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ScienceDirect
Nonlinear Analysis 66 (2007) 2255–2263
Nonlinear Analysis
www.elsevier.com/locate/na

# A quasilinearization method for elliptic problems with a nonlinear boundary condition

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Received 2 September 2005; accepted 9 March 2006

**Abstract**

We study a nonlinear elliptic second order problem with a nonlinear boundary condition. Assuming the existence of an ordered couple of a supersolution and a subsolution, we develop a quasilinearization method in order to construct an iterative scheme that converges to a solution. Furthermore, under an extra assumption we prove that the convergence is quadratic.


*Keywords:* Elliptic PDEs; Nonlinear boundary conditions; Quasilinearization method; Upper and lower solutions

## 1. Introduction

In this work, we study the following nonlinear elliptic boundary problem:

$$\begin{cases} \Delta u = f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = g(x,u) & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega$ is a bounded smooth domain of $\mathbb{R}^n$, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are continuous and twice continuously differentiable with respect to $u$.

Nonlinear boundary conditions of this kind appear for example when one considers the problem of finding extremals for the best constant in the Sobolev trace inequality (see e.g. [5]

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0362-546X/$ - see front matter 
doi:10.1016/j.na.2006.03.016

and [11]). On the other hand, for $n = 1$, the problem can be regarded as a mathematical model for the axial deformation of a nonlinear elastic beam, with two nonlinear elastic springs acting at the extremities according to the law $u'(0) = -g(u(0))$, $u'(T) = g(u(T))$, and the total force exerted by the nonlinear spring undergoing the displacement $u$ given by $f(t, u)$ [6,14].

The aim of this paper is to develop a quasilinearization technique for problem (1.1) assuming the existence of an ordered couple of a subsolution and a supersolution. More precisely, we construct an iterative scheme that converges to a solution. Furthermore, under an extra assumption we prove that the convergence is quadratic.

The method of supersolutions and subsolutions (definitions will be given in Section 2 below) is one of the most extensively used tools in nonlinear analysis, both for ODE and PDE problems. There exists a vast literature on this subject; see e.g. [4] for a survey. In particular, for elliptic problems with nonlinear boundary conditions such as (1.1), this method has been applied to obtain existence results for example in [7,12].

The quasilinearization method has been developed by Bellman and Kalaba [3], and generalized by Lakshmikantham [9,10]. It has been applied to different nonlinear problems in the presence of an ordered couple of a subsolution and a supersolution. In a recent work [8] it has been successfully applied for a second order ODE Neumann problem for the case in which the supersolution $\beta$ and the subsolution $\alpha$ present the reversed order, namely $\beta \le \alpha$.

Our main results read as follows.

**Theorem 1.1.** *Let $\alpha, \beta \in H^1(\Omega) \cap C(\overline{\Omega})$ be respectively a subsolution and a supersolution of (1.1) such that $\alpha \le \beta$. Furthermore, assume that*

$$\frac{\partial^2 f}{\partial u^2}(x, u) \le 0$$

*for $x \in \overline{\Omega}$ and $\alpha(x) \le u \le \beta(x)$, and*

$$\frac{\partial^2 g}{\partial u^2}(x, u) \ge 0$$

*for $x \in \partial\Omega$ and $\alpha(x) \le u \le \beta(x)$. Then the sequence defined below by (3.1) and (3.2) converges in $H^1(\Omega) \cap C(\overline{\Omega})$ to a solution of (1.1).*

The proof relies on an associated maximum principle and the unique solvability of the associated linear Robin problem.

Moreover, under a monotonicity condition on $f$ and $g$, we prove the quadratic convergence of the method.

**Theorem 1.2.** *Under the hypotheses of the previous theorem, assume furthermore that*

$$\frac{\partial f}{\partial u}(x, u) > 0$$

*for $x \in \overline{\Omega}$ and $\alpha(x) \le u \le \beta(x)$, and*

$$\frac{\partial g}{\partial u}(x, u) < 0$$

*for $x \in \partial\Omega$ and $\alpha(x) \le u \le \beta(x)$. Then the convergence of the sequence defined by (3.1) and (3.2) is quadratic for the $C(\overline{\Omega})$-norm.*

**Remark 1.1.** An analogous result may be obtained by a similar argument under the following non-local boundary condition:

$$u = c \quad \text{on } \partial\Omega, \qquad \int_{\partial\Omega} \frac{\partial u}{\partial \eta} = h(c)$$

where $c$ is a constant with unknown value (see e.g. [2]).

**Remark 1.2.** Our proof of Theorem 1.1 cannot be extended to $f = f(x, u, \nabla u)$, although the existence of solutions can be proved by fixed point arguments (see e.g. [7,12]).

This paper is organized as follows.

In Section 2 we give some preliminary results and definitions. In Section 3, we describe the method of quasilinearization. More precisely, we define iteratively a nondecreasing sequence of subsolutions that converges to a solution of the problem.

Finally, in Section 4 we prove the quadratic convergence of the scheme under a monotonicity condition on $f$ and $g$.

## 2. Definitions and preliminary results

We shall make use of the following maximum principle associated with our problem:

**Lemma 2.1.** *Let $\lambda > 0$, $\mu > 0$ and assume that $u \in H^2(\Omega)$ satisfies*

$$\begin{cases} \Delta u - \lambda u \geq 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} + \mu u \leq 0 & \text{on } \partial\Omega. \end{cases}$$

*Then $u \leq 0$.*

**Proof.** We multiply by $u^+ := \max\{u, 0\}$, and integrate by parts:

$$0 \leq \int_\Omega (\Delta u - \lambda u)u^+ = -\int_{\Omega^+} |\nabla u|^2 - \lambda \int_{\Omega^+} u^2 + \int_{\partial\Omega} u^+ \frac{\partial u}{\partial \eta}$$

where $\Omega^+ = \{x \in \Omega : u(x) > 0\}$. Moreover, as $\frac{\partial u}{\partial \eta} \leq -\mu u$ on $\partial\Omega$, then

$$\int_{\partial\Omega} u^+ \frac{\partial u}{\partial \eta} \leq -\mu \int_\Omega u^+ u \leq 0.$$

Hence, we conclude that $u^+ = 0$, and the proof is complete. □

We shall need the following basic existence and uniqueness result for the Robin problem:

**Lemma 2.2.** *Let $\varphi \in L^2(\Omega)$ and $\psi \in L^2(\partial\Omega)$. Then, for any $\lambda, \mu > 0$ the Robin problem:*

$$\begin{cases} \Delta u - \lambda u = \varphi & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} + \mu u = \psi & \text{on } \partial\Omega \end{cases}$$

*admits a unique solution $u \in H^2(\Omega)$.*

**Proof.** In order to show existence, we consider the functional $J : H^1(\Omega) \to \mathbb{R}$ given by

$$J(u) = \int_\Omega \left[ \frac{|\nabla u|^2}{2} + \lambda \frac{u^2}{2} - \varphi u \right] + \int_{\partial\Omega} \left[ \mu \frac{u^2}{2} - \psi u \right].$$

It is easy to see that $J$ is coercive and weakly lower semicontinuous; hence it achieves a minimum $u \in H^1(\Omega)$ (see e.g. [13], section 1.1), which is a weak solution. As $\|\Delta u\|_{L^2} = \|\lambda u + \varphi\|_{L^2} < \infty$, we conclude that $u \in H^2(\Omega)$. Uniqueness follows from the maximum principle. □

**Lemma 2.3.** *Let $\lambda, \mu > 0$, $\varphi \in L^\infty(\Omega)$ and $\psi \in L^\infty(\partial\Omega)$. Then there exists a constant $C$ such that if $u$ is a weak solution of*

$$\begin{cases} \Delta u - \lambda u = \varphi & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} + \mu u = \psi & \text{on } \partial\Omega \end{cases}$$

*then*

$$\|u\|_{L^\infty(\Omega)} \le C \left[ \|\varphi\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\partial\Omega)} \right].$$

**Proof.** Multiplying the equation by $u$ and integrating, it follows that

$$\|\varphi\|_{L^2} \|u\|_{L^2} \ge -\int_\Omega \varphi u = \|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 - \int_{\partial\Omega} u \frac{\partial u}{\partial \eta}.$$

Hence

$$\|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 \le \|\varphi\|_{L^2} \|u\|_{L^2} + \int_{\partial\Omega} u\psi - \mu \int_{\partial\Omega} u^2.$$

From the trace imbedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ (see e.g. [1], Theorem 5.22) it follows that

$$\|u\|_{H^1(\Omega)} \le c \left[ \|\varphi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\partial\Omega)} \right]$$

for some constant $c$.

Note that if $n = 1$, the result follows trivially. For $n > 1$, fix any $p > n$. As $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^p(\Omega)$, it suffices to show that

$$\|u\|_{W^{1,p}(\Omega)} \le c \left[ \|\varphi\|_{L^p(\Omega)} + \|\psi\|_{L^p(\partial\Omega)} \right]$$

for some constant $c$. By contradiction, suppose that there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that $\|u_k\|_{W^{1,p}(\Omega)} = 1$ and

$$\|\Delta u_k - \lambda u_k\|_{L^p(\Omega)} + \left\| \frac{\partial u_k}{\partial \eta} + \mu u_k \right\|_{L^p(\partial\Omega)} \to 0.$$

As $W^{m,p}(\Omega) \hookrightarrow H^m(\Omega)$, we get from the previous computations that $\|u_k\|_{H^1(\Omega)} \to 0$, and as $\Delta u_k - \lambda u_k \to 0$ for the $L^2$-norm it follows that $\|u_k\|_{H^2(\Omega)} \to 0$.

Then $\|u_k\|_{W^{1,2^*}(\Omega)} \to 0$, and repeating the previous argument we deduce that $\|u_k\|_{W^{2,2^*}(\Omega)} \to 0$. If $2^* \ge n$, then $W^{2,2^*}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ and $\|u_k\|_{W^{1,p}(\Omega)} \to 0$, a contradiction. Otherwise, repeating the argument a certain number of times we obtain that $\|u_k\|_{W^{2,q}(\Omega)} \to 0$ for some $q \ge n$, and the proof follows. □

Next, we recall the definition of the concept of supersolution and subsolution. We say that $\alpha \in H^1(\Omega) \cap C(\overline{\Omega})$ is a subsolution of problem (1.1) if it satisfies

$$\begin{cases} \Delta\alpha \geq f(x,\alpha) & \text{in } \Omega \\ \dfrac{\partial\alpha}{\partial\eta} \leq g(x,\alpha) & \text{on } \partial\Omega. \end{cases}$$

In the same way, $\beta \in H^1(\Omega) \cap C(\overline{\Omega})$ is a supersolution if it satisfies

$$\begin{cases} \Delta\beta \leq f(x,\beta) & \text{in } \Omega \\ \dfrac{\partial\beta}{\partial\eta} \geq g(x,\beta) & \text{on } \partial\Omega. \end{cases}$$

## 3. The quasilinearization method

In this section we define the quasilinearization method and give a proof of Theorem 1.1.

Assume that $\alpha$ and $\beta$ are respectively a subsolution and a supersolution of (1.1) with $\alpha \leq \beta$, and fix $\lambda, \mu > 0$ such that

$$\lambda > \max_{x\in\overline{\Omega},\alpha(x)\leq u\leq\beta(x)} \frac{\partial f}{\partial u}(x,u)$$

$$\mu > -\min_{x\in\partial\Omega,\alpha(x)\leq u\leq\beta(x)} \frac{\partial g}{\partial u}(x,u).$$

We define recursively a sequence $\{u_n\}$ in the following way. Set $u_0 = \alpha$, and assuming that $u_n$ is known, define $u_{n+1}$ as a solution to the following quasilinear Robin problem:

$$\Delta u_{n+1} - \lambda u_{n+1} = f(x,u_n) + \frac{\partial f}{\partial u}(x,u_n)[P_n(x,u_{n+1}) - u_n] - \lambda P_n(x,u_{n+1}) \tag{3.1}$$

in the domain $\Omega$, with the boundary condition

$$\frac{\partial u_{n+1}}{\partial\eta} + \mu u_{n+1} = g(x,u_n) + \frac{\partial g}{\partial u}(x,u_n)[P_n(x,u_{n+1}) - u_n] + \mu P_n(x,u_{n+1}) \tag{3.2}$$

on $\partial\Omega$, where

$$P_n(x,u) = \begin{cases} u_n(x) & \text{if } u < u_n(x) \\ u & \text{if } u_n(x) \leq u \leq \beta(x) \\ \beta(x) & \text{if } u > \beta(x). \end{cases}$$

We shall see in the proof of Theorem 1.1 that $u_n \leq \beta$ and consequently that $P_n$ is well defined. On the other hand, as $u_n(x) \leq P_n(x,u) \leq \beta(x)$, a straightforward application of the Schauder Theorem shows that (3.1) and (3.2) admits at least one solution and therefore $u_{n+1}$ is well defined.

Furthermore, from the fact that $\alpha \leq u_n \leq \beta$ it will follow that the sequence defined by (3.1) and (3.2) is indeed a Newton scheme (see (3.3) below). However, it is not possible to apply the Newton method directly since the linearized problem (3.3) might fail to have a unique solution.

**Proof of Theorem 1.1.** We will prove by induction that $u_n$ is a subsolution, and that $\alpha \leq u_n \leq u_{n+1} \leq \beta$.

First, observe that

$$\Delta u_{n+1} - \lambda u_{n+1} = f(x, u_n) + \left[\frac{\partial f}{\partial u}(x, u_n) - \lambda\right][P_n(x, u_{n+1}) - u_n] - \lambda u_n.$$

From our choice of $\lambda$, $\frac{\partial f}{\partial u}(x, u_n) - \lambda \le 0$, and by the inductive hypothesis $\Delta u_n \ge f(x, u_n)$. As $P_n(x, u_{n+1}) - u_n \ge 0$, it follows that

$$\Delta u_{n+1} - \lambda u_{n+1} \le \Delta u_n - \lambda u_n.$$

In a similar way, using the fact that $\frac{\partial u_n}{\partial \eta} + \mu u_n \le g(x, u_n)$, we conclude that

$$\begin{aligned}\frac{\partial u_{n+1}}{\partial \eta} + \mu u_{n+1} &= g(x, u_n) + \left[\frac{\partial g}{\partial u}(x, u_n) + \mu\right][P_n(x, u_{n+1}) - u_n] + \mu u_n \\ &\ge \frac{\partial u_n}{\partial \eta} + \mu u_n.\end{aligned}$$

From the maximum principle, it follows that $u_{n+1} \ge u_n$.

In order to show that $u_{n+1} \le \beta$, we use a Taylor expansion:

$$f(x, v) = f(x, u_n) + \frac{\partial f}{\partial u}(x, u_n)(v - u_n) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(x, \xi)(v - u_n)^2$$

for some $\xi$ between $u_n(x)$ and $v$.

Choosing $v = P_n(x, u_{n+1})$, as $\frac{\partial^2 f}{\partial u^2}(x, \xi) \le 0$ we deduce that

$$f(x, P_n(x, u_{n+1})) \le f(x, u_n) + \frac{\partial f}{\partial u}(x, u_n)(P_n(x, u_{n+1}) - u_n).$$

Hence, from the definition of $u_{n+1}$,

$$\Delta u_{n+1} - \lambda u_{n+1} \ge f(x, P_n(x, u_{n+1})) - \lambda P_n(x, u_{n+1}).$$

Moreover, from the choice of $\lambda$, the mapping $u \mapsto f(x, u) - \lambda u$ is nonincreasing and, therefore,

$$\Delta u_{n+1} - \lambda u_{n+1} \ge f(x, \beta) - \lambda\beta \ge \Delta\beta - \lambda\beta.$$

In a similar way,

$$\begin{aligned}\frac{\partial u_{n+1}}{\partial \eta} + \mu u_{n+1} &\le g(x, P_n(x, u_{n+1})) + \mu P_n(x, u_{n+1}) \\ &\le g(x, \beta) + \mu\beta \le \frac{\partial \beta}{\partial \eta} + \mu\beta.\end{aligned}$$

By the maximum principle, we conclude that $u_{n+1} \le \beta$.

Next we observe that, as it has been already proved that $\alpha \le u_{n+1} \le \beta$, the definition of $u_{n+1}$ reduces to a Newton iteration:

$$\begin{cases}\Delta u_{n+1} = f(x, u_n) + \frac{\partial f}{\partial u}(x, u_n)[u_{n+1} - u_n] & \text{in } \Omega \\ \frac{\partial u_{n+1}}{\partial \eta} = g(x, u_n) + \frac{\partial g}{\partial u}(x, u_n)[u_{n+1} - u_n] & \text{on } \partial\Omega.\end{cases} \tag{3.3}$$

Moreover, using the Taylor expansion as before we obtain

$$\Delta u_{n+1} = f(x, u_{n+1}) - \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(x, \xi)(u_{n+1} - u_n)^2 \ge f(x, u_{n+1}) \quad \text{in } \Omega,$$

and similarly

$$\frac{\partial u_{n+1}}{\partial \eta} \le g(x, u_{n+1}) \quad \text{on } \partial\Omega.$$

Hence, $u_{n+1}$ is a subsolution of the problem.

As $\{u_n\}$ is monotone nondecreasing, it converges pointwise to some function $u$, with $\alpha(x) \le u(x) \le \beta(x)$. Moreover, from the proof of Lemma 2.3

$$\|u_{n+1}\|_{H^1(\Omega)} \le C_1 \left\| f(x, u_n) + \frac{\partial f}{\partial u}(x, u_n)[u_{n+1} - u_n] - \lambda u_n \right\|_{L^2(\Omega)} + C_2 \left\| g(x, u_n) + \frac{\partial g}{\partial u}(x, u_n)[u_{n+1} - u_n] + \mu u_n \right\|_{L^2(\partial\Omega)}.$$

Hence, as $\alpha \le u_n \le \beta$, it follows that $\{u_n\}$ is bounded in $H^1(\Omega)$, and from (3.3) we deduce that it is bounded in $H^2(\Omega)$. Moreover, by the compactness of the imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ there exists a subsequence $u_{n_k}$ such that $u_{n_k} \to u$ in $H^1(\Omega)$. For any test function $\varphi \in H_0^1(\Omega)$,

$$-\int_\Omega \nabla u_{n_k} \nabla \varphi = \int_\Omega \left[ f(x, u_{n_k-1}) + \frac{\partial f}{\partial u}(x, u_{n_{k-1}})(u_{n_k} - u_{n_k-1}) \right] \varphi.$$

By dominated convergence, we conclude that $-\int_\Omega \nabla u \nabla \varphi = \int_\Omega f(x, u)\varphi$, and hence $u$ is a weak solution of

$$\Delta u = f(x, u).$$

Therefore $u \in H^2(\Omega)$. Then, if we consider again a test function $\varphi \in H^1(\Omega)$,

$$-\int_\Omega \nabla u_{n_k} \nabla \varphi + \int_{\partial\Omega} \left[ g(x, u_{n_k-1}) + \frac{\partial g}{\partial u}(x, u_{n_k-1})(u_{n_k} - u_{n_k-1}) \right] \varphi = \int_\Omega \left[ f(x, u_{n_k-1}) + \frac{\partial f}{\partial u}(x, u_{n_{k-1}})(u_{n_k} - u_{n_k-1}) \right] \varphi.$$

By dominated convergence,

$$-\int_\Omega \nabla u \nabla \varphi + \int_{\partial\Omega} g(x, u)\varphi = \int_\Omega f(x, u)\varphi.$$

Using the divergence theorem and the fact that the range of the trace operator $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ is $H^{1/2}(\partial\Omega)$ (see e.g. [1], 7.56), which is dense in $L^2(\partial\Omega)$, we conclude that $\frac{\partial u}{\partial \eta} = g(x, u)$ on $\partial\Omega$. □

## 4. Quadratic convergence

In this section we give a proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let us define the error $\mathcal{E}_n = u - u_n$. Using the Taylor expansion around $u_n$, it follows from (3.3) that

$$\begin{cases} \Delta \mathcal{E}_{n+1} = f(x, u) - f(x, u_{n+1}) + \dfrac{1}{2} \dfrac{\partial^2 f}{\partial u^2}(x, \xi)[u_{n+1} - u_n]^2 & \text{in } \Omega \\ \dfrac{\partial \mathcal{E}_{n+1}}{\partial \eta} = g(x, u) - g(x, u_{n+1}) + \dfrac{1}{2} \dfrac{\partial^2 g}{\partial u^2}(x, \theta)[u_{n+1} - u_n]^2 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

for some $\xi = \xi(x) \in [u_n(x), u_{n+1}(x)]$ for $x \in \overline{\Omega}$, and $\theta = \theta(x) \in [u_n(x), u_{n+1}(x)]$ for $x \in \partial\Omega$. Set

$$\lambda = \min_{x\in\overline{\Omega},\alpha(x)\le u\le\beta(x)} \frac{\partial f}{\partial u}(x,u) > 0,$$

$$\mu = -\max_{x\in\partial\Omega,\alpha(x)\le u\le\beta(x)} \frac{\partial g}{\partial u}(x,u) > 0.$$

As $\mathcal{E}_{n+1} \ge 0$, it follows that

$$f(x,u) - f(x,u_{n+1}) \ge \lambda\mathcal{E}_{n+1}$$

and

$$g(x,u) - g(x,u_{n+1}) \le -\mu\mathcal{E}_{n+1}.$$

Thus, we may define $\phi$ as the unique solution of the linear Robin problem

$$\begin{cases} \Delta\phi - \lambda\phi = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial u^2}(x,\xi)\mathcal{E}_n^2 & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial\eta} + \mu\phi = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial u^2}(x,\theta)\mathcal{E}_n^2 & \text{on } \partial\Omega. \end{cases}$$

Using the fact that $0 \le u_{n+1} - u_n \le \mathcal{E}_n$ and that $\frac{\partial^2 f}{\partial u^2}(x,\xi) \le 0 \le \frac{\partial^2 g}{\partial u^2}(x,\theta)$ we conclude

$$\Delta\mathcal{E}_{n+1} - \lambda\mathcal{E}_{n+1} \ge \Delta\phi - \lambda\phi \quad \text{in } \Omega,$$

$$\frac{\partial\mathcal{E}_{n+1}}{\partial\eta} + \mu\mathcal{E}_{n+1} \le \frac{\partial\phi}{\partial\eta} + \mu\phi \quad \text{on } \partial\Omega.$$

By the maximum principle, $\mathcal{E}_{n+1} \le \phi$, and from Lemma 2.3 we deduce that

$$\|\phi\|_{L^\infty(\Omega)} \le c\left\{\left\|\frac{\partial^2 f}{\partial u^2}(x,\xi)\mathcal{E}_n^2\right\|_{L^\infty(\Omega)} + \left\|\frac{\partial^2 g}{\partial u^2}(x,\theta)\mathcal{E}_n^2\right\|_{L^\infty(\partial\Omega)}\right\}.$$

Hence,

$$0 \le \mathcal{E}_{n+1} \le c\|\mathcal{E}_n\|^2_{L^\infty(\Omega)}$$

for some constant $c$ independent of $n$, and the proof is complete. □

## Acknowledgements

This work was partially supported by Fundación Antorchas and CONICET.

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