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THE WEIGHTED CORE WITH DISTINGUISHED COALITIONS

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In this paper we generalize the studies of Bondareva–Shapley for a general core having weights in the definition. These weights were introduced by Billera but not used in the form as here. Moreover we study such a core with conditions of equalities for some coalitions analogously as those obtained *a posteriori* in the assignment games due to Shapley and Shubik.

Keywords: Cooperative games; core; weight; distinguished coalitions; balanced collection.

1. Introduction

In the theory of game there were many sequential works related with the core. Let us state Bondareva [1962], Billera [1970, 1971], Scarf [1967] and Shapley [1970]. However even if a result of Billera [1970] tells us that every π-balanced game with $\pi \in \Pi$, has a nonempty core, where π is a suitable matrix, up today it is not found in the literature the weighted core with distinguished coalitions.

Let N be a finite set of players, $\#(N) = n$; $\gamma = (\gamma_{iS})$ be a matrix of order $n(2^n - 1)$ for each $i \in N, S \in \mathcal{P}(\mathcal{N}) - \{\emptyset\}$ and $\gamma_{iS} > 0$. We denote with v the characteristic function of the game $G = (N, v)$.

2. H−γ-core, γ-balanced Collections

Consider a family of nonempty subsets of N, $\mathcal{H} = \{H_1, \ldots, H_p\}$ such that $N \subset \cup_{S \notin \mathcal{H}}$.

Definition 2.1. The H-γ-core (or weighted core) is the set of vectors $(x_1, x_2, \ldots, x_n)$ such that

$$\sum_{i=1}^{n} \gamma_{iN} x_i = v(N)$$

$$\sum_{i \in S} \gamma_{iS} x_i = v(S) \quad \text{for } S \in \mathcal{H}$$

$$\sum_{i \in S} \gamma_{iS} x_i \geq v(S) \quad \text{for } S \notin \mathcal{H}\ S \subseteq N.$$

The notation for the H–γ-core will be $C_\gamma^{\mathcal{H}}(v)$.
It is of interest to determine the games with $C_\gamma^{\mathcal{H}}(v)$ nonempty.

As a simple interpretation we have for those coalitions in $\mathcal{H}$ they determine *a priori* which the coalitions for the maximal maching determine *a posteriori* in assignment game, which are fundamental. For this, the reader might read Shapley and Shubik [1972].

Secondly, the weights in left part of the equation of the core might be interpreted in the following way: consider a place where a player plays a n-person game weighed characteristic function v but with the condition that if he gets x_i he will pay to the owner of the agent in charge of the game the amount $(1-\gamma_i)x_i$ and he keeps $\gamma_i x_i$. Then the H–γ-core will be the imputations which cannot be improved upon. Similar interpretation might be done for the coefficient γ_{iS}.

In order to characterize the game with $C_\gamma^{\mathcal{H}}(v) \neq \emptyset$ we have that if and only if the linear program

$$\min \sum_{i=1}^{n} \gamma_{iN} x_i = z^* \tag{2.1}$$

subject to

$$\sum_{i\in S} \gamma_{iS} x_i = v(S) \quad \text{if } S \in \mathcal{H} \tag{2.2}$$

$$\sum_{i\in S} \gamma_{iS} x_i \geq v(S) \quad \text{if } S \notin \mathcal{H},\ \ S \subseteq N \tag{2.3}$$

has a minimum $z^* \leq v(N)$.

Now if one considers the dual program of (2.1); (2.3) which is

$$\max_y \sum_{S\subseteq N} y_S v(S) = q^* \tag{2.4}$$

subject to

$$\sum_{\substack{S \\ i\in S}} \gamma_{iS} y_S = \gamma_{iN} \qquad \text{for each } i \in N \tag{2.5}$$

$$y_S \geq 0 \quad \text{if } S \notin \mathcal{H},\ \ S \subseteq N \tag{2.6}$$

$$y_S \text{ unrestricted} \quad \text{if } S \in \mathcal{H}. \tag{2.7}$$

If both programs (2.1)–(2.3) and (2.4)–(2.7) are feasible, the minimum z^* must be equal to the maximum q^*, then $C_\gamma^{\mathcal{H}}(v) \neq \emptyset$ if and only if the maximum $q^* \leq v(N)$. In other words

Theorem 2.1. *A necessary and sufficient condition in order that the game* $G = (N, v)$ *has* H-γ-*core nonempty is that for each vector* $(y_S)_{S \subseteq N}$ *satisfying* (2.5)–(2.7) *verifies*

$$\sum_{S \subseteq N} y_S v(s) \leq v(N). \tag{2.8}$$

Definition 2.2. Let $\mathcal{C} = \{S_1, \ldots, S_m\}$ a collection of nonempty subsets of N such that

$$N \subset \cup \{S_j \in \mathcal{C} \mid S_j \notin \mathcal{H}\}. \tag{2.9}$$

We say that $\mathcal{C}$ is N–γ-balanced with respect to the collection $\mathcal{H}$, if there exists a vector $(y_{Sj})_{j=1}^m$ such that

$$\begin{cases} y_{S_j} > 0 & \text{if } S_j \notin \mathcal{H} \\ y_{S_j} \text{ unrestricted} & \text{if } S_j \in \mathcal{H} \\ \sum_{i \in S_j} \gamma_{iS_j} y_{S_j} = \gamma_{iN} & \forall i \in N. \end{cases} \tag{2.10}$$

The vector $(y_{S_j})_{j=1}^m$ is a γ-balanced vector for $\mathcal{C}$.

Examples:

(1) The collection $\{N\}$ is γ-balanced for any $\mathcal{H}$ and any matrix $\gamma = (\gamma_{iS})$. A γ-balanced vector is $y_N = 1$.

(2) The collection $\mathcal{C} = \{N, H_1, \ldots, H_l \mid H_i \in \mathcal{H}, 1 \leq i \leq l, N \notin \mathcal{H}\}$ is γ-balanced for any matrix $\gamma = (\gamma_{iS})$ such that

$$\frac{1}{\gamma_{iN}} \sum_{\substack{H_j \\ i \in H_j}} \gamma_{iH_j} y_{H_j} = \frac{1}{\gamma_{i'N}} \sum_{\substack{H_j \\ i' \in H_j}} \gamma_{i'H_j} y_{H_j} = (1 - y_N)$$

for each $i, i' \in N$.

(3) $\mathcal{C} = \{\{1\}, \{2\}, \ldots, \{n\}\}$ is γ-balanced for any matrix $\gamma = (\gamma_{iS})$ and any collection $\mathcal{H}$. A γ-balanced vector is

$$(y_{\{1\}}, \ldots, y_{\{n\}}) = \left(\frac{\gamma_{1N}}{\gamma_{1\{1\}}}, \ldots, \frac{\gamma_{nN}}{\gamma_{n\{n\}}}\right).$$

(4) $\mathcal{C} = \{N, \{1\}, \ldots, \{n\} \mid \{i\} \notin \mathcal{H}, 1 \leq i \leq n\}$ is γ-balanced for any matrix $\gamma = (\gamma_{iS})$. A γ-balanced vector for $\mathcal{C}$ is:

$$(y_N, y_{\{1\}}, \ldots, y_{\{n\}}) = \left(\epsilon, \frac{\gamma_{1N}(1-\epsilon)}{\gamma_{1\{1\}}}, \ldots, \frac{\gamma_{nN}(1-\epsilon)}{\gamma_{n\{n\}}}\right)$$

with $0 < \epsilon < 1$.

(5) Let $N = \{1, 2, 3\}$, $\mathcal{C} = \{\{1\}, \{1, 2\}, \{3\}\}$ and $\gamma = (\gamma_{iS})$ a matrix verifying $\gamma_{iN}\gamma_{2\{1,2\}} - \gamma_{1\{1,2\}}\gamma_{2N} < 0$. $\mathcal{C}$ is γ-balanced if $\{1\} \in \mathcal{H}$ and a γ-balanced vector for $\mathcal{C}$ is:

$$(y_{\{1\}}, y_{\{1,2\}}, y_{\{3\}}) = \left(\frac{\gamma_{1N}\gamma_{2\{1,2\}} - \gamma_{1\{1,2\}}\gamma_{2N}}{\gamma_{1\{1\}}\gamma_{2\{1,2\}}}, \frac{\gamma_{2N}}{\gamma_{2\{1,2\}}}, \frac{\gamma_{3N}}{\gamma_{3\{3\}}}\right).$$

The following properties of γ-balanced collections are of interest.

Theorem 2.2. *The union of γ-balanced collections is γ-balanced.*

Proof. Let $\mathcal{C} = \{S_1, \dots, S_m\}$ and $\mathcal{D} = \{T_1, \dots, T_k\}$ γ-balanced collections with γ-balanced vectors $(y_{S_1}, \dots, y_{S_m})$ and $(z_{T_1}, \dots, z_{T_k})$ respectively. Then

$$\mathcal{C} \cup \mathcal{D} = \{\mathcal{R}_1, \dots, \mathcal{R}_q\} \quad \text{with } q \leq m + k$$

for any $t, 0 < t < 1$, we define

$$w_{R_j} = \begin{cases} t y_{S_j} & \text{if } R_j = S_j \in \mathcal{C} - \mathcal{D} \\ (1-t) z_{T_p} & \text{if } R_j = T_p \in \mathcal{D} - \mathcal{C} \\ t y_{S_j} + (1-t) z_{T_p} & \text{if } R_j = S_l = T_p \in \mathcal{C} \cap \mathcal{D}. \end{cases}$$

It is easy to verify that the vector $(w_{R_1}, \dots, w_{R_q})$ is a γ-balanced vector for $\mathcal{C} \cup \mathcal{D}$ independently that $R_j \in \mathcal{H}$ or $R_j \notin \mathcal{H}$. □

From here, by induction, the union of any number of γ-balanced collections is γ-balanced.

Lemma 2.1. *Let $\mathcal{C} = \{S_1, S_2, \dots, S_k\}$ and $\mathcal{D} = \{S_1, S_2, \dots, S_k, \dots, S_m\}$ be two γ-balanced collections with γ-balanced vectors. $y = (y_{S_1}, \dots, y_{S_k})$ and $z = (z_{S_1}, \dots, z_{S_k}, \dots, z_{S_m})$ respectively such that*

$$\begin{cases} (a)\ \mathcal{C} \subset \mathcal{D} \quad \mathcal{C} \neq \mathcal{D} \\ (b)\ S_j \in \mathcal{H}, \quad S_j \in \mathcal{D} \Rightarrow S_j \in \mathcal{C} \\ (c) \displaystyle\sum_{\substack{S \subset \mathcal{H} \cap \mathcal{C} \\ i \in S}} \gamma_{iS} y_S = \sum_{\substack{S \in \mathcal{H} \cap \mathcal{D} \\ i \in S}} \gamma_{iS} z_S \quad \textit{for each, } i \in N. \end{cases} \tag{2.11}$$

Then there exists a γ-balanced collection $\mathcal{B}$ such that $\mathcal{B} \cup \mathcal{C} = \mathcal{D}$ but $\mathcal{D} \neq \mathcal{B}$.

Proof. For $t > 0$ let us define

$$\begin{aligned} w_{S_j} &= (1+t) z_{S_j} - t y_{S_j} \quad & j = 1, 2, \dots k \\ w_{S_j} &= (1+t) z_{S_j} & j = k+1, \dots, m. \end{aligned}$$

If $S \notin \mathcal{H}$ for $t > 0$ and small, it results $w_{S_j} > 0$. Moreover, for $i \in N$

$$\begin{aligned} \sum_{\substack{S_j \\ i \in S_j \in \mathcal{D}}} \gamma_{iS_j} w_{S_j} &= \sum_{\substack{S_j \\ i \in S_j \in \mathcal{C}}} \gamma_{iS_j} [(1+t) z_{S_j} - t y_{S_j}] + \sum_{\substack{S_j \\ i \in S_j \in \mathcal{D} - \mathcal{C}}} \gamma_{iS_j} (1+t) z_{S_j} \\ &= -t \sum_{\substack{S_j \\ i \in S_j \in \mathcal{C}}} \gamma_{iS_j} y_{S_j} + (1+t) \sum_{\substack{S_j \\ i \in S_j \in \mathcal{D}}} \gamma_{iS_j} z_{S_j} = \gamma_{iN}. \end{aligned}$$

In this way $w = (w_S)_{S \in \mathcal{D}}$ is a γ-balanced vector for $\mathcal{D}$. Since $w_{S_j} > z_{S_j} > 0$ for $k + 1 \leq j \leq m$, it is easy to see that z is not unique.

Besides, there exists an j which $1 \leq j \leq m, S_j \notin \mathcal{H}$ such that $z_{S_j} < y_{S_j}$. Suppose the contrary, then for each $j, 1 \leq j \leq k, z_{S_j} \geq y_{S_j}$ and

$$\begin{aligned}\gamma_{iN} &= \sum_{\substack{S_j \\ i \in S_j \in \mathcal{C}}} \gamma_{iS_j} y_{S_j} = \sum_{\substack{S_j \in \mathcal{C} \cap \mathcal{H} \\ i \in S_j}} \gamma_{iS_j} y_{S_j} + \sum_{\substack{S_j \in \mathcal{C} - \mathcal{H} \\ i \in S_j}} \gamma_{iS_j} y_{S_j} \\ &\leq \sum_{\substack{S_j \in \mathcal{C} \cap \mathcal{H} \\ i \in S_j}} \gamma_{iS_j} y_{S_j} + \sum_{\substack{S_j \in \mathcal{C} - \mathcal{H} \\ i \in S_j}} \gamma_{iS_j} z_{S_j} < \sum_{\substack{S_j \in \mathcal{D} \\ i \in S_j}} \gamma_{iS_j} z_{S_j} = \gamma_{iN}\end{aligned}$$

which is a contradiction. The last inequalities follow from the conditions (2.11) (b) and (c).

Let

$$\bar{t} = \min\left\{ \frac{z_{S_j}}{y_{S_j} - z_{S_j}} \mid y_{S_j} > z_{S_j}, S_j \notin \mathcal{H} \right\}$$

$$\mathcal{C}' = \{S_j \in \mathcal{C} \mid (1+\bar{t}) z_{S_j} = \bar{t} y_{S_j}\} \quad \text{and} \quad \mathcal{B} = \mathcal{D} - \mathcal{C}'.$$

Clearly $\mathcal{C}'$ is nonempty subcollection of $\mathcal{C}$ then $\mathcal{D} \neq \mathcal{B}$; $\mathcal{B} \cup \mathcal{C} = \mathcal{D}, w_{S_j} > 0$ for each $S_j \notin \mathcal{H}$ and it is easy to verify that (for $\bar{t}$) $(w_S)_{S \in \mathcal{B}}$ is a γ-balanced vector for the collection $\mathcal{B}$. □

Example. (illustrating Lemma 2.5).

Let $N = \{1,2,3\}$ and γ the matrix

	{1}	{2}	{3}	{1,2}	{2,3}	{1,3}	{N}
1	2	4	7	10	3	1	1/2
2	5	5	3	2/3	1/4	1/5	1/3
3	1	1/3	1/2	9	1	2	5

$$\mathcal{H} = \{\{1,2\}\} \quad \mathcal{C} = \{\{1\},\{2\},\{3\}\} \cup \mathcal{H} \quad \mathcal{C} \neq \mathcal{D}$$
$$\mathcal{D} = \{\{1\},\{2\},\{3\},\{2,3\}\} \cup \mathcal{H}.$$

$\mathcal{C}$ is γ-balanced with a γ-balanced vector

$$y = (21/4, 1/5, 10, -1)$$

and it is verified that

$$\sum_{\substack{S \in \mathcal{C} \cap \mathcal{H} \\ i \in S}} \gamma_{iS} y_S = c_i \quad \text{for each } i$$

and in our case

$$c_1 = -\gamma_{1\{1,2\}} = -10$$
$$c_2 = -\gamma_{2\{1,2\}} = -2/3,$$

$\mathcal{D}$ is γ-balanced with γ-balanced vector

$$\begin{aligned} z &= (21/4, 3/20, 8, 1, -1) \\ \bar{t} &= 3 \\ \mathcal{C}' &= \{S \in \mathcal{C} \mid (1+3)z_S = 3y_S\} = \{\{2\}\} \\ \mathcal{B} &= \mathcal{D} - \mathcal{C}' = \{\{1\}, \{3\}, \{2,3\}\} \cup \mathcal{H} \end{aligned}$$

and $w = (21/4, 6, 4, -1)$ is γ-balanced for $\mathcal{B}$.

Definition 2.3. A γ-balanced collection $\mathcal{B}$ with *a* γ-balanced vector $(y_S)_{S\in\mathcal{B}}$ is minimal if $\mathcal{B}' = \{S \in \mathcal{B} \mid S \notin \mathcal{H}\}$ does not have a proper sub collection $\mathcal{B}''$ such that $\mathcal{B}'' \cup (\mathcal{B} \cap \mathcal{H})$ is γ-balanced with γ-balanced vector $(z_S)_{S\in\mathcal{B}''\cup(\mathcal{B}\cup\mathcal{H})}$ which verifies

$$\sum_{\substack{S\in\mathcal{B}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} y_S = \sum_{\substack{S\in\mathcal{B}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} z_S$$

for each $i \in N$.

Theorem 2.3. *Any γ-balanced collection is union of γ-balanced minimal collections.*

Proof. By induction on m (the number of subsets of the collection). For $m = 1$, since the unique γ-balanced collection with one element is $\{N\}$ and this is minimal, the theorem is true.

Assume that the theorem is valid for all the collection with $(m-1)$ or less elements. Let $\mathcal{D}$ be *a* γ-balanced collection with m-elements. If $\mathcal{D}$ is minimal, then it is a union of γ-balanced minimal collection. If $\mathcal{D}$ is not minimal, then $\mathcal{D}' = \{S \in \mathcal{D} \mid S \notin \mathcal{H}\}$ contains a proper subcollection, $\mathcal{D}''$ such that $\mathcal{D}'' \cup (\mathcal{D} \cap \mathcal{H})$ is γ-balanced satisfying (2.11). By Lemma 2.5 there exists another proper γ-balanced collection $\mathcal{B}$ such that

$$\mathcal{B} \cup (\mathcal{D}'' \cup (\mathcal{D} \cap \mathcal{H})) = \mathcal{D}.$$

Since $\mathcal{B}$ and $\mathcal{D}'' \cup (\mathcal{D} \cap \mathcal{H})$ are proper subcollections of $\mathcal{D}$, they have $(m-1)$ or less elements and therefore each one of them, can be express as union of minimal sub collection and the theorem is true. □

Definition 2.4. A γ-balanced collection $\mathcal{C}$ has a distinguished γ-balanced vector if and only if for each pair (w_S) and (z_S) of γ-balanced vectors verifying

$$\sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} w_S = \sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} z_S$$

it holds $w_S = z_S$ for each $S \in \mathcal{C} - \mathcal{H}$.

Theorem 2.4. *Under the conditions of Lemma 2.5 a γ-balanced collection has a γ-balanced distinguished vector if and only if it is minimal.*

Proof. The Lemma 2.5 says that *a* γ-balanced vector is distinguished only for γ-balanced minimal collections.

In the reverse way, suppose that $\mathcal{C}$ does not have a γ-balanced distinguished vector, that is, there exists vectors y and z such that

$$\sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} y_S = \sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} z_S$$

and $y_S \neq z_S$ for at least a $S \in \mathcal{C} - \mathcal{H}$.

Without loss of generality we assume $y_S > z_S$. Choose $w = (1+\bar{t})z - \bar{t}y$ where

$$\bar{t} = \min\left\{\frac{z_S}{y_S - z_S} \mid y_S > z_S, S \in \mathcal{C} - \mathcal{H}\right\}.$$

Then w is *a* γ-balanced vector for

$$\mathcal{B} = \{S \in \mathcal{C} - \mathcal{H} \mid (1+\bar{t})z_S > \bar{t}y_S\} \cup (\mathcal{C} \cap \mathcal{H})$$

Since $\mathcal{B}$ is a proper sub collection of $\mathcal{C}$, and

$$\sum_{\substack{S\in\mathcal{B}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} w_S = \sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} w_S = \sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} z_S + \bar{t}\sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} z_S - \bar{t}\sum_{\substack{S\in\mathcal{C}\cap\mathcal{H}\\ i\in S}} \gamma_{iS} y_S$$

then $\mathcal{C}$ is not minimal. □

Theorem 2.5. *If y is an extreme point of the program* (2.5)–(2.7), *then it is a γ-balanced vector of a γ-balanced minimal collection.*

Proof. Let $(y_S)_{S\subset N}$ *a* vector verifying (2.5)–(2.7) of the program. Such a vector is of γ-balanced for the collection

$$\mathcal{C} = \{S \notin \mathcal{H} \mid y_S > 0\} \cup \mathcal{H}.$$

If $\mathcal{C}$ is not minimal let $\mathcal{B}$ *a* proper sub collection of $\mathcal{C}$ (as in the Definition 2.3) with γ-balanced vector z. If $z_S > 0, S \notin \mathcal{H}$ then $y_S > 0$ then for small values of t, both

$$w = (1-t)y + tz$$
$$w' = (1+t)y - tz$$

satisfy the conditions (2.5)–(2.7). Besides $w \neq w'$ and $w_S < w'_S$ for any $S \notin \mathcal{H}$ and $S \in \mathcal{C} - \mathcal{B}$. But $y = 1/2(w + w')$ and then y is not extreme of the program (2.5)–(2.7). □

Theorem 2.6. *If y is a vector γ-balanced of a γ-balanced minimal collection then y is extreme point of the program*

$$\max_y \sum_{\substack{S\subseteq N\\ S\notin\mathcal{H}}} y_S v(S) = q* \qquad (2.12)$$

subject to

$$\sum_{\substack{S \\ i \in S}} \gamma_{iS} y_S = \gamma_{iN} \qquad \textit{for each } i \in N \tag{2.13}$$

$$y_S \geq 0 \qquad \textit{if } S \notin \mathcal{H},\ S \subseteq N \tag{2.14}$$

$$\sum_{\substack{S \in \mathcal{H} \\ i \in S}} \gamma_{iS} y_S = c_i\ (cte) \qquad \textit{for each } i \in N. \tag{2.15}$$

Proof. Assume that $\mathcal{C}$ is a γ-balanced minimal collection with γ-balanced vector y. If y is not extreme let $y = 1/2(w + w')$ where $w \neq w'$ and both satisfy (2.13)–(2.15). By the nonnegativity conditions (2.14) it holds $w_S = w'_S = 0$ when $y_S = 0$ and $S \notin \mathcal{H}$.

Then w and w' will be γ-balanced vectors for $\mathcal{B}$ and $\mathcal{B}'$ respectively, where

$$\mathcal{B} = \{S \in \mathcal{C} - \mathcal{H} \mid w_S > 0\} \cup \mathcal{H}$$
$$\mathcal{B}' = \{S \in \mathcal{C} - \mathcal{H} \mid w'_S > 0\} \cup \mathcal{H}$$

and both are subcollections of $\mathcal{C}$. Since $\mathcal{C}$ is minimal then $\mathcal{B} = \mathcal{B}' = \mathcal{C}$ and by Theorem 2.9 for each $S \in \mathcal{C} - \mathcal{H}$ it is valid $w_S = w'_S = y_S$.

This contradiction proves that y must be extreme. □

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