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A system of coupled pendulii

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Abstract

We study the existence of solutions for a coupled system of n-dimensional pendulum equations under generalized periodic-type conditions. We obtain existence results under appropriate conditions using topological degree methods and a shooting type argument.

Keywords: Pendulum equation; Topological degree methods

1. Introduction

The existence and multiplicity of periodic solutions of one-dimensional pendulum-like equations

$$u'' + g(t, u) = p(t), \tag{1}$$

$$u(0) - u(T) = u'(0) - u'(T) = 0, \tag{2}$$

where g is T-periodic in u have been studied by many authors (see e.g. [5]; for the history and a survey of the problem see [8,9]).

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In order to generalize this problem to higher dimensions, we note that the boundary conditions can be written as

$$u(0) = u(T) = c, \qquad \int_0^T u'' = 0,$$

where c is a non-fixed constant. Thus, by the divergence theorem the periodic boundary conditions can be generalized to a boundary value problem for an elliptic PDE in the following way:

$$\begin{cases} \Delta u + g(x, u) = p(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = \text{constant}, \\ \int_{\partial\Omega} \dfrac{\partial u}{\partial \nu} = 0, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ domain, and $g \in L^\infty(\Omega \times \mathbb{R})$ is T-periodic in u.

This kind of problems have been considered for example in [3], where the authors study a model describing the equilibrium of a plasma confined in a toroidal cavity. Namely, they studied the boundary value problem

$$\begin{cases} \Delta u + g(x, u) = 0 & \text{in } \Omega \\ u = c & \text{on } \partial\Omega \\ \int_{\partial\Omega} \dfrac{\partial u}{\partial \eta} = h \end{cases} \tag{4}$$

with $h \leqslant 0$ and $g : \overline{\Omega} \times \mathbb{R} \to [0, +\infty)$ continuous and non-decreasing on u such that $g(x, u) = 0$ for $u \leqslant 0$. They proved that if

$$\lim_{u \to +\infty} \int_\Omega g(x, u)\, \mathrm{d}x + h > 0$$

and

$$\lim_{u \to +\infty} \frac{g(x, u)}{u^r} = 0$$

for some $r \in \mathbb{R}$ (with $r \leqslant n/(n-2)$ when $n > 2$) problem (4) admits at least one solution $u \in H^2(\Omega)$.

On the other hand, for the particular case $g(x, u) = [u]_+^p$ and $\Omega = B_1(0)$, Ortega has proved in [11] that if $n > 2$ and $p \geqslant n/(n-2)$ then there exists a finite constant h_p such that (4) has no solutions for $h < h_p$.

In [2] the authors proved the existence of a solution of (3) under a condition on the average of the forcing term p. Furthermore, they proved the existence of a compact interval $I_p \subset \mathbb{R}$ such that the problem is solvable for $\tilde{p}(x) = p(x) + c$ if and only if $c \in I_p$.

Using topological methods, the previous result can be extended to the more general problem

$$\begin{cases} \Delta u + \langle a(x), \nabla u \rangle + g(x, u) = p(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = \text{constant}, \\ \int_{\partial\Omega} \dfrac{\partial u}{\partial \nu} = 0, \end{cases} \tag{5}$$

where a is a C^1-field such that $\operatorname{div} a = 0$ [2].

However, for $a \neq 0$ the problem is no longer variational: in the particular case $n = 1$, it is well known that for the pendulum equation

$$u'' + au' + b \sin u = f(t),$$

where a is a positive constant, there exists a family of T-periodic functions f such that $\int_0^T f = 0$ for which the equation has no periodic solutions (see [1,10,12]).

Similar results for the one-dimensional case have been proved by Castro [4] using variational methods, and by Fournier and Mawhin [5], using topological methods.

This work is devoted to the study of a coupled system of n-dimensional pendulum equations under generalized periodic-type conditions, namely, the following elliptic system for a vector function $u : \overline{\Omega} \to \mathbb{R}^N$:

$$\begin{cases} \Delta u_i + \langle a, \nabla u_i \rangle + b_i \sin(u_i) = f_i(x, u) & \text{in } \Omega, \\ u_i = \text{constant} & \text{on } \partial\Omega, \\ \int_{\partial\Omega} \dfrac{\partial u_i}{\partial \nu} = 0. \end{cases} \tag{6}$$

We assume that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, and that $f = (f_1, \ldots, f_N) : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Thus, (6) can be regarded as a generalization of the periodic problem for a coupled system of one-dimensional pendulum equations. We shall prove the existence of solutions of (6) under different conditions on f.

We recall that these equations arise in a model that has been used in Physics to describe different phenomena, as the interaction of atoms in a lattice [6,13].

Our main results read as follows:

Theorem 1.1. *Let $R > 0$ and assume that*

$$\langle f(x, Ru), u \rangle > |b| \quad \forall u \in \partial B, \tag{7}$$

where $|\cdot|$ denotes the euclidean norm and $B \subset \mathbb{R}^N$ is the unit ball. Then problem (6) *admits at least one classical solution u with $\|u\|_{C(\overline{\Omega}, \mathbb{R}^N)} \leqslant R$.*

Remark 1.2. For example, condition (7) holds when f is superlinear, namely:

$$\lim_{u \to \infty} \frac{\langle f(x, u), u \rangle}{|u|^2} = +\infty.$$

Theorem 1.3. *For $i = 1, \ldots, N$ let $R_i^+ \geqslant R_i^-$ and assume that*

$$f_i(x, u_1, \ldots, R_i^+, \ldots, u_N) \geqslant b_i \sin(R_i^+),$$

$$f_i(x, u_1, \ldots, R_i^-, \ldots, u_N) \leqslant b_i \sin(R_i^-),$$

for every u_j such that $R_j^- \leqslant u_j \leqslant R_j^+$ $(j \neq i)$. Then problem (6) *admits at least one classical solution u with $R_i^- \leqslant u_i(x) \leqslant R_i^+$* for $i = 1, \ldots, N$, $x \in \overline{\Omega}$.

This paper is organized as follows. In the second section we present a proof of Theorem 1.1 using a topological degree argument [7].

Finally, in the third section we prove Theorem 1.3 by the method of upper and lower solutions.

2. A topological degree argument

In this section we give a proof of Theorem 1.1. Let $v \in C(\overline{\Omega}, \mathbb{R}^N)$ be fixed, and solve, for a constant $c \in \mathbb{R}^N$ and $\lambda \in [0, 1]$ the linear problem:

$$\Delta u_i + \langle a, \nabla u_i \rangle = \lambda[f_i(x, v) - b_i \sin(v_i)],$$

$$u_i|_{\partial\Omega} = c_i.$$

By standard results, the operator $T : C(\overline{\Omega}, \mathbb{R}^N) \times \mathbb{R}^N \times [0, 1] \to C(\overline{\Omega}, \mathbb{R}^N)$ given by $T(v, c, \lambda) = u$ is well defined and compact. Then, we may define the family of functions $F_\lambda : C(\overline{\Omega}, \mathbb{R}^N) \times \mathbb{R}^N \to C(\overline{\Omega}, \mathbb{R}^N) \times \mathbb{R}^N$ given by

$$F_\lambda(v, c) = \left(v - u, \lambda \int_{\partial\Omega} \frac{\partial u}{\partial \nu} + (1 - \lambda)c \right),$$

where $u = T(v, c, \lambda)$.

Let $B_R \subset C(\overline{\Omega}, \mathbb{R}^N) \times \mathbb{R}^N$ be defined by

$$B_R = \{(u, c) : \|u\|_{C(\overline{\Omega}, \mathbb{R}^N)} < R, |c| < R\},$$

and consider $(u, c) \in \partial B_R$. If $F_1(u, c) = 0$ for some $(u, c) \in \partial B_R$ the result trivially holds; thus, from now on we may assume that $F_1(u, c) \neq 0$ for all $(u, c) \in \partial B_R$. We claim that $F_\lambda(u, c) \neq 0$ for any $\lambda \in [0, 1)$. Indeed, for $\lambda = 0$ if $F_\lambda(u, c) = 0$ then $c = 0$, and $u = T(u, 0, 0) = 0$.

If $F_\lambda(u, c) = 0$ for $\lambda > 0$, define $\varphi(x) = |u(x)|^2$, then

$$\frac{\partial \varphi}{\partial x_j} = 2 \sum_{i=1}^{N} u_i \frac{\partial u_i}{\partial x_j},$$

and

$$\Delta\varphi = 2 \left(\langle \Delta u, u \rangle + \sum_{i=1}^{N} |\nabla u_i|^2 \right).$$

Thus, if $|u(x_0)| = R$ for some $x_0 \in \Omega$, it follows that

$$\sum_{i=1}^{N} \langle a, \nabla u_i(x_0) \rangle u_i(x_0) = \sum_{j=1}^{N} a_j \sum_{i=1}^{N} u_i(x_0) \frac{\partial u_i}{\partial x_j}(x_0) = 0,$$

and hence

$$\Delta\varphi(x_0) \geqslant 2\lambda \left[\langle f(x_0, u(x_0)), u(x_0) \rangle - \sum_{i=1}^{N} b_i \sin(u_i(x_0)) u_i(x_0) \right]$$
$$\geqslant 2\lambda R[\langle f(x_0, Rw), w \rangle - |b|] > 0,$$

where $w = u(x_0)/R$, a contradiction.

Next, assume that $|u| < R$ in Ω, and $|c| = R$. For $x \in \partial\Omega$, we have that

$$\frac{\partial \varphi}{\partial v}(x) \geqslant 0.$$

Then

$$0 \leqslant \sum_{i,j=1}^{N} c_i \frac{\partial u_i}{\partial x_j} v_j = \sum_{i=1}^{N} c_i \frac{\partial u_i}{\partial v}.$$

As $\lambda \int_{\partial\Omega} \partial u/\partial v + (1-\lambda)c = 0$, it follows that

$$\lambda \int_{\partial\Omega} \sum_{i=1}^{N} c_i \frac{\partial u_i}{\partial v} + (1-\lambda) R^2 = 0,$$

and hence $\lambda = 1$, a contradiction.

Finally, note that $F_0(u, c) = (u - c, c)$, from where it follows trivially that $deg(F_0, B_R, 0) = 1$, and from the homotopy invariance of the Leray-Schauder degree we conclude that $deg(F_1, B_R, 0) = 1$ and the proof is complete. □

3. Upper and lower solutions

In this section, we give a proof of Theorem 1.3. Let us define the function $P : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$P(u) = (P_1(u_1), \ldots, P_N(u_N)),$$

where

$$P_i(r) = \begin{cases} r & \text{if } R_i^- \leqslant r \leqslant R_i^+, \\ R_i^+ & \text{if } r > R_i^+, \\ R_i^- & \text{if } r < R_i^-. \end{cases}$$

For $\lambda > 0$ and $v \in C(\overline{\Omega}, \mathbb{R}^N)$ fixed, define $u = Tv$ as the unique solution of the linear system

$$\begin{cases} \Delta u_i + \langle a, \nabla u_i \rangle - \lambda u_i = f_i(x, P(v)) - b_i \sin(P_i(v_i)) - \lambda P_i(v_i) & \text{in } \Omega, \\ u_i = \text{constant} & \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial u_i}{\partial v} = 0. \end{cases}$$

By standard arguments, $T : C(\overline{\Omega}, \mathbb{R}^N) \to C(\overline{\Omega}, \mathbb{R}^N)$ is well defined and compact. Moreover, by Wirtinger inequality and elliptic estimates it follows that:

$$\|u_i\|_{W^{1,p}} \leqslant c \| f_i(x, P(v)) - b_i \sin(P_i(v_i)) - \lambda P_i(v_i)\|_{L^p} \leqslant C$$

for some constant C. Thus, taking $p > n$ we deduce that the range of T is bounded, and by Schauder Theorem T has a fixed point u. We claim that $R_i^- \leqslant u_i \leqslant R_i^+$ for $i = 1, \ldots, N$, and hence u is a solution of the problem. Indeed, if for example $u_i(x_0) > R_i^+$ with x_0 maximum, if $x_0 \in \Omega$ we have that

$$\Delta u_i(x_0) - \lambda u_i(x_0) = f_i(x, P_1(u_1(x_0)), \ldots, R_i^+, \ldots, P_N(u_N(x_0))) \\ - b_i \sin(R_i^+) - \lambda R_i^+ \geqslant -\lambda R_i^+$$

It follows that $\Delta u_i(x_0) \geqslant \lambda(u_i(x_0) - R_i^+) > 0$, a contradiction. On the other hand, if $u_i - R_i^+$ does not achieve a positive maximum in Ω, and $c_i = u_i|_{\partial\Omega} > R_i^+$, define $U = \{x \in \Omega : u_i(x) > R_i^+\}$. For $x \in U$ it follows as before that

$$\Delta u_i + \langle a, \nabla u_i \rangle - \lambda u_i \geqslant -\lambda R_i^+.$$

Let $v_i = u_i - R_i^+$, then

$$v_i \Delta v_i + v_i \langle a, \nabla v_i \rangle - \lambda v_i^2 \geqslant 0$$

in U, and as $v_i|_{\partial U - \partial\Omega} = 0$ and $\int_{\partial\Omega} v_i \partial v_i/\partial \nu = (c_i - R_i^+) \int_{\partial\Omega} \partial v_i/\partial \nu = 0$, we deduce that

$$-\int_U |\nabla v_i|^2 + \left\langle a, \int_U v_i \nabla v_i \right\rangle - \lambda \int_U v_i^2 \geqslant 0.$$

Furthermore,

$$\int_U v_i \nabla v_i = \frac{1}{2} \int_U \nabla(v_i^2) = 0,$$

and hence $|U| = 0$, a contradiction.

In the same way, we deduce that $u_i - R_i^-$ cannot achieve a negative minimum in $\overline{\Omega}$, and the claim is proved. □

Remark 3.1. For the (non-coupled) case $\tilde{f} = p(x) + c$, the existence of an N-dimensional compact interval $I_p \subset \mathbb{R}^N$ such that problem (6) is solvable if and only if $c \in I_p$ can be deduced as in [2]. This is an extension of previous results (see [4,5]).

Remark 3.2. Under appropriate conditions, Theorems 1.1 and 1.3 can be extended for f depending also on ∇u. For a system of ordinary equations, existence of solutions can be proved under Nagumo type conditions for f.

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