomial systems is at this point mostly theoretical, since its computational performance is not very good. Although in principle being able to work with any monomial order should have an advantage over costly elimination orders, in practice most computations lie beyond the capability of the standard computer algebra systems implemented in a personal computer. None of the results, old or new, mentioned in this note has a general efficient implementation yet. The main theoretical open problem is how to relate residues and resultants in the mixed case when the codimension is greater than 1. The resultant is still in the denominator of any non-zero residue, but it is not clear how to detect it with normal forms.

2. Sparse Resultant and Toric Residues. We first recall the definition of the unmixed sparse resultant associated to a family $A_0, \ldots, A_n$ of supports. Here, each $A_i \subset \mathbb{Z}^n$ is a finite subset of lattice points. Without loss of generality, we will assume that the affine lattice spanned by $\sum_{j=0}^{n} A_j$ is the whole $\mathbb{Z}^n$. Set $m_\ell := \mathrm{card}(A_\ell)$. Consider a system of $n+1$ generic Laurent polynomials in n variables $t = (t_1, \ldots, t_n)$ with supports in $A_0, \ldots, A_n$:

$$f_i(t) \quad = \quad \sum_{a \in A_i} c_{ia} t^a \tag{1}$$

for $i = 0, \ldots, n$. The vector $\mathbf{c}$ of all coefficients c_{ia} in (1) defines a point in the product of complex projective spaces $Y = \mathbb{P}^{m_0-1} \times \cdots \times \mathbb{P}^{m_n-1}$. Let $Z \subset Y$ denote the subset of coefficients for which (1) has a solution t in the torus $(\mathbb{C}^*)^n$ and let $\bar{Z}$ be its closure in Y. Then, the projective variety $\bar{Z}$ is irreducible and defined over $\mathbb{Q}$. When $\mathrm{codim}(\bar{Z}) = 1$, the *sparse resultant* $\mathrm{Res}_{A_0,\ldots,A_n} = \mathrm{Res}(f_0, \ldots, f_n)$ is the unique (up to sign) irreducible polynomial in $\mathbb{Z}[\ldots, c_{ia}, \ldots]$ which vanishes on the hypersurface $\bar{Z}$. If $\mathrm{codim}(\bar{Z}) \geq 2$ then $\mathrm{Res}_{A_0,\ldots,A_n}$ is defined to be the constant 1.

Sturmfels characterized those families of supports for which the sparse resultant is non-trivial [44]. To state his result, we need the following notion: a subcollection of supports $\{A_i\}_{i \in I}$ is said to be *essential* if $\mathrm{rank}(I) = \mathrm{card}(I) - 1$ and $\mathrm{rank}(J) \geq \mathrm{card}(J)$ for each proper subset $J \subset I$, where $\mathrm{rank}(J)$ denotes the dimension of the affine space generated by $\sum_{j \in J} A_j$. Then, $\mathrm{Res}_{A_0,\ldots,A_n} \neq 1$ if and only if there is a single subcollection $\{A_i\}_{i \in I}$ which is essential. In this case the sparse resultant $\mathrm{Res}_{A_0,\ldots,A_n}$ coincides with the sparse resultant associated to this subcollection.

We will thus assume in this note that $\{A_0, \ldots, A_n\}$ is essential. Let Q_i denote the convex hull of A_i. For all $i \in \{0, \ldots, n\}$ the degree of $\mathrm{Res}_{A_0,\ldots,A_n}$ in the i-th group of variables $\{c_{ia} \mid a \in A_i\}$ is a positive integer, equal to the mixed volume

$$MV\left(Q_0, \ldots, Q_{i-1}, Q_{i+1} \ldots, Q_n\right) = \sum_{J \subseteq \{0,\ldots,i-1,i+1\ldots,n\}} (-1)^{\mathrm{card}(J)} vol \left(\sum_{j \in J} Q_j \right).$$

One can associate, to the Minkowski sum $Q := \sum_{i=0}^{n} Q_i$, a complete toric variety $X = X_Q$ with an ample divisor D_Q. Note that it follows from the hypothesis of essentiality of the family of supports $A_0, \ldots, A_n$ that Q, and therefore X, are n-dimensional. Furthermore, one can associate to each Q_i a line bundle E_i in X which is generated by its sections. The variety X is described by the normal fan of Q, which is a refinement of the normal fan to each of $Q_0, \ldots, Q_n$.

Remark 2.1. One can also consider the resultant corresponding to the family of supports $Q_0 \cap \mathbb{Z}^n, \ldots, Q_n \cap \mathbb{Z}^n$. Since we are assuming that the affine lattice spanned by $\sum_{j \in J} A_j$ is the whole $\mathbb{Z}^n$, it follows from Theorem 1 in [39] that for polynomials $f_0, \ldots, f_n$ with supports in $A_0, \ldots, A_n$ as in (1),

$$\mathrm{Res}_{Q_0 \cap \mathbb{Z}^n, \ldots, Q_n \cap \mathbb{Z}^n}(f_0, \ldots, f_n) = \mathrm{Res}_{A_0, \ldots, A_n}(f_0, \ldots, f_n),$$

where it is understood as usual that the resultant polynomials are evaluated in the vector of coefficients $\mathbf{c}$ of $f_0, \ldots, f_n$. It can be seen as in [29, Prop. 1.5, Ch. 8] that the resultant $\mathrm{Res}_{Q_0 \cap Z^n, \ldots, Q_n \cap \mathbb{Z}^n}$ coincides with the resultant on X associated to $E_0, \ldots, E_n$.

Denote by $\eta_1, \ldots, \eta_s$ the primitive inner normal vectors to Q, or equivalently, the primitive generators of the one-dimensional cones in the normal fan. The homogeneous coordinate ring S of X is the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_s]$ [15]. Each variable x_j is associated to the generator η_j, and hence to a torus-invariant irreducible divisor D_j of X. The polynomial ring S is graded by the Chow group $A_{n-1}(X)$ of invariant Weil divisors in X as follows. Given $b \in \mathbb{Z}^s$, the degree of the (Laurent) monomial $x^b = \Pi_{j=1}^{s} x_j^{b_j}$ equals

$$\deg(x^b) := \left[\sum_{j=1}^{s} b_j D_j\right] \in A_{n-1}(X).$$

We denote by S_α the graded piece of S of polynomials of degree α. The associated divisor to each E_i has degree α_i given by:

$$\alpha_i = \left[\sum_{j=1}^{s} b_{ij} D_j\right],$$

where

$$b_{ij} = -\min_{m \in Q_i} \langle \eta_j, m \rangle.$$

The *critical degree* ρ is defined as:

$$\rho := \sum_{i=0}^{n} \alpha_i - \left[\sum_{j=0}^{s} D_j\right].$$

Given a polynomial $f(t) = \sum_{\alpha \in Q^\circ} c_\alpha t^\alpha$ supported in Q°, its Q°-*homogenization* is

$$F(x) \ = \ \sum_{\alpha \in Q} c_\alpha \prod_{j=1}^{s} x_j{}^{\langle \eta_j, a \rangle + b_j - 1}, \tag{2}$$

where $b_j = \sum_{i=0}^{n} b_{ij}$. Note that $F(x)$ is a multi-homogeneous polynomial of critical degree and that every element in S_ρ is of this form.

Similarly, for each $i = 0, \dots, n$, we associate to f_i its Q_i*-homogenization* $F_i \in S_{\alpha_i}$ given by:

$$F_i(x) \;=\; \sum_{a \in A_i} c_{ia} \prod_{j=1}^{s} x_j^{\langle \eta_j, a\rangle + b_{ij}}.$$

For each i, F_i defines a global section of E_i, and so its zero set is well-defined; indeed, $\{F_i = 0\}$ is the closure of the torus hypersurface defined by $\{f_i = 0\}$. Moreover, if the coefficients $\mathbf{c} = (c_{ia} \mid i = 0, \dots, n,\ a \in A_i)$ are such that

$$\mathrm{Res}_{Q_0 \cap \mathbb{Z}^n, \dots, Q_n \cap \mathbb{Z}^n}(f_0, \dots, f_n) = \mathrm{Res}_{A_0, \dots, A_n}(f_0, \dots, f_n) \neq 0, \tag{3}$$

the intersection $\cap_{i=0}^n \{F_i = 0\} = \emptyset$. Throughout the rest of this note we will assume that (3) holds.

Under these assumptions, the *toric residue* res_F is a particular non-zero morphism

$$\mathrm{res}_F : S_\rho / I_\rho \longrightarrow \mathbb{C}$$

from the degree-ρ piece of the quotient S/I, with $I = \langle F_0, \dots, F_n\rangle$, and defined by Cox [16] in cohomological terms.

Generically, the torus subset $V_k := \cap_{i \neq k} \{f_i = 0\}$ is finite and coincides with $\cap_{i \neq k}\{F_i = 0\}$. By Theorem 0.4 in [8], the toric residue has then the following affine interpretation. Given $H \in S_\rho$, it is the Q°-homogenization, as in (2), of a Laurent polynomial $h = \sum_{a \in Q^\circ} h_a t^a$ supported in the interior of Q. Since the intersection $\cap_{i=0}^n \{f_i = 0\}$ is empty, the rational function h/f_k is regular on V_k for every fixed $k \in \{0, \dots, n\}$. The toric residue equals

$$\mathrm{res}_F(H) \quad = \quad (-1)^k \sum_{\xi \in V_k} \mathrm{res}_\xi \left(\frac{h/f_k}{f_0 \cdots f_{k-1} f_{k+1} \cdots f_n} \, \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \right),$$

where the right-hand side is a sum of point Grothendieck residues [30, 47] relative to the divisors $\{f_i(t) = 0\} \subset (\mathbb{C}^*)^n$, with $i \neq k$. This is a linear function of the coefficients h_a of H (or h) and a rational function of the coefficients $\mathbf{c}$ of $F_0, \dots, F_n$ (or $f_0, \dots, f_n$).

We have the following generalization of Theorem 1.4 in [10], mimicking the proof of Proposition 3.5 in [23].

Theorem 2.2. *Let $H \in S_\rho$ (or h with support in Q°) and $F_0, \dots, F_n$ (or $f_0, \dots, f_n$) be as above. Then there exists a polynomial P_H in the coefficients $\mathbf{c}$ such that*

$$\mathrm{res}_F(H) = \frac{P_H(\mathbf{c})}{\mathrm{Res}_{A_0, \dots, A_n}(\mathbf{c})} = \frac{P_H(\mathbf{c})}{\mathrm{Res}_{A_0, \dots, A_n}(f_0, \dots, f_n)}.$$

This result generalizes the corresponding theorem for the classical case of homogeneous polynomials and projective space proved by Jouanolou [32]. If Σ denotes the unit simplex in $\mathbb{R}^n$, and $\deg(F_i) = d_i$, with $i = 0, \dots, n$, the classical case corresponds to the family of supports $(d_0.\Sigma) \cap \mathbb{Z}^n, \dots, (d_n.\Sigma) \cap \mathbb{Z}^n$. In this case, a theorem of Macaulay asserts that $\dim S_\rho / I_\rho = 1$ and the Jacobian J_F of $F_0, \dots, F_n$ defines a canonical homogeneous polynomial of critical degree with non-zero residue equal to the product of the degrees $d = \prod_{i=0}^n \deg(F_i)$. It follows that

$$P_{J_F}(c) = d.\mathrm{Res}_{(d_0.\Sigma) \cap \mathbb{Z}^n, \dots, (d_n.\Sigma) \cap \mathbb{Z}^n}(c),$$

from where

$$\mathrm{Res}_{(d_0.\Sigma)\cap\mathbb{Z}^n,\ldots,(d_n.\Sigma)\cap\mathbb{Z}^n} = \frac{1}{d}P_{J_F},$$

that is, the resultant can be computed, up to a constant, from the numerator of the residue of the Jacobian. Clearly, it can also be recovered from the numerator of the residue of any known polynomial $\Delta \in \mathbb{Z}[\mathbf{c}, x]$ of constant residue. Jouanolou also proved that the numerator of the residue is a subresultant, which can be computed from an associated complex; see [21] for a generalization to the toric ample case. We will discuss generalizations of Macaulay's and Jouanolou's results to the toric case, together with applications to the computation of toric resultants in the following section.

3. Resultants from residues. We recover the notations and hypotheses of the previous section. We will exploit now the relation between residues and resultants in the ρ piece S_ρ/I_ρ of the graded algebra S/I to propose an algorithm for the computation of resultants, under suitable assumptions.

Assume $\dim S_\rho/I_\rho = 1$. Given a term order $\prec$ in S, there will be a unique standard monomial of degree ρ, the smallest monomial x^{α_0}, relative to $\prec$, not in the ideal I. Consequently, for any $H \in S_\rho$, its normal form relative to the reduced Gröbner basis for $\prec$, will be a multiple of x^{α_0}.

Assume moreover that there exists a polynomial $\Delta \in \mathbb{Z}[\mathbf{c}, x_1, \ldots, x_s]$ of degree ρ in $(x_1, \ldots, x_s)$ such that $\mathrm{res}_F(\Delta) = 1$. We can write the *normalform* of Δ with respect to a Gröbner basis of I with respect to $\prec$:

$$\text{normalform}(\Delta) = R(\mathbf{c}) \cdot x^{\alpha_0}$$

where $R \in \mathbb{Q}(\mathbf{c})$ is a rational function of the coefficients of $F_0, \ldots, F_n$.

Theorem 3.1. *With notation and assumptions as above, write* $R = P/Q$, *with* $P, Q \in \mathbb{Z}[\mathbf{c}]$ *coprime and of content* 1. *Then,* $\mathrm{Res}_{A_0,\ldots,A_n}(\mathbf{c}) = P(\mathbf{c})$.

Proof. We have

$$1 = \mathrm{res}_F(\Delta) = \mathrm{res}_F\left(\frac{P(\mathbf{c})}{Q(\mathbf{c})} \cdot x^{\alpha_0}\right) = \frac{P(\mathbf{c})}{Q(\mathbf{c})}\mathrm{res}_F(x^{\alpha_0}).$$

By Theorem 2.2 we then have

$$1 = \frac{P(\mathbf{c})}{Q(\mathbf{c})}\frac{P_{x^{\alpha_0}}(\mathbf{c})}{\mathrm{Res}_{A_0,\ldots,A_n}(\mathbf{c})}.$$

Therefore $\mathrm{Res}_{A_0,\ldots,A_n}(\mathbf{c})Q(\mathbf{c}) = P(\mathbf{c})P_{x^{\alpha_0}}(\mathbf{c})$, but since the resultant $\mathrm{Res}_{A_0,\ldots,A_n}(\mathbf{c})$ is irreducible and coprime with $P_{x^{\alpha_0}}(\mathbf{c})$, this implies the statement. □

So, if we are able to assert that $\dim S_\rho/I_\rho = 1$ and to find an explicit element $\Delta = \Delta(F) \in S_\rho$ with toric residue with respect to $F_0, \ldots, F_n$ equal to 1, we could use the following algorithm for the computation of the sparse resultant.

Algorithm 3.2 (The resultant via the normalform).

(1) Pick any term order $\prec$ in the variables $x_1, \ldots, x_s$ and compute a Gröbner basis of I with respect to $\prec$.
(2) Compute the normalform $\text{normalform}(\Delta) = R(\mathbf{c}) \cdot x^{\alpha_0}$, where $R \in \mathbb{Q}(\mathbf{c})$ and x^{α_0} is the least monomial of degree ρ not in I.

(3) Then, $\mathrm{Res}_{A_0,\ldots,A_n}(f_0,\ldots,f_n)$ is equal to the numerator of R written in least terms.

Remark 3.3. In fact, a Gröbner basis computation is not needed, but only a normalform computation, that is we could start the algorithm in the second step and write $\Delta = R(\mathbf{c}) \cdot x^{\alpha_0}$ modulo I_ρ with any procedure.

Remark 3.4. This algorithm may also be used to compute resultants associated to parametric families of polynomials with support equal to $A_0,\ldots,A_n$.

4. Satisfying the assumptions. Our Algorithm 3.2 is predicated on two assumptions: that $\dim S_\rho/I_\rho = 1$ and that we can compute an explicit element $\Delta \in S_\rho$ of known constant residue. The results of Macaulay and Jouanolou guarantee that these results hold in the dense homogeneous case, while those in [16, 8] generalize them to the case when all the α_i are ample and Q is simplicial. We summarize in this section some recent works, which greatly extend these results in the toric case.

We point out that although many of the results discussed below are stated for the case of normal toric varieties, that is those associated with all the integer points in a polytope, Remark 2.1 implies that we can adapt ourselves to the normal case by considering the polynomial f_i as having support $Q_i \cap \mathbb{Z}^n$, with coefficient 0 for those monomials not in A_i.

Assume then that $A_i = Q_i \cap \mathbb{Z}^n$. Let $l^*(B) = \mathrm{card}(B^\circ \cap \mathbb{Z}^n)$ denote the number of lattice points in the relative interior of a polytope B. The following result generalizes Macaulay's Theorem as well as codimension results from [16, 8].

Theorem 4.1. [17, Theorem 2.2] *Assume as before that $A_0,\ldots,A_n$ is essential and $\mathrm{Res}_{A_0,\ldots,A_n}(F_0,\ldots,F_n) \neq 0$. Then,*

(1) *The codimension of I_ρ in S_ρ satisfies the inequalities*

$$1 + \sum_{\dim(Q_\ell)=1} l^*(Q_\ell) \leq \dim((S/I)_\rho) \leq 1 + \sum_{k=1}^{n-1} \sum_{\dim(Q_J)=|J|=k} l^*(Q_J),$$

where in the right-most sum we assume that $J \subset \{0,\ldots,n\}$ and Q_J denotes the Minkowski sum $\sum_{\ell \in J} Q_\ell$.

(2) *The codimension of I_ρ in S_ρ is given by*

$$\dim((S/I)_\rho) = 1 + \sum_{k=1}^{n-1} \sum_{\dim(Q_J)=|J|=k} l^*(Q_J)$$

if one of the following is satisfied for every J, with $1 \leq |J| \leq n-2$:

(a) $\dim(Q_J) \neq |J| + 1$;

(b) $\dim(Q_J) = |J| + 1$ *but Q_J has no interior lattice points;*

(c) $\dim(Q_J) = |J| + 1$ *but* $\dim(Q_{J\cup I}) > |J| + |I|$ *for non-empty subsets $I \subset \{0,\ldots,n\}$ such that $I \cap J = \emptyset$ and $|J| + |I| < n$.*

In particular, the codimension is always bigger than 1 if there exists at least one polytope Q_ℓ of dimension 1 and relative volume at least 2 (that is, with at least one interior lattice point). Conversely, it is always equal to 1 when all Q_ℓ are n-dimensional.

We concentrate now on the search for elements with known constant residue. Such elements generalize the classical Jacobian and the toric Jacobian studied in [16, 8, 9] in case the degrees α_i are equal (the *unmixed case*) or they are all positive integer multiples of a fixed ample α.

Consider the classical case of polynomials $F_0, \ldots, F_n$ homogeneous of positive degrees $d_0, \ldots, d_n$, as follows. For each $i = 0, \ldots, n$, write

$$F_i(x) = \sum_{j=0}^{n} A_{ij}(x) x_j, \tag{4}$$

where each A_{ij} is a homogeneous polynomial of degree $d_i - 1$. Note that these polynomials can be chosen in $\mathbb{Z}[\mathbf{c}, x]$. Define

$$\Delta(x) = \det((A_{ij})_{i,j=0,\ldots,n}). \tag{5}$$

Then, Δ has critical degree and since it is easily seen that $\mathrm{res}_{x_0,\ldots,x_n}(1) = 1$, the *transformation law* [8] yields $\mathrm{res}_F(\Delta) = 1$.

In generalizing this construction to the toric case, the first difficulty that arises is that the number s of variables is in general bigger than the number $n+1$ of polynomials. This can be dealt with in case all α_i are ample as follows; see [8] for the simplicial case and [21] for the general case.

Suppose Q is simplicial, choose the n-variables corresponding to the normals of the facets concurring at a vertex v of Q (labeled $x_1, \ldots, x_n$), and set $x_0 := \prod_{\ell=n+1}^{s} x_j$. Then, it is possible to find again multi-homogeneous polynomials A_{ij} such that the identity (4) holds and the corresponding polynomial Δ defined as in (5) has again critical degree and its toric residue $\mathrm{res}_F(\Delta) = 1$. In the non-simplicial case, it is possible to choose a complete flag of facets

$$\Sigma_0 = \emptyset \subset \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n$$

through v in such a way that the intersection of the facets in Σ_i is a face of codimension i (so Σ_n consists of all the facets through this vertex). Denote by Γ_i the set of those inner normal vectors η_ℓ to a facet in $\Sigma_i \setminus \Sigma_{i-1}$, for $i = 1, \ldots, n$, and by Γ_0 the set of those η_ℓ whose corresponding facet does not pass through v. Let $z_i = \prod_{\eta_\ell \in \Gamma_i} x_\ell$, with $i = 0, \ldots, n$. Then, it is shown in [21] that there exist multi-homogeneous polynomials A_{ij} such that (4) holds with x_i replaced by z_i and that the element Δ defined as in (5) has toric residue equal to 1.

In the general codimension-one case, the problem of constructing explicit elements of residue 1 remains open, even when all the Q_i's are n-dimensional. All known results are subsumed in the following recent construction by Khetan and Soprounov [35].

Definition 4.2. [35, Definition 5.1] Let $Q_0, \ldots, Q_n$ be lattice polytopes in $\mathbb{R}^n$. A *partition matrix* for $Q_0, \ldots, Q_n$ is a collection M_{ij}, with $0 \le i, j \le n$, such that:

(1) $M_{ij} \subset Q_i \cap \mathbb{Z}^n$ and $Q_i \cap \mathbb{Z}^n = M_{i0} \sqcup \cdots \sqcup M_{in}$;
(2) For any lattice point $a \in M_{ij}$, at least one vertex of the minimal face of Q_i containing a lies in M_{ij};
(3) For any permutation σ of $\{0, \ldots, n\}$, we have $\sum_{i=0}^{n} M_{\sigma(i)i} \subset Q^\circ$.

Given Laurent polynomials $f_0, \ldots, f_n$ as in (1), set, for $0 \le i, j \le n$,

$$f_{ij} = \sum_{a \in M_{ij}} c_a t^a.$$

The determinant $\delta = \det(f_{ij})$ is a Laurent polynomial supported on Q°. Denote by $\Delta \in S_\rho$ its Q°-homogenization. It is then shown in [35, Theorem 6.9] that the toric residue $\mathrm{res}_F(\Delta)$ is an integer that depends only on the combinatorics of the polytopes Q_i and the partitions of their lattice points. This integer is the combinatorial degree of an associated map.

When the polytopes Q_i share a complete flag of faces (in which case the system is called *locally unmixed*), Khetan and Soprounov give a combinatorial construction of δ (or Δ), which is a broad generalization of all previously known constructions. Moreover, they are able to construct elements of residue ± 1 for all essential configurations in the plane with one single exception. This construction is carried out in an algorithmic manner in [35, Theorem 8.2]. The exceptional case is when Q_0 and Q_1 are one-dimensional (with linearly independent directions) and Q_2 is a parallelogram with sides parallel to Q_0 and Q_1. We show how to compute the resultant associated with these configurations in Example 5.4.

5. Examples.

Example 5.1. Consider the following simple example of classical dense polynomials from [24, Chapter 1]:

$$\begin{cases} f_0 &= a_0 + a_1 t_1 + a_2 t_2; \\ f_1 &= b_0 + b_1 t_1 + b_2 t_2; \\ f_2 &= c_1 + c_2 t_1^2 + c_3 t_2^2 + c_4 t_1 + c_5 t_2 + c_6 t_1 t_2. \end{cases}$$

The corresponding Newton polytopes are $Q_0 = Q_1 = \Sigma_2$, $Q_2 = 2\Sigma_2$ and $Q = 4\Sigma_2$. The normal vectors to the facets of Q are $\eta_0 = (-1,-1)$, $\eta_1 = (1,0)$, and $\eta_2 = (0,1)$. Thus the Q_i-homogenizations are:

$$\begin{cases} F_0 &= a_0 x_0 + a_1 x_1 + a_2 x_2; \\ F_1 &= b_0 x_0 + b_1 x_1 + b_2 x_2; \\ F_2 &= c_1 x_0^2 + c_2 x_1^2 + c_3 x_2^2 + c_4 x_0 x_1 + c_5 x_0 x_2 + c_6 x_1 x_2. \end{cases}$$

In this case X_Q is $\mathbb{P}^2$ and $A_{n-1}(\mathbb{P}^2) \cong \mathbb{Z}$ with $\deg(x^b) = |b|$. Thus, the critical degree is $\rho = 1$ and we can choose

$$\Delta = \det \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_1 x_0 + c_4 x_1 + c_5 x_2 & c_2 x_1 + c_6 x_2 & c_3 x_2 \end{pmatrix}.$$

We can now read off the resultant $\mathrm{Res}_{\Sigma\cap\mathbb{Z}^2, \Sigma\cap\mathbb{Z}^2, (2\Sigma)\cap\mathbb{Z}^2}(F_0, F_1, F_2)$ (which agrees with the homogeneous resultant $\mathrm{Res}_{1,1,2}(F_0, F_1, F_2)$) from the normal form of Δ relative to any Gröbner basis of $I = \langle F_0, F_1, F_2 \rangle$. For example, computing relative to *grevlex* with $x_0 > x_1 > x_2$, we have:

$$\begin{aligned} \text{normalform}(\Delta) = \big(& (a_0^2 b_1^2 c_3 - a_0^2 b_1 b_2 c_6 + a_0^2 b_2^2 c_2 + a_0 a_1 b_0 b_2 c_6 - a_0 a_2 b_1^2 c_5 + \\ & a_0 a_1 b_1 b_2 c_5 - a_0 a_1 b_2^2 c_4 + a_0 a_2 b_0 b_1 c_6 - a_0 a_2 b_1 b_2 c_4 - 2 a_0 a_1 b_0 b_1 c_3 + a_1^2 b_0^2 c_3 - \\ & a_1^2 b_0 b_2 c_5 + a_1^2 b_2^2 c_1 - a_1 a_2 b_0^2 c_6 + a_1 a_2 b_0 b_1 c_5 + a_1 a_2 b_0 b_2 c_4 + 2 a_0 a_2 b_0 b_2 c_2 - \\ & 2 a_1 a_2 b_1 b_2 c_1 + a_2^2 b_0^2 c_2 - a_2^2 b_0 b_1 c_4 + a_2^2 b_1^2 c_1)/(a_0 b_1 - a_1 b_0) \big) x_2 \end{aligned}$$

and the numerator of the coefficient of x_2 in this expression is the resultant. Its denominator is the *s*ubresultant polynomial in the sense of [12], whose vanishing is equivalent to the condition $x_2 \in I$.

Example 5.2. Let $A_0 = A_1 = \{(0,0),(2,0),(0,1)\}$ and $A_2 = \{(1,0),(1,2),(0,2)\}$ be configurations in $\mathbb{Z}^2$. The Minkowski sum $Q = Q_1 + Q_2 + Q_3$ is the polytope in the plane with vertices

$$\mathrm{Vert}(Q) = \{(1,0),(5,0),(5,2),(1,4),(0,4),(0,2)\}.$$

The corresponding generic polynomials are

$$\begin{cases} f_0 &= a_0 + a_1 t_1^2 + a_2 t_2; \\ f_1 &= b_0 + b_1 t_1^2 + b_2 t_2; \\ f_2 &= c_0 t_1 + c_1 t_1 t_2^2 + c_2 t_2^2. \end{cases}$$

Following the notation of Section 2 we list the data necessary for homogenizing the polynomials f_i:

i	η_i	b_{i0}	b_{i1}	b_{i2}	b_i
1	$(1,0)$	0	0	0	0
2	$(2,1)$	0	0	2	2
3	$(0,1)$	0	0	0	0
4	$(-1,0)$	-2	-2	-1	-5
5	$(-1,-2)$	-2	-2	-5	-9
6	$(0,-1)$	-1	-1	-2	-4

Thus,

$$\begin{aligned} F_0(x_1,\dots,x_6) &= x_4^2 x_5^2 x_6 f_0\left(\tfrac{x_1 x_2^2}{x_4 x_5}, \tfrac{x_2 x_3}{x_5^2 x_6}\right) \\ &= a_0 x_4^2 x_5^2 x_6 + a_1 x_1^2 x_2^4 x_6 + a_2 x_2 x_3 x_4^2. \end{aligned}$$

Similarly,

$$\begin{cases} F_1(x_1,\dots,x_6) &= b_0 x_4^2 x_5^2 x_6 + b_1 x_1^2 x_2^4 x_6 + b_2 x_2 x_3 x_4^2; \\ F_2(x_1,\dots,x_6) &= c_0 x_1 x_5^4 x_6^2 + c_1 x_1 x_2^2 x_3^2 + c_2 x_3^2 x_4 x_5. \end{cases}$$

Since all three polytopes are two-dimensional, the codimension-one hypothesis is satisfied by Theorem 4.1. We then appeal to the algorithm described in [35, Theorem 8.2] to construct a partition matrix giving rise to an element of known constant residue. Since $Q_0 = Q_1$ but Q_2 has no common edges with them, the family of supports is, in the notation of [35], *partially unmixed* and of subtype (2a). Following their construction we then get that the polynomial

$$\begin{aligned} \delta &= \det \begin{pmatrix} a_0 & a_1 t_1^2 & a_2 t_2 \\ b_0 & b_1 t_1^2 & b_2 t_2 \\ c_0 t_1 & 0 & c_1 t_1 t_2^2 + c_2 t_2^2 \end{pmatrix} \\ &= (a_0 b_1 - a_1 b_0) c_1 t_1^3 t_2^2 + (a_0 b_1 - a_1 b_0) c_2 t_1^2 t_2^2 + (a_1 b_2 - a_2 b_1) c_0 t_1^3 t_2 \end{aligned}$$

is supported in Q°. Hence, its Q°-homogenization

$$\begin{aligned} \Delta &= ((a_0 b_1 - a_1 b_0) c_1 x_1 x_2^2 x_3 + (a_0 b_1 - a_1 b_0) c_2 x_3 x_4 x_5 \\ &\quad + (a_1 b_2 - a_2 b_1) c_0 x_1 x_2 x_5^2 x_6) x_4 x_5 x_6 x_1 x_2^3 \end{aligned}$$

is of critical degree and has toric residue ± 1. Using Maple we find that the reduced Gröbner basis for the ideal $I = \langle F_0, F_1, F_2 \rangle$ with respect to the graded reverse lexicographic ordering with $x_1 > \cdots > x_6$ consists of 14 polynomials. The smallest monomial of critical degree not in I is $x^{\alpha_0} = x_5 x_4^3 x_3^2 x_2^2$. Computing the normal form of Δ relative to the Gröbner basis we find a multiple of x^{α_0} whose numerator is the desired sparse resultant:

$$\left\{\begin{array}{l}
-4a_0^3b_1^3c_1^2a_1b_0^2a_2+c_0^2a_1^4a_2b_0b_2^4+a_0^4b_1^4c_1^2b_0a_2-c_0^2a_0a_1^4b_2^5+c_0^2a_2^5b_0b_1^4\\
-a_0^5b_1^4c_1^2b_2+a_1^4b_0^4c_2^2a_2b_1-a_1^5b_0^4c_2^2b_2-c_0^2a_0a_2^4b_1^4b_2+a_0^4b_1^5c_2^2a_2-a_0^4b_1^4c_2^2b_2a_1\\
-a_1^4b_0^4c_1^2a_0b_2+4a_0^4b_1^3c_1^2a_1b_0b_2+2a_1^2b_0^3a_2^3b_1^2c_0c_1-4a_0b_1^3a_2^3c_0c_1a_1b_0^2\\
+2a_0^2b_1^4a_2^3c_0c_1b_0-6a_0^3b_1^2a_1^2b_0^2c_1^2b_2+4a_0^2b_1a_1^3b_2^3c_0c_1b_0+4a_0^3b_1^3a_1a_2b_2^2c_0c_1\\
-4a_0b_1^2c_2^2b_0^3a_1^3a_2-4c_0^2a_1^3a_2^2b_0b_2^3b_1-6c_0^2a_0a_1^2a_2^2b_1^2b_2^3-6a_0^2b_1^2c_2^2b_0^2b_2a_1^3\\
-2a_0^3b_1^4a_2^2c_0c_1b_2+4c_0^2a_0a_2^3b_1^3b_2^2a_1+2a_1^4b_0^3b_2^2c_0c_1a_2-6a_0^2b_1^2a_1^2b_2^2c_0c_1b_0a_2\\
+6c_0^2a_1^2a_2^3b_0b_1^2b_2^2+4a_0b_1c_2^2b_0^3b_2a_1^4-2a_1^4b_0^2b_2^3c_0c_1a_0+6a_0b_1^2a_1^2a_2^2b_2c_0c_1b_0^2\\
-4a_1^3b_0^3a_2^2b_1b_2c_0c_1+4c_0^2a_0a_1^3b_2^4a_2b_1-4a_0^3b_1^4c_2^2b_0a_1a_2+4a_0^2b_1a_1^3b_0^3c_1^2b_2\\
+a_1^4b_0^5c_1^2a_2+4a_0^3b_1^3c_2^2b_0b_2a_1^2+6a_0^2b_1^3c_2^2b_0^2a_1^2a_2-4c_0^2a_1a_2^4b_0b_1^3b_2\\
+6a_0^2b_1^2a_1^2b_0^3c_1^2a_2-4a_0b_1a_1^3b_0^4c_1^2a_2-2a_0^3b_1^2a_1^2b_2^3c_0c_1.
\end{array}\right.$$

Example 5.3. Algorithm 3.2 may also be used to compute A-discriminants. Recall that given a configuration $A \subset \mathbb{Z}^n$ and its associated toric variety X_A, the *A-discriminant* D_A is the equation of the dual variety X_A^* if this is a hypersurface. Otherwise, we define $D_A = 1$. Hence, D_A is only defined up to constant though with a little more care it can be defined, up to sign, as a polynomial over $\mathbb{Z}$. We refer to [29] for more details.

The principal A-determinant $E_A(t)$ is defined as (see [29]):

$$E_A(t) := \mathrm{Res}_{A,\dots,A}\left(f_0, t_1\frac{\partial f_0}{\partial t_1}, \dots, t_n\frac{\partial f_0}{\partial t_n}\right),$$

where $f_0 = \sum_{a \in A} c_a t^a$ is a generic polynomial supported on A. The unique irreducible factor of $E_A(t)$ involving all the variables $\{c_a \mid a \in A\}$ is the discriminant D_A.

Discriminants appear in a variety of contexts in algebra, geometry, and analysis. As an illustration we will apply Algorithm 3.2 to compute an example from [2, Section 6].

Consider the following configuration of six points in $\mathbb{Z}^3$:

$$A = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(-1,-1,0),(1,1,-1)\}.$$

As the principal A-determinant and the A-discriminant are affine invariants of A we may translate A by the vector $(2,2,2)$ so as to make all the coordinates strictly positive. This implies that for all i, the Newton polytope of $f_i = t_i(\partial f_0/\partial t_i)$ agrees with that of f_0, which we denote by P. Thus our translated configuration is

$$A = \{(2,2,2),(3,2,2),(2,3,2),(2,2,3),(1,1,2),(3,3,1)\}.$$

Although this translation is not necessary, doing it simplifies the exposisition. The polytope P has 5 facets with inner-normals: $\eta_1 = (-1,-1,-1)$; $\eta_2 = (-1,2,-1)$; $\eta_3 = (2,-1,-1)$; $\eta_4 = (-1,2,2)$; and $\eta_5 = (2,-1,2)$.

It is then easy to check that the P-homogenizations of f_0, f_1, f_2 and f_3 are:

$$\left\{\begin{array}{lcl}
F_0 &=& a_0x_1x_2x_3x_4x_5+a_1x_3^3x_5^3+a_2x_2^3x_4^3+a_3x_4^3x_5^3+a_4x_1^3+a_5x_3^3x_2^3;\\
F_1 &=& 2a_0x_1x_2x_3x_4x_5+3a_1x_3^3x_5^3+2a_2x_2^3x_4^3+2a_3x_4^3x_5^3+a_4x_1^3+3a_5x_3^3x_2^3;\\
F_2 &=& 2a_0x_1x_2x_3x_4x_5+2a_1x_3^3x_5^3+3a_2x_2^3x_4^3+2a_3x_4^3x_5^3+a_4x_1^3+3a_5x_3^3x_2^3;\\
F_3 &=& 2a_0x_1x_2x_3x_4x_5+2a_1x_3^3x_5^3+2a_2x_2^3x_4^3+3a_3x_4^3x_5^3+2a_4x_1^3+a_5x_3^3x_2^3.
\end{array}\right.$$

Since P is simplicial, we can write $F_i(x)$ as in (4) with respect to the variables $\{x_3x_5, x_1, x_2, x_4\}$ and obtain an element of toric residue ± 1:

$$\begin{aligned}\Delta(x) \quad &= \quad a_1x_3^2x_5^5a_3x_4^3x_2^2\left(x_3x_5x_4^3a_0x_2a_2 + x_3^4x_5^2x_4a_0x_2a_5 + 3x_4^2a_4x_1^2a_2\right.\\ &\quad \left.+ 2x_3^3x_5a_4x_1^2a_5\right).\end{aligned}$$

Using Maple we find that the reduced Gröbner basis for the ideal $I = \langle F_0, F_1, F_2, F_3\rangle$ with respect to the graded reverse lexicographic ordering with $x_1 > \cdots > x_5$ consists of 80 polynomials. The smallest monomial of critical degree not in I is $x^{\alpha_0} = x_1^9$. Computing the normal form of Δ relative to the Gröbner basis we find a multiple of x^{α_0} whose numerator is a monomial times the desired A-discriminant:

$$D_A = 729a_4^2a_3^2a_5^2 + 54a_5a_3a_4a_0^3 - 1458a_5a_3a_2a_1a_4^2 + 729a_2^2a_1^2a_4^2 + a_0^6 + 54a_2a_1a_4a_0^3.$$

Example 5.4. We consider the exceptional case of [35] in dimension 2, that is, Q_0 and Q_1 are one-dimensional (with linearly independent directions) and Q_2 is a parallelogram with sides parallel to Q_0 and Q_1. Set $A_i = Q_i \cap \mathbb{Z}^2$. Without loss of generality, we can assume that Q_0 is a horizontal segment of length d_1 (that is, $f_0 = \sum_{i=0}^{d_1} a_it_1^i$ is a polynomial of degree d_1 in the variable t_1), Q_1 is a vertical segment of length d_2 (that is, $f_1 = \sum_{j=0}^{d_2} b_jt_2^j$ is a polynomial of degree d_2 in the variable t_2), and Q_2 is a rectangle with sides parallel to the axes of size $l_1 \times l_2$ $\left(\text{that is, } f_2 = \sum_{i=0}^{l_1}\sum_{j=0}^{l_2} c_{ij}t_1^it_2^j\right)$. After bihomogenization, we have

$$F_0 = \sum_{i=0}^{d_1} a_ix_1^iy_1^{d_1-i}, F_1 = \sum_{j=0}^{d_2} b_jx_2^jy_2^{d_2-j}, \text{ and } F_2 = \sum_{i=0}^{l_1}\sum_{j=0}^{l_2} c_{ij}x_1^iy_1^{l_1-i}x_2^jy_2^{l_2-j}.$$

The only case in which the codimension at the critical degree is 1 corresponds by Theorem 4.1 to the case $d_1 = d_2 = 1$. In this case, the resultant is simply obtained by substituting $t_1 = -a_0/a_1$ and $t_2 = -b_0/b_1$ in f_2 and clearing denominators.

The normal form algorithm could be adapted to the case of general $d_1, d_2 \in \mathbb{N}$, even if the codimension is greater than 1, as follows.

Lemma 5.5. *With the above notations, let $G_0 := y_2F_0$ and $G_1 := y_1F_1$. Then, G_0, G_1 and F_2 have ample degrees in $\mathbb{P}^1 \times \mathbb{P}^1$ and lie in the ideal generated by $\{y_1y_2, x_1, x_2\}$. Moreover, let Δ be defined as explained in Section 4 and compute $\operatorname{Res}_{A_0\times\{0,1\},\{0,1\}\times A_1,A_2}(G_0, G_1, F_2)$ (for instance using Algorithm 3.2). Then, the resultant of f_0, f_1, f_2 can be recovered as*

$$\operatorname{Res}_{A_0,A_1,A_2}(f_0, f_1, f_2) = \frac{\operatorname{Res}_{A_0\times\{0,1\},\{0,1\}\times A_1,A_2}(G_0, G_1, F_2)}{a_{d_1}^{l_2}\cdot b_{d_2}^{l_1}\cdot c_{d_1d_2}}. \qquad (6)$$

Proof. The equality (6) could be proven to be a consequence of Theorem 1.1 in [39], by considering the restrictions from the supports $A_0 \times \{0,1\}$ and $\{0,1\} \times A_1$ to $A_0 \times \{0\}$ and $\{0\} \times A_1$, which are affinely equivalent to A_0 and A_1. It could be directly deduced from the Poisson formula for sparse resultants in [40]. Note that the degrees of $\operatorname{Res}_{A_0,A_1,A_2}(f_0, f_1, f_2)$ and $\operatorname{Res}_{A_0\times\{0,1\},\{0,1\}\times A_1,A_2}(G_0, G_1, F_2)$ on the coefficients of f_0, f_1 and f_2 are $l_1d_2 + d_1l_2 + d_1d_2$ and $l_1d_2 + l_2 + d_1l_2 + l_1 + d_1 + d_2 + 1$ respectively. $\square$

Résumé substantiel en français. Les résultants fournissent des méthodes efficaces pour l'étude des systèmes d'équations. Ils ont des origines très classiques [3, 11, 25, 37, 38, 41, 46] et ils sont importants au niveau des applications [1, 4, 28, 31, 36], aussi bien que d'un point de vue théorique [29, 32, 26].

Dans cet article, nous considérons les résultants creux (ou toriques), qui permettent d'utiliser la connaissance *a priori* du support des équations [29]. Des méthodes différentes sont utilisées pour le calcul des résultants creux: des algorithmes à la Canny-Emiris basés sur des subdivisions mixtes des polytopes de Newton des polynômes donnés [6, 7, 18], des calculs basés sur des complexes résultants [29, 45, 48, 19, 20, 22, 33, 34], des algorithmes reposant sur la notion de Bézoutien (aussi appelés matrices de Dixon) [5, 27, 13, 14], et bien sûr, la méthode directe de calcul d'une base de Gröbner des polynômes donnés par rapport à un ordre monomial d'élimination.

Dans cette note, plutôt expositoire, nous considérons le résidu torique global associé à une famille générique de polynômes avec des supports donnés d'un polynôme de degré critique pour proposer une méthode différente de calcul des résultants creux. Ce résidu torique est une fonction rationnelle des coefficients des polynômes donnés. Notre travail est inspiré des résultats de Jouanolou dans le cas homogène [32] et par notre article en collaboration avec Sturmfels [10], où nous décrivons le résultant creux comme le dénominateur d'un résidu torique explicite. Ici, nous proposons d'utiliser un polynôme de degré critique avec résidu égal à 1 pour lire le résultant creux du numérateur de sa forme normale par rapport à une base de Gröbner arbitraire.

À la Section 2 nous rappelons les définitions et les faits de base sur les résultants et les résidus toriques, et nous rendons explicite leur relation dans le Théorème 2.2. À la Section 3 nous montrons dans le Théorème 3.1 comment le résultant creux peut être obtenu à partir d'un calcul de forme normale d'un élément de degré critique et résidu constant égal à 1. Comme corollaire, nous obtenons l'Algorithme 3.2 pour le calcul du résultant. Cet algorithme suppose, en principe, la validité de deux conditions. D'abord, la codimension de la partie de degré critique du quotient par l'idéal engendré par les polynômes donnés doit être égale à 1, ensuite, il faut connaître un polynôme de degré critique avec résidu torique constant. À la Section 4 nous rappelons des développements récents sur ces sujets [17, 35]. La dernière section est consacrée à développer des exemples.

L'intérêt de notre approche qui relie ces deux invariants importants des systèmes polynomiaux est jusqu'à maintenant plutôt théorique. Bien qu'en principe la possibilité de travailler avec un ordre monomial arbitraire devrait offrir des avantages sur les ordres d'élimination, très couteux au niveau calculatoire, dans la pratique la plupart des calculs effectifs surpassent la capacité des systèmes d'algèbre computationnelle standard sur un ordinateur personnel. Aucun des résultats, anciens ou nouveaux, mentionnés dans cette note ont en général une implantation efficace pour le moment. Le problème principal, ouvert du point de vue théorique, est celui de trouver la relation entre les résidus et les résultants toriques lorsque la codimension de la partie de degré critique du quotient par l'idéal engendré par les polynômes donnés est supérieure à 1. Le résultant est encore dans le dénominateur des résidus non nuls, mais on ne sait pas comment le détecter en utilisant un calcul de forme normale.

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