

Compact extra dimensions in cosmologies with $f(T)$ structure

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The presence of compact extra dimensions in cosmological scenarios in the context of $f(T)$ -like gravities is discussed. For the case of toroidal compactifications, the analysis is performed in an arbitrary number of extra dimensions. Spherical topologies for the extra dimensions are then carefully studied in six and seven spacetime dimensions, where the proper vielbein fields responsible for the parallelization process are found.

I. INTRODUCTION

In 1928 Einstein proposed an equivalent formulation of General Relativity (GR) nowadays known as the Teleparallel Equivalent of General Relativity (TEGR) [1, 2]. In this theory, the Weitzenböck connection is used instead of the Levi-Civita connection to define the covariant derivative. Weitzenböck connection has non-null torsion; however, it is curvatureless, which implies that this formulation of gravity exhibits only torsion. In D spacetime dimensions, the dynamical fields are the D linearly independent vielbeins and the torsion tensor is formed solely by them and the first derivatives of these objects. The Lagrangian density, which will be noted hereafter as T , is constructed from this torsion tensor, assuming invariance under general coordinate transformations and local Lorentz transformations, along with demanding that the Lagrangian density be quadratic in the torsion tensor [2]. In recent years an extension of the above Lagrangian density was constructed in [3–6], making the Lagrangian density a function of the scalar torsion T , the so-called $f(T)$ gravity. Much attention has been focused on this extended $f(T)$ gravity theory recently, because it exhibits interesting cosmological implications, as witnessed in the study of missing matter problems, as well as providing a new mechanism to explain the late acceleration of the universe based on a modification of the gravitational theory, instead of introducing an exotic content of matter (dark energy problem) [7–18]. It is also believed that $f(T)$ gravity could be a reliable approach to address the shortcomings of general relativity at high-energy scales [19]. For instance, in [4] a Born-Infeld $f(T)$ gravity Lagrangian was used to cure the physically inadmissible divergencies occurring in the Big Bang of standard cos-

mology, rendering spacetime geodesically complete and powering an inflationary stage without the introduction of an inflaton field.

We are now interested in studying this alternative theory of gravity in a higher dimensional context, and the basic features of cosmological behavior due to this extension. The conception of extra dimensions in physical theories has a long history that begins with the original ideas of Kaluza and Klein [20] and finds a new realization nowadays in modern string theory [21, 22]. One of the main motivations for studying these higher dimensional models is the chance of unifying the fundamental interactions of nature. In most extra-dimensional scenarios, the additional dimensions are assumed to be compactified on a very small (unobservable) internal submanifold, whereas the other spacetime dimensions constitute the 4-dimensional observable universe. Another mechanism of dimensional reduction studied in cosmological GR scenarios is dynamical compactification [23–25]. In this case, the internal dimensions evolve in time to very small scales as the external dimensions expand, and so the observable universe becomes effectively 4-dimensional. This type of reduction was first studied by Chodos and Detweiler in [23], who considered vacuum Einstein equations in five spacetime dimensions. These results were later extended, and a full classification of homogeneous eleven-dimensional cosmologies can be found in [26]. Dynamical reduction in multidimensional Bianchi type I models was studied in [27] and a study of multidimensional cosmological models as dynamical systems with topology $FRW^4 \times T^{D-4}$ (here T^{D-4} refers to the $(D-4)$ -torus) was performed in [28, 29]. More recently, and in a totally different context, GR in the presence of a large number of extra dimensions was carefully studied in [30, 31].

The specific topic of extra dimensions have scarcely been worked out in $f(T)$ theories, and it remits, in one hand, to some black hole solutions of $f(T)$ gravity in arbitrary spacetime dimensions [32], and on the other hand, to an analysis of four-dimensional $f(T)$ gravities as effective theories coming from five-dimensional Kaluza-Klein

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and Randall-Sundrum models [33]. In this work we investigate for the first time cosmological solutions in the context of $f(T)$ gravity in higher dimensions, considering that the extra dimensions are compactified in different ways. This investigation is motivated primarily by the increasing interest in multidimensional cosmological scenarios, particularly brane-world models [34]. In the present work we shall consider both toroidal and spherical topologies for the internal dimensions. With this purpose in mind we proceed first to obtain the proper parallel vector fields for the different topologies under consideration. This topic is a difficult one, and constitutes the starting point of any model describing extra dimensions in gravitational theories with absolute parallelism, $f(T)$ gravity being one of them. In particular we consider five, six, and seven-dimensional models and study all the possible compactifications of the extra dimensions using the right vielbein field that parallelizes the spacetime in each of these cases, and we then proceed to obtain the corresponding field equations. Then we show in general terms, that in some limited cases, simple analytical solutions describe physically relevant spacetimes representing different stages of cosmic evolution. The results presented here are mandatory aspects required to deal with more fundamental topics of physical cosmology such as structure formation and cosmological perturbations.

The paper is organized as follows: in section II we present a brief review of the fundamentals of $f(T)$ theories. In section III we study toroidal compactification for an arbitrary number of extra dimensions, and in section IV we study spherical compactifications with the emphasis on six and seven spacetime dimensions. Finally, in section V we present the conclusions.

II. FUNDAMENTALS OF $f(T)$ THEORIES

The extended gravitational schemes with absolute parallelism ($f(T)$ theories), take as a starting point the Teleparallel Equivalent of General Relativity. We will summarize here the basic elements needed to elaborate the ideas of the present work, leaving the details for the reader. For a thorough introduction to $f(T)$ gravity as well as to its mathematical basis the reader can consult for instance [35–39].

The spirit of the equivalence between the Riemann and Weitzenböck formulations of GR can be summarized in the equation

$$T = -R + 2 e^{-1} \partial_\nu (e T_\sigma^{\sigma\nu}) . \quad (1)$$

On the left hand side of Eq. (1) we have the so-called Weitzenböck invariant

$$T = S^\rho_{\mu\nu} T_\rho^{\mu\nu} , \quad (2)$$

where $T_\rho^{\mu\nu}$ are the components (on a coordinate basis) of the torsion two-form $T^a = de^a$ coming from the

Weitzenböck connection $\Gamma_{\nu\mu}^\lambda = e_a^\lambda \partial_\nu e_\mu^a$, and $S_{\lambda\mu\rho}$ is defined according to

$$S^\rho_{\mu\nu} = \frac{1}{4} (T^\rho_{\mu\nu} - T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho) + \frac{1}{2} \delta_\mu^\rho T_{\sigma\nu}{}^\sigma - \frac{1}{2} \delta_\nu^\rho T_{\sigma\mu}{}^\sigma . \quad (3)$$

Actually, T is the result of a very specific quadratic combination of irreducible representations of the torsion tensor under the Lorentz group $SO(1,3)$ [40]. Equation (1) simply says that the Weitzenböck invariant T differs from the scalar curvature R in a total derivative; therefore, both conceptual frameworks are totally equivalent at the time of describing the dynamics of the gravitational field.

$f(T)$ gravity can be viewed as a natural extension of Einstein gravity, and is ruled by the action in D spacetime dimensions

$$S = \frac{1}{16\pi G} \int d^D x e [f(T) + L_{matter}] , \quad (4)$$

being $e = \sqrt{\det(g_{\mu\nu})}$. Of course GR is contained in (4) as the particular case when $f(T) = T$. The dynamical equations in $f(T)$ theories, for matter coupled to the metric in the usual way, are

$$\left(e^{-1} \partial_\mu (e S_a^{\mu\nu}) + e_a^\lambda T^\rho_{\mu\lambda} S_\rho^{\mu\nu} \right) f'(T) + S_a^{\mu\nu} \partial_\mu (T) f''(T) - \frac{1}{4} e_a^\nu f(T) = -4\pi G e_a^\lambda T_\lambda{}^\nu \quad (5)$$

where the prime means derivative with respect to T and $T_\lambda{}^\nu$ is the energy-momentum tensor. It is worth insisting that equations (5) are second-order differential equations for the vielbein components. This implies a genuine advantage compared with other deformed gravitational schemes such as, for instance, the popular $f(R)$ gravities. This goodness, however, has consequences: the lack of local Lorentz invariance. If we perform on a given solution (let us say e^b), a local Lorentz transformation $e^a \rightarrow e^{a'} = \Lambda_b^{a'}(x) e^b$, we will find that $e^{a'}$ is not a solution of the field equations (5). The reason for this lack of local invariance is simple: under a local transformation the scalar (2) changes according to $T \rightarrow T' = T +$ surface term. So this surface term, which is harmless when $f(T) = T$, remains inside the function f , thus ruining the local invariance of the theory. This is because the theory picks up preferred referential frames that constitute the autoparallel curves of the given manifold [41]. In other words, the field equations (5) determine the full components of the vielbein and not just those of the metric tensor, related to the vielbein by means of

$$g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x) . \quad (6)$$

The preferred reference frames that solve the dynamical equations (5) constitute what we shall call the *proper frames*. These frames define the spacetime structure by means of a global set of basis covering the whole tangent bundle $\mathcal{T}(\mathcal{M})$, i.e., they *parallelize* the spacetime (see section IV for details). From the infinite set of vielbeins

giving a certain metric, the motion equations (5) choose those frames which act as well defined (smooth), non null fields on the whole tangent bundle. Equation (6) just says that these smooth basis fields are everywhere orthonormal.

An important fact sometimes omitted in the literature concerns the coupling with the matter fields. If we assume that the matter action depends only on the metric (and not on the whole vielbein), then the non-fermionic matter fields are unable to sense the lack of local Lorentz invariance by virtue of the invariant character of expression (6) under such transformations. Then, as far as coupling to (non-spinning) matter is concerned, the lack of local invariance should not be problematic. The effects of the additional degrees of freedom that certainly exist in $f(T)$ theories as a consequence of the breaking of the Lorentz invariance (see the analysis performed in [42]), must be found beyond the unspinning matter. Actually, it was found that on the flat FRW background with a scalar field, linear perturbation up to second order does not reveal any extra degree of freedom at all [37]. It is fair to say, though, that the nature of the additional degrees of freedom remains unknown.

III. TOROIDAL COMPACTIFICATIONS IN $D - 4$ INTERNAL DIMENSIONS

In this section we will study a D -dimensional cosmological model inside $f(T)$ -gravity theory. We will consider the simplest case where the additional internal dimensions are compactified in a torus, and in the next section we will discuss other topologies for the extra dimensions. The external spacetime is assumed to be spatially flat in accordance with the current experimental evidence. Thus, the topology of our cosmological model is described by $R \times R^3 \times S^1 \times \dots \times S^1$, where $S^1 \times \dots \times S^1$ is the $D - 4$ dimensional torus. Under these considerations the metric is written as

$$ds^2 = dt^2 - a_0^2(t)(dx^2 + dy^2 + dz^2) - \sum_{n=1}^{D-4} a_n(t) dX_n^2, \quad (7)$$

where $a_0(t)$ denotes the scale factor of the external space and $a_n(t)$ ($n = 1, \dots, D - 4$) the scale factors for the internal dimensions X_n . Based on these observations we know that the internal compact dimensions must be very small in order to be undetectable, so we are mainly interested in solutions where all the radii of the internal dimensions $a_n(t)$ ($n = 1, \dots, D - 4$) become smaller as cosmic time evolves. To proceed, we first need to find the correct parallelization of the spacetime under consideration. For the metric (7), and due to the fact that the circle S^1 is trivially parallelizable (just take any smooth, non null tangent vector field in $\mathcal{T}(S^1)$), a vielbein that parallelizes spacetime is a simple diagonal one

$$e_{\mu}^a = \text{diag}(1, a_0(t), a_0(t), a_0(t), a_1(t), \dots, a_{D-4}(t)). \quad (8)$$

Actually, we can think about the vielbein (8) as follows: unwrap the periodic coordinates X_n , and take the submanifold $t = \text{constant}$. After rescaling the coordinates according to

$$\begin{aligned} x &\rightarrow x' = a_0 x \\ y &\rightarrow y' = a_0 y \\ z &\rightarrow z' = a_0 z \\ X_n &\rightarrow X'_n = a_n X_n, \end{aligned}$$

we just get $D - 1$ Euclidian space. For this space, it is clear that the autoparallel lines are just the straight lines which can be generated by the basis ∂_i ($i : 1, \dots, D - 1$), the dual cobasis of which is dx_i . So, up to a time-dependent conformal factor, the frames describing the autoparallel lines are dx_i , and then the whole spacetime is parallelized as it is indicated in (8).

In order to obtain the motion equations, we shall work in the co-moving system, where the energy-momentum tensor for a perfect fluid reads

$$T_{\nu}^{\mu} = \text{diag}(\rho, -p_0, -p_0, -p_0, -p_1, \dots, -p_{D-4}). \quad (9)$$

In this way, the energy conservation equation takes the form

$$\dot{\rho} + (3H_0 + \sum_{n=1}^{D-4} H_n)\rho + 3H_0 p_0 + \sum_{n=1}^{D-4} H_n p_n = 0, \quad (10)$$

where a dot means derivative with respect to time and $H_0 = \dot{a}_0/a_0$ and $H_n = \dot{a}_n/a_n$ are the Hubble parameters of the submanifolds with the scale factors $a_0(t)$ and $a_n(t)$, respectively. We can now compute the Weitzenböck invariant (2), which is

$$T = -6 \left(H_0^2 + H_0 \sum_{n=1}^{D-4} H_n + \frac{1}{3} \sum_{n=1}^{D-4} H_n H_{n+1} \right). \quad (11)$$

The initial value equation, i.e., the equation coming from the variation of the action respect the e_0^0 component of the vielbein results

$$f - 2Tf' = 16\pi G\rho. \quad (12)$$

On the other hand, the action variation respect the spatial sector of the vielbein, leads to the following equations

$$\begin{aligned} 2f' \left(3H_0^2 + 2\dot{H}_0 + \sum_{n=1}^{D-4} (\dot{H}_n + 2H_0 H_n + H_n^2) - \frac{T}{2} \right) + \\ 2f''\dot{T} \left(2H_0 + \sum_{n=1}^{D-4} H_n \right) + f(T) = -16\pi G p_0, \end{aligned} \quad (13)$$

$$\begin{aligned} 2f' \left(6H_0^2 + 3\dot{H}_0 + \sum_{n=1, n \neq b}^{D-4} (\dot{H}_n + H_0 H_n + H_n^2) - \frac{T}{2} \right) + \\ 2f''\dot{T} \left(3H_0 + \sum_{n=1, n \neq b}^{D-4} H_n \right) + f(T) = -16\pi G p_b. \end{aligned} \quad (14)$$

Note that this last expression actually contains $D - 4$ equations. The full system of equations to be solved is given by Eqs. (12)-(14) with T given by (11). They contain the conservation equation (10) self-consistently.

Now we will investigate some simple limit cases, in order to analyze the behavior of the scale factors. In all of this section, we will consider that the Hubble parameters of the extra dimensions are equal ($H_1 = \dots = H_{D-4} \equiv H_1$) and the content of matter in the spacetime is assumed to be dust matter, i.e. $p_0 = p_n = 0$. In this way, Eqs. (11), (13), and (14) are simplified to

$$T = -6H_0^2 - 6(D-4)H_0H_1 - (D-4)(D-5)H_1^2, \quad (15)$$

$$2f' \left(3H_0^2 + 2\dot{H}_0 + (D-4)(\dot{H}_1 + 2H_0H_1 + H_1^2) - \frac{T}{2} \right) + 2f''\dot{T}(2H_0 + (D-4)H_1) + f(T) = 0, \quad (16)$$

$$2f' \left(6H_0^2 + 3\dot{H}_0 + (D-5)(\dot{H}_1 + H_0H_1 + H_1^2) - \frac{T}{2} \right) + 2f''\dot{T}(3H_0 + (D-5)H_1) + f(T) = 0. \quad (17)$$

First, we will analyze the limit case when both Hubble parameters, H_0 and H_1 , are constants; therefore, from Eq. (15) we know that T must be a constant too, and the above equations reduce to a more manageable form

$$2f' \left(3H_0^2 + (D-4)(2H_0H_1 + H_1^2) - \frac{T}{2} \right) + f(T) = 0, \quad (18)$$

$$2f' \left(6H_0^2 + (D-5)(H_0H_1 + H_1^2) - \frac{T}{2} \right) + f(T) = 0. \quad (19)$$

Therefore, the equations (18) and (19) yield the following condition

$$H_1 = -\frac{6H_0}{\sqrt{(D-3)^2 + 12} - (D-3)}. \quad (20)$$

Having imposed the constancy of H_0 and H_1 , we will see under what conditions the initial value and energy conservation equations are satisfied. In this case, the energy conservation equation can be written as

$$\dot{\rho} + (3H_0 + (D-4)H_1)\rho = 0. \quad (21)$$

Using the fact that $H_0 = \dot{a}_0(t)/a_0(t)$ and $H_1 = \dot{a}_1(t)/a_1(t)$, the energy density is integrated straightforwardly from the above expression, leading to

$$\rho = \frac{C}{a_0^3(t)a_1^{D-4}(t)}, \quad (22)$$

where C is an integration constant. The initial value equation (12) is satisfied if ρ is constant (due to the constancy of T), which implies that the term $a_0^3(t)a_1^{D-4}(t)$

must be constant. This condition actually means that the total volume of the higher dimensional space remains constant in time, which will ensure that a dynamical contraction of the internal dimensions yields an expanding external space. From Eq. (21) we obtain

$$H_0 = -\frac{(D-4)}{3}H_1. \quad (23)$$

This equation together with Eq. (20) are only satisfied for $D = 5$. However, in five dimensions we have $H_0 = -H_1/3$ and $T = 12H_0^2$. Thus, by replacing in Eq. (19), we obtain $f(T) = 0$. In this way, Eq. (12) yields $\rho = 0$ and therefore the constant C in (22) is null. Also, it is easy to see that for dimensions higher than five and for $p_0 = p_n = \rho = 0$ and a constant H_0 , the only possible solution is given by $H_0 = H_1 = 0$ with $f(T) = 0$. Therefore, we conclude that this compactification is not able to describe a de Sitter expansion in vacuum.

Another limit case worth mentioning is when the Hubble parameters of the extra dimensions vanish ($H_1 = \dots = H_{D-4} = 0$), which means that the internal space has a constant volume that should be very small. This represents a very simple model describing the asymptotic evolution of the additional dimensions to a final constant volume. It is easy to see that no solution of such type exists in this context. In contrast, we will see in section IV that this kind of behavior is possible for the extra dimensions in some spherical compactifications.

Finally, let's briefly comment on some additional exact solutions occurring in $D = 5$. Taking $\rho = 0$ and non-constant Hubble parameters in Eqs. (11), (13), and (14) we get

$$T = -6(H_0^2 + H_0H_1), \quad (24)$$

$$2f' \left(3H_0^2 + 2\dot{H}_0 + \dot{H}_1 + 2H_0H_1 + H_1^2 - \frac{T}{2} \right) + 2f''\dot{T}(2H_0 + H_1) + f(T) = 0, \quad (25)$$

$$2f' \left(6H_0^2 + 3\dot{H}_0 - \frac{T}{2} \right) + 6f''\dot{T}H_0 + f(T) = 0. \quad (26)$$

From Eq. (12), we obtain

$$\frac{f}{2f'} = T. \quad (27)$$

Therefore, T is a constant that depends on $f(T)$ [?], and by replacing it in Eq. (26), we get

$$\dot{H}_0 + 2H_0^2 + \frac{T}{6} = 0, \quad H_1 = -\frac{T}{6H_0} - H_0, \quad (28)$$

where we have used (24). By this manner we can distinguish three cases that depend on the sign of T .

- Case $T < 0$. Eq. (28) tells us in this case that the Hubble parameter H_0 yields

$$H_0 = \frac{1}{2} \sqrt{\frac{|T|}{3}} \tanh \left(\sqrt{\frac{|T|}{3}} (t + B) \right), \quad (29)$$

where B is an integration constant. It is worth noting that both scale factors, which are given by

$$a_0(t) = \alpha \cosh^{1/2} \left(\sqrt{\frac{|T|}{3}} (t + B) \right), \quad (30)$$

$$a_1(t) = \beta \sinh \left(\sqrt{\frac{|T|}{3}} (t + B) \right) \times \cosh^{-1/2} \left(\sqrt{\frac{|T|}{3}} (t + B) \right), \quad (31)$$

where α and β are constant, increase in time. This case, thus, seems to be unprovided of physical significance.

- Case $T > 0$. Now H_0 is given by

$$H_0 = -\frac{1}{2} \sqrt{\frac{T}{3}} \tan \left(\sqrt{\frac{T}{3}} (t + B) \right), \quad (32)$$

and the scale factors by

$$a_0(t) = \alpha \cos^{1/2} \left(\sqrt{\frac{T}{3}} (t + B) \right), \quad (33)$$

$$a_1(t) = \beta \sin \left(\sqrt{\frac{T}{3}} (t + B) \right) \times \cos^{-1/2} \left(\sqrt{\frac{T}{3}} (t + B) \right). \quad (34)$$

In order to be physically admissible, at least as a model of the early stages of cosmic evolution, we must bind the proper time to the interval $-\frac{\pi}{2} \sqrt{\frac{3}{T}} \leq t \leq 0$, having taken $B = 0$. In this way a_0 is an increasing function of time, while a_1 decreases from infinite at $t = -\frac{\pi}{2} \sqrt{\frac{3}{T}}$ to a null value at $t = 0$.

- Case $T = 0$. Finally, when the scalar torsion vanishes the scale factors are given by

$$a_0^2(t) = 2t + B, \quad (35)$$

$$a_1^2(t) = \frac{1}{2t + B}. \quad (36)$$

This behavior for the scale factors shows a suppression of the extra dimensions as the bulk expands linearly in time, and then, constitutes a physical admissible simple model of the early universe.

Note that in the three cases, the solutions are independent of the specific form of the function $f(T)$, in the sense that the sole role of $f(T)$ is to fix the constant value of T . We actually included them here as simple examples of exact states of the theory, and we hope that they might serve as a seed for more realistic solutions.

IV. SPHERICAL COMPACTIFICATIONS IN $D - 4$ INTERNAL DIMENSIONS

A. Statement of the problem

In theories relying on absolute parallelism, i.e., in theories where the whole vielbein components (not just the metric tensor) are determined by the field equations, the structure of the parallel vector fields describing the additional compact dimensions is highly non-trivial. In this section we investigate this issue further and discuss a number of topics essential to understanding the nature of the additional compact dimensions when they are other than the simple toroidal compactifications previously discussed. In particular, we shall now focus on extra compact dimensions with topology S^n , $n > 1$.

Our task is to find cosmological solutions of the $f(T)$ field equations (5) with $D - 4$ spherically compactified extra dimensions, i.e. a set of D one forms $e^a(x)$ that solve the equations and lead to the metric tensor

$$ds^2 = dt^2 - a_1^2(t) d\Omega_{(j)}^2 - a_2^2(t) d\Omega_{(k)}^2 - \dots - a_0^2(t) (dx^2 + dy^2 + dz^2), \quad (37)$$

where $d\Omega_{(j)}^2$ refers to the line element corresponding to the j sphere, so $j + k + \dots = D - 4$. In order to do so, we need first to propose a proper frame field for the topology $R \times S^j \times S^k \times \dots \times R^3$, i.e., D one forms $e^a(x)$ which turn equations (5) in a set of consistent equations for the unknown (time dependent) scale factors a_0, a_1, a_2, \dots . Once this is done, it will remain to solve the equations and to obtain the functional form of the scale factors. In section III we have explained how to do this for extra dimensions with toroidal topology, where the proper frame field was given in (8). Now we want to discuss this issue when the extra dimensions are spherically compactified.

We start noting that a remarkable result due to Stiefel states that every orientable three-dimensional manifold is also parallelizable [43], thus, so it is S^3 . This means that in every orientable three-manifold \mathcal{M}_3 there exist three independent smooth, non null vector fields covering the entire tangent bundle $\mathcal{T}(\mathcal{M}_3)$. Additionally, Kervaire and Milnor [44] have shown that, apart from S^1 and S^3 , the only other parallelizable sphere is S^7 . For our purposes this means that extra dimension spherical compactifications other than these will possess a rather complicated parallel vector field structure.

In general, a D -dimensional manifold \mathcal{M}_D will be parallelizable if a global basis exist in $\mathcal{T}(\mathcal{M}_D)$. This global

basis of the tangent bundle constitutes a preferred reference frame which can be used for defining the space structure. This is so because in this last case we can define a given spacetime as the pair $(\mathcal{T}(\mathcal{M}_D), e^a(x))$ instead of $(\mathcal{M}_D, g_{\mu\nu}(x))$. This preferred reference frame $e^a(x)$ and the solution of the equations (5), as we shall see, are closely related.

Remaining ever conscious of the subtleties involved, and in order to illustrate the problem, let us consider first two spherically compactified extra dimensions. In this case we need to find a parallelization $e^a(x)$ for the spacetime having the metric

$$ds^2 = dt^2 - a_1^2(t)(d\theta^2 + \sin^2 \theta d\phi^2) - a_0^2(t)(dx^2 + dy^2 + dz^2), \quad (38)$$

where (θ, ϕ) are standard spherical coordinates on the 2-sphere. It is intuitively obvious that the 6-dimensional manifold described by (38) is parallelizable. As a matter of fact, the spacetime topology is $\mathcal{M}_3 \times R^3$, where $\mathcal{M}_3 \approx R \times S^2$, so the full spacetime manifold can be written as a product of two 3-dimensional, orientable (i.e., parallelizable) manifolds. However, it is important to realize that the parallel vector fields of \mathcal{M}_3 will be highly non-trivial because S^2 itself is not parallelizable due to the hairy ball theorem, which states that S^2 (like all the even spheres) does not admit a global basis of the tangent bundle $\mathcal{T}(S^2)$ (see, for instance [45]). In order to better understand this point, Figure 1 below show two attempts of covering $\mathcal{T}(S^2)$ with a global, smooth, non null vector field. No matter what the arrangement of the field be, it will fail to be non null or smooth in at least one point of $\mathcal{T}(S^2)$.

Taking into account these facts, we can arrange the components of the vielbein corresponding to the metric (38) in a 6×6 matrix e^a_μ , so schematically we have

$$e^a_\mu = \left(\begin{array}{c|ccc} \mathbb{M}_3 & & \mathbb{O} & \\ \hline & a_0 & 0 & 0 \\ \mathbb{O} & 0 & a_0 & 0 \\ & 0 & 0 & a_0 \end{array} \right) \quad (39)$$

where \mathbb{O} is the 3×3 null matrix and \mathbb{M}_3 is the 3×3 matrix representing the parallel *triad* field of \mathcal{M}_3 . The main point we wish to emphasize is that \mathbb{M}_3 *cannot* be diagonal. If this were so, it would mean that the global vector fields which perform the parallelization of \mathcal{M}_3 are just $e^1 = dt$, $e^2 = a_1(t)d\theta$ and $e^3 = a_1(t)\sin\theta d\phi$, but this is impossible because in this case the fields e^2 and e^3 should be a parallelization of S^2 , which does not exist at all.

Even more, it is clear that \mathbb{M}_3 cannot have the *block* form

$$\mathbb{M}_3 = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ 0 & \mathbb{M}_2 & \end{array} \right), \quad (40)$$

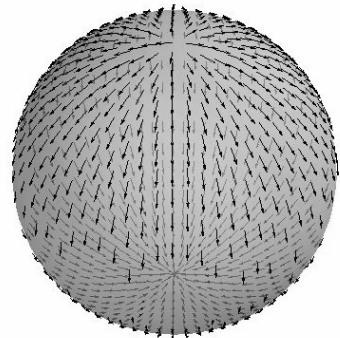
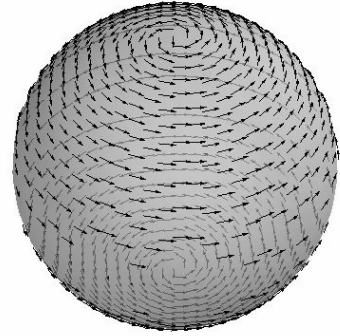


FIG. 1: Two infructuous attempts of parallelizing S^2 . In the upper panel the field is smooth but null at the poles. In the lower panel it is non null at the poles but instead it fails to be smooth there.

either, where \mathbb{M}_2 is some 2×2 matrix. In this case, we would have that \mathbb{M}_2 is related to the matrix $a_1(t)diag(d\theta, \sin\theta d\phi)$ by means of a local rotation. However, again, it is impossible to find a global basis for $\mathcal{T}(S^2)$. This means that the matrix \mathbb{M}_3 should be related to $diag(dt, a_1(t)d\theta, a_1(t)\sin\theta d\phi)$ by a Lorentz boost, and therefore the structure (40) can not be correct. Actually, we should mention that the proper (autoparallel) vector fields of the form (39) corresponding to the metric (38) are unknown at present, and the task of finding the parallelization of \mathcal{M}_3 remains as an open problem. Fortunately, a parallelization of the manifold (38) can be found in a somewhat different manner, and this will be matter for the paragraphs below. There we will find that the correct vector fields have the structure

$$e^a_\mu = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ 0 & \mathbb{M}_5 & \end{array} \right), \quad (41)$$

where \mathbb{M}_5 is the matrix representing the parallelization

of the five-dimensional manifold $S^2 \times R^3$.

As in the six-dimensional case, if one considers $D = 7$, the problem can be fully understood and this will also be explained in detail below. For extra dimensions higher than three more patience is required. If we take four extra dimensions, in addition to the trivial toroidal compactifications of the last section, we are led to the topologies $S^1 \times S^3$, $S^2 \times S^2$ or S^4 . With the exception of the first case, where the parallelization is provided by a simple extension of the procedure to be explained below for the S^3 compactification, the parallelization will be highly non-trivial because S^2 and S^4 do not admit a global basis. This problem turns out to be even more difficult as the dimension increases, because the number of possible compactifications increases as $D - 4$, the number of extra dimensions.

B. Complete characterization of $D = 7$

As a working example, let us now consider the various possibilities arising from the case $D = 7$. The first of these was worked with in the last section, where toroidal

compactifications were extensively discussed. Now we will consider the 7-dimensional metric

$$ds^2 = dt^2 - a_1^2(t)[d\psi^2 + \sin^2 \psi(d\theta^2 + \sin^2 \theta d\phi^2)] - a_0^2(t)(dx^2 + dy^2 + dz^2), \quad (42)$$

where we have supposed three additional dimensions spherically compactified. The manifold topology is $R \times S^3 \times R^3$, which is a product of parallelizable manifolds. Following the scheme (39) we will now have

$$e_\mu^a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & & \\ 0 & \mathbb{M}_3 & & & \mathbb{O} & & \\ 0 & & & & & & \\ 0 & & & a_0 & 0 & 0 & \\ 0 & \mathbb{O} & & 0 & a_0 & 0 & \\ 0 & & & 0 & 0 & a_0 & \end{pmatrix}, \quad (43)$$

where now \mathbb{M}_3 is the matrix representing the parallelization of S^3 . A global basis for $\mathcal{T}(S^3)$ has been obtained in [35], so we will just summarize the result here. For \mathbb{M}_3 we have

$$\mathbb{M}_3 = a_1(t) \begin{pmatrix} c(\theta) & -s(\psi)c(\psi)s(\theta) & -s^2(\psi)s^2(\theta) \\ s(\theta)c(\phi) & s(\psi)[s(\psi)s(\phi) + c(\psi)c(\theta)c(\phi)] & s(\psi)s(\theta)[s(\psi)c(\theta)c(\phi) - c(\psi)s(\phi)] \\ s(\theta)s(\phi) & s(\psi)[c(\psi)c(\theta)s(\phi) - s(\psi)c(\phi)] & s(\psi)s(\theta)[c(\psi)c(\phi) + s(\psi)c(\theta)s(\phi)] \end{pmatrix}, \quad (44)$$

where we have used $s \equiv \sin$ and $c \equiv \cos$. We can just add here that \mathbb{M}_3 can be obtained from the *naive* triad $a_1(t) \text{diag}(d\psi, \sin \psi d\theta, \sin \psi \sin \theta d\phi)$ by means of a local Euler rotation (see [35] for details). It is a simple exercise to check that the frame (43) with \mathbb{M}_3 given by (44) gives rise to the metric (42).

Having found the proper frame, we can now proceed to compute the motion equations. The vielbein (43) leads to the invariant

$$T = -6(H_0^2 + H_1^2 + 3H_0H_1 - a_1^{-2}). \quad (45)$$

The initial value equation then results in

$$f + 12f'(H_0^2 + H_1^2 + 3H_0H_1) = 16\pi G\rho. \quad (46)$$

On the other hand, by varying respect the spatial sector of the vielbein, we get two equations:

$$2f' \left(7H_0^2 + 4H_1^2 + 9H_0H_1 + 3\dot{H}_0 + 2\dot{H}_1 - \frac{T}{3} \right) + 2f''\dot{T}(3H_0 + 2H_1) + f = -16\pi Gp_1, \quad (47)$$

$$2f' \left(6H_0^2 + 9H_1^2 + 15H_0H_1 + 2\dot{H}_0 + 3\dot{H}_1 \right) + 2f''\dot{T}(2H_0 + 3H_1) + f = -16\pi Gp_0. \quad (48)$$

All the remaining equations are trivial or equal to these last two. This is proof that the frame (43) is actually a parallelization of (42), because it leads to a consistent set of motion equations.

It is interesting to compare this compactification with the toroidal ones (in $D = 7$) worked with in the previous section. A remarkable property of the system (46)-(48) is that it admits a de Sitter expansion for the bulk while the extra dimensions remains compactified with a constant scale factor, and that this happens for a pure vacuum. Actually, if we set H_0 as a constant and $H_1 = \rho = p_1 = p_2 = 0$, the system becomes (note that $T = -6(H_0^2 - a_1^{-2})$ is constant in this case)

$$f + 12H_0^2 f' = 0 \quad (49)$$

$$2f'(7H_0^2 - \frac{T}{3}) + f = 0, \quad (50)$$

equation (48) being equal to (49). Inserting the expression $T = -6(H_0^2 - a_1^{-2})$ and the value of f' from (49) in (50) we immediately get

$$H_0^2 = \frac{2}{3}a_1^{-2}, \quad (51)$$

which implies $T = 3H_0^2$. Note that $f(T) = T$ is not a solution of equations (49) and (50), showing that this kind of solution is absent in Einstein's theory. The result (51) is actually valid only for smooth deformations of GR, i.e., for $f(T)$ models of the form $f(T) = T + \beta g(T)$. Replacing this form of $f(T)$ in Eq. (49), we obtain a relationship between H_0 , β and the potential constants appearing in $g(T)$.

Also, note that at this point the constant value of H_0 (or a_1) remains undetermined. Nevertheless, given that the observational evidence suggests that the size of the extra dimensions, if non-null, should be very small at the present time, Eq. (51) focuses attention on the ultraviolet regime. This enables us to interpret inflation as a vacuum-driven expansion given by the small compact extra dimensions. For this purpose let us invoke, as an example, the high energy deformation of GR with the Born-Infeld structure worked in [3] and [4]. We thus have

$$f(T) = -\lambda [\sqrt{1 - 2\lambda^{-1}T} - 1], \quad (52)$$

where λ is the Born-Infeld constant (so, $\beta = -\lambda^{-1}$, $g(T) = T^2/2 + \lambda^2 \mathcal{O}(T/\lambda)^3$). In this manner we have that Einstein's gravity (in its absolute parallelism form) is recovered when $T/\lambda \ll 1$. The role of Eq. (49) will be to link the undetermined parameter H_0 (or a_1) with λ . Actually, replacing the functional form (52) in (49) we get the quadratic expression

$$(3u + 1)^2 = 1 + u, \quad u = -6\lambda^{-1}H_0^2, \quad (53)$$

which leads to the solutions $H_0 = 0$ and $H_0^2 = 5\lambda/54$. Of course, $H_0 = 0$ just says that Minkowski spacetime is a vacuum solution of the theory. In turn, $H_0^2 = 5\lambda/54$, which is very close to $H_0^2 = \lambda/12$, the value obtained in [4] for the inflationary era, represents a de Sitter accelerated expansion, the cause of which is the presence of the extra dimensions.

Let us go back now to the full characterization of $D = 7$. The remaining topology for the three additional dimensions is just $S^1 \times S^2$, so the full metric looks in this case like

$$ds^2 = dt^2 - a_2^2(t)d\Theta^2 - a_1^2(t)(d\theta^2 + \sin^2\theta d\phi^2) - a_0^2(t)(dx^2 + dy^2 + dz^2). \quad (54)$$

We proceed now to find the parallel fields for the geometry (54). For this task we focus on the embedding of

the submanifold $\mathcal{M}_3 = S^1 \times S^2$ in the four-dimensional space $S^1 \times R^3$. Installing coordinates (Θ, X, Y, Z) in this manifold, we have that a parallelization of $\mathcal{T}^*(\mathcal{M}_3)$ reads

$$\begin{aligned} E^1 &= a_2(t)Zd\Theta - a_1(t)(YdX - XdY) \\ E^2 &= a_2(t)Yd\Theta + a_1(t)(ZdX - XdZ) \\ E^3 &= a_2(t)Xd\Theta - a_1(t)(ZdY - YdZ). \end{aligned} \quad (55)$$

This base turns out to be a global parallelization for $S^1 \times S^2$ only if $a_1(t)$ and $a_2(t)$ are non-null, so a singularity in the Riemannian sense (i.e., when at least either $a_1(t)$ or $a_2(t)$ vanishes), actually represents a singularity in the parallelization process. In terms of the basis (Θ, θ, ϕ) , where $X = \sin\theta \cos\phi$, $Y = \sin\theta \sin\phi$ and $Z = \cos\theta$ we therefore have

$$\begin{aligned} E^1 &= a_2(t) \cos\theta d\Theta + a_1(t) \sin^2\theta d\phi \\ E^2 &= a_2(t) \sin\theta \sin\phi d\Theta + \\ &\quad a_1(t)(\cos\phi d\theta - \sin\theta \sin\phi \cos\theta d\phi) \\ E^3 &= a_2(t) \sin\theta \cos\phi d\Theta - \\ &\quad a_1(t)(\sin\phi d\theta + \cos\theta \sin\theta \cos\phi d\phi). \end{aligned} \quad (56)$$

Being $S^1 \times S^2$ a three dimensional manifold, we can only provide a glimpse of the structure underlying the fields (55); in every point of the submanifold $\Theta = \text{constant}$, we have

$$\begin{aligned} \tilde{E}^1 &= -YdX + XdY \\ \tilde{E}^2 &= ZdX - XdZ \\ \tilde{E}^3 &= -ZdY + YdZ, \end{aligned} \quad (57)$$

where $\tilde{E}^i = E^i/a_1(t)$. These fields, as viewed immersed in 3D-euclidian space, look similar to the ones of Figure 1 of Ref. [35], where the parallelization (44) is showed for $\psi = \pi/2$ (i.e. for the S^3 sphere hyper-equator). However, this resemblance is partial, and it is important to bear in mind that the fields (44) and (56) represent parallelizations of two genuinely different topological structures, S^3 and $S^1 \times S^2$ respectively.

In analogy with the analysis performed before, we can now obtain the motion equations. First, it is necessary to compute the scalar invariant, which reads

$$T = -2(3H_0^2 + H_1^2 + 6H_0H_1 + 2H_1H_2 + 3H_0H_2 - a_1^{-2}). \quad (58)$$

Note that just the scale factor corresponding to S^2 is present in T , as must be the case. Moreover, the scalars (45) and (58) are different even when $a_0 = a_1$, because they represent different geometries. The full set of motion equations are

$$f + 4f' (H_1^2 + 3H_2H_0 + 3H_0^2 + 2H_1H_2 + 6H_1H_0) = 16\pi G\rho, \quad (59)$$

$$f + 2f' (4H_1^2 + 2H_1H_2 + 9H_0^2 + 12H_1H_0 + 3H_0H_2 + 2\dot{H}_1 + 3\dot{H}_0) + 2f'' (2H_1 + 3H_0)\dot{T} = -16\pi G p_2, \quad (60)$$

$$f + 2f' (-a_1^{-2} + 2H_1^2 + H_2^2 + 9H_0^2 + 3H_1H_2 + 9H_1H_0 + 6H_2H_0 + \dot{H}_1 + \dot{H}_2 + 3\dot{H}_0) + 2f'' (H_1 + H_2 + 3H_0)\dot{T} = -16\pi G p_1, \quad (61)$$

$$f + 2f' (4H_1^2 + 6H_0^2 + H_2^2 + 4H_1H_2 + 10H_1H_0 + 5H_2H_0 + 2\dot{H}_1 + \dot{H}_2 + 2\dot{H}_0) + 2f'' (2H_1 + H_2 + 2H_0)\dot{T} = -16\pi G p_0. \quad (62)$$

A simple exercise shows that vacuum solutions with $H_1 = H_2 = 0$ and a non-null constant H_0 does not exist at all for this specific compactification. Imposing these conditions on (59) and (60) we are led to the inconsistent (when $H_0 \neq 0$) equations

$$\begin{aligned} f + 12f'H_0^2 &= 0, \\ f + 18f'H_0^2 &= 0. \end{aligned} \quad (63)$$

So, as far as vacuum-driven inflation given by the small compact extra dimensions is concerned, we see that the S^3 compactification is clearly favored.

C. Complete characterization of $D = 6$

In this subsection we consider the remaining topology (S^2) for the internal dimensions of the six-dimensional manifold. We are now ready to go back to the metric (38). The topology of this manifold is now described by $R \times S^2 \times R^3$, so we proceed to parallelize this submanifold using a similar method to that employed at the end of the previous subsection. The key point is that we can unwrap the periodic coordinate Θ in (56) and think about it as one of the bulk coordinates, let us say x . In this way, we obtain the submanifold $\mathcal{M}_3 = S^2 \times R$. Again, embedding this in order to obtain three-dimensional manifold in the four-dimensional space $R^3 \times R$ and installing coordinates (X, Y, Z, x) on it, we obtain that a parallelization of $\mathcal{T}^*(\mathcal{M}_3)$ reads

$$\begin{aligned} E^1 &= a_0(t) \cos \theta dx + a_1(t) \sin^2 \theta d\phi \\ E^2 &= a_0(t) \sin \theta \sin \phi dx + \\ &\quad a_1(t) (\cos \phi d\theta - \sin \theta \sin \phi \cos \theta d\phi) \\ E^3 &= a_0(t) \sin \theta \cos \phi dx - \\ &\quad a_1(t) (\sin \phi d\theta + \cos \theta \sin \theta \cos \phi d\phi). \end{aligned} \quad (64)$$

where again, $X = \sin \theta \cos \phi$, $Y = \sin \theta \sin \phi$ and $Z = \cos \theta$ was used. An observation is in order: nothing prevents thinking of the unwrapped Θ periodic coordinate now as the y or z coordinate of the bulk given the isotropy and homogeneity of the latter. This actually means that

we obtain three different but equivalent sets of parallel vector fields for (38).

With this vielbein in mind we can obtain the following torsion invariant and fields equations,

$$T = -2 (-a_1^{-2} + H_1^2 + 6H_0H_1 + 3H_0^2) \quad (65)$$

$$f + 4f' (H_1^2 + 6H_0H_1 + 3H_0^2) = 16\pi G\rho \quad (66)$$

$$f + 2f' (6H_0^2 + 10H_0H_1 + 4H_1^2 + 2\dot{H}_0 + 2\dot{H}_1) + 2f''\dot{T} (2H_0 + 2H_1) = -16\pi G p_0 \quad (67)$$

$$f + 2f' (-a_1^{-2} + 9H_0^2 + 9H_0H_1 + 2H_1^2 + 3\dot{H}_0 + \dot{H}_1) + 2f''\dot{T} (3H_0 + H_1) = -16\pi G p_1 \quad (68)$$

Again, we are interested in solutions with $\rho = p_0 = p_1 = H_1 = 0$ and constant H_0 . From (66) we immediately get

$$f + 12f'H_0^2 = 0, \quad (69)$$

which is consistent with (67). This expression replaced in (68) leads to

$$H_0^2 = \frac{1}{3}a_1^{-2}.$$

This last equation is the analog of (51). Note that this value of H_0 yields $T = 0$ by using (65), which turns out to be problematic. Actually, every ultraviolet deformation looks like $f(T) = T + O(T^2)$, so we have $f(0) = 0$, $f'(0) = 1$, and then (69) is satisfied only when $H_0 = 0$. We can therefore conclude that physically admissible models with this topology do not contain $H_1 = 0$ and a non-null constant H_0 as solution.

V. CONCLUDING COMMENTS

This work was devoted to the study of spacetimes with $FRW^4 \times \mathcal{M}^{D-4}$ topology in the context of the new

gravitational schemes with absolute parallelism known as $f(T)$ theories. We have assumed that FRW⁴ represents the spatially flat Friedmann-Robertson-Walker manifold corresponding to the four-dimensional space-time of standard cosmology, while \mathcal{M}^{D-4} refers to the $(D-4)$ -dimensional compact extra dimensions constituting internal space. On one hand we focussed the analysis on $(D-4)$ -dimensional manifolds consisting of $(D-4)$ copies of the torus, and on the other hand, several spherical compactifications were considered.

While obtaining the proper parallel vector fields for the toroidal compactifications realized in section III is trivial, (because the $(D-4)$ torus $T^{D-4} = S^1 \times \dots \times S^1$ is a product of trivial parallelizable manifolds), the corresponding characterization of the spherical topologies is highly non-trivial. Several aspects of this last issue were discussed in section IV, where the problem was posed in detail. Then, we characterized the six and seven-dimensional cases by finding the proper vielbeins for $\mathcal{M}^2 = S^2$ and $\mathcal{M}^3 = S^1 \times S^2$, $\mathcal{M}^3 = S^3$ respectively. These vielbeins constitute the starting point for *any* $f(T)$ cosmological model of compact extra dimensions, because they represent the basis responsible for the parallelization of the manifold under consideration, i.e., they define the spacetime structure.

In both sections, we discussed a number of simple exact solutions of the field equations. One of the simple solutions explored throughout the different compactifications considered in this work concerns the existence of a constant scale factor for the compact dimensions while the external scale factor expands exponentially in time. This situation corresponds to a vacuum-driven inflation powered by the presence of extra dimensions. We showed that for five and six spacetime dimensions, this kind of behavior does not exist at all. In turn, this problem finds a solution in $D = 7$ but not for any kind of compactification, just for S^3 . In this last case we found that for smooth deformations of GR, the relation between H_0 and a_1 is given by equation $H_0^2 = \frac{2}{3}a_1^{-2}$. Therefore, Hubble rates characterizing the inflationary era find their hugeness in the exceedingly tiny value of a_1 . We do not expect this exact solution to describe all the details in the compactification process, but merely the asymptotic stage of it. We think that a solution for the internal dimensions of

the form $a_{in} = a_1 + \alpha \text{Exp}(-H_0 t)$, with a constant α certainly do exist in theory, though it will probably require the presence of matter fields and the use of numerical techniques.

However, the existence of the exact solution in $D = 7$ with S^3 topology for the extra dimensions leads us to think that a similar exact solution will exist *only* in $D = 11$, where the seven additional dimensions will be compactified with topology S^7 . Note that S^7 is the only remaining parallelizable sphere, so vielbein fields with product structure FRW⁴ $\times S^7$ will constitute a parallelization of the entire manifold. In this respect, we proved in section III that toroidal compactifications certainly do not lead to this kind of solution in any spacetime dimension D . It will be a matter for future investigations to find the parallel vector fields of FRW⁴ $\times S^7$ and those of the other admissible eleven-dimensional spherical topologies, and confirm (or dismiss) this conjecture. The veracity of this statement may make it possible to detect interesting links between these $f(T)$ deformed gravitational schemes and the low energy limit of M-theory.

Much more work is certainly necessary to fully understand the peculiarities present in the parallelization process behind $f(T)$ theories. In particular, the characterization of all the spherical topologies in a cosmological context for a given spacetime dimension (with $D > 7$) remains entirely open. We would like to think that our work constitutes a first and significant step towards that goal.

Acknowledgments

F.F. is indebted to F. Canfora for many enlightening discussions about extra dimensions in cosmology. F.F. is supported by CONICET and Universidad de Río Negro. Y. V. is supported by FONDECYT grant 11121148, and by Dirección de Investigación y Desarrollo, Universidad de la Frontera, DIUFRO DI11-0071. P.G. and Y.V. acknowledge the hospitality of the Centro Atómico Bariloche where part of this work was undertaken.

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- [1] A. Unzicker and T. Case, arXiv:0503046.
 - [2] K. Hayashi and T. Shirafuji, *Phys. Rev.* **D19**, (1979) 3524, *Addendum-ibid.* **D24**, (1982) 3312.
 - [3] R. Ferraro and F. Fiorini, *Phys. Rev.* **D75** (2007) 084031.
 - [4] R. Ferraro, F. Fiorini, *Phys. Rev.* **D78**, (2008) 124019.
 - [5] G. R. Bengochea and R. Ferraro, *Phys. Rev.* **D79**, (2009) 124019.
 - [6] E. V. Linder, *Phys. Rev.* **D81**, (2010) 127301.
 - [7] R. Zheng, Q. -G. Huang, *JCAP* **1103**, (2011) 002.
 - [8] K. Bamba, C. -Q. Geng, C. -C. Lee, L. -W. Luo, *JCAP* **1101**, (2011) 021.
 - [9] Y. Zhang, H. Li, Y. Gong, Z. -H. Zhu, *JCAP* **1107**, (2011) 015.
 - [10] S. Capozziello, V. F. Cardone, H. Farajollahi, A. Ravanpak, *Phys. Rev.* **D84**, (2011) 043527.
 - [11] K. Bamba, C. -Q. Geng, *JCAP* **1111**, (2011) 008.
 - [12] H. Wei, *Phys. Lett.* **B712**, (2012) 430.
 - [13] C. -Q. Geng, C. -C. Lee, E. N. Saridakis, *JCAP* **1201**, (2012) 002.
 - [14] K. Karami and A. Abdolmaleki, *JCAP* **1204** (2012) 007.
 - [15] J. Yang, Y. -L. Li, Y. Zhong and Y. Li, *Phys. Rev.* **D85**, (2012) 084033.

- [16] C. Xu, E. N. Saridakis and G. Leon, *JCAP* **1207**, (2012) 005.
- [17] K. Bamba, R. Myrzakulov, S. Nojiri and S. D. Odintsov, *Phys. Rev.* **D85** (2012) 104036.
- [18] R. Ghosh, A. Pasqua and S. Chattopadhyay, *Eur. Phys. J. Plus* (2013) 128:12.
- [19] S. Capozziello and M. De Laurentis, *Phys. Rept.* **509**, (2011) 167.
- [20] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1921**, (1921) 966.
- [21] A. Font and S. Theisen, *Lect. Notes Phys.* **668**, (2005) 101.
- [22] C. S. Chan, P. L. Paul and H. L. Verlinde, *Nucl. Phys.* **B581**, (2000) 156.
- [23] A. Chodos and S. L. Detweiler, *Phys. Rev.* **D21**, (1980) 2167.
- [24] P. G. O. Freund, *Nucl. Phys.* **B209**, (1982) 146.
- [25] T. Dereli and R. W. Tucker, *Phys. Lett.* **B125**, (1983) 133.
- [26] M. Demianski, Z. Golda, L. M. Sokolowski, M. Szydłowski and P. Turkowski, *J. Math. Phys.* **28**, (1987) 171.
- [27] M. Biesiada, M. Szydłowski and J. Szczesny, *Acta Phys. Polon.* **B19**, (1988) 3.
- [28] M. Szydłowski, A. Lapeta and B. Lapeta, *Acta Phys. Polon.* **B21**, (1990) 627 .
- [29] M. Szydłowski, A. Lapeta and B. Lapeta, *Acta Phys. Polon.* **B21**, (1990) 643.
- [30] R. Emparan, R. Suzuki and K. Tanabe, *JHEP* **1306** (2013) 009.
- [31] G. Giribet, *Phys. Rev.* **D87** (2013) 107504.
- [32] S. Capozziello, P. A. Gonzalez, E. N. Saridakis and Y. Vasquez, *JHEP* **1302** (2013) 039.
- [33] K. Bamba, S. Nojiri and S. D. Odintsov, *Phys. Lett.* **B725** (2013) 368.
- [34] R. Maartens and K. Koyama, *Living. Rev. Rel.* **13** (2010) 5.
- [35] R. Ferraro and F. Fiorini, *Phys. Lett.* **B702** (2011) 75. See also R. Ferraro and F. Fiorini, *IJMP* (Conference Series), **3** (2011) 227.
- [36] R. Ferraro and F. Fiorini, *Phys. Rev.* **D84** (2011) 083518.
- [37] K. Izumi and Y. C. Ong, *JCAP* **1306** (2013) 029.
- [38] B. Li, T. P. Sotiriou and J. D. Barrow, *Phys. Rev.* **D83** (2011) 104017.
- [39] N. L. Youssef and W. A. Elsayed, arXiv:1209.1379 (gr-qc).
- [40] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, *Phys. Rep.* 258 (1995) 1.
- [41] N. Tamanini and C. G. Boehmer, *Phys. Rev.* **D86** (2012) 044009.
- [42] M. Li, R. X. Miao and Y. G. Miao, *JHEP* **1107** (2011) 108.
- [43] E. Stiefel, *Comm. Math. Helv.* **8** (1936) 305.
- [44] M. A. Kervaire and J. W. Milnor, *Annals of Math.* **77** (1963) 504.
- [45] M. Eisenberg and R. Guy, *Am. Math. Mon.* **86**, 7 (1979) 571.
- [46] Note that if a given function $f(T)$ fulfills Eqn. (27), then $\tilde{f}(T) = f(T) + \epsilon\sqrt{T}$, with a constant ϵ , also solves the equation. Then, the addition of the squared root term can always be dismissed in vacuum.