

AN OPTIMAL APPROACH TO THE MULTIPLE-DEPOT HETEROGENEOUS VEHICLE ROUTING PROBLEM WITH TIME WINDOW AND CAPACITY CONSTRAINTS

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Abstract. The vehicle routing problem (VRP) has become a crucial industrial issue for its impact on product distribution costs. Though quite important in practice, the time-constrained version of the VRP accounting for several types of vehicles and m-depots, called the extended VRP with time windows (*m*-VRPTW), has received less attention. Since it is an NP-hard problem, most of the current approaches to *m*-VRPTW are heuristic, thus providing good but not necessarily optimal solutions. This work presents a novel MILP mathematical framework for the m-depot heterogeneous-fleet VRPTW problem. The new optimization approach permits to find both the optimal vehicle route/schedule and the fleet size by choosing the best set of preceding nodes for each pick-up point. To get a significant reduction on the problem size to tackle larger m-VRPTW problems, some elimination rules have been embedded in the MILP framework. When applied to a pair of examples, it was observed a remarkable saving in computer costs with regards to prior VRPTW optimization methods.

Keywords: VRPTW, MILP formulation, soft/hard time windows

I. INTRODUCTION

In recent years, vehicle routing problems (VRP) have received much attention due to the importance of selecting efficient distribution strategies. Surveys on VRP problems can be found in Bodin *et al.* (1983) and Desrosiers *et al.* (1995). Vehicle routing problems with time windows (VRPTW) have recently become an area of intensive research since time windows naturally arise from realistic considerations and their impact on the optimal solution must be taken into account. Similarly to VRP, VRPTW problems are NP-hard too. Exact algorithms developed for solving VRPTW are described in Kolen *et al.* (1987), Desrochers *et al.* (1992) and Fisher *et al.* (1997). Though heuristic techniques still remain as the unique tool available for solving very large-scale VRP, optimization methods are becoming more effective to deal with problems of moderate size. Nevertheless, traditional optimizing formulations still require an exceptionally large number of variables and constraints to represent real routing problems. In order

to widen the scope for optimizing approaches, this work introduces a new MILP formulation for the *m*-depot VRPTW problem, also called *m*-VRPTW, based on a continuous time domain representation and a separate handling of vehicle assignment and node sequencing decisions.

II. PROBLEM FORMULATION

Let the road network be described by $\{I, P, A\}$ with $I = \{i_1, i_2, \dots, i_n\}$ denoting the set of nodes or customers, and $P = \{p_1, p_2, \dots, p_l\}$ representing the set of depots where the vehicles $V = \{v_1, v_2, \dots, v_m\}$ are housed. Nodes and depots are connected through a network of minimum cost arcs $A = \{d_{ij} / i, j \in (I \cup P)\}$. Each node represents a client that has a non-negative commodity production pr_i to be picked up and transported to a depot $p \in P$. There is a matrix of unit routing costs $C = \{c_{ij}^v\}$ and a matrix of vehicle speeds $\Gamma = \{\gamma_{ij}^v\}$ associated to the set A . Each node has its time window $[a_i, b_i]$, where a_i is the earliest arrival time and b_i is the latest visiting time. The service time on node i by vehicle v is $(tf_i + pr_i / r_v)$, where tf_i is the fixed stop time on node i and r_v is the loading rate of goods on vehicle v . The solution to the problem must comply the following constraints: (i) each route starts and ends on the same depot; (ii) each node belongs to exactly one route; (iii) the total amount of commodity assigned to vehicle v must never exceed its capacity q_v ; (iv) the duration of the trip for any vehicle v should be shorter than the maximum allowed routing time tv_v^{max} ; (v) the node i should be serviced within the time window $[a_i, b_i]$. The problem goal is to minimize the total cost of providing pickup service to every node. Three types of costs are considered in the objective function: fixed costs for using vehicles, traveling costs along the selected routes and penalty costs fining soft time-window and maximum routing time constraints violations.

III. THE MATHEMATICAL MODEL

Assignment of nodes to vehicles

Each node must belong to exactly one tour.

$$\sum_{v \in V} Y_{iv} = 1 \quad \forall i \in I \quad (1)$$

Assignment of vehicles to depots

Each vehicle can at most be assigned to a single depot.

$$\sum_{p \in P} X_{pv} \leq 1 \quad \forall v \in V \quad (2)$$

Capacity constraints

The total amount of commodity to be picked up by vehicle v must not exceed its capacity q_v .

$$\sum_{i \in I} pr_i Y_{iv} \leq q_v \sum_{p \in P} X_{pv} \quad \forall v \in V \quad (3)$$

Minimum routing cost for visiting node i

The routing cost from the starting depot p to node i by the assigned vehicle v takes a minimum value if one makes the trip all the way through the least cost route d_{pi} connecting depot p to node i .

$$C_i \geq d_{pi} c_{ip}^v - M_C (2 - X_{pv} - Y_{iv}) \quad (4)$$

$$\forall i \in I, p \in P, v \in V$$

Relationship between visiting costs for nodes (i, j)

The routing cost from the starting depot to node j by the assigned vehicle v must never be lower than the cost of visiting node i if both nodes are on the same tour ($Y_{iv} = Y_{jv} = 1$) and node i is visited earlier ($S_{ij} = 1$). Otherwise, the reverse statement will be true. A value of M_C equal to 1.2-1.3 times the maximum routing time tv_v^{max} multiplied by both the average vehicle speed and the average unit traveling cost is adopted.

$$C_j \geq C_i + d_{ij} c_{ij}^v - M_C (1 - S_{ij}) - M_C (2 - Y_{iv} - Y_{jv}) \quad (5.a)$$

$$C_i \geq C_j + d_{ij} c_{ij}^v - M_C S_{ij} - M_C (2 - Y_{iv} - Y_{jv}) \quad (5.b)$$

$$\forall i, j \in I, v \in V : i < j$$

Overall routing cost for used vehicle v

The overall traveling cost for the tour assigned to the used vehicle v must never be lower than the cost of visiting any particular node i on the tour plus the cost of returning from i to the starting depot p through the minimum cost route d_{ip} .

$$CV_v \geq C_i + d_{ip} c_{ip}^v - M_C (2 - X_{pv} - Y_{iv}) \quad (6)$$

$$\forall i \in I, p \in P, v \in V$$

Minimum visiting time for node i

The visiting time for node i by the assigned vehicle v takes a minimum value if one makes the trip all the way through the least cost route d_{pi} connecting depot p to node i .

$$T_i \geq \frac{d_{pi}}{\gamma_{pi}^v} - M_T (2 - X_{pv} - Y_{iv}) \quad (7)$$

$$\forall i \in I, p \in P, v \in V$$

Relationship between visiting times for nodes (i, j)

The time at which node j is visited by the assigned vehicle v must never be lower than the visiting time for node i if both nodes are on the same tour ($Y_{iv} = Y_{jv} = 1$) and node i is visited earlier ($S_{ij} = 1$). Otherwise, the reverse statement will be true. A value for the parameter M_T equal to the maximum routing time tv_v^{max} multiplied by a factor close to 1 (≈ 1.1 to 1.2) is adopted.

$$T_j \geq T_i + t_{ij}^v + \frac{pr_i}{r_v} + \frac{d_{ij}}{\gamma_{ij}^v} - M_T (1 - S_{ij}) - M_T (2 - Y_{iv} - Y_{jv}) \quad (8.a)$$

$$T_i \geq T_j + t_{ji}^v + \frac{pr_j}{r_v} + \frac{d_{ji}}{\gamma_{ji}^v} - M_T S_{ij} - M_T (2 - Y_{iv} - Y_{jv}) \quad (8.b)$$

$$\forall i, j \in I, v \in V : i < j$$

Overall routing time for used vehicle v

The overall routing time for the used vehicle v must never be lower than the visiting time for a particular node i on the tour plus both the loading time and the return time from i to the starting depot p through the minimum cost route d_{ip} .

$$TV_v \geq T_i + t_{ip}^v + \frac{pr_i}{r_v} + \frac{d_{ip}}{\gamma_{ip}^v} - M_T (2 - X_{pv} - Y_{iv}) \quad (9)$$

$$\forall i \in I, p \in P, v \in V$$

Time constraint violations due to early arrivals

$$\Delta a_i \geq a_i - T_i \quad \forall i \in I \quad (10)$$

Time constraint violation due to late arrivals

$$\Delta b_i \geq T_i - b_i \quad \forall i \in I \quad (11)$$

Routing time constraint violation

$$\Delta T_v \geq TV_v - tv_v^{max} \quad \forall v \in V \quad (12)$$

Objective function

$$\text{Min} \sum_{v \in V} \left(c_{fv} \sum_{p \in P} X_{pv} + CV_v + \rho_v \Delta T_v \right) + \sum_{i \in I} \rho_i (\Delta a_i + \Delta b_i) \quad (13)$$

Assignment constraints (1) and (2) together with routing constraints (4)-(6) define the feasible space for the traditional VRP. Timing constraints are considered through visiting time constraints (7)-(9) and the time constraint violations (10)-(12). No sub-tour breaking constraints are necessary. The major features of the proposed mathematical formulation are: (i) the use of a continuous representation for both the time domain and the routing cost domain; (ii) the separate handling of vehicle assignment and node sequencing decisions

through different sets of binary variables; (iii) the definition of 0-1 sequencing variables based on the concept of generalized predecessor. A comparative analysis of computational requirements for solving Example 1 using either the proposed model or the classical slot-based m -VRPTW formulation (Bodin *et al.*, 1983) is made in the next Section.

IV. EXAMPLE 1

Production at 10 nodes must be picked-up and transported to a processing plant P , with a fleet of at most 2 vehicles housed on P . Problem data are summarized in Table 1. Maximum routing time and

time windows are all considered as hard constraints by setting Δa , Δb and ΔT_v all equal to zero in eqns. (10) to (12).

Only traveling costs were included in the objective function. Fixed costs are also considered in Example 2. The best solution for Example 1 is shown in Table 2. Model sizes and computational requirements for this approach and the traditional m -VRPTW formulation (Bodin *et al.*, 1983) are compared in Table 3. It can be noted that the number of binary variables in the proposed MILP approach is reduced by more than three times and, consequently, the required CPU time decreases by a factor of 7.5.

Table 1. Data for Example 1

Nodes parameters											
	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	
$pr_i(lt)$	440	580	1370	820	2850	750	520	1480	2500	1940	
$a_i(h)$	0	0	0	0	1	1	1	2	2	2	
$b_i(h)$	1	1	1	1	2	2	2	3	3	3	
Distance matrix $d_{ij} [km]$											
	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	
T1	---	3.4	6.7	9.2	7.6	6.4	3.4	4.6	8.3	5.5	
T2		---	3.3	4.8	4.2	4.7	4	6.2	9.5	8	
T3			---	1.5	6.3	8.4	7.3	9.6	13.2	12.3	
T4				---	5.5	8.6	8.8	11.1	14.8	13.8	
T5					---	3.4	6.5	8.7	9.4	11.5	
T6						---	3.1	5.3	6	8.1	
T7							---	2.3	5.9	5	
T8								---	3.7	2.9	
T9									---	3.2	
T10										---	
P	6.2	9.5	8.2	9.7	14.8	12.7	9.6	8.6	12	10.7	
Other problem parameters											
Vehicle parameters					Route parameters						
$q_V(lt)$	7500		$r_v(lt/h)$		12000	$tf_i(h)$		0.15	$c_{ij}(\$/km)$		0.5
$cf_V(\$)$	30		$tv_v^{max}(h)$		4.0	$\gamma_{ij}(km/h)$		30			

Table 2. Optimal solution for Example 1

Vehicle	Node	Arrival time (h)	Departure time (h)	Node production (lt)	Utilized capacity (lt)	Vehicle routing time (h)
V1	T1	0.21	0.39	440	6880	3.66
	T7	1.00	1.19	520		
	T8	2.00	2.27	1480		
	T9	2.40	2.75	2500		
	T10	3.00	3.23	1940		
V2	T4	0.32	0.54	820	6370	2.44
	T3	0.59	0.86	580		
	T2	0.97	1.16	1370		
	T5	1.30	1.69	2850		
	T6	1.80	2.02	750		

V. ELIMINATION RULES FOR DISCARDING NON-OPTIMAL ROUTES

Though the proposed approach sharply reduces the problem size, VRP type problems are combinatorial. Nevertheless, problem data can be used to cut down the problem size. To this purpose, exact time window-based elimination rules and distance-based heuristic rules are presented.

A. Time window-based elimination rules

- If one vehicle cannot service a pair of nodes without violating time windows constraints, then two different vehicles must be assigned to service such nodes. Mathematically this condition is expressed as follows:

$$\forall i, j \in I, v \in V : i < j \wedge \left(a_i + tf_i + \frac{pr_i}{r_v} + \frac{d_{ij}}{\gamma_{ij}^v} \right) \geq b_j \wedge \left(a_j + tf_j + \frac{pr_j}{r_v} + \frac{d_{ij}}{\gamma_{ij}^v} \right) \geq b_i \Rightarrow Y_{jv} + Y_{iv} \leq 1 \quad (14)$$

Therefore ($Y_{jv} + Y_{iv} \leq 1$) and the vehicle v can service the node i or the node j but not both. Thus, for any pair of nodes $\{i, j\}$ satisfying condition (14), constraints (5.a)-(5.b) and (8.a)-(8.b) can be eliminated. Moreover, the sequencing variable S_{ij} is no longer needed.

- Let us suppose that nodes i and j are assigned to the same vehicle v and the sum of the earliest arrival time to node i (a_i) plus the travel time between such nodes, including the service time at node i , is higher than the latest arrival time to node j . Then, node i cannot be visited before node j and $S_{ij} = 0$ if both nodes are on the same tour. Consequently, the sequencing variable S_{ij} can be eliminated from the problem formulation.

$$\forall i, j \in I, v \in V : i < j \wedge \left(a_i + tf_i + \frac{pr_i}{r_v} + \frac{d_{ij}}{\gamma_{ij}^v} \right) > b_j \wedge \left(a_j + tf_j + \frac{pr_j}{r_v} + \frac{d_{ij}}{\gamma_{ij}^v} \right) \leq b_i \Rightarrow S_{ij} = 0 \quad (15)$$

Thus, inequalities (5.a) and (8.a) can be dropped from the model while constraints (5.b) and (8.b) reduce themselves to eqns. (16) and (17), respectively.

$$C_i \geq C_j + d_{ij} c_{ijv} - M_C (2 - Y_{iv} - Y_{jv}) \quad (16)$$

$$\begin{aligned} \forall i, j \in I, v \in V : i < j \\ T_i \geq T_j + tf_j + pr_j / r_v + d_{ij} / \gamma_{ij}^v - M_T (2 - Y_{iv} - Y_{jv}) \quad (17) \\ \forall i, j \in I, v \in V : i < j \end{aligned}$$

- Let us now suppose that the sum of the earliest arrival time to node j plus both the service time at node j and the travel time to node i , is higher than the latest arrival time to node i . Then, node j cannot be a

predecessor of node i and $S_{ij} = 1$ only if both nodes are on the same tour.

$$\forall i, j \in I, v \in V : i < j \wedge \left(a_i + tf_i + \frac{pr_i}{r_v} + \frac{d_{ij}}{\gamma_{ij}^v} \right) \leq b_j \wedge \left(a_j + tf_j + \frac{pr_j}{r_v} + \frac{d_{ij}}{\gamma_{ij}^v} \right) > b_i \Rightarrow S_{ij} = 1 \quad (18)$$

Thus, constraints (5.b) and (8.b) can be dropped from the model and inequalities (5.a) and (8.a) reduce to:

$$C_j \geq C_i + d_{ij} c_{ijv} - M_C (2 - Y_{iv} - Y_{jv}) \quad (19)$$

$$\begin{aligned} \forall i, j \in I, v \in V : i < j \\ T_j \geq T_i + tf_i + pr_i / r_v + d_{ij} / \gamma_{ij}^v - M_T (2 - Y_{iv} - Y_{jv}) \\ \forall i, j \in I, v \in V : i < j \end{aligned} \quad (20)$$

B. Distance-based heuristic rules

- Two far away nodes cannot be assigned to the same vehicle because it would lead to very high travelling costs, and possible violations of the routing time constraint. This condition can be stated as follows:

$$\forall i, j \in I, v \in V : i < j \wedge d_{ij} \geq d_{\max} \Rightarrow Y_{jv} + Y_{iv} \leq 1 \quad (21)$$

where d_{\max} is a model parameter selected by the user. Since ($Y_{jv} + Y_{iv} \leq 1$), then constraints (5.a)-(5.b) and (8.a)-(8.b) can be eliminated. The value of d_{\max} should be carefully chosen in such a way that a significant number of non-optimal routes could be eliminated without removing the optimum from the problem feasible region.

- A node i cannot be assigned to a far away depot. Then, the vehicle assigned to node i cannot be allocated to a depot that is far away from node i . This heuristic conditionally restricts the set of potential depots for a vehicle accounting for the route assigned to it. This condition can be so written:

$$\forall i \in I, p \in P, v \in V : d_{ip} \geq d_{\max} \Rightarrow X_{pv} + Y_{iv} \leq 1 \quad (22)$$

Then, constraints (4), (6), (7) and (9) can be eliminated.

C. The multi-heuristic case

Since the simultaneous use of several rules can either eliminate the problem optimal solution or generate redundant constraints of the type $Y_{iv} + Y_{jv} \leq 1$, the heuristics must sequentially be applied in a hierarchical way. Distance-based heuristics must first be enforced before using the exact time window-based elimination rules, since the former ones remove routes from the problem formulation while the others just take away and/or simplify sequencing constraints.

VI. EXAMPLE 2

Example 1 is revisited to now regard time-windows and maximum routing time as soft constraints that may be

violated. Such violations will be penalized and, consequently, the objective function (13) is being adopted. In order to generate routing time constraint violations, the maximum allowed routing time was reduced to $tv_v^{max} = 3.0$. Table 4 presents the new optimal solution. In turn, Table 3 shows the problem size and the computational requirement with/without applying distance-based and time window-based elimination rules.

It can be observed a significant reduction in the CPU time with regards to the classical formulation by a factor larger than 500 without using elimination rules and by 10,000 times with elimination rules. Savings in 0-1 variables and in CPU time are mostly brought about by the exact TW-based elimination rules. The distance-based heuristic rules cut the CPU time down by half but the optimal solution is still found (see Table 4).

Table 3. Computational requirements and models sizes for Examples 1 & 2

	Formulation	Binary variables	Cont. Variables	Constraints	Objective function	CPU time (s)**
Example 1	Traditional MILP approach	220	10	326	30.50	17.03
	This MILP approach	65	24	474	30.50	2.26
Example 2	Traditional MILP approach	220	32	326	94.62	9432
	This MILP approach	65	46	474	94.62	6.78
	This approach with rule V.1	37	46	354	94.62	2.31
	This approach with rules V.1&V.2*	37	46	345	94.62	1.20

* $d_{max} = 13.00$

** Seconds on Pentium III PC (733 MHz) with ILOG/CPLEX (Script mode)

Table 4. Optimal solution for Example 2

Vehicle	Node	Arrival time (h)	Departure time (h)	Earliness (h)	Lateness (h)	Node production (lt)	Utilized capacity (lt)	Vehicle routing time (h)
V1	T3	0.69	0.95	0.00	0.00	1370	7270	2.95
	T4	1.00	1.22	0.00	0.00	820		
	T5	1.50	1.89	0.00	0.00	2850		
	T6	2.00	2.21	0.00	0.00	750		
	T8	2.39	2.66	0.00	0.00	1480		
V2	T1	0.70	0.89	0.00	0.00	440	5980	3.00
	T2	1.00	1.20	0.00	0.00	580		
	T7	1.48	1.67	0.00	0.00	520		
	T9	1.87	2.22	0.13	0.00	2500		
	T10	2.33	2.64	0.00	0.00	1940		

VII. CONCLUSIONS

A novel continuous-time MILP formulation for the VRPTW problem based on the generalized precedence notion has been developed. In the proposed approach, a node precedes another one on the route if it is visited earlier by the same vehicle, but not necessarily immediately before. This allows to separately treating assignment and sequencing decisions through different sets of 0-1 variables. By so doing, a remarkable saving in binary variables is obtained. The proposed model is also capable of handling hard and soft time window constraints. The use of exact time window-based elimination permits to get an important additional reduction in the number of sequencing variables and constraints. Numerical experiments show very encouraging results.

NOMENCLATURE

Sets

- I Set of nodes
- V Set of vehicles
- P Set of depots

Parameters

- q_v Capacity of vehicle v
- cf_v Fixed cost for using vehicle v
- pr_i Production to be picked up at node i
- a_i Earliest arrival time in node i
- b_i Latest arrival time in node i
- tv_v^{max} Maximum routing time for vehicle v
- d_{ij} Length of minimum cost arc between nodes i and j
- d_{ip} Length of minimum cost arc between node i and depot p
- c_{ij}^v Unit routing cost for arc $i-j$ and vehicle v
- c_{ip}^v Unit routing cost for arc $i-p$ and vehicle v
- γ_{ij}^v Vehicle speed for arc $i-j$ and vehicle v
- γ_{ip}^v Vehicle speed for arc $i-p$ and vehicle v

r_v	Loading rate for vehicle v
tf_i	Fixed stop time on node i
ρ_l	Unit penalty cost due to violations on time windows constraints for node i
ρ_v	Unit penalty cost due to violations on the routing time constraint for vehicle v

Variables

X_{pv}	Binary variable denoting that vehicle v is assigned to depot p
Y_{iv}	Binary variable denoting that node i is visited by vehicle v
S_{ij}	Binary variable denoting that node i is visited before node j ($S_{ij}=1$) or after node j ($S_{ij}=0$) when both are on the same tour
C_i	Visiting cost at node i
CV_v	Traveling cost associated to vehicle v
T_i	Visiting time at node i
TV_v	Routing time for vehicle v
Δa_i	Time window constraint violation due to arrivals before a_i
Δb_i	Time window constraint violation due to arrivals after b_i
ΔT_v	Positive variable denoting routing time constraint violation for vehicle v

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