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U-CYCLES IN n-PERSON TU-GAMES WITH ONLY 1, $n-1$ AND n-PERSON PERMISSIBLE COALITIONS

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It has been recently proved that the non-existence of certain type of cycles of pre-imputation, fundamental cycles, is equivalent to the balancedness of a TU-games (Cesco (2003)). In some cases, the class of fundamental cycles can be narrowed and still obtain a characterization theorem. In this paper we prove that existence of maximal U-cycles, which are related to a transfer scheme designed for computing a point in the core of a game, is condition necessary and sufficient for a TU-game be non-balanced, provided $n-1$ and n-person are the only coalitions with non-zero value. These games are strongly related to games with only 1, $n-1$ and n-person permissible coalitions (Maschler (1963)).

Keywords: Non balanced games; cycles; transfer schemes.

1. Introduction

The Shapley-Bondareva's theorems (Bondareva (1963), Shapley (1967)) is the most well known characterization of games with transferable utility with non-empty core (balanced games). Recently, we proved an alternative equivalence result (Cesco (2003)). While Shapley-Bondareva's results belongs to the subject of duality theory, ours rests on very different ground. The key is the notion of cycle of pre-imputations, which is a finite sequence of pre-imputations, where each pair of neighbouring elements are interrelated to each other through a transfer of some amount of utility from members of a certain coalition to the members of the complementary coalition. This transfer is carried out with the understanding that individual gains or losses within any coalition are proportional to the number of members in the coalition.

Transfers of utility is a tool that has been extensively used to design bargaining dynamic processes, consistent with the standards of rational behavior, leading from an unacceptable payoff to an acceptable one (Billera and Wu (1977), Justman

(1977), Stearns (1968),Wu (1977)). In Cesco (1998) we introduce another transfer scheme converging to the core of a TU-game, provided this set is non-empty, which is, however, closely related to those in Wu (1977). A computational algorithm is derived which generates sequences of pre-imputations (maximal U-sequences) which converge to the core in balanced games. The cycles we study in this paper (maximal U-cycles) are closely related to that transfer scheme. The transfers of utility are maximal in some sense (cf. Sec. 3). Each time we run the algorithm on non-balanced games, we observed the numerical appearance of cycles of pre-imputations. The main objective of this paper is to prove that this fact characterizes the sub-class of monotonic n-person TU-games having 1, $n-1$ and n-person coalitions as the only permissible coalitions in the game (Maschler (1963)). An advantage of maximal U-cycles over the fundamental cycles we used in the full characterization of balanced games, is that there is an algorithm to approximate them. However we do not know yet if a general characterization can be obtained in terms of them. The results obtained in the subclass of games studied in this paper would provide some insight to get similar results for more general classes of games.

The paper is organized as follows. Preliminaries and some notation are set forth in the next section. In Sec. 3 we define cycles of pre-imputation and close it by showing that the existence of a certain class of cycles (U-cycles) in a TU-game implies the non-balancedness of it (Corollary 4). In Sec. 4 we prove a key auxiliary result which will be used latter to start U-cycles. In the last section we characterize balanced TU-games having $n-1$ and n-person coalitions as the only coalitions with non-zero values, as those for which do not exit a maximal U-cycles.

2. Preliminaries

A TU-game is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ represents the set of players and v the characteristic function. We assume that v is a real valued function defined on the family of subsets of $N, \mathcal{P}(N)$ satisfying $v(N) = 1$ and $v(\{i\}) = 0$ for each $i \in N$. The elements in $\mathcal{P}(N)$ are called coalitions.

The set of pre-imputations is defined by $E = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in N} x_i = 1\right\}$ and the set of imputations by $A = \{x \in E : x_i \geq 0 \text{ for all } i \in N\}$.

Given a coalition $S \in \mathcal{P}(N)$ and a pre-imputation x, the excess of the coalition S with respect to x is defined by $e(S, x) = v(S) - x(S)$,where $x(S) = \sum_{i \in S} x_i$ if $S \neq \Phi$ and 0 otherwise. The excess of a coalition S represents the aggregate gain (or loss, if negative) to its members if they depart from an agreement that yields x in order to form their own coalition. The core of a game (N, v) is defined by $C = \{x \in E : e(S, x) \leq 0 \text{ for all } S \in P(N)\}$.

The core of a game may be an empty set. The Shapley-Bondareva's theorem characterizes the sub class of TU-games with non-empty core. A central role is played by balanced families of coalitions. A family of non-empty coalitions $\mathcal{B} \subseteq \mathcal{P}(N)$ is called a *balanced* if there exists a set of positive real numbers $(\lambda_S)_{S \in \mathcal{B}}$ satisfying $\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \lambda_S = 1$, for all $i \in N$. The numbers $(\lambda_S)_{S \in \mathcal{B}}$ are called the balancing weights for $\mathcal{B}$. $\mathcal{B}$ is minimal balanced if there is no proper balanced subfamily of it. In this case, the set of balanced weights is unique. Equivalently, if $\chi_S \in \mathbb{R}^n$ denotes the characteristic vector defined by $(\chi_S)_i = 1$ if $i \in S$ and 0 if $i \in N \backslash S$, the family $\mathcal{B}$ is balanced if there exists a family of positive balancing weights $(\lambda_S)_{S\in\mathcal{B}}$, such that

$$\sum_{S\in\mathcal{B}} \lambda_S.\chi_S = \chi_N. \tag{2.1}$$

A well-known result establishes that

$$\sum_{S\in\mathcal{B}} \lambda_S.x(S) = x(N) \tag{2.2}$$

for all balanced family of coalitions. A game (N, v) is called balanced if

$$\sum_{S\in\mathcal{B}} \lambda_S.v(S) \leq v(N) \tag{2.3}$$

for all balanced family $\mathcal{B}$ with balancing weights $(\lambda_S)_{S\in\mathcal{B}}$. The Shapley-Bondareva's theorem states that a game (N, v) has non-empty core if and only if it is balanced. An *objectionable* family is a balanced family not satisfying (2.3).

In what follows, the notion of U-transfer will play a central role. Given $x \in E$ and a proper coalition S, the *U-transfer* from $N\backslash S$ to S with respect to x is the pre-imputation y defined by

$$y = x + e(S, x).\beta_S \tag{2.4}$$

Here $\beta_S = \frac{\chi_S}{|S|} - \frac{\chi_{N\backslash S}}{|N\backslash S|}$ if S is a proper coalition and the zero vector of $\mathbb{R}^n$ otherwise. $|S|$ indicates the number of players in S. The vector β_S describes a transfer of one unit of utility from the members of $N\backslash S$ to the members of S. The U-transfer called maximal if $e(S, x) \geq e(T, x)$ for all $T \in \mathcal{P}(N)$.

Remark 1. Vectors χ_S and β_S are strongly related. If we define the subspace $M = \left\{x \in \mathbb{R}^n : \sum_{i\in N} x_i = 0\right\}$, then, for any coalition S, the orthonormal projection of χ_S onto M, $P_M(\chi_S)$ is given by (Cesco (2003), Proposition 5)

$$P_M(\chi_S) = \frac{|S|.|N\backslash S|}{n}\beta_S$$

Remark 2. If $\mathcal{B}$ be a family of non-empty coalitions of N, and $\beta_{\mathcal{B}}$ the matrix having as columns the vectors $\beta_S, S \in \mathcal{B}$, then $\mathcal{B}$ is balanced if and only if the linear system $\beta_{\mathcal{B}}y = 0$ has a positive solution $y = (y_S)_{S\in\mathcal{B}}$ ($y_S > 0$ for all $S \in \mathcal{B}$. (Cesco (2003, Theorem 6))).

3. Cycles of Pre-Imputations

In this section we introduce two kind of cycles of pre-imputations and state, without proof several, results proved in Cesco and Aguirre (2002), Cesco (2003).

Definition 1. A *cycle* $\mathbf{c}$ in a TU-game (N, v) is a finite sequence of pre-imputations $(x^k)_{k=1}^m, m > 1$, such that there exist associated sequences of positive real numbers $(\mu_k)_{k=1}^m$ and $(S_k)_{k=1}^m$ of non-empty, proper coalitions of N (not necessarily all different) satisfying the neighbouring transfer properties

$$x^{k+1} = x^k + \mu_k.\beta_{S_k} \quad \text{for all } k = 1, \ldots, m \tag{3.1}$$

and

$$x^{m+1} = x^1 \tag{3.2}$$

as well.
A cycle is called *fundamental* if $\mu_k \leq e(S_k, x^k)$ for all $k = 1, \ldots, m$.
A cycle is called a *U-cycle* if $\mu_k = e(S_k, x^k)$ for all $k = 1, \ldots, m$.
A U-cycle is called *maximal* if for all $k = 1, \ldots, m, e(S_k, x^k) \geq e(S, x^k)$ for all coalition S.

Given a cycle $\mathbf{c} = (x^k)_{k=1}^m$, we denote the family $(S_k)_{k=1}^m$ by supp($\mathbf{c}$). Besides, we refer to the numbers $(\mu_k)_{k=1}^m$ as the *transfer amounts.*

The following result is a special case of a similar one proved in Cesco (2003, Theorem 1) for cycles.

Theorem 1. *Let* $\mathbf{c} = (x^k)_{k=1}^m$ *be a* U*-cycle in a* TU*-game* (N, v)*. Then supp* $(\mathbf{c})$ $= (S_k)_{k=1}^m$ *is a balanced family of coalitions with weights given by*

$$\lambda_{S_k} = \frac{n}{|S_k|.|N \setminus S_k|}.\frac{e(S_k, x_k)}{\sum_{q=1}^m \frac{e(S_q, x_q)}{|N \setminus S_q|}} \tag{3.3}$$

for all $k = 1, \ldots, m$.

The existence of cycles in a TU-game is strongly related to the non-existence of points in the core of the game. The results proved in Cesco (2003, Theorems 3 and 9) allow us to state the following theorem.

Theorem 2. *In a* TU*-game* (N, v) *the following assertions are equivalent*:

(i) *The game has non-empty core.*
(ii) *The game is balanced* (*Shapley-Bondareva's theorem*)
(iii) *There do not exist fundamental cycles in* (N, v).

The key to prove the only if part of (iii) is a good representation for the *worth* of a balanced family of coalitions $\mathcal{B}$ with respect to a set $(\lambda_k)_{k=1}^m$ of balancing weights for $\mathcal{B}$ in a game (N, v), which is the quantity defined by $\sum_{S \in \mathcal{B}} \lambda_S v(S)$. Below we outline the proof of this result in the case we restrict ourselves to U-cycles.

Proposition 3. *The worth of supp*$(\mathbf{c})$ *satisfies*

$$2.\left(\sum_{k=1}^m \lambda_{S_k}.v(S_k) - v(N)\right) = \sum_{k=1}^m \lambda_{S_k}.e(S_k, x^k) \tag{3.4}$$

for all U*-cycle* $\mathbf{c} = (x^k)_{k=1}^m$ *with balancing weights given by* (3.3).

Proof. It follows directly from part (v) of Lemma 2 in Cesco (2003) by taking into account that U-cycles are particular cases of the cycles for which $\mu_k = e(S_k, x^k)$ for all $k = 1, \ldots, m$. Besides, for U-cycles it holds that $e(S_k, x^{k+1}) = 0$ for all $k = 1, \ldots, m$. In particular, $e(S_m, x^1) = 0$. This completes the proof. □

Now, as a direct consequence of Proposition 3, we have the following:

Corollary 4. *If there exists a U-cycle* $\mathbf{c}$ *in a TU-game (N, v), the game is non-balanced.*

Proof. For a U-cycle $\mathbf{c}$, the right hand side of (3.4) is positive. This implies that supp($\mathbf{c}$) is an objectionable family and the game is non-balanced. □

4. An Auxiliary Lemma

The aim of the following two sections is to prove a converse of Corollary 4. We have not been successful in proving it in a general framework. In Cesco and Aguirre (2002) we get an existence result of U-cycles in non-balanced monotonic 3-person games. In the next section we address to this problem for n-person games, in where the set of permissible coalitions consists of the $1, n-1$, and n-person coalitions. These games have been studied in Maschler (1963) in connection to various bargaining sets and they resemble 3-person games in many ways. In this framework, we will explicitly construct U-cycles. The result we will prove next provides a starting point for such cycles. In the sequel, we will consider $n \geq 3$.

Lemma 5. *The linear system*

$$\tilde{x}_n = \tilde{x}_1 + (\mu - 1 + \tilde{x}_n)\frac{1}{n-1}$$

$$\tilde{x}_1 = \tilde{x}_2 + (\mu - 1 + \tilde{x}_n)\frac{1}{n-1}$$

$$\vdots \tag{4.1}$$

$$\tilde{x}_{n-2} = \tilde{x}_{n-1} + (\mu - 1 + \tilde{x}_n)\frac{1}{n-1}$$

$$1 = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n \tag{4.2}$$

has a unique solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ for all $\mu \in \mathbb{R}$. The solution $\tilde{x}$ satisfies

$$\tilde{x}_n > \tilde{x}_1 > \tilde{x}_2 > \cdots > \tilde{x}_{n-1} = 1 - \mu \tag{4.3}$$

if and only if $\mu > \frac{n-1}{n}$.

Proof. The system (4.1)–(4.2) can be written in matrix form like $Ax = b$ with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{n-2}{n-1} \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{n-1} \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & -\frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & -\frac{1}{n-1} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} \frac{1-\mu}{n-1} \\ -\frac{1-\mu}{n-1} \\ -\frac{1-\mu}{n-1} \\ \vdots \\ \vdots \\ \vdots \\ -\frac{1-\mu}{n-1} \\ 1 \end{pmatrix}$$

The matrix A is non-singular. In fact, the principal sub-matrix formed with the $n-1$ first rows and columns is lower triangular with all diagonal elements non-zero. Then, the rank of the submatrix formed by the $n-1$ first columns is $n-1$. We claim that the last column is linearly independent to the others. Otherwise, we would be able to compute the coefficients of the supposed linear combination $A_n = \sum_{i=1}^{n-1} \lambda_i A_i$. Here A_i denotes the i-column of A. The coefficients should be

$$\lambda_1 = -\frac{n-2}{n-1}$$
$$\lambda_2 = -\frac{n-3}{n-1}$$
$$\vdots$$
$$\lambda_{n-1} = -\frac{n-n}{n-1} = 0$$

if we expect to satisfy the equalities for the first $n-1$ components in the linear combination. But then, the last equality $\sum_{i=1}^{n-1} \lambda_i = 1$ cannot be satisfied since the $\lambda' s$ are all non-positive and some of them negative. Thus, we have shown that A is non-singular and (4.1)–(4.2) has a unique solution for all $\mu \in \mathbb{R}$.

Now, we compute the solution for a given μ. From the first Eq. of (4.1) we get that

$$\tilde{x}_n = \frac{1}{n-2}((n-1)\tilde{x}_1 - (1-\mu)). \tag{4.4}$$

From the second one, and taking into account the above expression, we obtain that

$$\tilde{x}_2 = \frac{1}{n-2}((n-3)\tilde{x}_1 + (1-\mu)).$$

By iterating this substitution process we get that

$$\tilde{x}_i = \frac{1}{n-2}((n-(i+1))\tilde{x}_1 + (i-1)(1-\mu)) \tag{4.5}$$

for all $2 \leq i \leq n-1$. In particular, $\tilde{x}_{n-1} = 1-\mu$. To determine $\tilde{x}_1$, we take into account that $\sum_{i=1}^{n} \tilde{x}_i = 1$ must hold. So

$$\begin{aligned}
1 &= \tilde{x}_1 + \tilde{x}_n + \sum_{i=2}^{n-1} \tilde{x}_i \\
&= \tilde{x}_1 + \frac{1}{n-2}((n-1)\tilde{x}_1 - (1-\mu)) \\
&\quad + \sum_{i=2}^{n-1} \frac{1}{n-2}((n-(i+1))\tilde{x}_1 + (i-1)(1-\mu)) \\
&= \frac{1}{n-2}\left(2n-3 + \sum_{i=2}^{n-1}((n-(i+1))\right)\tilde{x}_1 \\
&\quad + \frac{1-\mu}{n-2}\left(-1 + \sum_{i=2}^{n-1}(i-1)\right) \\
&= \frac{1}{2}\frac{n(n-1)}{n-2}\tilde{x}_1 + \frac{1}{2}\frac{n(n-3)}{(n-2)}(1-\mu).
\end{aligned} \tag{4.6}$$

We point out that the coefficient of $\tilde{x}_1$ in (4.6), is a positive number for $n \geq 3$. Then

$$\begin{aligned}
\tilde{x}_1 &= \frac{1 - \frac{1}{2}\frac{n(n-3)}{(n-2)}(1-\mu)}{\frac{1}{2}\frac{n(n-1)}{n-2}} \\
&= \frac{2(n-2) - n(n-3)(1-\mu)}{n(n-1)}.
\end{aligned} \tag{4.7}$$

Now, we claim that $\tilde{x}_n > \tilde{x}_1 > \tilde{x}_2 > \tilde{x}_3 > \cdots > \tilde{x}_{n-1} = 1-\mu$ if and only if $\mu > \frac{n-1}{n}$. To prove this, we first assume that $\tilde{x}_n > \tilde{x}_1 > \tilde{x}_2 > \tilde{x}_3 > \cdots > \tilde{x}_{n-1} = 1-\mu$. Then $\tilde{x}_i > 1-\mu$ for $i = 1, \ldots, n, i \neq n-1$, and

$$1 = \sum_{i=1}^{n} \tilde{x}_i > n(1-\mu).$$

From this we get that $\mu > \frac{n-1}{n}$.

Conversely, let us assume that $\mu > \frac{n-1}{n}$. From (4.1) we get that $\tilde{x}_n > \tilde{x}_1 > \tilde{x}_2 > \tilde{x}_3 > \cdots > \tilde{x}_{n-1} = 1-\mu$ if and only if $\mu - 1 + \tilde{x}_n$ is positive. But, because of (4.4) and (4.6) we have that

$$\begin{aligned}
\mu - 1 + \tilde{x}_n &= \mu - 1 + \frac{1}{n-2}.((n-1).\tilde{x}_1 - (1-\mu)) \\
&= \frac{n-1}{n-2}.\tilde{x}_1 - \frac{n-1}{n-2}.(1-\mu) \\
&= 2.\frac{1}{n} - 2.(1-\mu).
\end{aligned}$$

Since this last quantity is positive, $\mu - 1 + \tilde{x}_n > 0$. □

Remark 3. If $\frac{n-1}{n} < \mu \leq 1$,the solution satisfying (4.3) is also non-negative.

Remark 4. Is is easy to prove that the linear system (4.1)–(4.2) is equivalent to the following one:

$$(\tilde{x}_n, \tilde{x}_1, \ldots, \tilde{x}_{n-1}) = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) + (\mu - 1 + \tilde{x}_n)\left(\frac{1}{n-1}, \frac{1}{n-1}, \ldots, -1\right) \tag{4.8}$$

$$\sum_{i=1}^{n} \tilde{x}_i = 1 \tag{4.9}$$

5. Existence of *U*-Cycles

In what follows we will be concerned with an n-person TU-game $(N, \mathcal{C}, v)$ where N is the set of players, $\mathcal{C} \subseteq \mathcal{P}(N)$ is a family of permissible coalitions and v is the characteristic function defined on $\mathcal{C}$. From now on, $\mathcal{C}$ will be family containing the minimal balanced family of the coalitions having exactly $n-1$ players, the grand coalition N, and the unitary coalitions. As before, we still will assume that $v(N) = 1$, and $v(\{i\}) = 0$ for all $i \in N$. We recall that a game (N, v) is monotonic if $T \subseteq S$ implies that $v(T) \leq v(S)$. For further reference we denote by $\mathcal{B}_{n-1}$ the family $\{S \in \mathcal{P}(N) : |S| = n-1\}$. In the sequel, we will use the following notation for the coalitions in $\mathcal{B}_{n-1}$: $S_1 = N \backslash \{n\}$, and $S_k = N \backslash \{k-1\}$ for all $k = 2, \ldots, n$. The notion of cycles, as introduced in Definition 1, can be adapted to deal with games with a sub-family of permissible coalitions $\mathcal{C}$ by asking that supp($\mathbf{c}$)$\subseteq \mathcal{C}$.

Now we are ready to prove the first existence result of U-cycles in a game $(N, \mathcal{C}, v)$. We will make use of the linear shift operator $\mathbf{F}$ on $\mathbb{R}^n$ defined by

$$\mathbf{F}((x_1, x_2, \ldots, x_n)) = (x_n, x_1, \ldots, x_{n-1})$$

$\mathbf{F}^k$ will stand for k successive applications of the operator $\mathbf{F}$. As usual, $\mathbf{F}^0$ will represent the identity operator. The next result will be widely used.

Lemma 6. *Let* $(N, \mathcal{C}, v)$ *be a game with* $v(S) = \mu$ *for all* $S \in \mathcal{B}_{n-1}$. *Then, the following assertions are true*:

(i) $\mathbf{F}(\beta_{S_k}) = \beta_{S_{k+1}}$ *for all* $k = 1, \ldots, n-1$.
(ii) *If* $y = \mathbf{F}(x)$, *then* $e(S_k, y) = e(S_{k-1}, x)$ *for all* $S_k \in \mathcal{B}_{n-1}, x \in E$.

Proof. (i) follows directly from the definition of $\mathbf{F}$ and the fact that $\beta_{S_k} = \left(\frac{1}{n-1}, \frac{1}{n-1}, -1, \ldots, \frac{1}{n-1}\right)$, the -1 placed in the position n when $k = 1$ and in the $(k-1)$-th position if $k \geq 2$.

To prove (ii) we note that

$$\begin{aligned} e(S_k, y) &= \mu - \sum_{\substack{i=1 \\ i \neq k-1}}^{n} y_i \\ &= \mu - \sum_{\substack{i=1 \\ i \neq k-2}}^{n} x_i \\ &= e(S_{k-1}, x). \end{aligned}$$

Here we identify S_0 with S_n. □

Proposition 7. *Let $(N, \mathcal{C}, v)$ with $v(S) = \mu$ for all $S \in \mathcal{B}_{n-1}$. Then, there always exists a U-cycle in the game provided $\mu > \frac{n-1}{n}$.*

Proof. Let $\tilde{x}^1 = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ be the unique solution for (4.8)–(4.9). If $\mu > \frac{n-1}{n}$, $\tilde{x}$ is a pre-imputation satisfying (4.3). Then,

$$\begin{aligned} e(S_1, \tilde{x}^1) &= \mu - \sum_{i=1}^{n-1} \tilde{x}_i \\ &= \mu - 1 + \tilde{x}_n \\ &> 0. \end{aligned}$$

Let $\tilde{x}^2 = \tilde{x}^1 + e(S_1, \tilde{x}^1).\beta_{S_1}$. In the light of (4.8) we point out that $\tilde{x}^2 = \mathbf{F}(\tilde{x}^1)$. Lemma 6 (ii) shows that $e(S_2, \tilde{x}^2) = e(S_1, \tilde{x}^1) = \mu - 1 + \tilde{x}_n$. We now define, $\tilde{x}^k = \mathbf{F}(\tilde{x}^{k-1})$ for all $k \geq 3$. Clearly since $\tilde{x}^{n+1} = \mathbf{F}(\tilde{x}^n) = \mathbf{F}^n(\tilde{x}^1)$, we get that $\tilde{x}^{n+1} = \tilde{x}^1$. Besides, we claim that $\tilde{x}^{k+1} = \tilde{x}^k + e(S_k, \tilde{x}^k).\beta_{S_k}$ and $e(S_{k+1}, \tilde{x}^{k+1}) = e(S_k, \tilde{x}^k) = (\mu - 1 + \tilde{x}_n)$ for all $k = 1, \ldots, n$. To prove it we use finite induction. It is true for $k = 1$. Let us assume that it also holds for some $2 \leq k \leq n$. Then

$$\begin{aligned} \tilde{x}^{k+1} &= \mathbf{F}(\tilde{x}^k) \\ &= \mathbf{F}(\tilde{x}^{k-1}) + e(S_{k-1}, \tilde{x}^{k-1}).\mathbf{F}(\beta_{S_{k-1}}) \\ &= \tilde{x}^k + e(S_k, \tilde{x}^k).\beta_{S_k} \end{aligned}$$

since $\mathbf{F}(\beta_{k-1}) = \beta_k$ by Lemma 6 (i). Part (ii) of the same result guarantees that $e(S_{k+1}, \tilde{x}^{k+1}) = e(S_k, \tilde{x}^k) = (\mu - 1 + \tilde{x}_n) > 0$, the second equality being justified by the inductive hypothesis. Thus, the sequence

$$(\tilde{x}^k)_{k=1}^n = (F^k(\tilde{x}^1))_{k=0}^{n-1} \tag{5.1}$$

is a U-cycle of pre-imputations in the game $(N, \mathcal{C}, v)$. □

Remark 5. If in Proposition 7, $\mu \leq 1$ as well, all of the elements of the cycle (5.1) are, in fact, imputations (cf. Remark 3). We also point out that $y = x + e(S_k, x)\beta(S_k)$ is always an imputation whenever $x \in A, S_k \in \mathcal{B}_{n-1}$, and $e(S_k, x) > 0$.

The game studied in Proposition 7 is highly symmetric. It plays a central role in the general theory for the games having $\mathcal{C}$ as the family of permissible coalitions. The following result illustrates this point.

Definition 2. Let $(N, \mathcal{C}, v)$ a n-person game with characteristic function given by

$$v(N) = 1, \quad v(\{i\}) = 0 \quad \text{for all } i \in N \tag{5.2}$$

$$v(S_1) = \mu_n, \quad v(S_k) = \mu_{k-1}, \quad \text{for all } k = 2, \ldots, n \tag{5.3}$$

where μ_k is a real number for all $k = 1, \ldots, n$. The $\mathcal{B}$-equivalence of $(N, \mathcal{C}, v)$ is the n-person game $(N, \mathcal{C}, v_{\mathcal{B}})$ defined by

$$v_{\mathcal{B}}(N) = 1, \quad v_{\mathcal{B}}(\{i\}) = 0 \quad \text{for all } i \in N$$

$$v_{\mathcal{B}}(S_k) = \mu \quad \text{for all } S_k \in \mathcal{B}_{n-1}$$

with μ given by

$$\mu = \frac{1}{n} \sum_{k=1}^{n} \mu_k. \tag{5.4}$$

Remark 6. The worth of $\mathcal{B}_{n-1}$ is the same in both games, $(N, \mathcal{C}, v)$ and $(N, \mathcal{C}, v_{\mathcal{B}})$.

In the next result, we will use the fact that the matrix

$$\mathbf{E} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

is non singular. Indeed, it is easy to prove that it has $n-1$, -1, the latter repeated $n-1$ times, as eigenvalues. It is worth noting that the vector $(1, 1, \ldots, 1)'$ is an eigenvector for $\mathbf{E}$ associated with the eigen value $n-1$. Here x' stands for the transpose of the vector x.

Proposition 8. *Let $(N, \mathcal{C}, v)$ be a game with $v(S_1) = \mu_n, v(S_k) = \mu_{k-1}$ for all $k = 2, \ldots, n$. Then, there always exists a U-cycle in the game provided $\mu = \frac{1}{n}\sum_{k=1}^{n} \mu_k$ is greater than $\frac{n-1}{n}$.*

Proof. Let $(N, \mathcal{C}, v_{\mathcal{B}})$ be the $\mathcal{B}$-equivalence of the game $(N, \mathcal{C}, v)$ and $\tilde{\mathbf{c}} = (\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)$ the U-cycle in this game given by (5.1). Besides, let $t = (t_1, t_2, \ldots, t_n)$ be the unique solution of the linear system $\mathbf{E}t = b'$, with $b = (\mu_1 - \mu, \mu_2 - \mu, \ldots, \mu_n - \mu)$.

Since

$$\begin{aligned} (1, 1, \ldots, 1)\mathbf{E}t' &= (n-1) \sum_{k=1}^{n} t_k \\ &= \sum_{k=1}^{n} \mu_k - n\mu \\ &= 0 \end{aligned}$$

we conclude that

$$\sum_{k=1}^{n} t_k = 0.$$

We now define $x^i = \tilde{x}^i + t$ for all $i = 1, 2, \ldots, n$. We claim that $\mathbf{c} = (x^k)_{k=1}^n$ is a U-cycle of pre-imputations in $(N, \mathcal{C}, v)$. To prove this, we first note that

$$\begin{aligned} e(S_1, x^1) &= v(S_1) - \sum_{i=1}^{n-1} x_i \\ &= \mu_n - \sum_{i=1}^{n-1} \tilde{x}_i - \sum_{i=1}^{n-1} t_i \\ &= \mu - \sum_{i=1}^{n-1} \tilde{x}_i + \mu_n - \mu - \sum_{i=1}^{n-1} t_i \\ &= \mu - \sum_{i=1}^{n-1} \tilde{x}_i \\ &= v_{\mathcal{B}}(S_1) - \sum_{i=1}^{n-1} \tilde{x}_i = e(S_1, \tilde{x}^1). \end{aligned} \tag{5.5}$$

We point out here that $e(S, x)$ stands for $v(S) - x(S)$ and $e(S, \tilde{x}) = v_{\mathcal{B}}(S) - \tilde{x}(S)$. Therefore

$$\begin{aligned} x^1 + e(S_1, x^1)\beta_{S_1} &= \tilde{x}^1 + e(S_1, \tilde{x}^1)\beta_{S_1} + t \\ &= \tilde{x}^2 + t \\ &= x^2. \end{aligned}$$

With a similar argument than that used above, we are able prove that $e(S_k, x^k) = e(S_k, \tilde{x}^k)$ for all $k = 1, \ldots, n$, and therefore, that $x^{i+1} = x^i + e(S_i, x^i)\beta_{S_i}$ for all $i = 2, \ldots, n$. Since $x^{n+1} = x^1$ we conclude that $\mathbf{c}$ is a U-cycle in $(N, \mathcal{C}, v)$. □

Remark 7. It is simple to verify that $t_i = \mu - \mu_i$ for $i = 1, \ldots, n$.

Remark 8. A similar technique than that used to get (5.5) can be employed to show that $e(S_k, x^j) = e(S_k, \tilde{x}^j)$ for all $j = 1, \ldots, n, k = 1, 2, \ldots, n$.

Remark 9. We point out that if x is an imputation in a monotonic game (N, v), and $e(S, x) > 0$, for some $S \in \mathcal{B}_{n-1}$ then, $y = x + e(S, x).\beta_S$ is also an imputation. Indeed, for each $i \in S_k$, the component $y_i = x_i + \frac{e(S,x)}{n-1}$ is positive since $x_i \geq 0$ and $e(S, x) > 0$. On the other side, for the only $j \notin S$ we have that

$$\begin{aligned} y_j &= x_j - e\,(S, x) \\ &= x_j - v\,(S) + x\,(S) \\ &= x\,(N) - v\,(S) \end{aligned}$$

$$\geq 0.$$

Related to a TU-game (N,v) we will consider the restricted game $(N,\mathcal{C},\tilde{v})$, where $\tilde{v}$ is the restriction of v to $\mathcal{C}$.

Now, we are ready to state the main result of this paper.

Theorem 9. *Let (N,v) be a TU-game with characteristic function given by*

$$v(S_1) = \mu_n, \quad v(S_k) = \mu_{k-1} \text{ for all } k = 2,\ldots,n \tag{5.6}$$

$$v(N) = 1, \quad \text{and} \quad v(S) = 0 \quad \text{otherwise.} \tag{5.7}$$

If the game is monotonic, then there exists a maximal U-cycle in the game provided $\mu = \frac{1}{n}\sum_{k=1}^{n}\mu_k$ is greater than $\frac{n-1}{n}$.

Proof. Let $(N,\mathcal{C},\tilde{v})$ be the restricted game related to (N,v). It satisfies the hypothesis of Proposition 8. We claim that the U-cycle $\mathbf{c} = (x^k)_{k=1}^n$ exhibited there is maximal in (N,v). To prove this, we have to show that for each $k = 1,\ldots,n$, $e(S_k,x^k) \geq e(S,x^k)$ for all coalition S. We first note that x^1 is an imputation. In fact, according to Remark 7 and (4.3), $x_i \geq 1-\mu_i$ for $i = 1,\ldots,n$, and this quantities are non-negative since the game (N,v) is monotonic ($\mu_i \leq 1$). Moreover, according to Remark 8, we have that $e(S_1,x^1) > 0$. Thus, x^2 is also an imputation (Remark 9). An inductive argument shows that $x^k, k = 2,\ldots,n$ are imputations too. Thus, $e(S,x^k) \leq 0$ for all $S \notin \mathcal{B}_{n-1}$, and $k = 1,\ldots,n$.

On the other hand, let $\tilde{\mathbf{c}} = (\tilde{x}^k)_{k=1}^n$ be the U-cycle for the $\mathcal{B}$-equivalence $(N,\mathcal{C},\tilde{v}_{\mathcal{B}})$ of $(N,\mathcal{C},\tilde{v})$ given in Proposition 8. Then $\tilde{x}_n > \tilde{x}_i$ for all $i = 1,\ldots,n-1$, and we get that $e(S_1,\tilde{x}^1) = \mu - 1 + \tilde{x}_n > 0$. Now, successive applications of Lemma 6, and Remark 8 give that

$$e(S_k,x^k) = e(S_k,\tilde{x}^k) = \cdots = e(S_1,\tilde{x}^1) = \mu - 1 + \tilde{x}_n$$

for all $k = 1,\ldots,n$. On the other side, for each $S_r \in \mathcal{B}_{n-1}$ we have that

$$e(S_r,x^k) = e(S_r,\tilde{x}^k) = \mu + \tilde{x}_j - 1 \quad \text{for some } 1 \leq j \leq n$$

Thus we obtain that

$$\begin{aligned} e(S_r,x^k) &= \mu + \tilde{x}_j - 1 \\ &\leq \mu + \tilde{x}_n - 1 = e(S_k,x^k) \end{aligned}$$

for all $k = 1,\ldots,n$. Since, as we showed before, $e(S,x^k) \leq 0$ for all $S \notin \mathcal{B}_{n-1}$, we conclude that the cycle $\mathbf{c} = (x^k)_{k=1}^n$ is maximal in (N,v). □

Remark 10. The monotonicity condition can not be eliminated from Theorem 9. In the following 3-person game, the U-cycle given by Proposition 8 is not maximal. Let the characteristic function be $v(\{1,2\}) = \frac{1}{2}, v(\{1,3\}) = 0, v(\{2,3\}) = 2$ on the 2-person coalitions. It can be verified that the cycle $\tilde{\mathbf{c}}$ of the $\mathcal{B}$-equivalence of this game is

$$\tilde{x}^1 = \left(\mu - \frac{1}{3}, 1-\mu, \frac{1}{3}\right), \quad \tilde{x}^2 = \left(\frac{1}{3}, \mu - \frac{1}{3}, 1-\mu\right), \quad \tilde{x}^3 = \left(1-\mu, \frac{1}{3}, \mu - \frac{1}{3}\right)$$

with $\mu = \frac{5}{6}$. Therefore, $e(\{1,3\}, x^3) = \frac{1}{3}$, while $e(\{1\}, x^3) = 1$, since x^3 satisfies that $e(\{2,3\}, x^3) = 0$.

Lemma 10. *Let* (N, v) *be a TU-game with characteristic function given by* (5.6–5.7). *If* $\mathcal{B} \neq \mathcal{B}_{n-1}$ *is minimal balanced family of coalitions such that* $\sum_{S\in\mathcal{B}} \lambda_S.v(S) > 1$ (*objectionable family*), *then there exists* $S_k \in \mathcal{B}_{n-1}$ *such that* $v(S_k) > 1$.

Proof. Since the only proper coalitions with non-zero value are those in $\mathcal{B}_{n-1}$, and $\sum_{S\in\mathcal{B}} \lambda_S.v(S)$ must be grater than 1, $\mathcal{B} \cap \mathcal{B}_{n-1} \neq \Phi$. Moreover, because of the minimality of $\mathcal{B}, \mathcal{B} \cap \mathcal{B}_{n-1} = \{\bar{S}_1, \ldots, \bar{S}_s\}$, with $0 < s < n$. Therefore, there exists $i \in \bigcap_{j=1}^{s} \bar{S}_j$. This implies that $\sum_{j=1}^{s} \lambda_{\bar{S}_j} \leq 1$. If $v(\bar{S}_j) \leq 1$ for $j = 1, \ldots, s$, then $\sum_{S\in\mathcal{B}} \lambda_S.v(S) = \sum_{j=1}^{s} \lambda_{\bar{S}_j}.v(\bar{S}) \leq \sum_{j=1}^{s} \lambda_{\bar{S}_j} \leq 1$, and $\mathcal{B}$ would not be an objectionable family. □

The results presented in this paper can be summarized in the following theorem

Theorem 11. *Let* (N, v) *be a TU-game with characteristic function given by* (5.6–5.7). *Then the following statements are equivalent*

(i) *The game is balanced.*
(ii) *There does not exist a U-cycle in the game.*
If the game is monotonic, then (i) *is equivalent to*
(iii) *There does not exist a maximal U-cycle in the game.*

Proof. (i) ⇒ (ii) is, basically, Corollary 4. To prove the converse, assume that there exists an objectionable family $\mathcal{B}$ of coalitions. Then $\sum_{S\in\mathcal{B}} \lambda_S.v(S) > 1$. If $\mathcal{B} = \mathcal{B}_{n-1}$, and since $\lambda_S = \frac{1}{n-1}$ for all $S \in \mathcal{B}_{n-1}$, that inequality implies that $\frac{1}{n}\sum_{k=1}^{n} \mu_k > \frac{n-1}{n}$. But then, Proposition 8 assures the existence of a U-cycle, and thus, we get a contradiction to (ii). If this is not the case, according to Lemma 10, there exist $S \in \mathcal{B}_{n-1}$ such that $v(S) > 1$. Without lost of generality, we may assume that this coalition is S_1. Let x^2 be a pre-imputation satisfying $e(S_1, x^2) = 0$. It follows that $x_n^2 < 0$. Therefore, $e(\{n\}, x^2) > 0$. Now, let $x^1 = x^2 + e(\{n\}, x^2).\beta_{\{n\}}$. Since $e(\{n\}, x^1) = 0$, $\sum_{i=1}^{n-1} x_i^1 = 1$, and this implies that $e(S_1, x^1) > 0$. A simple computation shows that $x = x^1 + e(S_1, x^1).\beta_{S_1}$ coincides with x^2. Thus (x^1, x^2) is U-cycle and once more this contradicts (ii).

Corollary 4 also proves (i) ⇒ (ii) when the game is monotonic. In this case, no coalition in $\mathcal{B}_{n-1}$ can have value greater than one. This implies that $\mathcal{B}_{n-1}$ is the only objectionable family of coalitions in a non-balanced monotonic game. Therefore, (iii) ⇒ (i) is proved like the first part of the proof (ii) ⇒ (i). The existence of the maximal cycle is now guaranteed by Theorem 9. □

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