Algebras and Representation Theory (2025) 28:873–881
https://doi.org/10.1007/s10468-025-10329-6

CORRECTION

Correction: Universal Quantum (semi)groups and Hopf Envelopes: Erratum

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Received: 5 April 2023 / Accepted: 8 March 2025 / Published online: 16 May 2025

Mathematics Subject Classification (2010) 16T20 · 16T25 · 16T10 · 20G42

1 Error in "Universal Quantum(semi)groups and Hopf Envelopes"

For a bilinear form $b : V \times V \to k$ in a finite dimensional vector space V over a field k with basis x_μ, defined by coefficients

$$b_{\mu\nu} = b(x_\mu, x_\nu)$$

Dubois-Violette and Launer [1] define a Hopf algebra with generators t_λ^μ $(\lambda, \mu = 1, \ldots, \dim V)$ and relations

$$\sum_{\mu,\nu=1}^{\dim V} b_{\mu\nu} t_{\lambda\mu} t_{\rho\nu} = b_{\lambda\rho} 1 \tag{1}$$

$$\sum_{\mu,\nu=1}^{\dim V} b^{\mu\nu} t_{\mu\lambda} t_{\nu\rho} = b^{\lambda\rho} 1 \tag{2}$$

In [3] there is a Lemma 2.1 saying that Eq. 2 is redundant. Unfortunately the proof is incorrect. I thank Hongdi Huang and her collaborators Padmini Veerapen, Van Nguyen, Charlotte Ure, Kent Washaw and Xingting Wang for pointing me up the error. A lot of important consequences in [3] are derived form this lemma, mainly Sections 2 and 3:

- Corollary 2.2 in [3], saying that $A(b)$, the universal bialgebra associated to a bilinear form, is a Hopf algebra.
- Theorem 2.4 in [3], saying that a universal bialgebra associated to a specific bilinear form and a quotient of it is a Hopf algebra.

Presented by: Milen Yakimov

The original article can be found online at https://doi.org/10.1007/s10468-022-10122-9.

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- And the main result: Theorem 3.10 of [3], that says that $A(c)$ = the FRT construction associated to a solution $c : V \otimes V \otimes V \otimes V$ of the braid equation in V admitting a weakly graded-Frobenius algebra (WGF), becomes a Hopf algebra when localizing with respect to the quantum determinant associated that WGF algebra.

On the other hand, the general universal constructions of Section 1 is independent of Lemma 2.1 and the following parts are still safe:

- The bialgebraic nature of the construction (Theorem 1.1)
- Its universal property (Proposition 1.3).
- Example of computation 1.4 and Remark 1.5.
- Section 4: the locally finite graded case and comments on other related works.

2 The Mistake, and Alternatives to Lemma 2.1 in [3]

The mistake in the proof of Lemma 2.1 in [3] lies in the confusion of the matrix $\mathfrak{t}$ with entries $(\mathfrak{t})_{ij} = t_i^j$ between the inverse of $\mathfrak{t}$ and the inverse of the transposed matrix of $\mathfrak{t}$. Even though I do not have a concrete counter-example, I think Lemma 2.1 is false in its full generality. However, one can still view Dubois-Violette and Launer's Hopf algebra as a universal bialgebra construction. Recall briefly the universal construction in [3]:

If V is a finite dimensional vector space with basis $\{x_i\}_{i=1}^{\dim V}$ and $f : V^{\otimes n_1} \otimes V^{\otimes n_2}$ is a linear map, consider free generators t_{ij} $(i, j = 1, \dots, \dim V)$ and using multi-index notation

$$x_I := x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_\ell} \in V^{\otimes \ell}$$

$$t_I^J := t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_\ell j_\ell} \in k\{t_{ij}, i, j = 1, \dots, \dim V\}$$

write $f(x_I) = \sum_J f_I^J x_J$ and define the two-sided ideal $\mathcal{I}_f := \left\langle \sum_J (t_I^J f_J^K - f_I^J t_J^K) : \forall I, K \right\rangle$. We denote by $A(f)$ the algebra defined by

$$A(f) := k\{t_{ij} : i, j = 1 \dots, \dim V\}/\mathcal{I}_f$$

It is proven in [3, Theorem 1.1] that $A(f)$ is in fact a bialgebra, with comultiplication induced by

$$\Delta t_{ij} = \sum_{i=1}^{\dim V} t_{ik} \otimes t_{kj}$$

If $F = \{f_i : V^{\otimes n_1^i} \to V^{\otimes n_2^i}\}_{i \in I}$ is a family of linear maps indexed by a set I, define $\mathcal{I}_F := \sum_{i \in I} \mathcal{I}_{f_i}$ and $A(F)$ is the bialgebra defined by $A(F) := k\{t_{ij} : i, j = 1 \dots, \dim V\}/\mathcal{I}_F$.

2.1 Dubois-Violette and Launer's Hopf Algebra as a Universal Bialgebra

If $b : V \times V \to k$ is a non-degenerate bilinear form and using the notation $b_{ij} = b(x_i, x_j)$, we consider *two* linear maps

$$b : V^{\otimes 2} \to k = V^{\otimes 0}$$

$$x_i \otimes x_j \mapsto b_{ij}$$

and

$$i_b : k \to V^{\otimes 2}$$

$$1 \mapsto \sum_{i,j=1}^{\dim V} b^{ij} x_i \otimes x_j$$

where b^{ij} are the ij-entries of the inverse of the matrix $(B)_{ij} = b_{ij}$. If we denote $H_{DV-L}(b)$ the Dubois-Violette and Launer's Hopf algebra, one tautologically has that

$$H_{DV-L} = A(b, i_b),$$

but not necesarily $H_{DV-L} \stackrel{?}{=} A(b)$.

Let us denote B the matrix with indices $(B)_{\mu\nu} = b_{\mu\nu}$, and keep the "up convention" for $b^{\mu\nu} = (B^{-1})_{\mu\nu}$. Recall the equations for $H_{DV-L} = A(b, i_b)$ are:

$$\sum_{\mu,\nu=1}^{\dim V} b_{\mu\nu} t_{\lambda\mu} t_{\rho\nu} = b_{\lambda\rho} 1 \tag{3}$$

$$\sum_{\mu,\nu=1}^{\dim V} b^{\mu\nu} t_{\mu\lambda} t_{\nu\rho} = b^{\lambda\rho} 1 \tag{4}$$

If $\mathfrak{t}$ is the $n \times n$ matrix with entries $(\mathfrak{t})_{ij} = t_{ij}$ then the equations can be written in matrix form:

$$\mathfrak{t} B \mathfrak{t}^{tr} = B \tag{5}$$

$$\mathfrak{t}^{tr} B^{-1} \mathfrak{t} = B^{-1} \tag{6}$$

where $\mathfrak{t}^{tr}$ denotes the transposed matrix.

Equation 5 says that $\mathfrak{t}$ has a right inverse (and $\mathfrak{t}^{tr}$ has a left inverse), but it is not obvious that this single equation implies that $\mathfrak{t}$ has a left inverse (or that $\mathfrak{t}^{tr}$ has a right inverse). But clearly Eq. 6 says that $\mathfrak{t}$ has an inverse from the other side. The key point when proving both axioms of the antipode in the localization of the FRT construction at the quantum determinant: $A(c)[D^{-1}]$, is to prove that the matrix $\mathfrak{t}$ has both left and right inverse. So, recall the notation in [3], for $c : V^{\otimes 2} \to V^{\otimes 2}$ a solution of the braid equation, denote $A(c)$ its universal bialgebra, that is the algebra with generators t_{ij} and relations

$$\sum_{k,\ell=1}^{\dim V} c_{ij}^{k\ell} t_{kr} t_{\ell s} = \sum_{k,\ell=1}^{\dim V} t_{ik} t_{j\ell} c_{k\ell}^{rs} \qquad \forall\ 1 \leq i, j, r, s \leq n \quad \text{(FRT relations)}$$

that coincides with the FRT construction [2]. Recall that the non-commutative localization of $A(c)$ at an element $D \in A$, denote by $A(c)[D^{-1}]$ is, by definition, the algebra freely generated by $A(c)$ and the symbol "D^{-1}" with (the same relations as in $A(c)$ and)

$$DD^{-1} = 1 = D^{-1} D$$

The case of interest is when D is the particular group-like element comming from the $A(c)$-comodule structure of a Nichols algebra that we call the quantum determinant. It is discussed in [4] that $A(c)[D]$ is naturally a bialgebra with $\Delta(D^{-1}) = D^{-1} \otimes D^{-1}$. Actually, $A(c)[D^{-1}]$ is also co-quasi triangular, as it is $A(c)$.

Recall the following Proposition from [7]:

Proposition 2.1 *[7, Proposition 7.6.9] Suppose that H is a bialgebra over the field k. Then the following are equivalent:*

(a) For all $r > 0$, every $\mathcal{A} \in M_n(H)$ whose coefficients satisfy the comatrix identities has an inverse in $M_n(H)$.
(b) H is a Hopf algebra.

In our case, the matrix $\mathfrak{t} \in M_n(A(c)[D^{-1}])$ satisfies $\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$, so it is a necessary condition that $\mathfrak{t}$ becomes invertible when viewed inside any Hopf algebra, but we need to prove that it is also sufficient. That is, we will not prove that for all comatrix subcoalgebra the corresponding matrix is invertible, but just for a set of generators:

Lemma 2.2 *Assume H is a bialgebra generated by some invertible group-like element D and a set $\{t_{ij}, i, j = 1, \ldots, n\}$ with $\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$ and $\epsilon(t_{ij}) = \delta_{ij}$. Denote $\mathfrak{t} \in M_n(H)$ the $n \times n$ matrix with entries $(\mathfrak{t})_{ij} = t_{ij}$. If there exists an anti-algebra morphism $S : H \to H$ such that $S(D) = D^{-1}$ and $S(t_{ij}) = (\mathfrak{t}^{-1})_{ij}$ then S is an antipode, hence H is a Hopf algebra.*

Proof For group-like elements we know $\epsilon(D) = 1$, and assuming $S(D) = D^{-1}$ we have

$$m(S \otimes \mathrm{id})\Delta(D) = D^{-1}D = 1 = DD^{-1} = m(\mathrm{id} \otimes S)\Delta(D)$$

so, S satisfies the antipode axiom on D. Also, for t_{ij} we assume $\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$ and $\epsilon(t_{ij}) = \delta_{ij}$. So,

$$m(S \otimes \mathrm{id})\Delta(t_{ij}) = \sum_{k=1}^{n} S(t_{ik})t_{kj} = \sum_{k=1}^{n} (\mathfrak{t}^{-1})_{ik}t_{kj} = (\mathfrak{t}^{-1} \cdot \mathfrak{t})_{ij} = \delta_{ij} = \epsilon(t_{ij})$$

and similarly on the other side

$$m(\mathrm{id} \otimes S)\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik}S(t_{kj}) = \sum_{k=1}^{n} t_{ik}(t^{-1})_{kj} = (\mathfrak{t} \cdot \mathfrak{t}^{-1})_{ij} = \delta_{ij} = \epsilon(t_{ij})$$

Now the results follows because if an antihomomorphism verifies the antipode axiom on generators, it satisfies the antipode axioms on all elements (see also next Lemma). □

In a similar same way as we can reduce the checking of the antipode axiom only on algebra generators, we can prove the following Lemma that will be used later:

Lemma 2.3 *Let A, H be a bialgebras, $i : A \to H$ a bialgebra morphism and $S_0 : A \to H$ an anti-algebra map. For comultiplication, we will use "sumless Sweedler notation":*

$$\Delta a = a_{(1)} \otimes a_{(2)}$$

1. *If $a, b \in A$ are such that $S_0(a_{(1)})i(a_{(2)}) = \epsilon(a)$ and $S_0(b_{(1)})i(b_{(2)}) = \epsilon(b)$ then the same formula holds for $c = ab$.*

1'. *If $a, b \in A$ are such that $i(a_{(1)})S_0(a_{(2)}) = \epsilon(a)$ and $i(b_{(1)})S_0(b_{(2)}) = \epsilon(b)$ then the same formula holds for $c = ab$.*

2. *Let $\{t_i : i \in I\}$ a set of generators of A as an algebra. If $S_0\big((t_i)_{(1)}\big)i\big((t_i)_{(2)}\big) = \epsilon(t_i)$ for all $i \in I$ then $S_0(a_{(1)})i(a_{(2)}) = \epsilon(a)$ for all $a \in A$.*
   If $i\big((t_i)_{(1)}\big)S\big((t_i)_{(2)}\big) = \epsilon(t_i)$ for all $i \in I$ then $i(a_{(1)})S_0(a_{(2)}) = \epsilon(a)$ for all $a \in A$.
3. *If $S_0(a_{(1)})i(a_{(2)}) = \epsilon(a)$ for all $a \in A$ and $D \in A$ is a non-zero group-like element in A then $S_0(D)$ is a left inverse of $i(D)$.*
   If also $i(a_{(1)})S_0(a_{(2)}) = \epsilon(a)$ for all $a \in A$ then $S_0(D)$ is a right inverse of $i(D)$.

Proof 1 and 1' are similar, let us check 1: for $c = ab$, using that i is a bialgebra map and S_0 is anti-algebra map, we compute $S_0(c_{(1)})i(c_{(2)})$:

$$S_0(c_{(1)})i(c_{(2)}) = S_0((ab)_{(1)})i((ab)_{(2)}) = S_0(a_{(1)}b_{(1)})i((a_{(2)}b_{(2)})$$
$$= S_0(b_{(1)}) \overbrace{S_0(a_{(1)})i(a_{(2)})}^{\epsilon(a)} i(b_{(2)}) = \epsilon(a)S_0(b_{(1)})i(b_{(2)})$$
$$= \epsilon(a)\epsilon(b) = \epsilon(ab) = \epsilon(c).$$

2 is a straight consequence of 1. For 3, if D is a non-zero group-like element in A then $\epsilon(D) = 1$, so

$$1 = \epsilon(D) = S_0(D_{(1)})i(D_{(2)}) = S_0(D)i(D)$$

The second part of 3 is similar. □

Lemma 2.4 *Assume $0 \neq D \in A(c)$ is a group-like element such that the matrix $\mathfrak{t} \in M_n\big(A(c)\big)$ is invertible when viewed in $A(c)[D^{-1}]$, then $A(c)[D^{-1}]$ is a Hopf algebra.*

Proof We know $A(c)$ is generated by the t_{ij}'s, so $A(c)[D^{-1}]$ is generated by the t_{ij}'s and D^{-1}. We need to have a well-defined antialgebra map $A(c)[D^{-1}] \to A(c)[D^{-1}]$ satisfying conditions of the previous Lemma.

Recall we assume $\mathfrak{t}$ is an invertible matrix in $M_n(A(c)[D^{-1}])$; denote $\mathfrak{u}$ its inverse and $u_{ij} := (\mathfrak{u})_{ij}$. Let us prove that there exists a unique well-defined anti-algebra map

$$S_0 : A(c) \to A(c)[D^{-1}]$$
$$t_{ij} \mapsto u_{ij}$$

Since $A(c)$ is freely generated by the t_{ij} with relations

$$\sum_{k,\ell=1}^{\dim V} c_{ij}^{k\ell} t_{kr} t_{\ell s} = \sum_{k,\ell=1}^{\dim V} t_{ik} t_{j\ell} c_{k\ell}^{rs} \qquad \forall\, 1 \leq i, j, r, s \leq n. \tag{7}$$

one should check the opposite relation in $A(c)[D^{-1}]$

$$\sum_{k,\ell=1}^{\dim V} c_{ij}^{k\ell} u_{\ell s} u_{kr} \stackrel{?}{=} \sum_{k,\ell=1}^{\dim V} u_{j\ell} u_{ik} c_{k\ell}^{rs} \qquad \forall\, 1 \leq i, j, r, s \leq n.$$

But because the matrix $\mathfrak{t}$ is invertible in $M_n(A[D^{-1}])$,we apply the operator

$$\sum_{r,s,i,j=1}^{\dim V} t_{di} t_{cj} \big(- \big) t_{ra} t_{sb}$$

and we get the equivalent checking

$$\sum_{k,\ell,r,s,i,j=1}^{\dim V} t_{di} t_{cj} c_{ij}^{k\ell} u_{\ell s} u_{kr} t_{ra} t_{sb} \stackrel{?}{=} \sum_{k,\ell,r,s,i,j=1}^{\dim V} t_{di} t_{cj} u_{j\ell} u_{ik} c_{k\ell}^{rs} t_{ra} t_{sb} \qquad \forall\, 1 \leq i, j, r, s \leq n.$$

Using on LHS $\sum_{r,s=1}^{\dim V} u_{\ell s} u_{kr} t_{ra} t_{sb} = \delta_{\ell b} \delta_{ka}$ and, on RHS, $\sum_{ij=1}^{\dim V} t_{di} t_{cj} u_{j\ell} u_{ik} = \delta_{dk} \delta_{c\ell}$ we get

$$\sum_{i,j=1}^{\dim V} t_{di} t_{cj} c_{ij}^{ab} \stackrel{?}{=} \sum_{r,s=1}^{\dim V} c_{dc}^{rs} t_{ra} t_{sb}$$

and this is the same relation as Eq. 7, that is valid on $A(c)$, hence, it is valid in $A(c)[D^{-1}]$ as well.

Now we need to extend the map $S_0 : A(c) \to A(c)[D^{-1}]$ to a map

$$A(c)[D^{-1}] \to A(c)[D^{-1}]$$

It will be desirable that $S_0(D)$ computes the inverse of D in $A(c)[D^{-1}]$, in that case S_0 can be extended to $A(c)[D^{-1}]$ by declaring

$$D^{-1} \mapsto S_0(D),$$

giving a well-defined map $S : A(c)[D^{-1}] \to A(c)[D^{-1}]$. But recall D is a non-zero group-like, so, the fact that $S_0(D)$ is the inverse of D in $A(c)[D^{-1}]$ is a consequence of Lemma 2.3 for S_0 and $i : A(c) \to A(c)[D^{-1}]$ the cannonical map. □

3 Nichols Algebras and an Alternative to Theorem 3.10 of [3]

We will use the following well-known facts from finite dimensional Nichols algebras. Assume (V, c) a rigid solution of Braid equation and denote $\mathfrak{B} = \mathfrak{B}(V, c)$ the corresponding Nichols Algebra.

Fact 3.1 *$\mathfrak{B}$ is a graded algebra and coalgebra. It is not a Hopf algebra in the usual sense, but it is a Hopf algebra in the category of $A(c)$-comodules.*

This last fact plays a key role, but it is also very easy to see. Recall $A(c)$ is the universal bialgebra such that $c : V \otimes V \to V \otimes V$ is $A(c)$-colinear, and we can view V not only as an isolated braided vector space, but rather as an object in $A(c)$-comodules, where $A(c)$ is a coquasi-triangular bialgebra, so that the category of $A(c)$-comodules is braided, and the categorical braiding restricted to V is given by c. As a consequence, the construction of the Nichols algebra is done inside this category. Let us briefly recall one possible definition of $\mathfrak{B} = \mathfrak{B}(V, c)$: the quotient of the the tensor algebra TV by the Kernel of the quantum symmetrizer. The quantum symmetrizer is a graded map $TV \to TV$, and restricted to $V^{\otimes n}$ is defined by a sum over the symmetric group

$$QS(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in \mathbb{S}_n} (-1)^{\sigma} L_{\sigma}(v_1 \otimes \cdots \otimes v_n)$$

where L is a set theoretical lifting -using minimal lenght- from $\mathbb{S}_n$ into the braided group $\mathbb{B}_n$ and L_σ is the action of the lifted element associated to σ, acting on $V^{\otimes n}$. The lifting is such that the trasposition $(i, i+1)$ acting on the i and $i+1$ tensor factors is given by $\mathrm{Id}^{\otimes i-1} \otimes c \otimes \mathrm{Id}^{n-i-2}$. For example

$$QS(v \otimes w) = v \otimes w - c(v \otimes w)$$

Is is clear that the Quantum Symmetrizer is $A(c)$-colinear, because it is constructed using c and tensor products of c with identity maps, hence $\mathfrak{B}$ is an $A(c)$-comodule algebra. Also, one can give TV the structure of a bialgebra on the category of $A(c)$-comodules by means of the map

$$V \to V \otimes k \oplus k \otimes V \subset TV \otimes TV$$

$$v \mapsto v \otimes 1 + 1 \otimes v$$

and extending it multiplicatively, where the algebra structure on $TV \otimes TV$ *is not* the standar one, but the one ussing the braiding in order to interchange factors. Checking on generators it is an easy and standard fact to see that TV, with that braided-defined comultiplication, is (not a Hopf algebra in the usual sense but) a Hopf algebra in the braided category of $A(c)$-comodules, and its quotient $\mathfrak{B}$ is also a Hopf algebra in the same category.

In particular, multiplication and comultiplication is $A(c)$-colinear, and since the Quantum Symmetrizer is homogeneous, $\mathfrak{B}$ is also graded, and each homogeneous component is a subcomodule. Hence, multiplication and comultiplication is also colinear when restricted / projected to particular homogeneous parts of $\mathfrak{B}$ or $\mathfrak{B} \otimes \mathfrak{B}$.

Fact 3.2 *Assume c is rigid and $\mathfrak{B}$ is finite dimensional. Denoting $\mathfrak{B}^{top}$ the highest non-zero degree of $\mathfrak{B}$, one has $\dim \mathfrak{B}^{top} = 1$, say $\mathfrak{B}^{top} = k\mathfrak{b}$ for a choice of a non-zero element $\mathfrak{b} \in \mathfrak{B}^{top}$. The projection into the coefficient of $\mathfrak{b}$*

$$\mathfrak{B} \ni \omega = \omega_0 + \omega_1 + \cdots + \omega_{top} = \omega_0 + \omega_1 + \cdots + \lambda\mathfrak{b} \quad \longmapsto \quad \lambda \in k$$

is an integral of the braided Hopf algebra $\mathfrak{B}$. This is a classical fact, for a modern exposition see [5, Theorem 4.4.13]. Also, because of $\dim \mathfrak{B}^{top} = 1$, the $A(c)$-comodule structure of $\mathfrak{B}^{top}$ gives a non trivial grouplike element D determined by

$$\rho(\mathfrak{b}) = D \otimes \mathfrak{b}$$

(see [4, Section 2]).

Another usefull fact, also discussed in [4], is the non degeneracy of the product map, a classical fact that goes back to the original work of Nichols [6], see also the survey of M. Takeuchi [8] in the language of braided Hopf algebras:

Fact 3.3 *Assume c is rigid and $\mathfrak{B}$ is finite dimensional. For each degree p, the multiplication map induces a non-degenerate pairing*

$$m| : \mathfrak{B}^p \otimes \mathfrak{B}^{top-p} \to \mathfrak{B}^{top}$$

In particular, since $\mathfrak{B}^1 = V$, the restriction of the multiplication

$$V \otimes \mathfrak{B}^{top-1} \to \mathfrak{B}^{top} = \mathfrak{b}k$$

gives a non-degenerate pairing.

In terms of basis, if ω^i is a basis of $\mathfrak{B}^{top-1}$ and x_i is a basis of V, then write

$$\omega^i x_j = m_{ij}\mathfrak{b}, \quad m_{ij} \in k$$

and the matrix (m_{ij}) is invertible, and one can choose a "dual basis" ω^i such that

$$x_i \omega^j = \delta_{ij}\mathfrak{b}$$

Fact 3.4 *If (V, c) is rigid then so is (V^*, c^*), and $\mathfrak{B}(V^*, c^*)$ is the graded dual (as algebra and coalgebra) of $\mathfrak{B}(V, c)$ (see [5, Section 7.2]).*

Fact 3.5 *By duality, and using Fact 3.3, denoting* coev *the comultiplication composed with projection*

$$\begin{tikzcd} \mathfrak{B}^{top} \arrow[r, "\Delta"] \arrow[rr, bend right, "\mathrm{coev}"'] & \bigoplus_p \mathfrak{B}^{top-p} \otimes \mathfrak{B}^p \arrow[r, "\pi"] & \mathfrak{B}^{top-1} \otimes V \end{tikzcd},$$

it is non-degenerate, in the sense that if x_i, $\widehat{\omega}^i$ are bases of V and $\mathfrak{B}^{top-1}$ respectively, and

$$\mathrm{coev}(\mathfrak{b}) = \sum_{i,j=1}^{\dim V} \mathrm{coev}_{ij}\widehat{\omega}^i \otimes x_j, \quad \mathrm{coev}_{ij} \in k$$

then the matrix (coev_{ij}) is invertible. In particular, there exists a basis $\widehat{\omega}^i$ such that

$$\mathrm{coev}(\mathfrak{b}) = \sum_{i=1}^{\dim V} \widehat{\omega}^i \otimes x_i$$

We remark that even if we call "coevaluation", we will not need to prove any categorical coevaluation property, we will only use that the $n \times n$ matrix with coeficients in k given by coev_{ij} is invertible in $M_n(k)$. This invertibility conditon is a clear consequence of Fact 3.4 because the map in 3.5 is the dual of multiplication map as in 3.3, but corresponding to V^* instead of V.

Fact 3.6 *The maps in 3.3 and 3.5 are $A(c)$-colinear.*

This tast fact is clear because $\mathfrak{B}$ is a graded Hopf algebra in the category of $A(c)$-comodules. Now denote $\rho : \mathfrak{B} \to A(c) \otimes \mathfrak{B}$ the comodule structure map and write $\rho(\omega^j) = T_{jk} \otimes \omega^k$ and $\rho(\widehat{\omega}^j) = \widehat{T}_{jk} \otimes \widehat{\omega}^k$, where $\{\omega^j\}$ and $\{\widehat{\omega}^j\}$ are basis of $\mathfrak{B}^{top-1}$ as in 3.3 and 3.5. Using that $\mathfrak{B}$ is an $A(c)$-comodule algebra we have:

$$D \otimes \delta_{ij}\mathfrak{b} = \rho(\delta_{ij}\mathfrak{b}) = \rho(x_i\omega^j) = \sum_{k,l=1}^{\dim V} t_{ik}T_{jl} \otimes x_k\omega^l = \sum_{k,l=1}^{\dim V} t_{ik}T_{jl} \otimes \delta_{kl}\mathfrak{b}$$

$$= \sum_{k=1}^{\dim V} t_{ik}T_{jk} \otimes \mathfrak{b} \Rightarrow \sum_{k=1}^{\dim V} t_{ik}T_{jk} = D\delta_{ij}$$

$$\iff \mathfrak{t} \cdot T^{tr} = D\,\mathrm{id}$$

By the $A(c)$-colinearity of ρ:

$$\rho(\sum_i \widehat{\omega}^i \otimes x_i) = \rho(\mathrm{coev}(\mathfrak{b})) = (1 \otimes \mathrm{coev})\rho(\mathfrak{b})$$

$$= (1 \otimes \mathrm{coev})(D \otimes \mathfrak{b}) = D \otimes \sum_i (\widehat{\omega}^i \otimes x_i)$$

but also

$$\rho(\sum_i \widehat{\omega}^i \otimes x_i) = \sum_{i,j,k} \widehat{T}_{ij}t_{ik} \otimes \widehat{\omega}^j \otimes x_k$$

This proves

$$\sum_i \widehat{T}_{ij}t_{ik} = D\delta_{jk} \Rightarrow \widehat{T}^{tr} \cdot \mathfrak{t} = D\,\mathrm{id}$$

hence, $D^{-1}\widetilde{T}^{tr}$ is a left inverse of $\mathfrak{t}$, that is, $\mathfrak{t}$ is invertible in $M_n\big(A(c)[D^{-1}]\big)$.

Using Lemma 2.4 and observing that $A(c) = A(qc)$ for any $0 \neq q \in k$, one can conclude the following main result, that is an alternative to Theorem 3.10 in [3]:

Theorem 1 *Let V be a finite dimensional vector space, $c : V^{\otimes 2} \to V^{\otimes 2}$ a rigid solution of the braid equation and assume there is a non-zero scalar $0 \neq q \in k$ such that $\mathfrak{B} := \mathfrak{B}(V, cq)$ is finite dimensional. Denote D the associated group-like element in $A(c)$ coming from the $A(c)$-comodule structure of $\mathfrak{B}^{top}$. Then $A(c)[D^{-1}]$ is a Hopf algebra.*

Acknowledgements I thank the referee for careful reading of this manuscript and his/her suggestions that improves the presentations.

Author Contributions In this work I answer in an affirmative way the question of whether the localization of the FRT construction with respect to a quantum determinant is a Hopf algebra, assuming that the Nichols algebra associated to the braiding is finite dimensional.

Funding Partially supported by UBACyT "K-teoría y bialgebras en álgebra, geometría y topología" and PICT 2018-00858 "Aspectos algebraicos y analíticos de grupos cuánticos".

Data Availability This manuscript has no associated data.

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare no competing interests.

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