Fermionic van der Waals and Casimir-Polder interactions

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We formulate fermionic versions, for any number of spatial dimensions, of the van der Waals and Casimir-Polder interactions and study their properties. In both cases, the systems we introduce contain localized "atoms": two-level systems, coupled to a vacuum Dirac field. This Dirac field plays here a role akin to the electromagnetic field in the van der Waals case. In this context, bag model conditions for the Dirac field serve as the analog of the "mirror" in the Casimir-Polder effect. We found that, in this case, the resulting interaction is repulsive.

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I. INTRODUCTION

Some distinctive effects due to the quantum vacuum [1] have become the subject of intensive research, partly due to the fact that those effects may become relevant, even dominant, at the increasingly accessible nanoscopic scales [2]. Among them is the van der Waals interaction, by which two electrically neutral microscopic objects may interact, as a consequence of the quantum fluctuations of their respective dipole operators. They fluctuate with zero average, but the electromagnetic (EM) field coupling leads to a nonvanishing interaction energy, which depends on the quantum average of the *square* of each dipole moment. A closely related phenomenon is the Casimir-Polder effect, by which one of those microscopic objects facing a reflecting wall experiences an attractive force. This force may be interpreted as due to the presence of an image atom, which also fluctuates, albeit in coordination with the real one.

The Casimir effect is sometimes explained as the result of the effect of van der Waals interactions between the constituents of the media which compose the "mirrors". It is, however, clear, that those interactions do *not* obey a superposition principle [3], and as a consequence, the total effect cannot be obtained, except in very special circumstances, as a sum of the pairwise interactions.

Quantum interactions involving fermionic fields are less explored compared to their electromagnetic counterparts. This work aims to bridge this gap by formulating and analyzing fermionic analogs of van der Waals and Casimir-Polder interactions. In the fermionic Casimir effect, a vacuum Dirac field is in the presence of surfaces where the normal component of the current vanishes.

In this paper, we study that kind of phenomenon, namely, how localized objects can be coupled to a fermionic field, leading to a quantum interaction similar to its known EM counterpart. More explicitly, we consider either one or two two-level systems, which represent localized degrees of freedom, coupled to a massless Dirac field. In the case of just one two-level system, it will be facing a wall which imposes bag model boundary conditions on the fermionic field.

Realizations of this kind of system might arise, for instance, in graphene, where a Dirac field in $2+1$ dimensions appears as an effective description. Here, different kinds of localized defects, affecting the Dirac field, naturally appear [4].

In a rather different context, interesting phenomena on the spin-dependent interactions between fermions, due to the exchange of virtual particles, have been studied in [5,6], and in [7], an example where virtual neutrinos induce Casimir and Casimir-Polder like effects has been presented.

In this paper, we carry out a study of this kind of interaction in different numbers of dimensions, showing that the quantum vacuum fluctuations lead to an effective interaction energy which becomes entirely analogous, albeit in a fermionic realm, to what happens in the van der Waals interaction and the Casimir-Polder effect.

This paper is organized as follows: in Sec. II, we introduce the elements of the system considered in this paper. Then, in Sec. III, we evaluate the effective action corresponding to two atoms, to determine the resulting interaction energy, which is presented in Sec. IV. Analogous results, but for the interaction between an atom and a wall are presented in Sec. V. Finally, in Sec. VI, we present the conclusions.

II. THE SYSTEM

Throughout this paper, we consider either one or two localized two-level systems, at fixed positions, coupled to a vacuum Dirac field. Each two-level system is described by a complex multicomponent Grassmann field: $\theta = (\theta_a)$, with $a = 1, \ldots, n$, and $\bar{\theta} = \theta^\dagger$, in such a way that its free Euclidean action is given by

$$\mathcal{S}_0^a(\theta, \bar{\theta}) = \int d\tau \bar{\theta}(\tau)(\partial_\tau + \Omega)\theta(\tau), \tag{1}$$

where τ denotes the imaginary time and Ω the energy of the excited state, relative to the ground state energy. The number of components of θ is assumed to match that of a Dirac field (in the irreducible representation of the respective Clifford algebra) in $d+1$ space-time dimensions, whose free action is assumed to be

$$\mathcal{S}_0^D(\psi, \bar{\psi}) = \int d^{d+1}x \bar{\psi}(x) \not{\partial} \psi(x), \tag{2}$$

with $\not{\partial} \equiv \gamma_\mu \partial_\mu$ denoting the Dirac operator in $d+1$ dimensions. Dirac γ matrices are Hermitian and assumed satisfy the anticommutation relations,

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \tag{3}$$

where $\mu, \nu = 0, 1, \ldots, d$. They are chosen according to the conventions used in [8], but it should be noted that the results presented in this work are independent of the specific representation of the γ matrices.

We use natural units, whereby $\hbar$ and the velocity corresponding to the dispersion relation for the fermions, v_F, are both equal to 1. Space-time coordinates are denoted by x_μ, and we occasionally use $\tau = x_0$ and $\mathbf{x}$ for the spatial coordinates. The metric is Euclidean: $g_{\mu\nu} = \mathrm{diag}(1, \ldots, 1)$, and no distinctions are meant between upper and lower indices, due their positions to notational convenience. The Einstein summation convention for repeated indices in monomial expressions is also assumed, unless explicitly stated otherwise.

The total action $\mathcal{S}$, in the case of two atoms, at the positions $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$ is then introduced, with the following form:

$$\mathcal{S} = \mathcal{S}_0^a(\theta^{(1)}, \bar{\theta}^{(1)}) + \mathcal{S}_0^a(\theta^{(2)}, \bar{\theta}^{(2)}) + \mathcal{S}^D(\psi, \bar{\psi}; \theta, \bar{\theta}; \mathbf{r}^{(1)}, \mathbf{r}^{(2)}), \tag{4}$$

where $\theta = (\theta^{(1)}, \theta^{(2)})$ denotes the Grassmann degrees of freedom sitting on two spatial positions, and $\mathcal{S}^D$ is the Dirac field action, now including coupling to the two-level systems,

$$\mathcal{S}^D(\psi, \bar{\psi}; \theta, \bar{\theta}; \mathbf{r}^{(1)}, \mathbf{r}^{(2)}) = \mathcal{S}_0^D(\psi, \bar{\psi}) + \lambda \sum_{\alpha=1}^{2} \int d\tau \big(\bar{\psi}(\tau, \mathbf{r}^{(\alpha)}) \theta^{(\alpha)}(\tau) + \bar{\theta}^{(\alpha)}(\tau) \psi(\tau, \mathbf{r}^{(\alpha)}) \big), \tag{5}$$

where the constant λ sets the strength of the coupling between the atoms and the field.

It is worth noting here some features of the proposed system, in the context of the phenomena we want to describe: we first note that the effect of the atoms, when integrating out their degrees of freedom, would be to introduce a kind of frequency dependent mass term, for the Dirac field. That mass goes like the square of the constant λ and is localized on the position (center of mass) of the atom. We recall that the bag model conditions may be obtained by introducing mass terms localized on the boundaries [9]; thus, the model we consider here appears to be an acceptable microscopic version of the EM Casimir effect.

III. EFFECTIVE ACTION

We introduce the effective action Γ_{eff} which results from the functional integration of all the degrees of freedom in the system,

$$e^{-\Gamma_{\mathrm{eff}}(\mathbf{r}^{(1)}, \mathbf{r}^{(2)})} \equiv \frac{1}{\mathcal{N}} \mathcal{Z}(\mathbf{r}^{(1)}, \mathbf{r}^{(2)}),$$
$$\mathcal{Z}(\mathbf{r}^{(1)}, \mathbf{r}^{(2)}) = \int \mathcal{D}\theta \mathcal{D}\bar{\theta} \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-\mathcal{S}(\theta, \bar{\theta}, \psi, \bar{\psi}; \mathbf{r}^{(1)}, \mathbf{r}^{(2)})}. \tag{6}$$

Since the atoms are static, this would be proportional to the vacuum energy and the total evolution time.

The interaction energy of the system, subtracting the self-energy contributions, is obtained by a careful choice of the normalization constant $\mathcal{N}$, namely, such that the interaction energy vanishes in the limit of infinite distance between the atoms,

$$\mathcal{N} \equiv [\mathcal{Z}(\mathbf{r}^{(1)}, \mathbf{r}^{(2)})]_{|\mathbf{r}^{(1)} - \mathbf{r}^{(2)}| \to \infty}. \tag{7}$$

Note that with this in mind, we may discard terms that do not depend on the positions $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$ in the intermediate calculations of the effective action.

Functionally integrating the Dirac field leads us to an intermediate effective action, denoted by $\mathcal{S}_{\mathrm{eff}}$,

$$e^{-\mathcal{S}_{\mathrm{eff}}(\theta, \bar{\theta}; \mathbf{r}^{(1)}, \mathbf{r}^{(2)})} \equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\mathcal{S}(\theta, \bar{\theta}; \psi, \bar{\psi})}$$
$$\mathcal{S}_{\mathrm{eff}}(\theta, \bar{\theta}; \mathbf{r}^{(1)}, \mathbf{r}^{(2)}) = \mathcal{S}_0^a(\theta^{(1)}, \bar{\theta}^{(1)}) + \mathcal{S}_0^a(\theta^{(2)}, \bar{\theta}^{(2)}) - \int_{x,x'} \bar{\eta}(x) S_F^{(0)}(x - x') \eta(x'), \tag{8}$$

where $S_F^{(0)}(x-x')$ is the free propagator of the Dirac field, and we have introduced

$$\eta(x) \equiv \lambda\big(\delta^d(\mathbf{x}-\mathbf{r}^{(1)})\theta^{(1)}(\tau)+\delta^d(\mathbf{x}-\mathbf{r}^{(2)})\theta^{(2)}(\tau)\big), \tag{9}$$

and its Dirac adjoint.

Substituting (9) into Eq. (8), we observe that the intermediate effective action contains a diagonal part in the atomic fields, $\mathcal{S}_{\text{eff}}^{\text{d}}$, and a nondiagonal part, $\mathcal{S}_{\text{eff}}^{\text{nd}}$, in which the fields are mixed,

$$\mathcal{S}_{\text{eff}} = \mathcal{S}_{\text{eff}}^{\text{d}} + \mathcal{S}_{\text{eff}}^{\text{nd}}. \tag{10}$$

The diagonal part takes the form,

$$\mathcal{S}_{\text{eff}}^{\text{d}} = \sum_{\alpha}\int_{\tau}\bar{\theta}^{(\alpha)}(\tau)(\partial_\tau+\Omega)\theta^{(\alpha)}(\tau)+[\mathcal{S}_{\text{eff}}^{\text{d}}]_{\text{div}}. \tag{11}$$

The first term in this expression represents the sum of the free actions of the atoms. The second term, arising from their self-energies, is a divergent contribution that leads to a renormalization of the frequency Ω. Since this divergent term is independent of the positions $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$, it can be discarded following a procedure analogous to that described in [10].

The nondiagonal part, which describes the coupling between the fluctuations of the two atoms, is

$$\mathcal{S}_{\text{eff}}^{\text{nd}} = -\lambda^2\sum_{\alpha\neq\beta}\int_{\tau,\tau'}\bar{\theta}^{(\alpha)}(\tau)S_F^{(0)}(\tau-\tau',\mathbf{r}^{(\alpha)}-\mathbf{r}^{(\beta)})\theta^{(\beta)}(\tau'). \tag{12}$$

By combining the diagonal and off-diagonal terms and performing a Fourier transform to the frequency domain, we have

$$\mathcal{S}_{\text{eff}} = \sum_{\alpha,\beta}\int\frac{d\nu}{2\pi}\tilde{\bar{\theta}}^{(\alpha)}(\nu)M^{(\alpha\beta)}(\nu,\mathbf{r})\tilde{\theta}^{(\beta)}(\nu), \tag{13}$$

where the matrix $M^{(\alpha\beta)}(\nu,\mathbf{r})$ is defined as

$$M^{(\alpha\beta)}(\nu,\mathbf{r}) \equiv \delta^{(\alpha\beta)}\tilde{\Delta}^{-1}(\nu,\Omega)-\lambda^2\sigma^{(\alpha\beta)}\tilde{S}_F^{(0)}(\nu;\mathbf{r}^{(\alpha)}-\mathbf{r}^{(\beta)}), \tag{14}$$

with $\sigma^{(\alpha\beta)} \equiv \delta^{(\alpha1)}\delta^{(\beta2)}+\delta^{(\alpha2)}\delta^{(\beta1)}$, $\mathbf{r}\equiv\mathbf{r}^{(1)}-\mathbf{r}^{(2)}$, and

$$\tilde{\Delta}(\nu,\Omega) \equiv \frac{i\nu+\Omega}{\nu^2+\Omega^2}. \tag{15}$$

Finally, by integrating out the atomic fields, we obtain the full effective action for the system,

$$e^{-\Gamma_{\text{eff}}(\mathbf{r}^{(1)},\mathbf{r}^{(2)})} = \det[M^{(\alpha\beta)}(\nu,\mathbf{r})], \tag{16}$$

where the determinant must be taken over frequencies, atomic indices, and Dirac matrix indices.

IV. INTERACTION ENERGY

Having obtained the effective action of the system, we can now compute the interaction energy $E_I(\mathbf{r})$ between the atoms. We find interaction energy to be

$$E_I(\mathbf{r}) = \lim_{T\to\infty}\frac{\Gamma_{\text{eff}}(\mathbf{r})}{T} = -\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\log\det[\mathbb{1}-\mathbb{T}(\nu,\mathbf{r})], \tag{17}$$

where T is the Euclidean time extent, and the matrix $\mathbb{T}$ encodes the correlations between the quantum fluctuations of the atoms, mediated by the Dirac field. This matrix is defined as

$$\mathbb{T}(\nu,\mathbf{r}) \equiv \lambda^4\tilde{\Delta}(\nu,\Omega)\tilde{S}_F^{(0)}(\nu;\mathbf{r})\tilde{\Delta}(\nu,\Omega)\tilde{S}_F^{(0)}(\nu;-\mathbf{r}). \tag{18}$$

The free fermion propagator $\tilde{S}_F^{(0)}(\nu;\mathbf{r})$, appearing in the expression for $\mathbb{T}$, takes the general form,

$$\tilde{S}_F^{(0)}(\nu;\mathbf{r}) = a(\nu,r)i\gamma_0+\frac{1}{r}b(\nu,r)r_i\gamma_i, \tag{19}$$

valid for any dimension d. The functions $a(\nu,r)$ and $b(\nu,r)$ encapsulate the spatial dimensional dependence, with their explicit forms provided in Appendix A.

Substituting the propagator into the determinant of (17), the resulting expression is

$$\begin{aligned} d(\nu,r) &\equiv \det\left[\mathbb{1}-\mathbb{T}(\nu,\mathbf{r})\right] \\ &= \big[1+2\lambda^4\tilde{\Delta}^2(\nu,\Omega)\big(a^2(\nu,r)+b^2(\nu,r)\big) \\ &\quad +\lambda^8\tilde{\Delta}^4(\nu,\Omega)(a^2(\nu,r)-b^2(\nu,r))^2\big]^{\lfloor\frac{d+1}{2}\rfloor}, \end{aligned} \tag{20}$$

where $\lfloor\cdot\rfloor$ denotes the floor function.

Since $d^*(\nu,r)=d(-\nu,r)$, the interaction energy simplifies to

$$E_I(r) = -\int_0^{\infty}\frac{d\nu}{2\pi}\log(|d(\nu,r)|^2), \tag{21}$$

which is clearly real, indicating the stability of the two-atom system. In contrast, in the model considered in [10], where the atoms have internal structure, an imaginary component appears in the interaction energy, signaling the instability of the system. However, in the present case, since the atoms are modeled as pointlike, such instability cannot arise.

To simplify the expression further, we perform a change of variables, $\nu=\frac{u}{r}$, and introduce the dimensionless parameter $x\equiv\Omega r$, which denotes the interatomic distance in units of the atomic transition wavelength $\sim\Omega^{-1}$. Also, we rescale the coupling as $\lambda=\tilde{\lambda}\Omega^{1-d/2}$, thereby providing a

dimensionless description of the interaction strength. The interaction energy can then be expressed as $E_I(r) = \Omega\mathcal{E}_I(\Omega r)$, where the dimensionless interaction energy $\mathcal{E}_I(x)$ is given by

$$\mathcal{E}_I(x) = -\frac{1}{x}\int_0^\infty \frac{du}{2\pi}\log(|D(u,x)|^2), \tag{22}$$

where

$$D(u,x) \equiv \left[1 + 2\tilde{\lambda}^4 x^{2(2-d)}\tilde{\Delta}^2(u,x)\left(a^2(u,1)+b^2(u,1)\right) + 4\tilde{\lambda}^8 x^{4(2-d)}\tilde{\Delta}^4(u,x)\left(a^2(u,1)-b^2(u,1)\right)^2\right]^{\lfloor\frac{d+1}{2}\rfloor}. \tag{23}$$

We analyze this expression in detail under various assumptions regarding the system's parameters.

A. Weak coupling limit

To further understand the behavior of the system, we analyze the interaction energy under the weak coupling limit. This regime simplifies the equations, offering insight into both short- and long-distance behaviors.

In the weak coupling regime, characterized by $\tilde{\lambda}^4 \ll 1$, the interaction energy simplifies to a logarithmic form, $\log(1+z)$, where the terms contributing to z are proportional to powers of $\tilde{\lambda}$.

At large distances ($x \gg 1$), the interaction energy becomes exponentially suppressed, indicating a weakening of the interaction as the separation between the atoms increases. In the short-distance limit ($x \to 0$), the behavior of the interaction varies with the spatial dimension d. For $d = 1$ and $d = 2$, the argument of the logarithm remains finite as $x \to 0$, allowing for a straightforward expansion in powers of $\tilde{\lambda}^4$. However, in $d = 3$, the argument of the logarithm develops divergences as $x \to 0$, requiring the additional condition $\tilde{\lambda}^4/x^2 \ll 1$ to ensure the validity of the expansion and to avoid such divergences.

In this regime, the logarithmic term $\log(1+z)$ can be approximated by z when $|z| \ll 1$, further simplifying the interaction energy. Applying this approximation, the interaction energy reduces to

$$\mathcal{E}_I(x) \sim -\tilde{\lambda}^4 x^{3-2d} 2^{1+\lfloor\frac{d+1}{2}\rfloor} \times \int_{-\infty}^\infty \frac{du}{2\pi}\frac{(x^2-u^2)(a^2(u,1)+b^2(u,1))}{(u^2+x^2)^2}. \tag{24}$$

Table I presents the explicit results of the integral for different spatial dimensions d, illustrating how the dimensionless interaction energy $\mathcal{E}_I(x)$ depends on the interatomic distance x in each case.

Based on the results and the behavior of the functions presented in Table I, we conclude that the interaction energy is negative for all spatial dimensions d and at any distance x. Moreover, the magnitude of the energy decreases monotonically as the distance x increases, indicating that the force between the atoms is consistently attractive across all regimes considered. This attractive force is stronger at short distances and gradually weakens as the atoms move farther apart, in line with the expected behavior in the weak coupling regime.

TABLE I. Dimensionless interaction energy $\mathcal{E}_I(x)$ between the two fermionic atoms in the weak coupling regime, as a function of the dimensionless distance x and for different spatial dimensions d. In this context, we use the auxiliary function $g(z) \equiv -\mathrm{Ci}(z)\cos(z) - \mathrm{si}(z)\sin(z)$, where $\mathrm{si}(z) \equiv \mathrm{Si}(z) - \frac{\pi}{2}$, with $\mathrm{Ci}(z)$ and $\mathrm{Si}(z)$ being the cosine and sine integrals, respectively [11]. Additionally, G corresponds to the Meijer G-function [12,13].

d	$\mathcal{E}_I(x)$
1	$-\frac{\tilde{\lambda}^4}{\pi} 2xg(2x)$
2	$-\frac{\tilde{\lambda}^4}{2\pi x}\frac{\sqrt{\pi}}{4x\pi^2}\left\{x^2\left[G^{2,0}_{0,3}\left(x^2\middle|\begin{matrix}-\frac{1}{2}\\0,0,0\end{matrix}\right) + x^2 G^{2,0}_{1,4}\left(x^2\middle|\begin{matrix}-\frac{3}{2},\frac{1}{2}\\-\frac{1}{2},0,0,0\end{matrix}\right)\right] - \left[G^{2,0}_{0,3}\left(x^2\middle|\begin{matrix}-\frac{1}{2}\\-1,0,1\end{matrix}\right) + x^2 G^{2,0}_{1,4}\left(x^2\middle|\begin{matrix}-\frac{3}{2},\frac{1}{2}\\-1,-\frac{1}{2},0,1\end{matrix}\right)\right]\right\}$
3	$-\frac{\tilde{\lambda}^4}{4\pi^3 x^3}(1-(2x)^2 g(2x))$

We now proceed to analyze the behavior of the interaction energy in both the long-distance and short-distance limits, providing insights into how the energy scales as a function of the interatomic separation in these two regimes.

1. Long-distance limit ($x \gg 1$)

When the separation between the atoms is much larger than the characteristic wavelength of atomic transitions, $\sim\Omega^{-1}$, i.e., $r \gg \Omega^{-1}$, retardation effects become significant due to the finite propagation speed of the interaction. As the distance between the atoms increases, the interaction energy decays more rapidly, with the asymptotic behavior dominated by the leading-order terms in $1/x$.

To capture this behavior, we analyze the asymptotic form of the functions appearing in the expression for the interaction energy, as summarized in Table I. The results for various spatial dimensions are shown in Table II, which highlights the power-law dependence of the interaction energy at large distances.

2. Short-distance limit ($x \ll 1$)

At very short distances, $r \ll \Omega^{-1}$, the separation between the atoms is much smaller than the characteristic wavelength of atomic transitions, and retardation effects become negligible. In this regime, known as the *London limit* [14], the interaction can be treated as essentially instantaneous.

TABLE II. Dimensionless interaction energy $\mathcal{E}_I(x)$ between the fermionic atoms in the weak coupling regime, as a function of the dimensionless distance x, for different spatial dimensions d, in the short-distance ($x \ll 1$) and long-distance ($x \gg 1$) limits.

	$\mathcal{E}_I(x)$	
d	$x \gg 1$	$x \ll 1$
1	$-\frac{1}{2\pi}\frac{\tilde{\lambda}^4}{x}$	$-\frac{1}{4}\tilde{\lambda}^4$
2	$-\frac{1}{16\pi}\frac{\tilde{\lambda}^4}{x^3}$	$-\frac{1}{8\pi^2}\frac{\tilde{\lambda}^4}{x^2}$
3	$-\frac{24}{(4\pi)^3}\frac{\tilde{\lambda}^4}{x^5}$	$-\frac{1}{(4\pi)^2}\frac{\tilde{\lambda}^4}{x^4}$

To evaluate the interaction energy, we approximate $x \simeq 0$ in the numerator of the integrand in (24), which is valid due to the small atomic separation.

The results are summarized in Table II, where the power-law dependence of the interaction energy at short distances shows a slower decay compared to the long-distance limit.

V. ATOM-MIRROR INTERACTION

Let us study here the interaction between an atom and a perfectly reflecting boundary, modeled using bag model boundary conditions. The wall is located at $x_d = 0$, what is obtained by equipping the Dirac field ψ with the action

$$\mathcal{S}_0^D(\psi,\bar{\psi}) = \int d^{d+1}x\bar{\psi}(x)\big(\not{\partial} + V(x)\big)\psi(x). \quad (25)$$

Here, the presence of $V(x) = g\delta(x_d)$ enforces bag model boundary conditions when $g = 2$. For any other value of g, the boundary conditions become imperfect, leading to partial transmission of the fermionic current through the wall [9,15].

The interaction between the two-level system and the Dirac field occurs at a fixed position $\mathbf{r} = (0,\ldots,a) \in \mathbb{R}^d$, with a being the separation between the atom and the reflecting surface. This interaction is described by

$$\mathcal{S}_I = \lambda \int d\tau\big(\bar{\psi}(\tau,\mathbf{r})\theta(\tau) + \bar{\theta}(\tau)\psi(\tau,\mathbf{r})\big), \quad (26)$$

where λ determines the strength of the coupling between the atom and the fermionic field.

The total action for the system, which includes the two-level atom, the Dirac field with the boundary condition, and their interaction, is given by

$$\mathcal{S} = \mathcal{S}_0^a(\theta,\bar{\theta}) + \mathcal{S}_0^D(\psi,\bar{\psi}) + \mathcal{S}_I(\theta,\bar{\theta},\psi,\bar{\psi};a). \quad (27)$$

Now, we examine the effective description of the system alongside the interaction energy.

A. Effective action and interaction energy

To compute the interaction energy between the atom and the mirror, we first derive the effective action by integrating out the Dirac field. The effective action depends on the distance a between the atom and the mirror and is given by

$$e^{-\Gamma_{\text{eff}}(a)} \equiv \frac{1}{\mathcal{N}}\mathcal{Z}(a), \quad (28)$$

$$\mathcal{Z}(a) = \int \mathcal{D}\theta\mathcal{D}\bar{\theta}\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\mathcal{S}(\theta,\bar{\theta},\psi,\bar{\psi};a)}. \quad (29)$$

By performing the functional integral over the Dirac field, we obtain the intermediate effective action,

$$\mathcal{S}_{\text{eff}}(\theta,\bar{\theta};a) = \mathcal{S}_0^a(\theta,\bar{\theta}) - \lambda^2 \lim_{\mathbf{x}'_\parallel \to \mathbf{x}_\parallel} \int_{\tau,\tau'} \bar{\theta}(\tau) S_F(\tau-\tau', \mathbf{x}_\parallel - \mathbf{x}'_\parallel; a, a)\theta(\tau'), \quad (30)$$

where $S_F(\tau-\tau', \mathbf{x}_\parallel - \mathbf{x}'_\parallel; x_d, x'_d)$ is the propagator of the Dirac field in the presence of the mirror. This propagator can be decomposed as

$$S_F(x;x') = S_F^{(0)}(x-x') + T_F(x;x'), \quad (31)$$

where $S_F^{(0)}(x-x')$ is the free propagator, and $T_F(x;x')$ accounts for the mirror's contribution. A detailed derivation of this propagator can be found in Appendix B.

As in the atom-atom interaction case, the free propagator $S_F^{(0)}(x-x')$ introduces a divergent term related to the atom's self-energy. This divergence, which occurs when $\mathbf{x}'_\parallel \to \mathbf{x}_\parallel$, leads to a renormalization of the energy Ω, and since it is independent of the atom's position relative to the mirror, we discard it using the same procedure as before.

The relevant term for computing the interaction energy is the mirror-dependent contribution $T_F(\tau-\tau', \mathbf{0}_\parallel; a, a)$, which explicitly depends on the distance a and takes the form

$$T_F(\tau-\tau', \mathbf{0}_\parallel; a, a) = S_F^{(0)}(\tau-\tau', \mathbf{0}_\parallel, 2a)\gamma_d. \quad (32)$$

To simplify further calculations, we transform the effective action to the frequency domain via a Fourier transform, resulting in

$$\mathcal{S}_{\text{eff}}(\tilde{\theta},\tilde{\bar{\theta}};a) = \int \frac{d\nu}{2\pi}\tilde{\bar{\theta}}(\nu)(\tilde{\Delta}^{-1}(\nu,\Omega) - \lambda^2\tilde{T}_F(\nu;\mathbf{0}_\parallel;a,a))\tilde{\theta}(\nu), \quad (33)$$

where $\tilde{T}_F(\nu;\mathbf{0}_\parallel;a,a)$ is the Fourier transform of the mirror's contribution.

Finally, we obtain the full effective action by integrating over the atomic degrees of freedom,

$$e^{-\Gamma_{\rm eff}(a)} = \frac{1}{\mathcal{N}} \int \mathcal{D}\tilde{\theta}\mathcal{D}\tilde{\bar{\theta}} e^{-\mathcal{S}_{\rm eff}(\tilde{\theta},\tilde{\bar{\theta}};a)} = \det[\mathbb{1} - \mathbb{T}(\nu,a)], \quad (34)$$

where the matrix $\mathbb{T}(\nu,a)$ is defined as

$$\mathbb{T}(\nu,a) \equiv \lambda^2 \tilde{\Delta}(\nu,\Omega)\tilde{S}_F^{(0)}(\nu;\mathbf{0}_\parallel,2a)\gamma_d. \quad (35)$$

The interaction energy can then be expressed similarly to Eq. (16), but with the matrix $\mathbb{T}$ specific to this scenario. Using dimensionless quantities, we obtain a result analogous to Eq. (22), with the function $D(u,x)$ defined as

$$D(u,x) \equiv \big[1 - 2\tilde{\lambda}^2 x^{2-d}\tilde{\Delta}(u,x)b(u,2) - \tilde{\lambda}^4 x^{2(2-d)}\tilde{\Delta}^2(u,x)\big(a^2(u,2) - b^2(u,2)\big)^2\big]^{\lfloor\frac{d+1}{2}\rfloor}. \quad (36)$$

The interaction energy is, once again, real for all distances x, confirming the stability of the system across the entire range of distances and coupling strengths.

B. Weak coupling limit

In the weak coupling regime, the analysis is analogous to the case of the two-atom interaction discussed in Sec. IV. In this case, the expression for the interaction energy is

$$\mathcal{E}_I(x) \sim 2^{1+\lfloor\frac{d+1}{2}\rfloor}\tilde{\lambda}^2 x^{2-d}\int_{-\infty}^{\infty}\frac{du}{2\pi}\frac{b(u,2)}{u^2+x^2}. \quad (37)$$

From this result, we observe that, unlike the two-atom case, the interaction energy is positive. Since the energy monotonically decreases with distance x, this indicates the presence of a repulsive force between the atom and the conducting plane.

In Table III, we present the expressions obtained for the interaction energy in the weak coupling limit, and in Table IV, we show the results in the long- and short-distance limits.

TABLE III. Dimensionless interaction energy $\mathcal{E}_I(x)$ between the plane and the atom in the weak coupling regime, as a function of the dimensionless distance x and for different spatial dimensions d. In this context, we use the auxiliary function $f(z) \equiv \mathrm{Ci}(z)\sin(z) - \mathrm{si}(z)\cos(z)$, along with the previously defined function $g(z)$. Additionally, $Y_\nu(z)$ corresponds to the Bessel function of the second kind, and $\mathbf{H}_\nu(z)$ is the Struve function [11].

d	$\mathcal{E}_I(x)$
1	$\frac{1}{\pi}\tilde{\lambda}^2 f(2x)$
2	$\frac{1}{4}\tilde{\lambda}^2[Y_{-1}(2x) - \mathbf{H}_{-1}(2x)]$
3	$\frac{1}{(2\pi x)^2}\tilde{\lambda}^2[f(2x) + 2xg(2x)]$

TABLE IV. Dimensionless interaction energy $\mathcal{E}_I(x)$ between the fermionic atom and the perfectly reflecting plane in the weak coupling regime, as a function of the dimensionless distance x, for different spatial dimensions d, in the short-distance ($x \ll 1$) and long-distance ($x \gg 1$) limits.

	$\mathcal{E}_I(x)$	
d	$x \gg 1$	$x \ll 1$
1	$\frac{1}{2\pi}\frac{\tilde{\lambda}^2}{x}$	$\frac{1}{2}\tilde{\lambda}^2$
2	$\frac{1}{8\pi}\frac{\tilde{\lambda}^2}{x^2}$	$\frac{1}{4\pi}\frac{\tilde{\lambda}^2}{x}$
3	$\frac{1}{(2\pi)^2}\frac{\tilde{\lambda}^2}{x^3}$	$\frac{1}{4\pi}\frac{\tilde{\lambda}^2}{x^2}$

VI. CONCLUSIONS

In this work, we have introduced and studied the fermionic counterparts of the van der Waals and Casimir-Polder interactions in two distinct setups. First, we analyzed the interaction energy between two fermionic two-level atoms mediated by a massless fermionic field, deriving integral expressions for the interaction energy in 1, 2, and 3 spatial dimensions. Our results show that the interaction is universally attractive, becoming stronger at short distances and decaying more rapidly at large distances, in line with the expected power-law behavior.

We then examined the interaction between a fermionic two-level atom and a perfectly reflecting fermionic mirror, modeled using bag boundary conditions. In contrast to the two-atom case, this interaction is repulsive. We again derived integral expressions for the interaction energy in various dimensions, observing that the repulsive force dominates at short distances and weakens as the separation increases, following a characteristic decay.

It is at this point worth pointing out that the validity of the approximations, whereby one regards a material medium as a plane where bag boundary conditions are imposed, is strongly dependent on the kind of (microscopic) system involved. First, the distance between the atoms in the microscopic model should be much shorter than the distance between the two bodies, since the latter determines the wavelength of the most relevant modes of the quantum field. Therefore, the density of atoms has a twofold relevance in this context: on the one hand, it should be larger than a certain scale to justify a continuous description. On the other, the boundary condition itself is also sensitive to the value of the density (as it is the case, for instance, for conducting media in the electromagnetic case).

Therefore, since the existence of the repulsive force in our model appears when a strong boundary condition is imposed, the possibility of having a transition between a

nonrepulsive and a repulsive phase is to be expected, for instance, as a function of the density, or any other relevant parameters, in the medium constituting the wall.

We conclude by mentioning that our findings underscore the relevance of fermionic field vacuum fluctuations in quantum interactions. These could pave the way for exploring analogous phenomena in experimental setups, such as in graphene or other condensed matter systems.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: FREE DIRAC PROPAGATOR

Here, we present the Euclidean propagator for a free Dirac field. This propagator satisfies the following differential equation:

$$(\not{\partial})S_F^{(0)}(x;x') = \delta^{d+1}(x-x'). \tag{A1}$$

Owing to the invariance of the system under space-time translations, the propagator can be expressed in terms of its Fourier transform,

$$S_F^{(0)}(x-x') = \int \frac{d\nu}{2\pi}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\tilde{S}_F^{(0)}(\nu;\mathbf{r}), \tag{A2}$$

where, in an arbitrary spatial dimension d, $\tilde{S}_F^{(0)}$ has the form

$$\tilde{S}_F^{(0)}(\nu;\mathbf{r}) = a(\nu,r)i\gamma_0 + \frac{1}{r}b(\nu,r)r_i\gamma_i, \tag{A3}$$

with the functions $a(\nu,r)$ and $b(\nu,r)$ depending explicitly on d. The explicit forms of these functions for $d=1,2,3$ are provided in Table V.

TABLE V. Functions $a(\nu,r)$ and $b(\nu,r)$ for different spatial dimensions d. Here, K_ν denotes the modified Bessel function of the second kind.

d	$a(\nu,r)$	$b(\nu,r)$
1	$\frac{1}{2}\mathrm{sign}(\nu)e^{-r\lvert\nu\rvert}$	$\frac{1}{2}e^{-r\lvert\nu\rvert}$
2	$\frac{\nu}{2}K_0(r\lvert\nu\rvert)$	$\frac{\nu}{2}K_1(r\lvert\nu\rvert)$
3	$\frac{\nu}{4\pi r}e^{-r\lvert\nu\rvert}$	$\frac{1+r\lvert\nu\rvert}{4\pi r^2}e^{-r\lvert\nu\rvert}$

APPENDIX B: DIRAC PROPAGATOR IN THE PRESENCE OF A WALL

In this appendix, we derive the Euclidean Dirac propagator in the presence of a wall located at $x_d=0$. With this setup, the propagator satisfies the following differential equation:

$$\big(\not{\partial}+g\delta(x_d)\big)S_F(x;x') = \delta^{d+1}(x-x'). \tag{B1}$$

Given that the system is time invariant and translation invariant along the directions parallel to the wall, $\mathbf{x}_\parallel$, the propagator can be expressed in terms of its Fourier transform along those coordinates,

$$\begin{aligned}S_F(x;x') &= S_F(\tau-\tau',\mathbf{x}_\parallel-\mathbf{x}'_\parallel;x_d,x'_d)\\ &= \int\frac{d\nu}{2\pi}\int\frac{d^{d-1}\mathbf{k}_\parallel}{(2\pi)^{d-1}}e^{-i\nu(\tau-\tau')-i\mathbf{k}_\parallel(\mathbf{x}_\parallel-\mathbf{x}'_\parallel)}\\ &\quad\times\tilde{S}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d).\end{aligned} \tag{B2}$$

In Fourier space, the propagator satisfies

$$\big(-i\not{\nu}-i\not{\mathbf{k}}_\parallel+\not{\partial}_{x_d}+g\delta(x^d)\big)\tilde{S}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d)=\delta(x_d-x'_d). \tag{B3}$$

To solve this equation, we apply the free propagator $\tilde{S}_F^{(0)}$ to both sides from the left,

$$\begin{aligned}\tilde{S}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d) &= \tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;x_d,x'_d)\\ &\quad-g\tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;x_d,0)\tilde{S}_F(\nu,\mathbf{k}_\parallel;0,x'_d).\end{aligned} \tag{B4}$$

We observe that (B4) is an implicit equation for $\tilde{S}_F$, as it depends on $\tilde{S}_F(\nu,\mathbf{k}_\parallel;0,x'_d)$. By evaluating at $x_d=0$, we can explicitly solve for $\tilde{S}_F(\nu,\mathbf{k}_\parallel;0,x'_d)$, yielding

$$\tilde{S}_F(\nu,\mathbf{k}_\parallel;0,x'_d) = \big(\mathbb{1}+g\tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;0,0)\big)^{-1}\tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;0,x'_d). \tag{B5}$$

Substituting this result into (B4), we arrive at

$$\tilde{S}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d) = \tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;x_d,x'_d)+\tilde{T}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d), \tag{B6}$$

$$\begin{aligned}\tilde{T}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d) &= -g\tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;x_d,0)\\ &\quad\times[\mathbb{1}+g\tilde{S}_F(\nu,\mathbf{k}_\parallel;0,0)]^{-1}\\ &\quad\times\tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;0,x'_d),\end{aligned} \tag{B7}$$

where the inverse of the term in brackets is given by

$$[\mathbb{1}+g\tilde{S}_F(\nu,\mathbf{k}_\parallel;0,0)]^{-1}=\frac{1}{1+(\frac{g}{2})^2}\left[\mathbb{1}-i\frac{g}{2}\frac{\not{\nu}+\not{\mathbf{k}}_\parallel}{\sqrt{\nu^2+\mathbf{k}_\parallel^2}}\right]. \tag{B8}$$

Finally, for $x_d, x'_d > 0$, which corresponds to the region where the atom is located, we find

$$\tilde{T}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d)=\frac{g}{1+(\frac{g}{2})^2}\tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;x_d,-x'_d)\gamma_d. \tag{B9}$$

In the case where the mirror perfectly reflects the normal component of the fermionic current, corresponding to setting $g=2$, the expression simplifies to

$$\tilde{T}_F(\nu,\mathbf{k}_\parallel;x_d,x'_d)=\tilde{S}_F^{(0)}(\nu,\mathbf{k}_\parallel;x_d,-x'_d)\gamma_d. \tag{B10}$$

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