

Exact Sequences of Tensor Categories

Alain Bruguières¹ and Sonia Natale²

¹Département de Mathématiques, Université Montpellier II, Place Eugène Bataillon, 34 095 Montpellier, France and ²Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, CIEM – CONICET, Ciudad Universitaria, 5000 Córdoba, Argentina

Correspondence to be sent to: e-mail: natale@famaf.unc.edu.ar/sonia_natale@yahoo.com.ar

We introduce the notions of normal tensor functor and exact sequence of tensor categories. We show that exact sequences of tensor categories generalize strictly exact sequences of Hopf algebras as defined by Schneider, and in particular, exact sequences of (finite) groups. We classify exact sequences of tensor categories $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ (such that \mathcal{C}' is finite) in terms of normal, faithful Hopf monads on \mathcal{C}'' and also in terms of self-trivializing commutative algebras in the center of \mathcal{C} . More generally, we show that, given any dominant tensor functor $\mathcal{C} \rightarrow \mathcal{D}$ admitting an exact (right or left) adjoint, there exists a canonical commutative algebra (A, σ) in the center of \mathcal{C} such that F is tensor equivalent to the free module functor $\mathcal{C} \rightarrow \text{mod}_{\mathcal{C}}(A, \sigma)$, where $\text{mod}_{\mathcal{C}}(A, \sigma)$ denotes the category of A -modules in \mathcal{C} endowed with a monoidal structure defined using σ . We re-interpret equivariantization under a finite group action on a tensor category and, in particular, the modularization construction, in terms of exact sequences, Hopf monads, and commutative central algebras. As an application, we prove that a braided fusion category whose dimension is odd and square-free is equivalent, as a fusion category, to the representation category of a group.

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1 Introduction

Tensor categories are abelian categories over a field \mathbb{k} having finite-dimensional Hom spaces and objects of finite length, endowed with a rigid (or autonomous) structure, that is, a monoidal structure with duals, such that the monoidal tensor product is \mathbb{k} -bilinear and the unit object $\mathbf{1}$ is simple ($\text{End}(\mathbf{1}) = \mathbb{k}$). A fusion category is a split semisimple tensor category having finitely many isomorphism classes of simple objects. A tensor functor is a strong monoidal \mathbb{k} -linear functor between tensor categories.

In this paper, we introduce and study the notion of exact sequence of tensor categories, defined as follows. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor. Then F is *dominant* if any object Y of \mathcal{C}' is a subobject of $F(X)$ for some X in \mathcal{C} . It is *normal* if any object X of \mathcal{C} admits a subobject X_0 such that $F(X_0)$ is the largest subobject of $F(X)$ which is trivial, that is, isomorphic to $\mathbf{1}^n$. Denote by \mathfrak{Ker}_F the full subcategory of \mathcal{C} whose objects have a trivial image under F .

An exact sequence of tensor categories is a diagram of tensor functors

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

such that

- (a) F is normal;
- (b) F is dominant;
- (c) i induces an equivalence between \mathcal{C}' and $\mathfrak{Ker}_F \subset \mathcal{C}$.

This notion extends the notion of strictly exact sequence of Hopf algebras introduced by Schneider in [31]. Indeed, any strictly exact sequence of Hopf algebras

$$K \xrightarrow{i} H \xrightarrow{p} H'$$

gives rise to an exact sequence of tensor categories:

$$\text{comod-}K \longrightarrow \text{comod-}H \longrightarrow \text{comod-}H',$$

and, if H is finite dimensional, to a second exact sequence of tensor categories,

$$H'\text{-mod} \longrightarrow H\text{-mod} \longrightarrow K\text{-mod}.$$

For instance, an exact sequence of groups $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ yields an exact sequence of tensor categories:

$$\mathcal{C}(G') \longrightarrow \mathcal{C}(G) \longrightarrow \mathcal{C}(G''),$$

(where $\mathcal{C}(G)$ denotes the tensor category of G -graded vector spaces) and if G is finite, a second exact sequence of tensor categories is

$$\text{rep } G'' \longrightarrow \text{rep } G \longrightarrow \text{rep } G'.$$

In particular, we study exact sequences of fusion categories, and we show that the Frobenius–Perron dimension is multiplicative, that is, given an exact sequence of fusion categories $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$, we have

$$\text{FPdim } \mathcal{C} = \text{FPdim } \mathcal{C}' \text{FPdim } \mathcal{C}''.$$

We show that an exact sequence of pointed categories is classified by an exact sequence of finite groups $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ together with a cohomology class $\alpha \in H^3(G'', \mathbb{k}^\times)$. Also, generalizing a well-known result for semisimple Hopf algebras, we show that a dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{C}''$ of Frobenius–Perron index 2 between fusion categories is normal, and therefore, gives rise to an exact sequence: $\text{rep } \mathbb{Z}_2 \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$.

Can we interpret an exact sequence of tensor categories

$$(\mathcal{E}) \quad \mathcal{C}' \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

in terms of ‘algebraic’ data on \mathcal{C}'' , or on \mathcal{C} ? For technical reasons, we assume that \mathcal{C}' is finite, that is, F has adjoints. Then we show that (\mathcal{E}) is encoded by a certain Hopf monad on \mathcal{C}'' , and also, if the right adjoint of F is exact, by a certain commutative algebra in the center $\mathcal{Z}(\mathcal{C})$ of \mathcal{C} .

A Hopf monad on a rigid category \mathcal{D} (as defined in [7, Subsection 3.3]) is an algebra T in the monoidal category $\text{End}(\mathcal{D})$ of endofunctors of \mathcal{D} , which is also a comonoidal functor in a compatible way, and possesses left and right antipodes.

An exact sequence of tensor categories over a field \mathbb{k}

$$(\mathcal{E}) \quad \mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

defines a fiber functor $\omega = \text{Hom}(1, Ff) : \mathcal{C}' \rightarrow \text{vect}_{\mathbb{k}}$; hence by Tannaka reconstruction, a Hopf algebra $H = L(\omega)$, called the induced Hopf algebra of (\mathcal{E}) , such that $\mathcal{C}' \simeq \text{comod-}H$. The Hopf algebra H is finite dimensional if and only if the tensor functor F has adjoints.

A \mathbb{k} -linear right exact Hopf monad T on a tensor category \mathcal{C} is *normal* if $T(1)$ is a trivial object. If T is such a Hopf monad, and if in addition T is faithful, then it gives rise to an exact sequence of tensor categories

$$\text{comod-}H \rightarrow \mathcal{C}^T \rightarrow \mathcal{C},$$

where H is the induced Hopf algebra of T , isomorphic to $\text{Hom}(T(1), 1)$.

We show that, given tensor categories \mathcal{C}' and \mathcal{C}'' , with \mathcal{C} finite, exact sequences (\mathcal{E}) are classified by \mathbb{k} -linear, normal, faithful Hopf monads T on the tensor category \mathcal{C}'' whose induced Hopf algebra H is such that $\text{comod-}H$ is tensor equivalent to \mathcal{C}' .

Equivariantization provides examples of exact sequences of tensor categories. Indeed, if a finite group G acts on a tensor category \mathcal{C} by tensor autoequivalences, then the equivariantization \mathcal{C}^G is again a tensor category, and we have an exact sequence of tensor categories as follows:

$$\text{rep } G \rightarrow \mathcal{C}^G \rightarrow \mathcal{C}.$$

An action ρ of a finite group G on a tensor category \mathcal{C} by tensor autoequivalences can be encoded in the form of a Hopf monad T^ρ on \mathcal{C} , defined by

$$T^\rho = \bigoplus_{g \in G} \rho(g)$$

as an endofunctor of \mathcal{C} , so that \mathcal{C}^G is the category of T^ρ -modules \mathcal{C}^{T^ρ} .

We show that a Hopf monad T on a tensor category \mathcal{C} is of the form T^ρ for some group action ρ if and only if T is \mathbb{k} -linear right exact, faithful, normal, and cocommutative (see Definition 5.20) and its induced Hopf algebra is split semisimple.

A special case of an equivariantization is given by the modularization procedure [6, 23]. A premodular category \mathcal{C} over an algebraically closed field \mathbb{k} of characteristic zero is modularizable if there exists a dominant ribbon tensor functor $F : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}}$ is a modular category. Such is the case if and only if the tensor subcategory of transparent objects \mathcal{T} of \mathcal{C} is tannakian, that is, $\mathcal{T} \simeq \text{rep } G$ as a symmetric tensor category, for some finite group G (see [6]). In that case, we have an exact sequence of fusion categories

$$\text{rep } G \rightarrow \mathcal{C} \xrightarrow{F} \tilde{\mathcal{C}},$$

where F is the modularization functor and G acts on $\tilde{\mathcal{C}}$ by braided tensor equivalences, so that F is an equivariantization. Conversely, given a modular category \mathcal{D} , we classify premodular categories admitting \mathcal{D} as a modularization in terms of \mathbb{k} -linear, semisimple, faithful, normal Hopf monads on \mathcal{D} which are compatible with the ribbon structure.

We also interpret exact sequences of tensor categories in terms of commutative central algebras using results of [9]. If \mathcal{C} is a tensor category and (A, σ) is a commutative algebra in the categorical center $\mathcal{Z}(\mathcal{C})$ of \mathcal{C} , then the \mathbb{k} -linear abelian category $\text{mod}_{\mathcal{C}}(A, \sigma)$ of right A -modules in \mathcal{C} admits a monoidal structure involving the half-braiding σ , so that the free module functor $F_A : \mathcal{C} \rightarrow \text{mod}_{\mathcal{C}} A$, $X \mapsto X \otimes A$ is strong monoidal. We show that, given a dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ admitting an exact right adjoint R , the object $A = R(1)$ admits a canonical structure of commutative algebra in the center of \mathcal{C} such that $\text{mod}_{\mathcal{C}}(A, \sigma)$ is a tensor category, and $F = F_A$ up to a tensor equivalence $\mathcal{D} \simeq \text{mod}_{\mathcal{C}} A$. The central algebra (A, σ) is called the *induced central algebra of F* . We show that F is normal if and only if A is *self-trivializing*, that is, $F_A(A)$ is trivial. Then the induced Hopf algebra of F is $\text{Hom}(1, F(A))$.

Thus, an exact sequence of tensor categories $\mathcal{C}' \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}''$ such that F has an exact right adjoint is equivalent to $\langle A \rangle \longrightarrow \mathcal{C} \longrightarrow \text{mod}_{\mathcal{C}}(A, \sigma)$, where (A, σ) is the induced central algebra of F and $\langle A \rangle$ denotes the smallest abelian subcategory of \mathcal{C} containing A and stable by direct sums, subobjects, and quotients. Moreover, we show that F is an equivariantization if and only if $F(\sigma)$ is trivial and the induced Hopf algebra of F is split semisimple.

We introduce the notions of *simple* fusion category and *normal* fusion subcategory arising naturally from the definition of an exact sequence. If \mathcal{C} is a fusion category and $\mathcal{C}' \subset \mathcal{C}$ is a fusion subcategory, we say that $\mathcal{C}' \subset \mathcal{C}$ is normal if it fits in an exact sequence of fusion categories $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$. We say that \mathcal{C} is simple if it has no nontrivial normal fusion strict subcategory. We characterize normal fusion subcategories in terms of commutative central algebras, and show that our notion of simplicity differs from that introduced in [17]. If G is a finite group, then the simplicity of $\text{rep } G$ is equivalent to the simplicity of G and also to the simplicity of the fusion category $\mathcal{C}(G)$ of G -graded vector spaces.

As an application of the notion of exact sequence of fusion categories, we show the following classification result.

Theorem 1.1. Let \mathcal{C} be a braided fusion category over an algebraically closed field \mathbb{k} of characteristic 0. If $\dim \mathcal{C}$ is odd and square-free, then \mathcal{C} is equivalent to $\text{rep } \Gamma$ as fusion categories for some finite group Γ . \square

The proof relies on modularization and on the fact that a quasitriangular Hopf algebra whose dimension is odd and square-free is in fact a group algebra [26].

The paper is organized as follows: general definitions and classical results are recalled in Section 2, which also contains elementary facts about Hopf monads. In Section 3, we define dominant and normal tensor functors, and exact sequences of tensor categories. We prove several fundamental results and study the relations between strictly exact sequences of Hopf algebras as defined by Schneider and exact sequences of tensor categories. In Section 4, we study exact sequences of fusion categories. Section 5 is devoted to the classification of exact sequences of tensor categories in terms of Hopf monads, as well as equivariantization and the special case of modularization. In Section 6, we revisit tensor functors and exact sequences of tensor categories in terms of commutative central algebras, and study normal fusion subcategories and simple fusion categories. Section 7 is devoted to the proof of Theorem 1.1.

2 Tensor Categories and Hopf Monads

2.1 Conventions and notations

Monoidal categories will be strict, unless otherwise specified, and the unit object will be denoted by 1 . A monoidal category is *rigid* (or *autonomous*) if any object admits a left dual and a right dual. If such is the case, the left dual and right dual functors are denoted ${}^\vee?$ and $?^\vee$, respectively.

Most of the time, we work over a base field \mathbb{k} . If \mathcal{A} is an abelian \mathbb{k} -linear category, we say that an object X of \mathcal{A} is *scalar* if $\text{End}(X) = \mathbb{k} \text{id}_X$. The category \mathcal{A} is *split semisimple* if there is a set Λ of scalar objects in \mathcal{A} such that every object of \mathcal{A} is a finite direct sum of elements of Λ , and such that $\text{Hom}_{\mathcal{C}}(X, Y) = 0$ for $X \neq Y$ in Λ . Note that if \mathcal{A} is split semisimple, scalar objects and simple objects coincide in \mathcal{A} .

An abelian \mathbb{k} -linear category \mathcal{A} is *finite* if it is \mathbb{k} -linearly equivalent to the category of finite-dimensional left modules over a finite-dimensional \mathbb{k} -algebra. A \mathbb{k} -linear functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between finite abelian \mathbb{k} -linear categories has a left (respectively right) adjoint if and only if it is left exact (respectively right exact).

2.2 Tensor categories and tensor functors

A *tensor category over \mathbb{k}* is a \mathbb{k} -linear abelian rigid monoidal category where Hom spaces are finite dimensional, all objects have finite length, the monoidal product is \mathbb{k} -linear in each variable, and the unit object 1 is scalar. In a tensor category, the monoidal product is exact in each variable and the unit object is simple [4, Lemma 2.4.], see also [11].

A tensor category over \mathbb{k} is *finite* if it is finite as a \mathbb{k} -linear abelian category.

A *tensor functor* is a \mathbb{k} -linear exact strong monoidal functor between two tensor categories. A tensor functor preserves duals. Moreover, it is automatically faithful. Indeed, we have the following lemma, which generalizes [5, Proposition 5.2.2].

Lemma 2.1. Let \mathcal{A}, \mathcal{B} be two abelian monoidal categories, and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a strong quasimonoidal additive exact functor. Assume that

- (1) \mathcal{A} is rigid,
- (2) the unit object of \mathcal{A} is simple,
- (3) the unit object of \mathcal{B} is nonzero.

Then F is faithful. □

Here *strong quasimonoidal* means that F is endowed with natural isomorphisms $F(a) \otimes F(a') \simeq F(a \otimes a')$ and with an isomorphism $1_{\mathcal{B}} \simeq F(1_{\mathcal{A}})$.

Proof. Since F is additive exact, it is enough to show that the image of a nonzero object a of \mathcal{A} is nonzero. Now if $a \neq 0$, the evaluation $e : a \otimes a^{\vee} \rightarrow 1_{\mathcal{A}}$ is nonzero (because $0 \otimes a = 0$), so it is epic since $1_{\mathcal{A}}$ is simple. Since F preserves epimorphisms, the morphism $F(e) : F(a) \otimes F(a^{\vee}) \simeq F(a \otimes a^{\vee}) \rightarrow 1_{\mathcal{B}}$ is epic and since $1_{\mathcal{B}}$ is nonzero, $F(a) \otimes F(a^{\vee}) \neq 0$, so we have $F(a) \neq 0$ (because $0 \otimes F(a^{\vee}) = 0$). ■

If H is a Hopf algebra over \mathbb{k} , $H\text{-mod}$ denotes the tensor category of finite-dimensional representations of H , that is, finite-dimensional left H -modules. Similarly, $\text{comod-}H$ denotes the tensor category of finite-dimensional right H -comodules. In particular, if G is a finite group, $\text{rep } G := \mathbb{k} G\text{-mod} \simeq \text{comod-}\mathbb{k}^G$ is the category of finite-dimensional representations of G , whereas $\text{comod-}\mathbb{k} G$ is the category of finite-dimensional G -graded vector spaces.

A morphism of Hopf algebras $f : H \rightarrow H'$ defines two tensor functors:

$$f_* : \begin{cases} \text{comod-}H & \rightarrow \text{comod-}H' \\ (M, \delta) & \mapsto (M, (\text{id}_M \otimes f)\delta), \end{cases}$$

$$f^* : \begin{cases} H'\text{-mod} & \rightarrow H\text{-mod} \\ (M, r) & \mapsto (M, r(f \otimes \text{id}_M)). \end{cases}$$

A *fiber functor* for a tensor category \mathcal{C} over \mathbb{k} is a tensor functor $\omega : \mathcal{C} \rightarrow \text{vect}_{\mathbb{k}}$. By Tannaka theory, given a fiber functor ω for a tensor category \mathcal{C} over \mathbb{k} , the coend $L(\omega) = \int^{X \in \mathcal{C}} \omega(X) \otimes_{\mathbb{k}} \omega(X)^\vee$ is a Hopf algebra over \mathbb{k} , and we have a canonical equivalence of tensor categories $\mathcal{C} \xrightarrow{\sim} \text{comod-}L(\omega)$.

Let \mathcal{C} be a tensor category. For a finite-dimensional vector space E and an object $X \in \mathcal{C}$, their tensor product, denoted by $E \otimes X \in \mathcal{C}$ is defined by the adjunction

$$\text{Hom}_{\mathcal{C}}(E \otimes X, Y) \simeq \text{Hom}_{\mathbb{k}}(E, \text{Hom}_{\mathcal{C}}(X, Y)).$$

The assignment $(E, X) \mapsto E \otimes X$ makes \mathcal{C} a left $\text{vect}_{\mathbb{k}}$ -module category. In particular, the functor $\text{vect}_{\mathbb{k}} \rightarrow \mathcal{C}$, $E \mapsto E \otimes 1$ is a tensor functor from $\text{vect}_{\mathbb{k}}$ to \mathcal{C} , and in fact the only such functor up to tensor isomorphism. It is right adjoint to the *global section functor* $\Gamma : \mathcal{C} \rightarrow \text{vect}_{\mathbb{k}}$, defined by $\Gamma(X) = \text{Hom}_{\mathbb{k}}(1, X)$.

If X is an object or set of objects of a tensor category \mathcal{C} , we denote by $\langle X \rangle$ the smallest abelian subcategory of \mathcal{C} containing X and stable by direct sums, subobjects, and quotients.

An object of a tensor category \mathcal{C} is *trivial* if it is isomorphic to 1^n for some natural integer n . The full subcategory of trivial objects of \mathcal{C} is $\langle 1 \rangle \subset \mathcal{C}$. It is a tensor category equivalent to $\text{vect}_{\mathbb{k}}$ via the tensor functor $X \mapsto \text{Hom}_{\mathcal{C}}(1, X)$. A tensor category \mathcal{C} is *trivial* if $\mathcal{C} = \langle 1 \rangle$, that is, if \mathcal{C} is tensor equivalent to $\text{vect}_{\mathbb{k}}$.

2.3 Existence of adjoints

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a strong monoidal functor between rigid categories. According to [8, Lemma 3.4], if F has a left adjoint G , then it has a right adjoint R , and conversely. In that case, R and G are related thus:

$$R(X) \simeq {}^V G(X^\vee) \simeq G({}^V X)^\vee \quad \text{and} \quad R(X) \simeq G({}^V X)^\vee \simeq {}^V G(X^\vee) \quad (\text{for } X \text{ in } \mathcal{C}).$$

In that case we say that F *has adjoints*. A tensor functor between finite tensor categories has adjoints. In general, a tensor functor need not have adjoints; for instance, a fiber functor for a tensor category \mathcal{C} has adjoints if and only if \mathcal{C} is finite.

However, a tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ has an Ind-adjoint and a Pro-adjoint because it is exact [3]. In other words, the functor $\text{Ind } F : \text{Ind } \mathcal{C} \rightarrow \text{Ind } \mathcal{D}$ obtained by extending F to the categories of Ind-objects of \mathcal{C} and \mathcal{D} has a right adjoint $R : \text{Ind } \mathcal{D} \rightarrow \text{Ind } \mathcal{C}$, called the Ind-adjoint of F , and, dually, the functor $\text{Pro } F : \text{Pro } \mathcal{C} \rightarrow \text{Pro } \mathcal{D}$ obtained by extending

F to the categories of Pro-objects of \mathcal{C} and \mathcal{D} has a left adjoint $G : \text{Pro } \mathcal{D} \rightarrow \text{Pro } \mathcal{C}$, called the Pro-adjoint of F .

2.4 Fusion categories

A *fusion category* over \mathbb{k} is a split semisimple finite tensor category over \mathbb{k} . Note that if \mathbb{k} is algebraically closed, a tensor category \mathcal{C} is split semisimple if and only if it is semisimple. See for instance [33, Section 2].

Let \mathcal{C} be a fusion category over \mathbb{k} . The finite set of isomorphism classes of simple (or scalar) objects in \mathcal{C} will be denoted by $\Lambda_{\mathcal{C}}$. The class of an object X of \mathcal{C} in the Grothendieck ring $K_0(\mathcal{C})$ will be denoted by $[X]$. If $X \in \Lambda_{\mathcal{C}}$ and Y is an object of \mathcal{C} , denote by $m_X(Y)$ the multiplicity of X in Y , that is, $m_X(Y) = \dim \text{Hom}_{\mathcal{C}}(X, Y) = \dim \text{Hom}_{\mathcal{C}}(Y, X)$, so that we have

$$Y \simeq \bigoplus_{X \in \Lambda_{\mathcal{C}}} X^{m_X(Y)}.$$

The *Frobenius–Perron dimension* $\text{FPdim } X$ of $X \in \Lambda_{\mathcal{C}}$ is the largest positive eigenvalue of the matrix of left multiplication by X in the Grothendieck ring of \mathcal{C} with respect to the basis $\Lambda_{\mathcal{C}}$. It is a real nonnegative algebraic integer. The *Frobenius–Perron dimension* of \mathcal{C} is $\text{FPdim } \mathcal{C} := \sum_{X \in \Lambda_{\mathcal{C}}} (\text{FPdim } X)^2$.

See [16, 24] for a survey and a general reference on fusion categories.

2.5 Monads

A *monad* on a category \mathcal{A} is an algebra T in the monoidal category $\text{End}(\mathcal{A})$ of endofunctors of \mathcal{A} . In other words, it is an endofunctor T of \mathcal{A} endowed with natural transformations $\mu : T^2 \rightarrow T$ and $\eta : \text{id}_{\mathcal{A}} \rightarrow T$ (the multiplication and unit of T , respectively), satisfying the associativity and unit axioms in $\text{End}(\mathcal{A})$.

Let T be a monad on \mathcal{A} . A *T -module in \mathcal{A}* (often called a *T -algebra*) is a pair (M, r) , where $M \in \mathcal{D}$ and $r : T(M) \rightarrow M$ is a morphism in \mathcal{D} , such that

$$rT(r) = r\mu_M, \quad r\eta_M = \text{id}_M. \quad (2.1)$$

A morphism $f : (M', r') \rightarrow (M, r)$ between T -modules (M', r') and (M, r) is a morphism $f : M' \rightarrow M$ in \mathcal{D} such that $fr' = rT(f)$. This defines a category \mathcal{A}^T of T -modules in \mathcal{A} . Let $\mathcal{U} : \mathcal{A}^T \rightarrow \mathcal{A}$ denote the forgetful functor: $\mathcal{U}(M, r) = M$. Then \mathcal{U} admits a left adjoint $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{A}^T$, defined by $\mathcal{L}(X) = (T(X), \mu_X)$. We have $T = \mathcal{U}\mathcal{L}$.

If T and T' are monads on \mathcal{C} , a *morphism of monads* $f: T' \rightarrow T$ is a natural transformation such that $f\mu'_X = \mu_X f_{T(X)} T'(f_X)$ and $f_X \eta'_X = \eta_X$, for all object X of \mathcal{C} .

A monad T will be called *faithful* (respectively, *left/right exact*), if its underlying functor is faithful (respectively, left/right exact).

We will require the following characterization of faithful monads.

Lemma 2.2. Let T be a monad on a category \mathcal{A} . The following assertions are equivalent:

- (i) For any X object of \mathcal{A} , there exists a T -module (M, r) and a monomorphism $X \rightarrow M$;
- (ii) The unit η of T is monomorphic;
- (iii) The monad T is faithful;
- (iv) The free module functor $\mathcal{L}: \mathcal{A} \rightarrow \mathcal{A}^T$ is faithful. □

Proof. Recall that \mathcal{L} is left adjoint to the forgetful functor $\mathcal{U}: \mathcal{A}^T \rightarrow \mathcal{A}$, and $T = \mathcal{U}\mathcal{L}$. Clearly, we have (ii) \Rightarrow (i). Conversely, (i) \Rightarrow (ii): if $f: X \rightarrow M$ is a monomorphism, (M, r) being a T -module, then η_X is a monomorphism because we have $f = r\eta_M f = rT(f)\eta_X$. Now, since $T = \mathcal{U}\mathcal{L}$ and \mathcal{U} is faithful, we have (iv) \iff (iii). Also (iii) \Rightarrow (ii): let $u, v: Y \rightarrow X$ be morphisms such that $\eta_X u = \eta_X v$. Then, $T(u) = \mu_X T(\eta_X)T(u) = \mu_X T(\eta_X)T(v) = T(v)$, hence $u = v$ if T is faithful. Finally, (ii) \Rightarrow (iii). Indeed, let $u, v: X \rightarrow Y$ be morphisms such that $T(u) = T(v)$. We have $\eta_Y u = T(u)\eta_X = T(v)\eta_X = \eta_Y v$, so $u = v$ if η_Y is a monomorphism. ■

2.6 Hopf monads

Let \mathcal{C} be a monoidal category. A monad on \mathcal{C} is a *bimonad* when the category \mathcal{C}^T is endowed with a monoidal structure such that the forgetful functor $\mathcal{U}: \mathcal{C}^T \rightarrow \mathcal{C}$ is monoidal strict. This is equivalent to saying that the monad T is endowed with a structure of comonoidal endofunctor, that is, a natural transformation

$$T_2(X, Y): T(X \otimes Y) \rightarrow T(X) \otimes T(Y) \quad (X, Y \text{ in } \mathcal{C})$$

and a morphism $T_0: T(1) \rightarrow 1$, satisfying

$$(T_2(X, Y) \otimes \text{id}_{T(Z)})T_2(X \otimes Y, Z) = (\text{id}_{T(X)} \otimes T_2(Y, Z))T_2(X, Y \otimes Z),$$

$$(\text{id}_{T(X)} \otimes T_0)T_2(X, 1) = \text{id}_{T(X)} = (T_0 \otimes \text{id}_{T(X)})T_2(1, X);$$

and such that the product μ and the unit η are monoidal transformations, that is,

$$T_2(X, Y)\mu_{X \otimes Y} = (\mu_X \otimes \mu_Y)T_2(T(X), T(Y))T(T_2(X, Y)),$$

$$T_0\mu_1 = T_0T(T_0), \quad T_2(X, Y)\eta_{X \otimes Y} = \eta_X \otimes \eta_Y, \quad T_0\eta_1 = \text{id}_1.$$

Bimonads are introduced in [20] under the name Hopf monads.

If \mathcal{C} is rigid, a bimonad T on \mathcal{C} is a *Hopf monad* if \mathcal{C}^T is rigid; this is equivalent to saying that T has a left and a right antipode (see [7, Subsection 3.3]) Hopf monads on arbitrary monoidal categories are defined in [9].

A *morphism of bimonads* or *Hopf monads* is a comonoidal morphism of monads between bimonads or Hopf monads.

2.7 Monadicity

Let $(G : \mathcal{A} \rightarrow \mathcal{B}, F : \mathcal{B} \rightarrow \mathcal{A})$ be an adjunction, with unit $\eta : \text{id}_{\mathcal{A}} \rightarrow FG$ and counit $\epsilon : GF \rightarrow \text{id}_{\mathcal{B}}$. Then $T = FG$ is a monad on \mathcal{A} , and there exists a unique functor $\kappa : \mathcal{B} \rightarrow \mathcal{A}^T$ such that $\mathcal{U}\kappa = F$ and $\kappa G = \mathcal{L}$. The functor κ , called the *comparison functor* of the adjunction, is given by $\kappa(X) = (F(X), F(\epsilon_X))$.

The adjunction $G \vdash F$ is called *monadic* if the comparison functor κ is an equivalence. Necessary and sufficient conditions for an adjunction to be monadic are given by Beck's Theorem, see [21, Chapter VI.7]. In particular, if \mathcal{A} and \mathcal{B} are abelian and the functor F is additive and faithful exact, then $G \vdash F$ is monadic.

Now let \mathcal{C}, \mathcal{D} be monoidal categories and $F : \mathcal{D} \rightarrow \mathcal{C}$ be a strong monoidal functor. Assume that F has a left adjoint G . Then, ref. [7, Theorem 9.1] asserts that the functor G is comonoidal, the monad FG in \mathcal{D} has a canonical structure of a bimonad, and the comparison functor $\kappa : \mathcal{C} \rightarrow \mathcal{D}^T$ is strong monoidal. Moreover, $\mathcal{U}\kappa = F$ as monoidal functors and $\kappa G = \mathcal{L}$ as comonoidal functors. Finally, if \mathcal{C} and \mathcal{D} are rigid, then T is a Hopf monad.

2.8 Hopf monads and tensor categories

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories over a field \mathbb{k} . Assume that F admits a left adjoint G (which is then unique up to unique isomorphism). Then, F being faithful exact, the adjunction $G \vdash F$ is monadic. The monad $T = FG$ of this adjunction is a Hopf monad on \mathcal{D} , which is called the Hopf monad of F . It is \mathbb{k} linear right exact, and we have $\mathcal{C} \simeq \mathcal{D}^T$ as tensor categories.

Note that if F is a tensor functor between finite tensor categories (such as fusion categories), then F admits a left adjoint and so, it is monadic.

Proposition 2.3. Let \mathcal{C} be a tensor category over a field \mathbb{k} , and let T be a \mathbb{k} -linear right exact Hopf monad on \mathcal{C} . Then, \mathcal{C}^T is a tensor category over \mathbb{k} , and the forgetful functor $\mathcal{U} : \mathcal{C}^T \rightarrow \mathcal{C}$ is a tensor functor. \square

Proof. It is a general fact that the category of modules over a \mathbb{k} -linear right exact monad on an abelian category is \mathbb{k} -linear abelian and the forgetful functor is \mathbb{k} -linear exact. Applying this to T , the category \mathcal{C}^T is \mathbb{k} -linear abelian and rigid; its tensor product is \mathbb{k} -linear; and its unit object $(1, T_0)$ is scalar. Moreover, the forgetful functor \mathcal{U} is monoidal strict and \mathbb{k} -linear, faithful, exact; in particular, in \mathcal{C}^T , Hom's are finite dimensional and objects have finite length. Thus, \mathcal{C}^T is a tensor category over \mathbb{k} , and \mathcal{U} is a tensor functor. \blacksquare

Example 2.4. A Hopf algebra H in a braided autonomous category \mathcal{B} defines a Hopf monad $H \otimes ?$ on \mathcal{B} , see [7, Example 3.10] and [8, Example 2.8]. In particular, a finite-dimensional Hopf algebra H over \mathbb{k} defines a Hopf monad $H \otimes ?$ on the category $\text{vect}_{\mathbb{k}}$ of finite-dimensional vector spaces. It is the monad of the forgetful functor

$$H\text{-mod} \rightarrow \text{vect}_{\mathbb{k}}.$$

□

In fact, \mathbb{k} -linear Hopf monads on trivial tensor categories are just finite-dimensional Hopf algebras:

Lemma 2.5. Let \mathcal{C} be a trivial tensor category. If H is a finite-dimensional Hopf algebra over \mathbb{k} , then $H \otimes ?: \mathcal{C} \rightarrow \mathcal{C}$ admits a natural structure of \mathbb{k} -linear Hopf monad on \mathcal{C} . The assignment $H \mapsto H \otimes ?$ defines an equivalence of categories between finite-dimensional Hopf algebras over \mathbb{k} and \mathbb{k} -linear Hopf monads on \mathcal{C} . \square

Proof. A trivial tensor category, being by definition tensor equivalent to $\text{vect}_{\mathbb{k}}$, admits a unique braiding and is symmetric. The tensor functor $\text{vect}_{\mathbb{k}} \rightarrow \mathcal{C}$, $E \mapsto E \otimes 1$, is symmetric and sends a finite-dimensional Hopf algebra H over \mathbb{k} to a Hopf algebra $H \otimes 1$ in \mathcal{C} . Thus, $H \otimes ?: \mathcal{C} \rightarrow \mathcal{C}$ is \mathbb{k} -linear Hopf monad on \mathcal{C} . Now if T is a \mathbb{k} -linear Hopf monad on \mathcal{C} , set $H = \text{Hom}_{\mathcal{C}}(1, T(1))$. We have a canonical isomorphism $a_1 : H \otimes 1 \xrightarrow{\sim} T(1)$, which extends uniquely to a natural isomorphism $a : H \otimes ? \xrightarrow{\sim} T$ because \mathcal{C} is semisimple and $\Lambda_{\mathcal{C}} = \{1\}$. One verifies that there exists a unique structure of Hopf algebra on H such that

when $H \otimes ?$ is endowed with the corresponding structure of Hopf monad, a becomes an isomorphism of Hopf monads on \mathcal{C} . \blacksquare

3 Exact Sequences of Tensor Categories

In this section, we introduce the notions of normal tensor functor and of exact sequences of tensor categories over a field \mathbb{k} .

3.1 Dominant functors, normal functors, and exact sequences

Lemma 3.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories. The following assertions are equivalent:

- (i) Any object Y of \mathcal{D} is a subobject of $F(X)$ for some object X of \mathcal{C} ;
- (ii) Any object Y of \mathcal{D} is a quotient of $F(X)$ for some object X of \mathcal{C} ;
- (iii) The Pro-adjoint of F is faithful;
- (iv) The Ind-adjoint of F is faithful.

□

Proof. Assume (i) and let Y be an object of \mathcal{D} . There exists an object X of \mathcal{C} and a monomorphism $i : Y^\vee \rightarrow F(X)$. In a rigid category, the left and right dual functors are quasi-inverse contravariant equivalences, and strong monoidal functors preserve duals. Hence, $\vee i : F(\vee X) \simeq \vee F(X) \rightarrow \vee Y^\vee \simeq Y$ is an epimorphism, hence (i) \Rightarrow (ii); the converse is proved similarly. Hence (i) \Leftrightarrow (ii).

Now, let us prove (i) \Leftrightarrow (iii). The Pro-adjoint functor of F is the left adjoint functor G of the functor $\text{Pro } F : \text{Pro } \mathcal{C} \rightarrow \text{Pro } \mathcal{D}$. $\text{Pro } \mathcal{C}$ and $\text{Pro } \mathcal{D}$ are abelian categories and $\text{Pro } F$ is \mathbb{k} -linear, faithful, and exact. By Beck's theorem, the adjunction $G \vdash \text{Pro } F$ is monadic, with monad $T = \text{Pro } F G$, so $\text{Pro } \mathcal{C}$ is equivalent to $(\text{Pro } \mathcal{D})^T$ via the comparison functor, F being the forgetful functor. Denote by η the unit of the monad T . Assume (i) holds. Then for any object Y in $\text{Pro } \mathcal{D}$, η_Y is a monomorphism. Indeed, if Y is in $\mathcal{D} \subset \text{Pro } \mathcal{D}$, then there exists X in \mathcal{C} and a monomorphism $i : Y \rightarrow F(X)$. Since \mathcal{C} is a full subcategory of $\text{Pro } \mathcal{C} \simeq (\text{Pro } \mathcal{D})^T$, we may view X as a T -module, with action $r : TF(X) \rightarrow F(X)$ and $i = rT(i)\eta_Y$, so η_Y is a monomorphism in that case. In general, an object Y of $\text{Pro } \mathcal{D}$ is of the form " \lim " Y_i for some filtering system (Y_i) of objects of \mathcal{D} , and η_{Y_i} is a monomorphism for all i , hence η_Y is a monomorphism. By Lemma 2.2, this implies that G is faithful. Conversely, if G is faithful, again by Lemma 2.2 for any object Y of $\text{Pro } \mathcal{D}$ there exists a filtering system (X_i) of objects of \mathcal{C} such that Y is a subobject of $\text{Pro } F(" \lim " X_i) = " \lim " F(X_i)$. If Y is in

\mathcal{D} , it has finite length so there exists i such that $Y \rightarrow F(X_i)$ is a monomorphism. Hence (i) \Leftrightarrow (iii).

Finally, (ii) \Leftrightarrow (iv) results from (i) \Leftrightarrow (iii) applied to the opposite functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$, hence the lemma is proved. \blacksquare

Definition 3.2. A tensor functor $F : \mathcal{C} \rightarrow \mathcal{C}''$ is *dominant* if it satisfies the equivalent conditions of Lemma 3.1. \square

Remark 3.3. A dominant tensor functor is surjective in the sense of [15]. \square

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories. We denote by \mathfrak{Ker}_F the full tensor subcategory $F^{-1}(\langle 1 \rangle) \subseteq \mathcal{C}$ of objects X of \mathcal{C} such that $F(X)$ is a trivial object of \mathcal{C} .

Definition 3.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories. Then F is *normal* if for any object X of \mathcal{C} , there exists a subobject $X_0 \subset X$ such that $F(X_0)$ is the largest trivial subobject of $F(X)$. \square

Proposition 3.5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories.

- (1) If F admits a right adjoint R , or, equivalently, a left adjoint G , then F is normal if and only if $G(1)$ belongs to \mathfrak{Ker}_F , if and only if $R(1)$ belongs to \mathfrak{Ker}_F .
- (2) If \mathcal{C} and \mathcal{D} are fusion categories, F is normal if and only if any simple object X of \mathcal{C} such that $\text{Hom}(1, F(X)) \neq 0$ belongs to \mathfrak{Ker}_F . \square

Proof. Let us prove Part (1). For X in \mathcal{C} , denote by $X_0 \subset X$ the largest subobject of X belonging to \mathfrak{Ker}_F , which exists because objects have finite length in \mathcal{C} . We have a commutative diagram

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{D}}(1, F(X_0)) & \xrightarrow{a_X} & \text{Hom}_{\mathcal{D}}(1, F(X)) \\
 \downarrow \sim & & \downarrow \sim \\
 \text{Hom}_{\mathcal{C}}(G(1), X_0) & \xrightarrow{b_X} & \text{Hom}_{\mathcal{C}}(G(1), X),
 \end{array}$$

where the vertical arrows are the adjunction isomorphisms, and the horizontal arrows a and b are induced by the inclusion $X_0 \subset X$. If $G(1)$ belongs to \mathfrak{Ker}_F , then for all X , b_X

is bijective, so a_X is bijective, which means that $F(X_0)$ is the largest trivial subobject of $F(X)$. Hence, F is normal. Conversely, suppose F is normal, and let $X_0 \subseteq X$ be such that $F(X_0)$ is the largest trivial subobject of $F(X)$. Thus a_X is bijective, so b_X is bijective, for all object X of \mathcal{C} . In particular, $b_{G(1)}$ is bijective, so $G(1)_0 = G(1)$, hence $G(1)$ belongs to \mathfrak{Ker}_F . Thus, F is normal $\iff G(1)$ belongs to $\mathfrak{Ker}_F \iff R(1) = G(1)^\vee$ belongs to \mathfrak{Ker}_F .

Let us now prove Part (2). If \mathcal{C} and \mathcal{D} are fusion categories, F admits a left adjoint G , and F is normal if and only if $G(1)$ is in \mathfrak{Ker}_F by Part (1). By adjunction, we have $m_X G(Y) = m_Y F(X)$ for all $X \in \Lambda_{\mathcal{C}}$ and $Y \in \Lambda_{\mathcal{D}}$. In particular, $m_1 F(X) > 0$ if and only if $m_X G(1) > 0$, hence Part (2). \blacksquare

Lemma 3.6. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories. Then:

- (1) The functor F is an equivalence if and only if it is full and dominant.
- (2) The functor F is full if and only if F is normal and \mathfrak{Ker}_F is trivial; \square

Proof. Part (1): clearly, if F is an equivalence, it is both full and dominant. Conversely, assume F is full and dominant. Let Y be an object of \mathcal{D} . By Lemma 3.1, there exist X_1, X_2 in \mathcal{C} , an epimorphism $p : F(X_1) \rightarrow Y$ and a monomorphism $i : Y \rightarrow F(X_2)$, and Y is (isomorphic to) the image of ip . Since F is full, there exists $\pi : X_1 \rightarrow X_2$ such that $F(\pi) = ip$. Let X be the image of π ; since F is exact, it preserves images so $F(X) \simeq Y$. Thus, F is essentially surjective, and is therefore an equivalence.

Part (2): if F is full, it is normal and \mathfrak{Ker}_F is trivial. Conversely, assume F is normal and \mathfrak{Ker}_F is trivial. Then for any X in \mathcal{C} we have a subobject $X_0 \subset X$ such that $F(X_0)$ is the largest trivial subobject of $F(X)$. In particular, X_0 is trivial, so F induces an isomorphism $\text{Hom}_{\mathcal{C}}(1, X) \rightarrow \text{Hom}_{\mathcal{D}}(1, F(X))$. Since, in a rigid category, $\text{Hom}(X, X') \simeq \text{Hom}(1, X^\vee \otimes X')$, we conclude that F is fully faithful. \blacksquare

Definition 3.7. Let \mathcal{C}' , \mathcal{C} , and \mathcal{C}'' be tensor categories over \mathbb{k} . A sequence of tensor functors

$$\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}'' \tag{3.1}$$

is an *exact sequence of tensor categories* if the following conditions hold:

- (1) The tensor functor F is dominant and normal;
- (2) The tensor functor f is a full embedding;
- (3) The essential image of f is \mathfrak{Ker}_F ;

Two exact sequences of tensor categories

$$\mathcal{C}' \xrightarrow{f_1} \mathcal{C}_1 \xrightarrow{F_1} \mathcal{C}'' \quad \text{and} \quad \mathcal{C}' \xrightarrow{f_2} \mathcal{C}_2 \xrightarrow{F_2} \mathcal{C}''$$

are *equivalent* if there exists a tensor equivalence $\lambda: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that $F_1 \simeq F_2 \lambda$ and $f_2 \simeq \lambda f_1$ as tensor functors, that is, such that the diagram

$$\begin{array}{ccccc} \mathcal{C}' & \xrightarrow{f_1} & \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{C}'' \\ \downarrow = & & \downarrow \lambda & & \downarrow = \\ \mathcal{C}' & \xrightarrow{f_2} & \mathcal{C}_2 & \xrightarrow{F_2} & \mathcal{C}''. \end{array}$$

is commutative up to monoidal isomorphism.

A exact sequence $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ is called an *extension of \mathcal{C}'' by \mathcal{C}'* ; we also say that \mathcal{C} is an extension of \mathcal{C}'' by \mathcal{C}' . \square

Note that a normal dominant tensor functor $F: \mathcal{C} \rightarrow \mathcal{C}''$ between tensor categories defines an exact sequence of tensor categories

$$\mathfrak{Ker}_F \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}''.$$

Proposition 3.8. If $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ is an exact sequence of tensor categories, then:

- (1) The tensor functor F is an equivalence if and only if \mathcal{C}' is trivial;
- (2) The tensor functor f is an equivalence if and only if \mathcal{C}'' is trivial.

\square

Proof. Part (1): if F is an equivalence, then $\mathfrak{Ker}_F = \langle 1 \rangle$, that is, \mathcal{C}' is trivial. Conversely, if \mathfrak{Ker}_F is trivial, then by Lemma 3.6 F is an equivalence.

Part (2): if \mathcal{C}'' is trivial, $\mathfrak{Ker}_F = \mathcal{C}$, hence f is an equivalence. Conversely, if f is an equivalence, $\mathfrak{Ker}_F = \mathcal{C}$. Hence, $F(\mathcal{C}) \subset \langle 1 \rangle$, and, F being dominant, $\mathcal{C}'' = \langle 1 \rangle$, that is, \mathcal{C}'' is trivial. \blacksquare

3.2 Exact sequences of tensor categories from Hopf algebras

Strictly, exact sequences of Hopf algebras as defined in [31] give rise to exact sequences of tensor categories. We always assume that Hopf algebras have a bijective antipode. If H is a Hopf algebra, we denote by $H^+ \subset H$ the *augmentation ideal* $H^+ = \{x \in H \mid \varepsilon(x) = 0\}$.

A Hopf subalgebra $K \subset H$ is normal if it is a submodule of H for the adjoint action of H on itself, defined by $x.y = x_{(1)}yS(x_{(2)})$, and a Hopf ideal $I \subset H$ is normal if it is a subcomodule of H for the right coadjoint coaction of H on itself, defined by $x \mapsto x_{(2)} \otimes S(x_{(1)})x_{(3)}$.

In the category of Hopf algebras over a field \mathbb{k} , the trivial Hopf algebra \mathbb{k} is a zero object, that is, it is both initial and final. A morphism $f: H \rightarrow H'$ of Hopf algebras over \mathbb{k} admits a categorical kernel and a categorical cokernel, defined by

$$\ker(f) = \{x \in H \mid x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes 1 \otimes x_{(2)}\}$$

in Sweedler's notation, and

$$\text{coker}(f) = H'/H'f(H^+)H'.$$

Observe that $\ker(f)$ is not $f^{-1}(0)$ and $\text{coker}(f)$ is not $H'/f(H)$.

A *strictly exact sequence of Hopf algebras* is a diagram

$$K \xrightarrow{i} H \xrightarrow{p} H'$$

where i, p are morphisms of Hopf algebras such that

- (a) K is a normal Hopf subalgebra of H ;
- (b) H is right faithfully flat over K ;
- (c) p is a categorical cokernel of i ,

or, equivalently, setting $I = p^{-1}(0)$, such that

- (a') I is a normal Hopf ideal of H ;
- (b') H is right faithfully coflat over H' ;
- (c') i is a categorical kernel of p .

Proposition 3.9. A strictly exact sequence $K \xrightarrow{i} H \xrightarrow{p} H'$ Hopf algebras over a field gives rise to an exact sequence of tensor categories:

$$\text{comod-}K \xrightarrow{i_*} \text{comod-}H \xrightarrow{p_*} \text{comod-}H'$$

and also, if H is finite dimensional, to a second exact sequence of tensor categories

$$H'\text{-mod} \xrightarrow{P^*} H\text{-mod} \xrightarrow{i^*} K\text{-mod}.$$

□

Proof. Let $f: H \rightarrow H'$ be a morphism of Hopf algebras over \mathbb{k} . Denote, respectively, by $H^{\text{co} H'}$ and ${}^{\text{co} H'} H$ the subalgebras of H of right and left H' coinvariants, that is,

$$H^{\text{co} H'} = \{h \in H \mid (\text{id}_H \otimes f)\Delta(h) = h \otimes 1\}, \quad {}^{\text{co} H'} H = \{h \in H \mid (f \otimes \text{id}_H)\Delta(h) = 1 \otimes h\}.$$

Lemma 3.10. Let $f: H \rightarrow H'$ be a morphism of Hopf algebras over a field, and consider the tensor functors

$$f_*: \text{comod-}H \rightarrow \text{comod-}H' \quad \text{and} \quad f^*: H'\text{-mod} \rightarrow H\text{-mod}.$$

Then,

- (1) We have $\text{Ker } f_* = \text{comod-ker}(f)$. Moreover, the tensor functor f_* is normal if and only if $H^{\text{co} H'} = {}^{\text{co} H'} H$. If such is the case, $\text{ker}(f) = H^{\text{co} H'}$.
- (2) We have $\text{Ker } f^* = \text{coker}(f)\text{-mod}$. Moreover, the tensor functor f^* is normal if $f(H)$ is a normal Hopf subalgebra of H' . If such is the case, $\text{coker}(f) = H'/H' f(H^+)$. □

Proof. Part (1). If (M, δ) is a right H -comodule, then $f_*(M, \delta) = (M, (\text{id}_M \otimes f)\delta)$, and the largest trivial subobject of $f_*(M, \delta)$ is

$$M^{\text{co} H'} = \{x \in M \mid (\text{id}_M \otimes f)\delta(x) = x \otimes 1\}.$$

We have $M^{\text{co} H'} = M$ if and only if M is a $\text{ker}(f)$ -comodule. That is, $\text{Ker } f_* = \text{comod-ker}(f)$.

According to Definition 3.4, f_* is normal if and only if for all finite-dimensional right H -comodule M , $M^{\text{co} H'} \subset M$ is a subcomodule. This is equivalent to saying that for all right H -comodule M , $M^{\text{co} H'} \subset M$ is a subcomodule, because any comodule is locally finite.

Now assume f_* is normal. Then, $H^{\text{co} H'}$ is a subcomodule of the right comodule $H = (H, \Delta)$. If $h \in H^{\text{co} H'}$, we have in Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$, with $h_{(1)} \in H^{\text{co} H'}$. Thus, $h_{(1)} \otimes f(h_{(2)}) \otimes h_{(3)} = h_{(1)} \otimes 1 \otimes h_{(2)}$, and so $f(h_{(1)}) \otimes h_{(2)} = 1 \otimes h$, and $h \in {}^{\text{co} H'} H$. Thus $H^{\text{co} H'} \subset {}^{\text{co} H'} H$. The reverse inclusion follows from the fact that the antipode of H ,

being an anti-bialgebra isomorphism by assumption, exchanges $H^{\text{co } H'}$ and ${}^{\text{co } H'} H$. Hence, $H^{\text{co } H'} = {}^{\text{co } H'} H$.

Conversely, assume $H^{\text{co } H'} = {}^{\text{co } H'} H$. Let (M, δ) be a right H -comodule. We have $\delta(M^{\text{co } H'}) \subseteq M \otimes H^{\text{co } H'}$. Indeed, for $x \in M$, let $\delta(x) = x_{(0)} \otimes x_{(1)}$ in Sweedler's notation. If $x \in M^{\text{co } H'}$, we have

$$x_{(0)} \otimes (\text{id}_M \otimes f) \Delta(x_{(1)}) = \delta x_{(0)} \otimes f(x_{(1)}) = \delta(x) \otimes 1 = x_{(0)} \otimes x_{(1)} \otimes 1.$$

Now by assumption $H^{\text{co } H'} = {}^{\text{co } H'} H$, so for $x \in M^{\text{co } H'}$ we have

$$(\text{id}_M \otimes f) \delta(x_{(0)}) \otimes x_{(1)} = x_{(0)} \otimes (f \otimes \text{id}_H) \Delta(x_{(1)}) = x_{(0)} \otimes 1 \otimes x_{(1)},$$

so $M^{\text{co } H'}$ is a subcomodule of M , hence f_* is normal.

Note that if f_* is normal, $H^{\text{co } H'} = {}^{\text{co } H'} H = \ker(f)$.

Part (2). Let (M, r) be a finite-dimensional left H' -module. The largest trivial subobject of $f^*(M, r) = (M, r(f \otimes \text{id}_M))$ is

$$M_0 = \{m \in M \mid \forall x \in H, f(x)m = \varepsilon(x)m\} = \{m \in M \mid f(H^+)m = 0\}.$$

In particular, $M_0 = M$ if and only if M is a $\text{coker}(f)$ -module.

If $f(H)$ is a normal Hopf subalgebra of H' , we have $f(H^+)H' = H'f(H^+)$; hence, M_0 is a H' -submodule of M and so, f^* is normal, and $\text{coker}(f) = H'/H'f(H^+)$. ■

Lemma 3.11. Let $f: H \rightarrow H'$ be a morphism of Hopf algebras over a field. Then

- (1) The tensor functor $f_*: \text{comod-}H \rightarrow \text{comod-}H'$ is dominant if and only if ${}^? \square^{H'} H: \text{Comod-}H' \rightarrow \text{Comod-}H$ is faithful;
- (2) The tensor functor $f^*: H'\text{-mod} \rightarrow H\text{-mod}$ is dominant if f is injective and H' is finite dimensional. □

Remark 3.12. In particular, if f_* is dominant, then f is surjective. Conversely, if f is surjective and H is right H' coflat, then f_* is dominant, with exact Ind-adjoint. If H is finite dimensional, it is right H' coflat; and in that case, f_* is dominant if and only if f is surjective. □

Proof. Part (1). If C is a coalgebra over a field, the category of Ind-objects of the category of finite-dimensional right C -comodules is the category $\text{Comod-}C$ of all right C -comodules. The Ind-adjoint R of f_* is the right adjoint of

$$\text{Ind}(f_*): \begin{cases} \text{Comod-}H & \rightarrow \text{Comod-}H' \\ (X, \delta) & \mapsto (X, (\text{id}_X \otimes f)\delta), \end{cases}$$

that is, $R = ? \square^{H'} H$. By Lemma 3.1, f_* is dominant if and only if R is faithful.

Part (2). If H' is finite dimensional and f is injective, then H' is a free left H -module [28]. If M is a finite-dimensional left H' -module, then $f \otimes_{H'} M : M \simeq H' \otimes_{H'} M \rightarrow H \otimes_{H'} M$ is a monomorphism $M \rightarrow f_*(H \otimes_{H'} M)$, hence f_* is dominant. ■

Now let us conclude the proof of Proposition 3.9. Consider a strictly exact sequence $K \xrightarrow{i} H \xrightarrow{p} H'$ of Hopf algebras over a field \mathbb{k} . By Lemma 3.11, p_* is dominant because H is right faithfully coflat over H' . Set $I = p^{-1}(0)$. The fact that I is, by assumption, a normal Hopf ideal of H means that the morphism $p : H \rightarrow H' \simeq H/I$ is conormal. By [31, Lemma 1.3] we have $H^{\text{co } H'} = {}^{\text{co } H'} H$, so by Lemma 3.10 the tensor functor p_* is normal, and $\mathfrak{Ker}_{p_*} = \text{comod-}K$ since $K = \ker(p)$. Hence, $\text{comod-}K \rightarrow \text{comod-}H \rightarrow \text{comod-}H'$ is an exact sequence of tensor categories. Now assume H is finite dimensional. Then, by Lemma 3.11, the tensor functor $i^* : H\text{-mod} \rightarrow K\text{-mod}$ is dominant because i is injective, and by Lemma 3.10 it is normal, with $\mathfrak{Ker}_{i^*} = H'\text{-mod}$ because $H' = \text{coker}(i)$. Hence $H'\text{-mod} \rightarrow H\text{-mod} \rightarrow K\text{-mod}$ is an exact sequence of tensor categories.

Remark 3.13. Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an exact sequence of tensor categories over a field \mathbb{k} such that the Ind-adjoint of F is exact. Moreover, assume that \mathcal{C}'' admits a fiber functor $\omega : \mathcal{C}'' \rightarrow \text{vect}_{\mathbb{k}}$. Then, setting $H' = L(\omega)$, $H'' = L(\omega F)$, and $K = L(\omega F f)$, and denoting by $i : K \rightarrow H$ and $p : H \rightarrow H'$, the Hopf algebra morphisms induced by f and F , respectively, we obtain a strictly exact sequence of Hopf algebras over \mathbb{k} :

$$K \xrightarrow{i} H \xrightarrow{p} H',$$

and we have an isomorphism of exact sequences of tensor categories

$$\begin{array}{ccccc} & f & & F & \\ \mathcal{C}' & \xrightarrow{\quad} & \mathcal{C} & \xrightarrow{\quad} & \mathcal{C}'' \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \text{comod-}K & \xrightarrow{i_*} & \text{comod-}H & \xrightarrow{p_*} & \text{comod-}H'' \end{array}$$

Note that we must assume that the Ind-adjoint of F is exact because the definition of strictly exact sequence of tensor categories implies that H is right coflat over H' . \square

Applying the proposition to exact sequences of group algebras, we obtain the following corollary.

Corollary 3.14. An exact sequence of finite groups $1 \rightarrow G'' \xrightarrow{\iota} G \xrightarrow{\pi} G' \rightarrow 1$ gives rise to an exact sequence of tensor categories

$$\text{rep } G' \xrightarrow{\pi^*} \text{rep } G \xrightarrow{\iota^*} \text{rep } G''. \quad \square$$

3.3 Induced Hopf algebras

Let $F : \mathcal{C} \rightarrow \mathcal{C}''$ be a tensor functor between tensor categories over a field \mathbb{k} . The functor

$$\omega_F : \begin{cases} \mathfrak{Ker}_F & \rightarrow \text{vect}_{\mathbb{k}}, \\ X & \mapsto \text{Hom}(1, F(X)) \end{cases}$$

is a fiber functor for \mathfrak{Ker}_F because $F(\mathfrak{Ker}_F) \subset \langle 1 \rangle$. Then

$$L(\omega_F) = \int^{X \in \mathfrak{Ker}_F} \omega_F(X) \otimes_{\mathbb{k}} \omega_F(X)^{\vee}$$

is a Hopf algebra, and we have a canonical tensor equivalence

$$\mathfrak{Ker}_F \xrightarrow{\sim} \text{comod-}L(\omega_F).$$

If F is a normal tensor functor, $L(\omega_F)$ is called the *induced Hopf algebra of F* .

If $\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ is an exact sequence of tensor categories over \mathbb{k} , we have a canonical tensor equivalence $\mathcal{C}' \xrightarrow{\sim} \text{comod-}L(\omega_F)$. Then, $L(\omega_F)$ is called the *induced Hopf algebra of the short exact sequence*.

Proposition 3.15. Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an exact sequence of tensor categories, with induced Hopf algebra H . The following assertions are equivalent:

- (i) The functor F has adjoints;
- (ii) The tensor category \mathcal{C}' is finite;

(iii) The Hopf algebra H is finite dimensional.

In particular, if \mathcal{C}' and \mathcal{C}'' are finite, so is \mathcal{C} . □

Proof. We may assume $\mathcal{C}' = \mathfrak{Ker}_F$, f being the inclusion. We have (ii) \Leftrightarrow (iii) because $\mathfrak{Ker}_F \simeq \text{comod-}H$. If F has a right adjoint R , then the fiber functor $\omega = \text{Hom}(1, Ff) : \mathcal{C}' \rightarrow \text{vect}_{\mathbb{k}}$ has a right adjoint $? \otimes R(1)$, hence by adjunction

$$H = \int^{X \in \mathcal{C}'} \omega(X)^\vee \otimes \omega(X) \simeq \int^{E \in \text{vect}_{\mathbb{k}}} E^\vee \otimes E \otimes \omega R(1) = \text{Hom}_{\mathcal{C}''}(1, FR(1))$$

is finite dimensional. Hence, (i) \Rightarrow (iii). Conversely, assume H is finite dimensional, and denote by $R : \text{Ind } \mathcal{C}'' \rightarrow \text{Ind } \mathcal{C}$ the Ind-adjoint of F , that is, the right adjoint of $\text{Ind } F$. The functor F has a right adjoint if and only if, for any X in \mathcal{C}'' , $R(X)$ is isomorphic to an object of \mathcal{C} . Now $\text{Ind } F$ is strong monoidal, and in particular comonoidal, so its right adjoint R is monoidal; hence, we have a natural transformation $R_2(X, Y) : R(X) \otimes R(Y) \rightarrow R(X \otimes Y)$. Let X be in \mathcal{C} and let

$$H_X = R_2(1, F(X))(\text{id}_{R(1)} \otimes h_X) : R(1) \otimes X \rightarrow RF(X),$$

where h denotes the evaluation of the adjunction $\text{Ind } F \vdash R$. We claim that H_X is an isomorphism. In order to verify this, it is enough to check that for all Y in \mathcal{C} , H_X induces a bijection $\text{Hom}_{\mathcal{C}}(Y, R(1) \otimes X) \simeq \text{Hom}_{\mathcal{C}}(Y, RF(X))$. We have $\text{Hom}_{\mathcal{C}}(Y, R(1) \otimes X) \simeq \text{Hom}_{\mathcal{C}}(Y \otimes X^\vee, R(1)) \simeq \text{Hom}_{\mathcal{C}''}(F(Y \otimes X^\vee), 1) \simeq \text{Hom}_{\mathcal{C}''}(F(Y), F(X)) \simeq \text{Hom}_{\mathcal{C}}(Y, RF(X))$, and this bijection is the natural map induced by H_X . Hence, H_X is an isomorphism. Let Y be an object of \mathcal{C}'' . Then, F being dominant, Y is a subobject of $F(X)$ for some X in \mathcal{C} . Since $RF(X) \simeq R(1) \otimes X$, $RF(X)$ belongs to \mathcal{D} because $R(1) \simeq H \otimes 1$, and $R(X)$, being a subobject of $RF(X)$ because R is left exact, is isomorphic to an object of \mathcal{D} , so F has a right adjoint; it also has a left adjoint by Remark 2.3, hence (iii) \Rightarrow (i). ■

4 Exact Sequences of Fusion Categories

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between fusion categories. Then F is dominant if and only if any simple object Y of \mathcal{D} is a direct factor of $F(X)$ for some simple object X of \mathcal{C} , and F is normal if and only if, for any simple object X of \mathcal{C} , $F(X)$ contains a copy of the unit of 1 then $F(X)$ is trivial.

Example 4.1. Let $K \subset H$ be an inclusion of finite-dimensional split semisimple Hopf algebras. Then, the restriction functor from $\mathcal{C} = H\text{-mod}$ to $\mathcal{D} = K\text{-mod}$ is a dominant tensor functor between fusion categories. We have $\text{FPdim } \mathcal{D} = \dim K$ and $\text{FPdim } \mathcal{C} = \dim H$, so that the quotient $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D} = \dim H / \dim K$ is a natural integer, called the index of K in H . \square

This example motivates the following definition.

Definition 4.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories. The *Frobenius–Perron index* of F is the ratio $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$; we denote it by $\text{FPind}(F)$ or $\text{FPind}(\mathcal{C} : \mathcal{D})$. \square

It follows from [16, Corollary 8.11] that if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a dominant tensor functor, then $\text{FPdim } \mathcal{D}$ divides $\text{FPdim } \mathcal{C}$, that is, the Frobenius–Perron index of a dominant tensor functor F between fusion categories is always an algebraic integer. More precisely, we have the following proposition.

Proposition 4.3. If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a dominant tensor functor between fusion categories, then

$$\text{FPind}(F) = \text{FPdim } G(1),$$

where G is a left (or a right) adjoint of F . \square

Proof. For any fusion category \mathcal{C} , let $R(\mathcal{C}) = K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the \mathbb{R} -algebra obtained by extension of scalars from the Grothendieck ring $K_0(\mathcal{C})$ of \mathcal{C} . Consider the element

$$R_{\mathcal{C}} := \sum_{X \in A_{\mathcal{C}}} \text{FPdim } X [X] \in R(\mathcal{C}).$$

We have $\text{FPdim } R_{\mathcal{C}} = \text{FPdim } \mathcal{C}$.

Lemma 4.4. [16] Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories. Then

$$F_!(R_{\mathcal{C}}) = \frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{D}} R_{\mathcal{D}}, \quad (4.1)$$

where $F_! : R(\mathcal{C}) \rightarrow R(\mathcal{D})$ is the algebra map induced by F , and for all $Y \in \Lambda_{\mathcal{D}}$,

$$\sum_{X \in \Lambda_{\mathcal{C}}} m_Y(F(X)) \text{FPdim } X = \frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{D}} \text{FPdim } Y. \quad (4.2)$$

□

Proof. Equation (4.1) is proved in [16, Proposition 8.8], and (4.2) is obtained by comparing the multiplicities of Y in both sides of (4.1). ■

Now for any simple object X of \mathcal{C} , if G is a right adjoint of F , we have by adjunction $\text{Hom}_{\mathcal{C}}(G(1), X) \simeq \text{Hom}_{\mathcal{D}}(1, F(X))$, hence $m_1 F(X) = m_X G(1)$. The same equation holds if G is a left adjoint. Thus, Equation (4.2) for $Y = 1$ reads: $\text{FPind}(F) = \sum_{X \in \Lambda_{\mathcal{C}}} \text{FPdim } X m_X G(1) = \text{FPdim } G(1)$. ■

Lemma 4.5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between fusion categories and X be an object of \mathcal{C} . Then, X belongs to \mathfrak{Ker}_F if and only if $\text{FPdim } X = m_1 F(X)$, and in this case $F(X) \simeq \mathbf{1}^{\text{FPdim } X}$. □

Proof. An object Y of \mathcal{D} is trivial if and only if $\text{FPdim}(Y) = m_1(Y)$. Besides, F preserves Frobenius–Perron dimensions. Thus, X is in \mathfrak{Ker}_F if and only if $\text{FPdim}(X) = \text{FPdim } F(X) = m_1 F(X)$. ■

4.1 Exact sequences of pointed categories

Recall that a *pointed category* is a fusion category \mathcal{C} whose simple objects are invertible, so that $G = \Lambda_{\mathcal{C}}$ is a finite group for the tensor product called the *Picard group* of \mathcal{C} and denoted by $\text{Pic } \mathcal{C}$. If \mathcal{C} is pointed, then \mathcal{C} is equivalent to the category $\mathcal{C}(G, \alpha)$ of G -graded vector spaces with associativity constraint given by a three-cocycle in a class $\alpha \in H^3(G, \mathbb{k}^\times)$. It is well known that this correspondence yields a bijection $(G, \alpha) \rightarrow \mathcal{C}(G, \alpha)$ between pairs (G, α) , where G is a finite group and $\alpha \in H^3(G, \mathbb{k}^\times)$, and pointed categories over \mathbb{k} up to equivalences of tensor categories. Note that if a pointed category \mathcal{C} is tensor equivalent to H -mod for some Hopf algebra H , then in fact $H \simeq \mathbb{k}^G$, where $G = \text{Pic } \mathcal{C}$, so that $\mathcal{C} \simeq \mathcal{C}(G, 1) = \mathbb{k}^G$ -mod.

Proposition 4.6. Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an exact sequence of fusion categories over a field \mathbb{k} , and assume that \mathcal{C} is pointed. Then, \mathcal{C}' and \mathcal{C}'' are pointed too, and we have an exact

sequence of groups

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1,$$

where G , G' , and G'' denote the Picard groups of \mathcal{C} , \mathcal{C}' , and \mathcal{C}'' , respectively. Moreover, up to equivalence, such an exact sequence of fusion categories is of the form

$$\mathcal{C}(G', 1) \rightarrow \mathcal{C}(G, \text{infl}(\alpha)) \rightarrow \mathcal{C}(G'', \alpha),$$

where $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ is an exact sequence of finite groups, α is a cohomology class in $H^3(G'', \mathbb{k}^\times)$, and $\text{infl}: H^3(G'', \mathbb{k}^\times) \rightarrow H^3(G, \mathbb{k}^\times)$ denotes the inflation map. \square

Proof. The category \mathcal{C} is of the form $\mathcal{C}(G, \beta)$, where $G = \text{Pic } \mathcal{C}$ and $\beta \in H^3(G, \mathbb{k}^\times)$. Being a full tensor subcategory of \mathcal{C} , \mathcal{C}' is of the form $\mathcal{C}(G', \beta')$ where $\beta' \in H^3(G', \mathbb{k}^\times)$ is the image of β under the restriction morphism. Since $\mathcal{C}' = \text{Ker}_F$, it is of the form H , H being the Hopf algebra associated with our exact sequence of fusion categories (see Section 3.3). Therefore, $\beta' = 1$ and $\mathcal{C}' = \mathcal{C}(G', 1)$.

On the other hand, tensor functors preserve invertible objects. Since the tensor functor $F: \mathcal{C} \rightarrow \mathcal{C}''$ is dominant and \mathcal{C} is pointed, so is \mathcal{C}'' . Therefore, \mathcal{C}'' is of the form $\mathcal{C}(G'', \alpha)$ for some finite group G'' and $\alpha \in H^3(G'', \mathbb{k}^\times)$. Since F preserves the monoidal structures, we have $\beta = \text{infl}(\alpha)$.

Conversely, given an exact sequence of finite groups $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ and a class $\alpha \in H^3(G'', \mathbb{k}^\times)$, we have a tensor functor $F: \mathcal{C} = \mathcal{C}(G, \text{infl}(\alpha)) \rightarrow \mathcal{C}(G'', \alpha)$ such that $\text{Ker}_F = \mathcal{C}(G', 1)$, hence an exact sequence of fusion categories $\mathcal{C}(G, 1) \rightarrow \mathcal{C}(G, \text{infl}(\alpha)) \rightarrow \mathcal{C}(G'', \alpha)$. \blacksquare

Remark 4.7. In an exact sequence of fusion categories $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$, such that \mathcal{C}' and \mathcal{C}'' are pointed, \mathcal{C} need not be pointed. Indeed, let Γ and K be abelian groups endowed with a nontrivial action of K on Γ by automorphisms, and let $G = \Gamma \rtimes K$ be the semidirect product (for instance, take G to be a dihedral group). Then the category $\text{rep } G$ is not pointed, since G is not abelian. On the other hand, there exists an exact sequence of groups $1 \rightarrow \Gamma \rightarrow G \rightarrow K \rightarrow 1$, which gives rise to an exact sequence of fusion categories $\mathcal{C}(\Gamma, 1) \rightarrow \text{rep } G \rightarrow \mathcal{C}(K, 1)$ by Proposition 3.9. \square

Remark 4.8. Let $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ be an exact sequence of finite groups such that the inflation map $H^3(G'', \mathbb{k}^\times) \rightarrow H^3(G, \mathbb{k}^\times)$ is not injective, and let $\alpha \in H^3(G'', \mathbb{k}^\times)$ be a nontrivial element of its kernel. Then, by Proposition 4.6, we have an exact sequence of

fusion categories:

$$\mathcal{C}(G', 1) \rightarrow \mathcal{C}(G, 1) \rightarrow \mathcal{C}(G'', \alpha)$$

and $\mathcal{C}(G'', \alpha)$ does not admit a fiber functor. \square

Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an exact sequence of fusion categories. If \mathcal{C}'' admits a fiber functor, then so does \mathcal{C} . The converse is not true, as exemplified by the previous remark. However, we have the following proposition.

Proposition 4.9. Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an exact sequence of fusion categories. Then, \mathcal{C} is tensor equivalent to the category of representations of a quasi-Hopf algebra if and only if \mathcal{C}'' is. \square

Proof. A fusion category is tensor equivalent to the representation category of a quasi-Hopf algebra if and only if the Frobenius–Perron dimensions of its objects are natural integers, see [16, Theorem 8.33]. Hence, the “if” Part because a tensor functor preserves Frobenius–Perron dimensions (or by Tannaka reconstruction), and the “if” Part by [16, Corollary 8.36]. \blacksquare

4.2 Multiplicativity of Frobenius–Perron dimensions

Proposition 4.10. Let \mathcal{C} , \mathcal{C}' , and \mathcal{C}'' be fusion categories and $i: \mathcal{C}' \rightarrow \mathcal{C}$, $F: \mathcal{C} \rightarrow \mathcal{C}''$ be tensor functors. Assume that F is dominant, i is full and $i(\mathcal{C}) \subset \mathfrak{Rep}_F$. Then, $\text{FPdim } \mathcal{C} \geq \text{FPdim } \mathcal{C}' \text{ FPdim } \mathcal{C}''$. Moreover, the diagram

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

is an exact sequence if and only if $\text{FPdim } \mathcal{C} = \text{FPdim } \mathcal{C}' \text{ FPdim } \mathcal{C}''$. If such is the case, then for all simple object $Y \in \mathcal{C}''$,

$$\text{FPdim } Y = \frac{1}{\text{FPdim } \mathcal{C}'} \sum_{X \in \Lambda_{\mathcal{C}}} m_Y(F(X)) \text{FPdim } X. \quad (4.3)$$

\square

Proof. First, notice that since the tensor functor $i_0: \mathcal{C}' \rightarrow \mathfrak{Rep}_F$ is full, we have $\text{FPdim } \mathcal{C}' \leq \text{FPdim } \mathfrak{Rep}_F$, with equality if and only if i_0 is an equivalence. Thus, we may assume that $\mathcal{C}' = \mathfrak{Rep}_F$, i being the inclusion.

Equation (4.2), applied to the dominant functor $F : \mathcal{C} \rightarrow \mathcal{C}''$ and the simple object $Y = 1$, yields

$$\sum_{X \in \Lambda_{\mathcal{C}}} m_1(F(X)) \text{FPdim } X = \frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{C}''}. \quad (4.4)$$

Since $\mathcal{C}' = \text{Ker}_F$ is a full subcategory of \mathcal{C} , we may choose representatives of classes of simple objects so that $\Lambda_{\mathcal{C}'} \subset \Lambda_{\mathcal{C}}$. For $X \in \Lambda_{\mathcal{C}'}$, we have $F(X) = 1^{\text{FPdim } X}$, and therefore $\sum_{X \in \Lambda_{\mathcal{C}'}} m_1(F(X)) \text{FPdim } X = \text{FPdim } \mathcal{C}'$. Thus,

$$\frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{C}''} = \text{FPdim } \mathcal{C}' + E. \quad (4.5)$$

where $E = \sum_{X \in \Lambda_{\mathcal{C}} \setminus \Lambda_{\mathcal{C}'}} m_1(F(X)) \text{FPdim } X$. Now we have $E \geq 0$, and $E = 0$ if and only if F is normal, hence the proposition is proved. \blacksquare

Remark 4.11. Relation (4.3) implies the identity $F_!(R_{\mathcal{C}}) = \frac{1}{\text{FPdim } \mathcal{C}'} R_{\mathcal{C}''}$. \square

Corollary 4.12. Consider a diagram of tensor functors between fusion categories with exact rows, and commutative up to tensor isomorphisms:

$$\begin{array}{ccccc} \mathcal{C}' & \xrightarrow{f} & \mathcal{C} & \xrightarrow{F} & \mathcal{C}'' \\ \downarrow l & & \downarrow \lambda & & \downarrow r \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} & \xrightarrow{G} & \mathcal{D}'' \end{array}$$

If l and r are equivalences, then λ is an equivalence. \square

Proof. Since r and l are equivalences, we may assume that they are identities, $\mathcal{C}' = \mathcal{D}'$, $\mathcal{C}'' = \mathcal{D}''$, and the diagram commutes. By Proposition 4.10, we have $\text{FPdim } \mathcal{C} = \text{FPdim } \mathcal{D}$. Now denote by \mathcal{E} the dominant image of λ , that is, the full subcategory of \mathcal{D} whose objects are direct factors of objects belonging to the image of λ . The tensor functor λ factors as $j\lambda_0$, where $\lambda_0 : \mathcal{C} \rightarrow \mathcal{E}$ is dominant, and $j : \mathcal{E} \rightarrow \mathcal{D}$ is a replete inclusion. In particular, $\text{FPdim } \mathcal{E} \leq \text{FPdim } \mathcal{D}$. In the diagram of tensor functors

$$\mathcal{C}' \xrightarrow{\lambda_0 f} \mathcal{E} \xrightarrow{Gj} \mathcal{C}'',$$

the functor Gj is dominant because $F = Gj\lambda_0$ is dominant, $\lambda_0 f$ is fully faithful because $g = j\lambda_0 f$ and j are fully faithful, and $\lambda_0 f(\mathcal{C}'_1) \subset \mathfrak{Ker}_{Gj}$ because $Gj\lambda_0 f = Ff$. By Proposition 4.10, we have $\text{FPdim } \mathcal{E} \geq \text{FPdim } \mathcal{C}' \text{ FPdim } \mathcal{C}'' = \text{FPdim } \mathcal{D}$. Hence, $\text{FPdim } \mathcal{E} = \text{FPdim } \mathcal{D}$, so that $\mathcal{E} = \mathcal{D}$ and therefore λ is dominant. Applying again Proposition 4.10, this time to the sequence

$$\mathfrak{Ker}_\lambda \longrightarrow \mathcal{C} \xrightarrow{\lambda} \mathcal{D}$$

we find that $\text{FPdim } \mathfrak{Ker}_\lambda \leq 1$, hence $\mathfrak{Ker}_\lambda = \langle 1 \rangle$, and the sequence is exact, so λ is an equivalence. \blacksquare

4.3 Functors of Frobenius–Perron index 2

It is well known that if H is a split semisimple Hopf algebra and K is a Hopf subalgebra of index 2, then K is a normal Hopf subalgebra and there is a cocentral exact sequence of Hopf algebras $K \rightarrow H \rightarrow \mathbb{k}\mathbb{Z}_2$. See [22, 25, Corollary 1.4.3]. Hence, an exact sequence of fusion categories $\text{rep } \mathbb{Z}_2 \rightarrow H\text{-mod} \rightarrow K\text{-mod}$.

We extend this result to general fusion categories. Recall that the the Frobenius–Perron index $\text{FPind } F$ of a dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between fusion categories is the ratio $\text{FPind } F = \text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$.

Proposition 4.13. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor of Frobenius–Perron index 2 between fusion categories. Then F is normal, and we have an exact sequence of fusion categories

$$\text{rep } \mathbb{Z}_2 \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

□

Proof. Equation (4.2), applied to $Y = 1$, yields

$$\sum_{X \in \Lambda_{\mathcal{C}}} m_1(F(X)) \text{FPdim } X = \frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{D}} = 2.$$

Since $\text{FPdim } X \geq 1$ for any simple object X , we conclude that there is exactly one element $J \in \Lambda_{\mathcal{C}}$, $J \not\simeq 1$, such that $F(J)$ contains 1 . Moreover, we have $\text{FPdim } F(J) = \text{FPdim } J = 1$ so $F(J) \simeq 1$. Thus F is normal, and we have an exact sequence of fusion categories $\mathfrak{Ker}_F \rightarrow \mathcal{C} \rightarrow \mathcal{D}$. Note that J is invertible. On the other hand, \mathfrak{Ker}_F is a pointed category whose group of invertibles is of order 2 (because J is invertible). Therefore, $\mathfrak{Ker}_F \simeq H$,

where H is a split semisimple Hopf algebra H of dimension 2, that is, $H \simeq \mathbb{k}\mathbb{Z}_2$, hence the proposition holds. \blacksquare

5 Hopf Monads and Exact Sequences

In this section, we study tensor functors and, in particular, exact sequences of tensor categories in terms of Hopf monads. Recall that a tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is monadic if and only if it has a left adjoint G (whose existence is equivalent to that of a right adjoint, see Section 2.3). If such is the case, the monad $T = FG$ of G is a Hopf monad on \mathcal{D} , and \mathcal{C} is tensor equivalent to the category \mathcal{D}^T of T -modules in \mathcal{D} .

Proposition 5.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories over a field \mathbb{k} , and assume F admits a left adjoint G . Let $T = FG$ be the Hopf monad of F . Then the following assertions are equivalent:

- (i) The functor F is dominant;
- (ii) The unit η of the monad T is a monomorphism;
- (iii) The monad T is faithful;
- (iv) The left adjoint of F is faithful;
- (v) The right adjoint of F is faithful.

\square

Proof. The equivalence of the first four assertions results immediately from Lemma 2.2, considering that, since F is monadic, we may assume that $\mathcal{C} = \mathcal{D}^T$, F is the forgetful functor $\mathcal{U} : \mathcal{D}^T \rightarrow \mathcal{D}$ and G is the free module functor. The right adjoint of F is $R \simeq {}^\vee G({}^\vee ?)$, hence (iv) \Leftrightarrow (v). \blacksquare

Proposition 5.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between fusion categories over a field \mathbb{k} , and let T be the Hopf monad of F . Then $\text{FPind}(\mathcal{C} : \mathcal{D}) \leq \text{FPdim } T(\mathbf{1})$, and F is dominant if and only if

$$\text{FPind}(\mathcal{C} : \mathcal{D}) = \text{FPdim } T(\mathbf{1}).$$

If such is the case, then for all $X \in \Lambda_{\mathcal{C}}$, $\text{FPdim } T(X) = \text{FPdim } T(\mathbf{1}) \text{ FPdim } X$. \square

Proof. Let G be a left adjoint of F . If F is dominant, we have $\text{FPdim } \mathcal{C} = \text{FPdim } T(\mathbf{1}) \text{ FPdim } \mathcal{D}$ by Proposition 4.3, noting that $T(\mathbf{1}) = G(\mathbf{1})$. In general, let \mathcal{E} be the dominant image of F , that is, the full subcategory of \mathcal{D} whose objects are direct factors of objects belonging to the image of F . Denote by F_0 the dominant tensor functor $\mathcal{C} \rightarrow \mathcal{E}$

induced by F , and i the full embedding $\mathcal{E} \subset \mathcal{D}$. Then $\mathcal{L}|_{\mathcal{E}}$ is left adjoint to F_0 so we have $\text{FPdim } \mathcal{C} = \text{FPdim } T(\mathbf{1}) \text{FPdim } \mathcal{E}$. We also have $\text{FPdim } \mathcal{E} \leq \text{FPdim } \mathcal{D}$, with equality if and only if $\mathcal{E} = \mathcal{D}$, that is, F is dominant. This proves both the inequality and the equivalence of the proposition.

Now assume F is dominant. For $X \in \Lambda_{\mathcal{D}}$, we have by Equation (4.2):

$$\begin{aligned} \text{FPdim}(X) &= \frac{\dim \mathcal{D}}{\dim \mathcal{C}} \sum_{Y \in \Lambda_{\mathcal{C}}} m_Y(F(X)) \text{FPdim } Y \\ &= \frac{1}{\text{FPdim } T(\mathbf{1})} \sum_{Y \in \Lambda_{\mathcal{C}}} m_X(G(Y)) \text{FPdim } Y \quad (\text{by adjunction}) \\ &= \frac{1}{\text{FPdim } T(\mathbf{1})} \text{FPdim } G(X) = \frac{1}{\text{FPdim } T(\mathbf{1})} \text{FPdim } T(X), \end{aligned}$$

hence the last assertion of the proposition holds. ■

Definition 5.3. Let \mathcal{C} be a tensor category. A \mathbb{k} -linear right exact Hopf monad T on \mathcal{C} is *normal* if $T(\mathbf{1})$ is a trivial object of \mathcal{C} . □

Recall that a tensor functor F gives rise to a fiber functor $\omega_F : \mathfrak{Ker}_F \rightarrow \text{vect}_{\mathbb{k}}$, $X \mapsto \text{Hom}(\mathbf{1}, F(X))$, hence a Hopf algebra $L(\omega_F)$ such that there exists a canonical tensor equivalence $c : \mathfrak{Ker}_F \xrightarrow{\sim} \text{comod-}L(\omega_F)$ (see Section 3.3).

Lemma 5.4. Let T be a \mathbb{k} -linear right exact normal Hopf monad on a tensor category \mathcal{C} , with forgetful functor $\mathcal{U} : \mathcal{C}^T \rightarrow \mathcal{C}$. Then the Hopf algebra $L = L(\omega_{\mathcal{U}})$ is finite-dimensional, and we have a canonical tensor equivalence

$$\mathfrak{Ker}_{\mathcal{U}} \xrightarrow{\sim} \text{comod-}L.$$

The Hopf algebra $L(\omega_{\mathcal{U}})$ is called the *induced Hopf algebra of the normal Hopf monad T* .

Proof. First, we show that $L(\omega)$ is finite dimensional. Since T is normal, we have $T(\langle \mathbf{1} \rangle) \subset \langle \mathbf{1} \rangle$, so that T restricts to a Hopf monad $T|_{\langle \mathbf{1} \rangle}$ on $\langle \mathbf{1} \rangle$, and $\mathfrak{Ker}_{\mathcal{U}} = \langle \mathbf{1} \rangle^{T|_{\langle \mathbf{1} \rangle}}$. Thus, we may assume $\mathcal{C} = \langle \mathbf{1} \rangle$ (that is, $\mathcal{C} \simeq \text{vect}_{\mathbb{k}}$). Then we have $\mathcal{C}_{\mathcal{U}} = \mathcal{C}^T$, and by Lemma 2.5, there exists a finite-dimensional Hopf algebra H on \mathbb{k} such that $T = H \otimes ?$. We have a

commutative triangle of tensor functors

$$\begin{array}{ccc}
 \mathcal{C}^T & \xrightarrow{\sim} & H\text{-mod} \\
 \omega_{\mathcal{U}} \searrow & & \swarrow \\
 & \text{vect}_{\mathbb{k}} &
 \end{array}$$

whose horizontal is an equivalence, hence $L(\omega_{\mathcal{U}}) \simeq H^*$ is finite dimensional. \blacksquare

Remark 5.5. The induced Hopf algebra of a normal Hopf monad T is the induced algebra of the restriction of T to the trivial tensor subcategory $\langle 1 \rangle \subset \mathcal{C}$. \square

We have by definition the following.

Proposition 5.6. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor admitting a left adjoint. Then F is normal if and only if its Hopf monad T is normal. \square

Example 5.7. Let \mathcal{C} be a fusion category over a field \mathbb{k} . It is shown in [8] that the forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ is monadic, with monad Z defined by

$$Z(X) = \sum_{Y \in \Lambda_{\mathcal{C}}} {}^Y Y \otimes X \otimes Y.$$

The unit of the monad Z is the 1-component inclusion $X \hookrightarrow Z(X)$; it is therefore a monomorphism. Hence, \mathcal{U} is dominant by Proposition 5.1. However, \mathcal{U} is normal if and only if \mathcal{C} is a pointed category, because $Z(1) = \sum_{Y \in \Lambda_{\mathcal{C}}} {}^Y Y \otimes Y$ is trivial exactly in that case. \square

5.1 Exact sequences and normal, faithful Hopf monads

The following theorem classifies extensions of tensor categories in terms of normal, faithful Hopf monads.

Theorem 5.8. Let \mathcal{C}' , \mathcal{C}'' be tensor categories over a field \mathbb{k} , and assume that \mathcal{C}' is finite. Then the following data are equivalent:

- (1) A normal, faithful, \mathbb{k} -linear right exact Hopf monad T on \mathcal{C}'' , with induced Hopf algebra H , endowed with a tensor equivalence $K : \mathcal{C}' \xrightarrow{\sim} \text{comod-}H$;
- (2) An extension $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ of \mathcal{C}'' by \mathcal{C}' . \square

Proof. Let T be a normal, faithful, \mathbb{k} -linear right exact Hopf monad on \mathcal{C}'' with induced Hopf algebra H . Denote by $\mathcal{U} : \mathcal{C}''^T \rightarrow \mathcal{C}''$ the forgetful functor. Let q be a quasi-inverse of the canonical tensor equivalence $\mathfrak{Ker}_{\mathcal{U}} \rightarrow \text{comod-}H$. According to Corollary 5.14, we have an exact sequence:

$$\text{comod-}H \xrightarrow{iq} (\mathcal{C}'')^T \xrightarrow{\mathcal{U}} \mathcal{C}'',$$

hence an extension $\mathcal{C}' \xrightarrow{iqK} (\mathcal{C}'')^T \xrightarrow{\mathcal{U}} \mathcal{C}''$ of \mathcal{C}'' by \mathcal{C}' .

Conversely, let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an extension of \mathcal{C}'' by \mathcal{C}' . Then F has a left adjoint by Proposition 3.15. Let T be the Hopf monad of F . Then T is \mathbb{k} -linear, faithful (because F is dominant) and normal (because F is normal). The tensor functor f induces a tensor equivalence $f_0 : \mathcal{C}' \rightarrow \mathfrak{Ker}_F$, and we also have a canonical tensor equivalence $c : \mathfrak{Ker}_F \xrightarrow{\sim} \text{comod-}H$, where H is the induced Hopf algebra of the Hopf monad T of F , hence a tensor equivalence $K = cf_0 : \mathcal{C}' \xrightarrow{\sim} \text{comod-}H$.

These two constructions are mutually quasi-inverse. Indeed, if T and K are as in (1), then the Hopf monad of the forgetful functor $\mathcal{U} : \mathcal{C}''^T \rightarrow \mathcal{C}''$ is T and the reconstructed tensor equivalence $\mathcal{C}' \rightarrow \text{comod-}H$ is $cqK \simeq K$.

On the other hand, given an extension $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$, and denoting by T and K the corresponding Hopf monad and tensor equivalence, we have an equivalence of extensions:

$$\begin{array}{ccccc} & f & & F & \\ \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' \\ = \downarrow & & \kappa \downarrow & & \downarrow = \\ \mathcal{C}' & \xrightarrow{iqK} & \mathcal{C}''^T & \xrightarrow{\mathcal{U}} & \mathcal{C}'', \end{array}$$

where \mathcal{U} denotes the forgetful functor and κ is the comparison functor of the monadic functor F . Indeed, κ is a tensor equivalence, we have $\mathcal{U}\kappa = F$ and $iqK = iqcf_0 \simeq if_0 = f$. ■

Combining Theorem 5.8 with Proposition 4.13, we obtain the following corollary.

Corollary 5.9. Let $F : \mathcal{C} \rightarrow \mathcal{C}''$ be a dominant tensor functor between fusion categories \mathcal{C} and \mathcal{C}'' such that $\text{FPind } F = 2$. Then there exists a \mathbb{k} -linear, faithful, normal semisimple Hopf monad T on \mathcal{C}'' having induced Hopf algebra $\mathbb{k}^{\mathbb{Z}_2}$, such that $\mathcal{C} \simeq (\mathcal{C}'')^T$ as fusion categories. □

Example 5.10. Let $1 \rightarrow G'' \xrightarrow{\iota} G \xrightarrow{\pi} G' \rightarrow 1$ be an exact sequence of finite groups. Then we have an exact sequence of tensor categories

$$\text{rep } G' \xrightarrow{\pi^*} \text{rep } G \xrightarrow{\iota^*} \text{rep } G''$$

as in Example 3.14. Let us describe the normal Hopf monad T on $\text{rep } G''$ associated with this exact sequence. We may assume without loss of generality that G'' is a normal subgroup of G , and that $G' = G/G''$, ι being the inclusion and π the canonical surjection. The induction functor $\text{Ind}_{G'}^G : \text{rep } G'' \rightarrow \text{rep } G$ is left adjoint to the restriction functor $\iota^* = \text{Res}_{G''}^G$. Let Y be a $\mathbb{k}G''$ -module. As a consequence of Mackey's Subgroup Theorem, there is a natural isomorphism

$$\text{Res}_{G''}^G \text{Ind}_{G''}^G(Y) \simeq \bigoplus_{\gamma \in G/G''} {}^\gamma Y,$$

where ${}^\gamma Y$ denotes the $\mathbb{k}G''$ -module conjugated to Y under the action of an element $\gamma \in G$ representing the class γ . See [10, Remark (10.11)]. Then the Hopf monad T is given, as an endofunctor of $\text{rep } G'$, by

$$T(Y) = \bigoplus_{\gamma \in G'} {}^\gamma Y.$$

In fact, conjugation under elements $\gamma \in G/G''$ defines an action of the group G' on $\text{rep } G''$ by tensor autoequivalences. In subsequent subsections, we will study group actions on tensor categories in terms of Hopf monads. \square

5.2 Semisimplicity

A monad T on a category \mathcal{A} is said to be *semisimple* if any T -module is a T -linear retract of a free T -module, that is, of $(T(X), \mu_X)$, for some $X \in \mathcal{A}$. We have the following analogue of Maschke's semisimplicity criterion for Hopf monads.

Theorem 5.11 ([7, Theorem 6.5.]). Let T be a Hopf monad on a rigid category \mathcal{C} . Then T is semisimple if and only if there exists a morphism $\Lambda : 1 \rightarrow T(1)$ such that $\mu_1 \Lambda = \Lambda T_0$ and $T_0 \Lambda = \text{id}_1$. \square

Corollary 5.12. Let T be a Hopf monad on a rigid category \mathcal{C} and let $\mathcal{C}' \subset \mathcal{C}$ be a full rigid subcategory of \mathcal{C} such that $T(\mathcal{C}') \subset \mathcal{C}'$. Then T is semisimple if and only if its restriction $T|_{\mathcal{C}'}$ to \mathcal{C}' is semisimple. \square

Proof. Apply the theorem to T and $T|_{\mathcal{C}'}$, which is a Hopf monad on \mathcal{C}' . ■

Proposition 5.13. Let \mathcal{C} be a semisimple tensor category over a field \mathbb{k} , and let T be a \mathbb{k} -linear Hopf monad on \mathcal{C} . Then the tensor category \mathcal{C}^T is semisimple if and only if the monad T is semisimple. In particular, if \mathbb{k} is algebraically closed and \mathcal{C} is a fusion category, then \mathcal{C}^T is a fusion category if and only if T is semisimple. □

Proof. Observe that, since \mathcal{C} is semisimple, T is exact. Let $\mathcal{U} : \mathcal{C}^T \rightarrow \mathcal{C}$ be the forgetful functor, and $\mathcal{L} : \mathcal{C} \rightarrow \mathcal{C}^T$ be the free module functor, defined by $\mathcal{L}(X) = (T(X), \mu_X)$, which is left adjoint to \mathcal{U} . If X is an object of \mathcal{C} , then $\mathcal{L}(X)$ is a projective object of \mathcal{C}^T . Indeed, let $p : (Y, r) \rightarrow \mathcal{L}(X)$ be an epimorphism in \mathcal{C}^T ; then p is an epimorphism in \mathcal{C} , so it has a section $i : TX \rightarrow Y$ in \mathcal{C} because \mathcal{C} is semisimple. Then $rT(i)T(\eta_X)$ is a T -linear section of p .

In particular, if T is semisimple, any object of \mathcal{C}^T is projective, being a direct factor of a projective object, so \mathcal{C}^T is semisimple.

Conversely, assume \mathcal{C}^T is semisimple, and let (X, r) be a T -module. Then $r : T(X) \rightarrow X$ is an epimorphism because $r\eta_X = \text{id}_X$. It is also a morphism of T -modules from to (X, r) , which is an epimorphism in \mathcal{C}^T because the forgetful functor $\mathcal{U} : \mathcal{C}^T \rightarrow \mathcal{C}$ is faithful exact. By semisimplicity of \mathcal{C}^T , r has a T -linear section so (X, r) is a direct factor of $\mathcal{L}(X)$. Hence, T is semisimple. ■

Corollary 5.14. Let \mathcal{C} be a fusion category over an algebraically closed field \mathbb{k} , and let T be a \mathbb{k} -linear, faithful, normal, semisimple Hopf monad on \mathcal{C} , with induced Hopf algebra H . Then we have an exact sequence of fusion categories

$$\text{comod-}H \rightarrow \mathcal{C}^T \xrightarrow{\mathcal{U}} \mathcal{C}.$$
□

Lemma 5.15. Let \mathcal{C} be a tensor category over a field \mathbb{k} , and let T be a \mathbb{k} -linear right exact normal dominant Hopf monad on \mathcal{C} with induced Hopf algebra H . Then T is semisimple if and only if H is cosemisimple. □

Proof. Let T_0 be the restriction of T to $\langle 1 \rangle$. Then T is semisimple if and only if T_0 is semisimple by Corollary 5.12. On the other hand, $\langle 1 \rangle^{T_0} = \mathfrak{Ker}_{\mathcal{U}} \simeq \text{comod-}H$, where \mathcal{U} is the forgetful functor, so T_0 is semisimple $\iff \langle 1 \rangle^{T_0}$ is semisimple (by Proposition 5.13) $\iff H$ is cosemisimple. ■

Corollary 5.16. An extension of fusion categories over an algebraically closed field is a fusion category. □

Proof. Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an exact sequence of fusion categories. Then \mathcal{C}' is finite, so by Proposition 3.15, F has a left adjoint and is therefore monadic, and \mathcal{C} is finite. Its monad T is normal, dominant and its induced Hopf algebra H is cosemisimple because $\mathcal{C}' \simeq \text{comod-}H$, so T is semisimple by the previous lemma and \mathcal{C} is semisimple. Since \mathbb{k} is algebraically closed, \mathcal{C} is a fusion category. \blacksquare

5.3 Equivariantization

Let \mathcal{C} be a tensor category over a field \mathbb{k} . Denote by $\underline{\text{End}}_{\otimes} \mathcal{C}$ the monoidal category whose objects are tensor endofunctors of \mathcal{C} , the morphisms being monoidal natural transformations, the monoidal product being the composition \circ of tensor functors, and the unit object, the identity functor $\text{id}_{\mathcal{C}}$.

If G is a group, denote by \underline{G} the strict monoidal category whose objects are the elements of G and morphisms are identities, the monoidal product being the multiplication of G .

Definition 5.17. An *action of a group G on a tensor category \mathcal{C} (by tensor autoequivalences)* is a strong monoidal functor

$$\rho : \underline{G} \rightarrow \underline{\text{End}}_{\otimes} \mathcal{C}. \quad (5.1)$$

In other words, it consists of the following data:

- (1) For each $g \in G$, a tensor endofunctor $\rho^g : \mathcal{C} \rightarrow \mathcal{C}$;
- (2) For each pair $g, h \in G$, a monoidal isomorphism $\rho_2^{g,h} : \rho^g \rho^h \xrightarrow{\sim} \rho^{gh}$;
- (3) A monoidal isomorphism $\rho_0 : \text{id}_{\mathcal{C}} \xrightarrow{\sim} \rho^1$;

such that for any g, h, k in G the following diagrams commute:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \rho^g \rho^h \rho^k & \xrightarrow{\rho_2^{g,h} \rho^k} & \rho^{gh} \rho^k \\
 \downarrow \rho^g \rho_2^{h,k} & & \downarrow \rho_2^{gh,k} \\
 \rho^g \rho^{hk} & \xrightarrow{\rho_2^{g,hk}} & \rho^{ghk}
 \end{array} & \quad &
 \begin{array}{ccc}
 \rho^g & \xrightarrow{\rho^g \rho_0} & \rho^g \rho^1 \\
 \downarrow \rho_0 \rho^g & \searrow \text{=} & \downarrow \rho_2^{g,1} \\
 \rho^1 \rho^g & \xrightarrow{\rho_2^{1,g}} & \rho^g.
 \end{array}
 \end{array}$$

Observe that if G is a monoidal action of G on \mathcal{C} , the ρ^g 's are in fact tensor autoequivalences of \mathcal{C} , $\rho^{g^{-1}}$ being quasi-inverse to ρ^g for all $g \in G$. \square

Let $\rho: \underline{G} \rightarrow \text{Aut}_{\otimes} \mathcal{C}$ be an action of a group G on a tensor category \mathcal{C} . A G -equivariant object in \mathcal{C} is a pair (X, u) , where X is an object of \mathcal{C} , and u is a family $(u^g)_{g \in G}$, where for each $g \in G$, $u^g: \rho^g X \rightarrow X$ is a morphism, satisfying

$$u^g \rho^g(u^h) = u^{gh} \rho_{2_X}^{g,h} \quad \text{for all } g, h \in G \quad \text{and} \quad u_1 \rho_{0_X} = \text{id}_X. \quad (5.2)$$

Note that the morphisms u^g are then actually isomorphisms.

A G -equivariant morphism $f: (X, u) \rightarrow (Y, v)$ between G -equivariant objects is a morphism $f: X \rightarrow Y$ in \mathcal{C} such that $f u_g = v_g f$ for all $g \in G$.

The *G -equivariantization* of \mathcal{C} , denoted \mathcal{C}^G , is, by definition, the category of G -equivariant objects and G -equivariant morphisms [2, 18, 29, 32]. It is a tensor category, with monoidal product defined as follows: if (X, u) and (Y, v) are G -equivariant objects, then

$$(X, u) \otimes (Y, v) = (X \otimes Y, w), \quad \text{where} \quad w = (w^g = (u^g \otimes v^g) \rho_{2_X, Y}^{g-1})_{g \in G},$$

the unit object being $(1, (\rho_0^{g-1})_{g \in G})$.

Moreover, if \mathcal{C} is a fusion category, G is finite, \mathbb{k} is algebraically closed and $\text{char}(\mathbb{k})$ does not divide the order of G , then \mathcal{C}^G is a fusion category. In that case, it is shown in [29] that \mathcal{C}^G is dual to a crossed product fusion category $\mathcal{C} \rtimes G$ with respect to the indecomposable module category \mathcal{C} .

Example 5.18. Let G be a group and let ρ be the trivial action of G on $\text{vect}_{\mathbb{k}}$. Then $(\text{vect}_{\mathbb{k}})^G = \text{rep } G$. □

Definition 5.19. A tensor functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between tensor categories is an *equivariantization* if there exists a finite group G acting on \mathcal{D} by tensor equivalences, and a tensor equivalence $\mathcal{C} \simeq \mathcal{D}^G$ over \mathcal{D} , that is, such that the following triangle of tensor functors commutes up to a natural isomorphism of tensor functors:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathcal{D}^G \\ & \searrow F & \downarrow \mathcal{U} \\ & & \mathcal{D}, \end{array}$$

where $\mathcal{U}: \mathcal{D}^G \rightarrow \mathcal{D}$ is the forgetful functor. □

5.4 Characterization of equivariantizations in terms of Hopf monads

In this section, we show that group actions on tensor categories and equivariantization can be interpreted in the language of Hopf monads.

Definition 5.20. A normal \mathbb{k} -linear right exact Hopf monad T on a tensor category \mathcal{C} is *cocommutative* if for any morphism $g: T(\mathbf{1}) \rightarrow \mathbf{1}$ and any object X of \mathcal{C} , the following square is commutative:

$$\begin{array}{ccc}
 T(X) & \xrightarrow{T_2(X, \mathbf{1})} & T(X) \otimes T(\mathbf{1}) \\
 \downarrow T_2(\mathbf{1}, X) & & \downarrow \text{id}_{T(X)} \otimes g \\
 T(\mathbf{1}) \otimes T(X) & \xrightarrow{g \otimes \text{id}_{T(X)}} & T(X).
 \end{array}$$

□

Theorem 5.21. Let \mathcal{C} be a tensor category over a field \mathbb{k} , and let ρ be an action of a finite group G on \mathcal{C} by tensor autoequivalences. Then,

- (1) The \mathbb{k} -linear exact endofunctor

$$\mathbb{T}^\rho = \bigoplus_{g \in G} \rho^g$$

admits a canonical structure of Hopf monad on \mathcal{C} ;

- (2) There is a canonical isomorphism of categories:

$$\mathcal{C}^G \simeq \mathcal{C}^{\mathbb{T}^\rho}$$

over \mathcal{C} , where \mathcal{C}^G denotes the equivariantization of \mathcal{C} under G ;

- (3) The Hopf monad \mathbb{T}^ρ is faithful, normal, and cocommutative.
- (4) The induced Hopf algebra of T is \mathbb{k}^G . In particular, \mathbb{T}^ρ is semisimple if and only if $\text{char}(\mathbb{k})$ does not divide the order of G . □

Proof. The endofunctor \mathbb{T}^ρ is \mathbb{k} -linear exact by construction. Define natural transformations

$$\mu: (\mathbb{T}^\rho)^2 = \bigoplus_{g, g'} \rho^g \rho^{g'} \rightarrow \mathbb{T}^\rho = \bigoplus_{h \in G} \rho^h \quad \text{and} \quad \eta: \text{id}_{\mathcal{C}} \rightarrow \mathbb{T}^\rho = \bigoplus_{g \in G} \rho^g$$

componentwise by the collection of morphisms $\rho_2^{g,g'} : \rho^g \rho^{g'} \rightarrow \rho^{gg'}$ and by the morphism $\rho_0 : \text{id}_{\mathcal{C}} \rightarrow \rho^1$, respectively. The axioms of a group action imply that $(\mathbb{T}^\rho, \mu, \eta)$ is a monad on \mathcal{C} . Given an object X of \mathcal{C} , the canonical bijection

$$\prod_{g \in G} \text{Hom}_{\mathcal{C}}(\rho^g X, X) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}\left(\bigoplus_{g \in G} \rho^g X, X\right)$$

restricts to a bijection between families $u = (u^g)$ such that (X, u) is an object of \mathcal{C}^G on one hand, and actions $r : \mathbb{T}^\rho(X) \rightarrow X$ of the monad \mathbb{T}^ρ on X , and this induces an isomorphism of categories $\kappa : \mathcal{C}^G \rightarrow \mathcal{C}^{\mathbb{T}^\rho}$ over \mathcal{C} . In particular, $\mathcal{C}^{\mathbb{T}^\rho}$ is a tensor category over \mathbb{k} , and the forgetful functor $\mathcal{U} : \mathcal{C}^{\mathbb{T}^\rho} \rightarrow \mathcal{C}$ is a tensor functor. This implies that \mathbb{T}^ρ is a Hopf monad on \mathcal{C} . The comonoidal structure of \mathbb{T}^ρ is as follows:

$$\mathbb{T}_2^\rho(X, Y) : \bigoplus_{g \in G} \rho^g(X \otimes Y) \rightarrow \bigoplus_{g', g'' \in G} \rho^{g'} X \otimes \rho^{g''} Y \quad \text{and} \quad \mathbb{T}_0^\rho : \bigoplus_{g \in G} \rho^g \mathbf{1} \rightarrow \mathbf{1}$$

are given componentwise by the strong (co)monoidal structure of the tensor functors ρ^g , that is, by

$$\rho_{2X,Y}^{g-1} : \rho^g(X \otimes Y) \xrightarrow{\sim} \rho^g X \otimes \rho^g Y \quad \text{and} \quad \rho_0^{g-1} : \rho^g \mathbf{1} \xrightarrow{\sim} \mathbf{1}.$$

Hence, Parts (1) and (2). Now \mathbb{T}^ρ is faithful because η is a monomorphism. It is normal because we have $\mathbb{T}^\rho(\mathbf{1}) \xrightarrow{\sim} \mathbf{1}^G$. The cocommutativity of \mathbb{T}^ρ results from the fact that the endofunctor ρ^g being strong monoidal for all $g \in G$, for any object X of \mathcal{C} the diagram of isomorphisms:

$$\begin{array}{ccc} \rho^g X & \xrightarrow{\rho_{2X,1}^{g-1}} & \rho^g X \otimes \rho^g \mathbf{1} \\ \rho_{21,X}^{g-1} \downarrow & & \downarrow \text{id}_{\rho^g X} \otimes \rho_0^{g-1} \\ \rho^g \mathbf{1} \otimes \rho^g X & \xrightarrow{\rho_0^{g-1} \otimes \text{id}_{\rho^g X}} & \rho^g X \end{array}$$

is commutative, hence Part (3).

Let L be the induced Hopf algebra of \mathbb{T}^ρ . As noted previously (see Remark 5.5), L is also the induced Hopf algebra of the restriction of \mathbb{T}^ρ to $\langle \mathbf{1} \rangle$, that is, of $\mathbb{T}^{\rho'}$, where ρ' is the restriction to $\langle \mathbf{1} \rangle$ of the action of G on \mathcal{C} . In order to compute L we may therefore assume that \mathcal{C} is a trivial fusion category (that is, all objects of \mathcal{C} are trivial). In that case, the category of tensor endofunctors of \mathcal{C} is equivalent to the point; we may consequently

assume that ρ is the trivial action. We have a commutative triangle of tensor functors

$$\begin{array}{ccc}
 \mathcal{C}^G & \xrightarrow{\sim} & \text{vect}_{\mathbb{k}}^G = \text{rep } G \\
 & \searrow \text{Hom}_{\mathcal{C}}(1, \mathcal{U}) & \swarrow \\
 & \text{vect}_{\mathbb{k}} &
 \end{array}$$

whose horizontal arrow is an equivalence, hence $L = L(\text{Hom}_{\mathcal{C}}(1, \mathcal{U})) \simeq \mathbb{k}^G$. In particular, \mathbb{T}^ρ is semisimple \iff $\text{rep } G$ is semisimple \iff $\text{char}(\mathbb{k})$ does not divide the order of G . Hence, Part (4). \blacksquare

The Hopf monad \mathbb{T}^ρ is called the *monad of the group action* ρ . We also denote it by \mathbb{T}^G when the action is clear from the context.

Corollary 5.22. An action of a finite group G on a tensor category \mathcal{C} by tensor autoequivalences gives rise to an exact sequence of tensor categories:

$$\text{rep } G \rightarrow \mathcal{C}^G \rightarrow \mathcal{C}.$$

It is an exact sequence of fusion categories if \mathbb{k} is algebraically closed, \mathcal{C} is a fusion category, and $\text{char}(\mathbb{k})$ does not divide the order of G . \square

Proof. Results from Theorem 5.21 and Corollary 5.14. \blacksquare

Remark 5.23. The fusion category \mathcal{C}_p constructed in [29, Subsection 4.1] is group theoretical and admits an action of the group \mathbb{Z}_2 , such that $\mathcal{C}_p^{\mathbb{Z}_2}$ is not group theoretical [29, Corollary 4.6]. The resulting exact sequence $\text{rep } \mathbb{Z}_2 \rightarrow \mathcal{C}_p^{\mathbb{Z}_2} \rightarrow \mathcal{C}_p$ shows that an extension of group-theoretical categories need not be group-theoretical. \square

The converse of Theorem 5.21 is true.

Theorem 5.24. Let T be a \mathbb{k} -linear right exact, faithful, normal, cocommutative Hopf monad on a tensor category \mathcal{C} over a field \mathbb{k} , whose induced Hopf algebra H is split semisimple. Then H is isomorphic to \mathbb{k}^G for some finite group G , and there exists an action of G on \mathcal{C} by tensor autoequivalences $\rho : \underline{G} \rightarrow \underline{\text{End}}_{\otimes} \mathcal{C}$ such that $T \simeq \mathbb{T}^\rho$. \square

Proof. The restriction T_0 of T to $\langle 1 \rangle$ is isomorphic to $L \otimes ?$ as a Hopf monad, where L is the Hopf algebra H^* . Since T is cocommutative, so is L , that is, H is commutative. Being split semisimple, H is of the form $\mathbb{k}G$ for some finite group G .

For $g \in G$, denote by $e^g : T(1) \rightarrow 1$ the morphism corresponding to the map $\mathbb{k}G \rightarrow \mathbb{k}$, $h \mapsto \delta_{g,h}$ via an isomorphism $T_0 \simeq \mathbb{k}G \otimes ?$

The morphisms $e^g : T(1) \rightarrow 1$ satisfy the following equations:

$$(1) \quad (e^g \otimes e^h)T_2(1, 1) = \delta_{g,h}e^g,$$

$$(2) \quad \sum_g e^g = T_0,$$

$$(3) \quad e^g \mu_1 = \sum_{g'g''=g} e^{g'} T(e^{g''}),$$

$$(4) \quad e^g \eta_1 = \delta_{g,1} \text{id}_1,$$

which reflect the Hopf algebra structure of $\mathbb{k}G$.

For $g \in G$, define a natural endomorphism π^g of T by setting

$$\pi_X^g = (e^g \otimes \text{id}_{T(X)})T_2(1, X), \quad \text{for } X \text{ object of } \mathcal{C}.$$

Note that we also have $\pi_X^g = (\text{id}_{T(X)} \otimes e^g)T_2(X, 1)$, T being cocommutative.

Using Equations (1) and (2) above and the comonoidality of T , one verifies easily the following equations:

$$(5) \quad \pi_X^g \pi_X^h = \delta_{g,h} \pi_X^g, \quad (6) \quad \sum_g \pi_X^g = \text{id}_{T(X)},$$

$$(7) \quad T_2(X, Y) \pi_{X \otimes Y}^g = (\pi_X^g \otimes \text{id}_{T(Y)})T_2(X, Y) = (\text{id}_{T(X)} \otimes \pi_Y^g)T_2(X, Y),$$

the last equation resulting from cocommutativity of T .

By Equations (5) and (6) above, for X object of \mathcal{C} , the family $(\pi_X^g)_{g \in G}$ is a complete orthogonal system of idempotents of $T(X)$. Denote by ρ_X^g the image of the idempotent π_X^g . This defines an endofunctor ρ^g of \mathcal{C} , and we have

$$T = \bigoplus_{g \in G} \rho^g.$$

The point is now to show that $\rho : g \mapsto \rho^g$ is an action of G on \mathcal{C} by tensor autoequivalences, so that $T = \mathbb{T}^\rho$. From Equations (3) and (4) above, we deduce

$$(8) \quad \pi_X^g = \sum_{g'g''=g} \mu_X \pi_{T(X)}^{g'} T(\pi_X^{g''}), \quad (9) \quad \pi_X^g \eta_X = \delta_{g,1} \eta_X.$$

Indeed, we have

$$\begin{aligned}\pi_X^g \mu_X &= (e^g \otimes \text{id}_{T(X)}) T_2(1, X) \mu_X = (e^g \mu_1 \otimes \mu_X) T_2(T(1), T(X)) T(T_2(1, X)) \\ &= \sum_{g'g''=g} (e^{g'} T(e^{g''}) \otimes \mu_X) T_2(T(1), T(X)) T(T_2(1, X)) = \sum_{g'g''=g} \mu_X \pi_{T(X)}^{g'} T(\pi_X^{g''}); \\ \pi_X^g \eta_X &= (e^g \otimes \text{id}_{T(X)}) T_2(1, X) \eta_X = e^g \eta_1 \otimes \eta_X = \delta_{g,1} \eta_X.\end{aligned}$$

Thus $\mu : \bigoplus_{g,h} \rho^g \circ \rho^h \rightarrow \bigoplus_k \rho^k$ is given componentwise by morphisms

$$\rho_2^{g,h} : \rho^g \circ \rho^h \rightarrow \rho^{gh},$$

and $\eta : \text{id}_{\mathcal{C}} \rightarrow \bigoplus_g \rho^g$, by a morphism $\rho_0 : \text{id}_{\mathcal{C}} \rightarrow \rho^1$. Since T is a monad, we have

$$\rho_2^{g,hk} \rho^g \rho_2^{h,k} = \rho_2^{gh,k} \rho_2^{g,h} \rho^k \quad \text{and} \quad \rho^{g,1} \rho^g \rho_0 = \text{id}_{\rho^g} = \rho^{1,g} \rho_0 \rho^g.$$

Using Equations (5) and (7) above, we also have

$$(\pi_X^g \otimes \pi_Y^h) T_2(X, Y) = \delta_{g,h} T_2(X, Y) \pi_{X \otimes Y}^g.$$

Thus $T_2(X, Y) : \bigoplus_g \rho^g(X \otimes Y) \rightarrow \bigoplus_{h,k} \rho^h(X) \otimes \rho^k(Y)$ is given componentwise by morphisms

$$f_2^g(X, Y) : \rho^g(X \otimes Y) \rightarrow \rho^g(X) \otimes \rho^g(Y).$$

Finally, $T_0 : \bigoplus_g \rho^g(1) \rightarrow 1$ is given by morphisms $f_0^g : \rho^g(1) \rightarrow 1$. The fact that T is a bimonad implies that (ρ^g, f_2^g, f_0^g) is a comonoidal endofunctor of \mathcal{C} for all $g \in G$, and that the natural transformations $\rho_2^{g,h} : \rho^g \rho^h \rightarrow \rho^{gh}$ and $\rho_0 : \text{id}_{\mathcal{C}} \rightarrow \rho^1$ are comonoidal.

Next we show that the structure morphisms $\rho_2^{g,h}$, ρ_0 , f_2^g , f_0^g are isomorphisms. The left fusion operator H^l of the bimonad T , introduced in [9] and defined as

$$H_{X,Y}^l = (\text{id}_{T(X)} \otimes \mu_Y) T_2(X, T(Y)) : T(X \otimes T(Y)) \rightarrow T(X) \otimes T(Y),$$

is an isomorphism by [9, Theorem 3.10], because T is a Hopf monad. We have $T(X \otimes T(Y)) = \bigoplus_{g,h \in G} \rho^g(X \otimes \rho^h Y)$ and $T(X) \otimes T(Y) = \bigoplus_{m,n \in G} \rho^m X \otimes \rho^n Y$, and $H_{X,Y}^l$ is defined

componentwise by isomorphisms

$$\omega_{X,Y}^{g,h} = (\text{id}_{\rho^g X} \otimes \rho_2^{gh}) f_2^g(X, \rho^h(Y)) : \rho^g(X \otimes \rho^h(Y)) \xrightarrow{\sim} \rho^g X \otimes \rho^{gh} Y$$

Now for $g \in G$, f_0^g is an isomorphism because $T_0 = \sum_g e_g$. So

$$\rho_2^{gh} Y = (f_0^g \otimes \text{id}_{\rho_Y^{gh}}) \omega_{1,Y}^{g,h}$$

is an isomorphism too. Now we check that $\rho_0 : \text{id}_{\mathcal{C}} \rightarrow \rho^1$ is an isomorphism. Observe first that ρ_0 is a monomorphism because, T being faithful, η is a monomorphism, and η factors through ρ_0 . In particular, ρ^1 is faithful; it is also \mathbb{k} -linear right exact since it is a direct summand of T . We have $\rho_2^{11} \rho^1(\rho_0) = \text{id}_{\rho^1}$, so $\rho^1(\rho_0) = \rho_2^{11-1}$ is an isomorphism. Since ρ^1 is faithful right exact, ρ_0 is an epimorphism. The category \mathcal{C} being abelian, ρ_0 is an isomorphism.

In particular, ρ^g is a \mathbb{k} -linear autoequivalence of \mathcal{C} , with quasi-inverse $\rho^{g^{-1}}$.

Now $f_2^g(X, Y)$ is an isomorphism, because $Y \simeq \rho^1 Y$, $\omega^{g,1}$ and f_2^g are isomorphisms. Thus (ρ^g, f_2^g, f_0) is a strong comonoidal functor, that is, $(\rho^g, (f_2^g)^{-1}, (f_0^g)^{-1})$ is a strong monoidal functor. It is therefore a tensor autoequivalence of \mathcal{C} . We have shown that ρ is an action of G on \mathcal{C} by tensor autoequivalences, and $T = \mathbb{T}^\rho$. ■

Corollary 5.25. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories over a field \mathbb{k} admitting a left adjoint. Then F is an equivariantization if and only if F is dominant normal, its Hopf monad T is cocommutative, and its induced Hopf algebra is split semisimple. □

Proof. The functor F is monadic, with monad T . We may therefore assume $\mathcal{C} = \mathcal{D}^T$, F being the forgetful functor \mathcal{U} . Let L be the induced Hopf algebra of T . We conclude by Theorem 5.24, noting that if L is commutative, then it is the function algebra of a finite group if and only if it is split semisimple. ■

Example 5.26. Let Γ be a finite group and let L be a finite-dimensional Hopf algebra over an algebraically closed field \mathbb{k} endowed with a Γ -graduation $L = \bigoplus_{g \in \Gamma} L_g$ such that $L_g \neq 0$ for all $g \in \Gamma$. Let $H = L^*$. The Γ -graduation on L translates into an injective Hopf algebra morphism $i : \mathbb{k}^\Gamma \rightarrow H$ whose image is central in H . This morphism is characterized by

$$\langle i(\varphi), \lambda \rangle = \sum_{g \in \Gamma} \varphi(g) \varepsilon(\lambda_g)$$

for all $\phi \in \mathbb{k}^\Gamma$, $\lambda \in L$, ε denoting the counit of L . We have $H = \bigoplus_{g \in \Gamma} H_g$, where $H_g = L_g^*$. The dominant tensor functor $\text{Res}_{\mathbb{k}^\Gamma}^H : H\text{-mod} \rightarrow \mathbb{k}^\Gamma\text{-mod}$ is monadic, with Hopf monad $\mathcal{T} = \text{Res}_{\mathbb{k}^\Gamma}^H \text{Ind}_{\mathbb{k}^\Gamma}^H$. The tensor category $\mathbb{k}^\Gamma\text{-mod}$ is the pointed category $\mathcal{C}(\Gamma)$ of Γ -graded vector spaces, whose simple objects are indexed by the elements of Γ , and we have $\mathcal{T}(g) = H_g \otimes g$; in particular, \mathcal{T} is normal, with induced Hopf algebra H_1 . In particular, if L is cosemisimple, H is semisimple and we have an exact sequence of fusion categories

$$H_1\text{-mod} \rightarrow H\text{-mod} \rightarrow \mathcal{C}(\Gamma).$$

Moreover, \mathcal{T} is cocommutative if and only if L_1 is contained in the center of L . Now let Γ act on a finite group G in a nontrivial way by $\Gamma \times G \rightarrow G$, $(x, g) \mapsto x.g$, and let L be the abelian extension $L = \mathbb{k}^G \# \mathbb{k}\Gamma$ corresponding with this action. Then the multiplication and comultiplication in L are given by

$$(e_g \# x)(e_h \# y) = \delta_{g, h} e_g \# xy, \quad \Delta(e_g \# x) = \sum_{st=g} e_s \# x \otimes e_t \# x = \Delta(e_g) \Delta(x),$$

where $e_g \in \mathbb{k}^G$ are defined as $e_g(h) = \delta_{g, h}$, $g, h \in G$. Thus $L_1 = \mathbb{k}^G$ is a Hopf subalgebra which is not central in L and \mathbb{k}^Γ is a central Hopf subalgebra in $H = L^*$.

Then L is a cosemisimple Γ -graded Hopf algebra, and $L_1 = \mathbb{k}^G$ is commutative, but not central in L , hence for $H = L^*$ there is an exact sequence of fusion categories

$$\text{rep } G \rightarrow H\text{-mod} \rightarrow \mathcal{C}(\Gamma)$$

which is not induced by an action of the group G on $\mathcal{C}(\Gamma)$ by tensor autoequivalences. \square

5.5 The braided case: modularization revisited

Definition 5.27. A bimonad T on a braided category \mathcal{C} is *braided* if the following diagram is commutative:

$$\begin{array}{ccc} T(X \otimes Y) & \xrightarrow{T_2(X, Y)} & T(X) \otimes T(Y) \\ \downarrow T(c_{X, Y}) & & \downarrow c_{T(X), T(Y)} \\ T(Y \otimes X) & \xrightarrow{T_2(Y, X)} & T(Y) \otimes T(X) \end{array}$$

for any objects X, Y of \mathcal{C} , where c denotes the braiding of \mathcal{C} . \square

Remark 5.28. This is equivalent to saying that $R_{X,Y} = (\eta_Y \otimes \eta_X)c_{X,Y}$ defines a R -matrix for T . \square

Proposition 5.29. Let T be a bimonad on a braided category \mathcal{C} , with forgetful functor $\mathcal{U} : \mathcal{C}^T \rightarrow \mathcal{C}$. There exists a braiding on \mathcal{C}^T such that \mathcal{U} is braided if and only if T is braided. \square

Proof. Since \mathcal{U} is faithful, a braiding \tilde{c} on \mathcal{C}^T such that \mathcal{U} is braided is necessarily given by

$$\tilde{c}_{(M,r),(N,s)} = c_{M,N} \quad \text{for any } T\text{-modules } (M, r), (N, s).$$

Now the morphism \tilde{c} so defined is T -linear if and only if T is braided; and if such is the case, it is a braiding on \mathcal{C}^T . \blacksquare

Proposition 5.30. A braided \mathbb{k} -linear right exact normal Hopf monad on a braided tensor category \mathcal{C} over a field \mathbb{k} is cocommutative. \square

Proof. Denoting by c the braiding of \mathcal{C} , we have $T_2(X, 1) = c_{T(1), T(X)}T_2(1, X)$ for any object X of \mathcal{C} , hence $(g \otimes \text{id}_{T(X)})T_2(1, X) = (\text{id}_{T(X)} \otimes g)T_2(X, 1)$ for any $g : T(1) \rightarrow 1$ by functoriality of c . \blacksquare

Corollary 5.31. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a braided tensor functor between braided tensor categories over an algebraically closed field \mathbb{k} admitting a left adjoint, and assume that F is normal and dominant. Assume that $\text{char } \mathbb{k}$ does not divide $\text{FPInd } F$. Then F is an equivariantization. More precisely, there exists a finite group G acting on \mathcal{D} by braided tensor autoequivalences and a braided tensor equivalence $\mathcal{C} \rightarrow \mathcal{D}^G$ over \mathcal{D} , that is, such that the following triangle commutes up to a natural isomorphism of tensor functors:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathcal{D}^G \\ & \searrow_F & \downarrow \mathcal{U} \\ & & \mathcal{D}, \end{array}$$

where $\mathcal{U} : \mathcal{D}^G \rightarrow \mathcal{D}$ is the forgetful functor. \square

Proof. The tensor functor F being monadic, with Hopf monad T , we have an equivalence of tensor categories $\mathcal{C} \rightarrow \mathcal{D}^T$ over \mathcal{C} . The Hopf monad T is \mathbb{k} -linear right exact, faithful, normal, and by Proposition 5.30, it is cocommutative. By Corollary 5.25, T is the Hopf monad of an action of a finite group G on \mathcal{D} by tensor autoequivalences and we have $\mathcal{C} \simeq \mathcal{D}^G$. Moreover, T is braided, which means that G acts by braided autoequivalences. \blacksquare

Example 5.32. Let $1 \rightarrow G'' \xrightarrow{i} G \xrightarrow{\pi} G' \rightarrow 1$ be an exact sequence of finite groups, with its associated exact sequence of tensor categories

$$\mathrm{rep} G' \xrightarrow{\pi^*} \mathrm{rep} G \xrightarrow{i^*} \mathrm{rep} G''.$$

The tensor functor $i_* : \mathrm{rep} G \rightarrow \mathrm{rep} G''$ is symmetric, so its monad T is cocommutative and i^* is an equivariantization functor. In fact, T is the monad on $\mathrm{rep} G''$ introduced in Example 5.10, and it is the monad of the action of G' on $\mathrm{rep} G''$ by conjugation. The tensor equivalence $\mathrm{rep} G \simeq (\mathrm{rep} G'')^{G'}$ is a special case of the one established in [27] for cocentral extensions of finite-dimensional Hopf algebras. \square

Example 5.33. Let \mathcal{C} be a premodular category, that is, a ribbon fusion category. In particular, \mathcal{C} is braided. See Section 7.1. Recall that an object X of \mathcal{C} is called *transparent* if and only if $c_{Y,X}c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all objects Y of \mathcal{C} , where c denotes the braiding in \mathcal{C} . Let $\mathcal{T} \subseteq \mathcal{C}$ be the category of transparent objects of \mathcal{C} . Assume \mathcal{C} is modularizable, and let $F : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ be its modularization (see [6]). The modularization functor is dominant and normal, and we have $\mathrm{Ker}_F = \mathcal{T}$ (see [6, Propositions 2.3 and 3.2]), hence an exact sequence of fusion categories

$$\mathcal{T} \longrightarrow \mathcal{C} \xrightarrow{F} \tilde{\mathcal{C}}.$$

Moreover, F is a braided functor; it is therefore an equivariantization by Corollary 5.31. In fact, \mathcal{T} is a tannakian category, so that we have a symmetric tensor equivalence $\mathcal{T} \simeq \mathrm{rep} G$, G being a finite group, G acts on $\tilde{\mathcal{C}}$ and $\mathcal{C} = \tilde{\mathcal{C}}^G$. Modularization is therefore a special case of the de-equivariantization procedure described in [17, Subsection 2.6]. \square

Proposition 5.34. Let \mathcal{D} be a modular category over an algebraically closed field \mathbb{k} of characteristic 0, with twist θ . The following data are equivalent:

- (A) A premodular category \mathcal{C} and a modularization functor $\mathcal{C} \rightarrow \mathcal{D}$;

(B) A \mathbb{k} -linear, faithful, normal braided semisimple Hopf monad T on \mathcal{D} preserving the twist, that is, such that $\theta_T = T(\theta)$. \square

Proof. Let T be a \mathbb{k} -linear, faithful, normal braided semisimple Hopf monad on \mathcal{D} , and let $\mathcal{U} : \mathcal{D}^T \rightarrow \mathcal{D}$ be the forgetful functor. Then \mathcal{D}^T is a braided fusion category, and \mathcal{U} is a dominant braided tensor functor. One verifies easily that the condition that T preserves the twist is equivalent to saying that there exists a twist $\tilde{\theta}$ on \mathcal{D}^T which is preserved by \mathcal{U} . So if T preserves the twist, \mathcal{D}^T is premodular and \mathcal{U} is a modularization. Conversely, let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a modularization. Its monad T is a \mathbb{k} -linear, faithful, normal braided semisimple Hopf monad, and since F preserves twist, T preserves the twist of \mathcal{D} . \blacksquare

6 Exact Sequences and Commutative Central Algebras

The modularization $F : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ of a modularizable premodular category \mathcal{C} is constructed in [6] as the free module functor $\mathcal{C} \rightarrow \text{mod}_{\mathcal{C}}(A)$, A being a commutative algebra in the braided category \mathcal{C} . More precisely, A is a trivializing algebra of the full subcategory $\mathcal{T} \subset \mathcal{C}$ of transparent objects of \mathcal{C} .

In this section, we show that, more generally, any dominant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between tensor categories admitting an exact right adjoint is, up to tensor equivalence, a free module functor $\mathcal{C} \rightarrow \text{mod}_{\mathcal{C}}(A)$, A being a certain commutative algebra in the center of \mathcal{C} called the induced central algebra of F . Such a functor is normal if and only if A is self-trivializing.

6.1 Induced central algebra of a tensor functor

If A is an algebra in a tensor category \mathcal{C} over \mathbb{k} , with product $m : A \otimes A \rightarrow A$ and unit $u : 1 \rightarrow A$, we denote by $\text{mod}_{\mathcal{C}} A$ the abelian \mathbb{k} -linear category of right A -modules in \mathcal{C} . The forgetful functor $V_A : \text{mod}_{\mathcal{C}} A \rightarrow \mathcal{C}$ is \mathbb{k} -linear exact, and has a left adjoint, namely the *free A -module functor* $F_A : \mathcal{C} \rightarrow \text{mod}_{\mathcal{C}} A$ defined by $X \mapsto (X \otimes A, \text{id}_X \otimes m)$.

We say that A is *semisimple* in \mathcal{C} if every right A -module in \mathcal{C} is a direct factor of $F(X)$ for some object X of \mathcal{C} . Note that if A is semisimple and \mathcal{C} is semisimple, then $\text{mod}_{\mathcal{C}} A$ is semisimple too.

A *central algebra* of \mathcal{C} is an algebra A in \mathcal{C} endowed with a half-braiding $\sigma : A \otimes \text{id}_{\mathcal{C}} \xrightarrow{\sim} \text{id}_{\mathcal{C}} \otimes A$ such that the pair (A, σ) is an algebra in the categorical center $\mathcal{Z}(\mathcal{C})$ of \mathcal{C} . This means that the product m and unit u of A are morphisms of half-braidings,

that is

$$\sigma_X(m \otimes \text{id}_X) = (\text{id}_X \otimes m)(\sigma_X \otimes \text{id}_A)(\text{id}_A \otimes \sigma_X) \quad \text{and} \quad \sigma_X(u \otimes \text{id}_X) = \text{id}_X \otimes u.$$

A central algebra (A, σ) is *commutative* if $m\sigma_A = m$.

Now let (A, σ) be a commutative central algebra of \mathcal{C} . We define a tensor product \otimes_A on $\text{mod}_{\mathcal{C}} A$, as follows. Given two right A -modules M and N , with A -actions $r: M \otimes A \rightarrow M$ and $s: N \otimes A \rightarrow N$, the tensor product $M \otimes_A N$ is the coequalizer of the pair of morphisms

$$(r \otimes \text{id}_N, \text{id}_M \otimes s\sigma_N): M \otimes A \otimes N \rightrightarrows M \otimes N.$$

It is a right A -module, with action $t: M \otimes_A N \otimes A \rightarrow M \otimes_A N$ defined by $t(\pi \otimes A) = \pi(\text{id}_M \otimes s)$, where π is the canonical epimorphism $M \otimes N \rightarrow M \otimes_A N$.

One verifies that \otimes_A defines a monoidal structure on $\text{mod}_{\mathcal{C}} A$, with unit object $F(1) = A$. We denote by $\text{mod}_{\mathcal{C}}(A, \sigma)$ this monoidal category. The functor F_A admits a natural structure of strong monoidal functor from \mathcal{C} to $\text{mod}_{\mathcal{C}}(A, \sigma)$, which we denote by $F_{A, \sigma}$.

A tensor functor admitting a right adjoint defines a central coalgebra.

Proposition 6.1. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories over a field \mathbb{k} , admitting a right adjoint R . Then $A = R(1)$ has a natural structure of commutative central algebra in \mathcal{C} , with half-braiding denoted by σ . Moreover, if R is faithful exact, then $\text{mod}_{\mathcal{C}}(A, \sigma)$ is a tensor category and we have a tensor equivalence $K: \mathcal{D} \rightarrow \text{mod}_{\mathcal{C}}(A, \sigma)$ such that the following triangle of tensor functors commutes up to tensor isomorphism:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow F_{A, \sigma} & \downarrow K \\ & & \text{mod}_{\mathcal{C}}(A, \sigma) \end{array}$$

□

Definition 6.2. The commutative central algebra (A, σ) associated with a tensor functor F admitting a right adjoint is called the *induced central algebra of F* . □

Proof. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a strong monoidal functor between rigid categories, and let $R: \mathcal{D} \rightarrow \mathcal{C}$ be a right adjoint of F . Note that R is unique up to unique isomorphism. Then the adjunction $F \vdash R$ is monoidal, which means that R has a natural structure of monoidal functor such that the adjunction morphisms are monoidal. Considering the opposite

monoidal categories \mathcal{C}^{op} and \mathcal{D}^{op} (with opposite composition and tensor products), we have a comonoidal adjunction $R^{\text{op}} \vdash F^{\text{op}}$, which is in fact a Hopf adjunction because \mathcal{C}^{op} and \mathcal{D}^{op} are rigid. The induced coalgebra of this Hopf adjunction is, by [9, Theorem 6.5], a cocommutative central coalgebra in \mathcal{C}^{op} , that is, a commutative central algebra (A, σ) in \mathcal{C} .

As an algebra, $A = R(1)$ with product $R_2(1, 1)$ and unit R_0 , where (R_2, R_0) denotes the monoidal structure of R . The half-braiding $\sigma : A \otimes \text{id}_{\mathcal{C}} \xrightarrow{\sim} \text{id}_{\mathcal{C}} \otimes A$ is defined by the following commutative diagram:

$$\begin{array}{ccccc}
 R1 \otimes X & \xrightarrow{\sigma_X} & X \otimes R1 & & \\
 \downarrow \text{id}_{R1} \otimes \eta_X & \searrow \sim & \downarrow \eta_X \otimes \text{id}_{R1} & & \\
 R1 \otimes RFX & \xrightarrow{R_2(1, F X)} & RFX & \xleftarrow{R_2(F X, 1)} & RFX \otimes R1,
 \end{array}$$

where X is an object of \mathcal{C} , $\eta_X : X \rightarrow RFX$ is the adjunction unit, and the slanted arrows are the Hopf isomorphisms of the Hopf adjunction, see [9].

Now assume that R is faithful exact. In particular, R and R^{op} are conservative. By [9, Theorem 6.6], \mathcal{D} is monoidally equivalent to $\text{mod}_{\mathcal{C}}(A, \sigma)$ via the \mathbb{k} -linear functor $K : \mathcal{D} \rightarrow \text{mod}_{\mathcal{C}}(A, \sigma)$, $Y \mapsto (R(Y), R_2(Y, 1))$, and $F_{A, \sigma} \simeq KF$ as tensor functors. ■

Example 6.3. Let H be a finite-dimensional Hopf algebra over a field \mathbb{k} . The forgetful functor $U : \text{comod-}H \rightarrow \text{vect}_{\mathbb{k}}$ admits a right adjoint $R = ? \otimes H$. The induced central algebra (A, σ) is a commutative algebra in $\mathcal{Z}(H\text{-mod})$, that is, a commutative algebra in $D(H)\text{-mod}$. As an algebra in $\text{comod-}H$, $A = H$ with right coaction Δ . The half-braiding σ is defined, in Sweedler's notation, by

$$\sigma_{(V, \partial)} : \begin{cases} A \otimes V & \rightarrow V \otimes A \\ h \otimes x & \mapsto x_{(0)} \otimes S(x_{(1)}) h x_{(2)} \end{cases}$$

for any right H -comodule (V, ∂) . We have $\text{mod}_{\text{comod-}H}(A, \sigma) \simeq \text{vect}_{\mathbb{k}}$ as tensor categories. □

Example 6.4. Let $f : H \rightarrow H'$ be a surjective morphism between finite-dimensional Hopf algebras over a field \mathbb{k} , and denote by $F = f_* : \text{comod-}H \rightarrow \text{comod-}H'$ the dominant tensor functor defined by f . It has a right adjoint $R = ? \square^{H'} H$, which is exact because H is H' -coflat. The induced central algebra (B, σ') of F is a commutative algebra in $\mathcal{Z}(\text{comod-}H)$. We have $B = R(1) = \mathbb{k} \square^{H'} H = H^{\text{co}H'} \subset H$, where H is seen as a commutative central algebra of $\text{comod-}H$ (see Example 6.3). According to Proposition 6.1, we have a tensor

equivalence:

$$\mathrm{mod}_{\mathrm{comod}-H}(B, \sigma') \simeq \mathrm{comod}-H'.$$

See also [30, Theorem II]. □

Commutative central algebras define tensor functors.

Proposition 6.5. Let \mathcal{C} be a tensor category over a field \mathbb{k} , and let (A, σ) be a commutative central algebra of \mathcal{C} such that $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) = \mathbb{k}$. Then:

- (1) If $\mathrm{mod}_{\mathcal{C}}(A, \sigma)$ is rigid, it is a tensor category and the free module functor $F_{A, \sigma} : \mathcal{C} \rightarrow \mathrm{mod}_{\mathcal{C}}(A, \sigma)$ is a dominant tensor functor, whose induced central algebra is (A, σ) .
- (2) If A is semisimple as an algebra in \mathcal{C} , then $\mathrm{mod}_{\mathcal{C}}(A, \sigma)$ is rigid.
- (3) If \mathcal{C} is a fusion category, A is semisimple and \mathbb{k} is algebraically closed, then $\mathrm{mod}_{\mathcal{C}}(A, \sigma)$ is a fusion category. □

Proof. Since (A, σ) is a commutative algebra in $\mathcal{Z}(\mathcal{C})$, $\mathrm{mod}_{\mathcal{C}} A$ admits a monoidal structure denoted by $\mathrm{mod}_{\mathcal{C}}(A, \sigma)$, with tensor product \otimes_A and unit object $F(\mathbf{1}) = A$, hence a strong monoidal functor $F_{A, \sigma} : \mathcal{C} \rightarrow \mathrm{mod}_{\mathcal{C}}(A, \sigma)$. Set $\mathcal{D} = \mathrm{mod}_{\mathcal{C}}(A, \sigma)$.

The category \mathcal{D} is abelian \mathbb{k} -linear, it has finite-dimensional Homs, and objects have finite length in \mathcal{D} . Moreover, its tensor product \otimes_A is \mathbb{k} -bilinear, and $\mathrm{End}_{\mathcal{D}}(\mathbf{1}) = \mathrm{Hom}_{\mathrm{mod}_{\mathcal{C}} A}(A, A) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \simeq \mathbb{k}$ by assumption. If \mathcal{D} is rigid, it is a tensor category and $F_{A, \sigma}$ is a dominant tensor functor. Its right adjoint R is the forgetful functor. Its induced central algebra is $R(\mathbf{1}) = A$, with the half-braiding defined in Proposition 6.1, which is in fact σ , hence Part (1).

Since $F_{A, \sigma}$ is strong monoidal, all objects of the form $F_{A, \sigma}(X)$ have a left and a right dual. If A is semisimple, then any object of \mathcal{D} is a direct factor of $F_{A, \sigma}(X)$ for some X in \mathcal{C} , so it also has a left and a right dual, so \mathcal{D} is rigid, hence Part (2).

If \mathbb{k} is algebraically closed, \mathcal{C} is a fusion category, and A is semisimple, then \mathcal{D} is semisimple and finite because $F_{A, \sigma}$ is dominant, so it is a fusion category, hence Part (3). ■

Corollary 6.6. Let \mathcal{C} be a fusion category over an algebraically closed field \mathbb{k} .

Then the following data are equivalent:

- (A) A commutative central algebra (A, σ) of \mathcal{C} such that A is a semisimple algebra in \mathcal{C} and $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) = \mathbb{k}$;
- (B) A dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$, where \mathcal{D} is a fusion category over \mathbb{k} . □

Proof. According to Proposition 6.5, a commutative central algebra (A, σ) as in (A) gives rise to a dominant tensor functor $F_{A, \sigma} : \mathcal{C} \rightarrow \text{mod}_{\mathcal{C}}(A, \sigma)$, and $\text{mod}_{\mathcal{C}}(A, \sigma)$ is a fusion category over \mathbb{k} . Conversely, let \mathcal{D} be a fusion category and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor. Then F admits a right adjoint R , which is exact because \mathcal{D} is semisimple, and faithful because F is dominant. Thus we may apply Proposition 6.1. Let (A, σ) be the induced central algebra of F . Then F is equivalent to $F_{A, \sigma}$, in the sense that there exists a tensor equivalence $K : \mathcal{D} \rightarrow \text{mod}_{\mathcal{C}}(A, \sigma)$ such that $F_{A, \sigma} \simeq KF$ as tensor functors. ■

6.2 Normal functors and trivializing algebras

We have seen that dominant tensor functors between fusion categories are classified by their induced central algebras. We now characterize similarly normal tensor functors between fusion categories.

Let A be an algebra in a tensor category \mathcal{C} . We say that A *trivializes an object* X of \mathcal{C} if $F_{A, \sigma}(X) \simeq F_{A, \sigma}(1)^n$ for some natural integer n , and A is *self-trivializing* if it trivializes its underlying object.

The notion of trivializing algebra plays a central role in Deligne's internal construction of symmetric fiber functors [12] and supersymmetric fiber functors [13].

Proposition 6.7. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an exact tensor functor between tensor categories admitting an exact right adjoint, and let (A, σ) be its induced central algebra (in \mathcal{C}). Then F is normal if and only if the algebra A is self-trivializing. If such is the case, $\text{Ker}_F = \langle A \rangle \subset \mathcal{C}$ so that we have an exact sequence of tensor categories:

$$\langle A \rangle \longrightarrow \mathcal{C} \longrightarrow \mathcal{D}.$$

□

Proof. Denote by R a left adjoint of F , so that A is isomorphic to $R(1)$. According to Proposition 3.5, F is normal $\iff F(R(1))$ is trivial $\iff F_A(A)$ is trivial by Proposition 6.1. Now assume A is self-trivializing. Since Ker_F is an abelian subcategory of \mathcal{C} containing A and stable under subobjects and quotients, it contains $\langle A \rangle$ by definition. Conversely, if X is in $\text{Ker}_F = \text{Ker}_{F_{A, \sigma}}$, then $X \otimes A \simeq A^n$ and $1 \hookrightarrow A$ so $X \hookrightarrow A^n$, hence X is in $\langle A \rangle$. ■

Corollary 6.8. An exact sequence of tensor categories $\mathcal{C}' \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}''$ such that F has a faithful right adjoint is equivalent to the exact sequence of tensor categories

$$\langle A \rangle \longrightarrow \mathcal{C} \xrightarrow{F_{A, \sigma}} \text{MOD}_{\mathcal{C}}(A, \sigma),$$

where (A, σ) denotes the induced central algebra of F . □

Proposition 6.9. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between fusion categories, and let (A, σ) be its induced central algebra. Then any simple object of \mathcal{Ker}_F is a direct factor of A . The functor F is normal if and only if A is self-trivializing, and if such is the case, then we have

$$A \simeq \bigoplus_{X \in \Lambda_{\mathcal{Ker}_F}} X^{\text{FPdim } X},$$

and in particular $\text{FPdim}(\mathcal{Ker}_F) = \text{FPdim}(A)$. □

Proof. The functor F can be decomposed as $F = iF_0$, where $i : \mathcal{C} \subset \mathcal{D}$ is the inclusion of the dominant image of \mathcal{E} , that is, the full subcategory of \mathcal{D} whose objects are direct summands of elements of the image of F , and $F_0 : \mathcal{C} \rightarrow \mathcal{E}$ is the dominant tensor functor induced by F . Then F is normal if and only if F_0 is normal, and (A, σ) is the induced central algebra of F_0 , so by Proposition 6.7, F is normal if and only if A is self-trivializing.

For $X \in \Lambda_{\mathcal{C}}$ we have by adjunction $\text{Hom}_{\mathcal{D}}(F(X), 1) = \text{Hom}_{\mathcal{C}}(X, A)$. In particular, if F trivializes X , then X is a direct factor of A . Assume that F is normal. Then F trivializes A , so for any simple direct factor X of A , we have $F(X) \simeq 1^n$, with $n = \text{FPdim } X$. Also, $n = \dim \text{Hom}(F(X), 1) = \dim \text{Hom}(X, A) = m_{X, A}$. Hence, $A = \sum_X X^{\text{FPdim } X}$, where X ranges over the set of classes of simple factors of A , that is, $\Lambda_{\mathcal{Ker}_F}$. ■

Corollary 6.10. Let \mathcal{C} be a finite tensor category. The following are equivalent:

- (A) Fiber functors for \mathcal{C} ;
- (B) Commutative central algebras (A, σ) of \mathcal{C} such that A trivializes all objects of \mathcal{C} and satisfies $\text{Hom}_{\mathcal{C}}(1, A) = \mathbb{k}$. □

Proof. A fiber functor for \mathcal{C} is just a tensor functor $\omega : \mathcal{C} \rightarrow \text{vect}_{\mathbb{k}}$. Such a tensor functor is automatically dominant and normal; it admits a right adjoint R , \mathcal{C} being finite; and R is faithful (because ω is dominant) and exact, $\text{vect}_{\mathbb{k}}$ being semisimple. Thus, by Proposition 6.1, ω is classified by its induced central algebra (A, σ) . Conversely, let (A, σ) be a commutative central algebra which trivializes all objects of \mathcal{C} and satisfies $\text{Hom}(1, A) = \mathbb{k}$. Then $\mathcal{D} = \text{MOD}_{\mathcal{C}}(A, \sigma)$ is tensor equivalent to $\text{vect}_{\mathbb{k}}$, and

$$\omega : \begin{cases} \mathcal{C} & \rightarrow \text{vect}_{\mathbb{k}}, \\ X & \mapsto \text{Hom}_{\mathcal{D}}(1, F_{A, \sigma}(X)) \simeq \text{Hom}_{\mathcal{C}}(1, X \otimes A) \end{cases}$$

is a fiber functor for \mathcal{C} . ■

Remark 6.11. In particular, let H be a finite-dimensional Hopf algebra, with product m , unit u , coproduct Δ , and counit ε . The forgetful functor $\mathcal{U} : \text{comod-}(H) \rightarrow \text{vect}_{\mathbb{k}}$ is a fiber functor for $\text{comod-}(H)$. The induced central algebra (A, σ) of \mathcal{U} is H_H , seen as a central algebra of $\text{comod-}H$ (see Example 6.3), and we have $\mathcal{U} \simeq \text{Hom}_{\text{comod-}H}(\mathbf{1}, ? \otimes H_H)$ as fiber functors. \square

6.3 Equivariantizations in terms of central commutative algebras

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant functor between tensor categories admitting an exact right adjoint, and let (A, σ) be its induced central algebra, so that $\mathcal{D} \simeq \text{mod}_{\mathcal{C}}(A, \sigma)$. Denote by $G = \text{Aut}(A, \sigma)$ the group of automorphisms of the central algebra (A, σ) (that is, algebra automorphisms compatible with the half-braiding σ). The group G acts on the category $\text{mod}_{\mathcal{C}}(A, \sigma)$ by tensor autoequivalences (see [17, Proposition 2.10]), setting:

$$\rho^g : \begin{cases} \text{mod}_{\mathcal{C}}(A, \sigma) & \rightarrow \text{mod}_{\mathcal{C}}(A, \sigma) \\ (X, r) & \mapsto (X, r(\text{id}_X \otimes a_g^{-1})). \end{cases}$$

Moreover, for each $g \in G$, let $u^g = \text{id}_X \otimes a(g) : \rho^g F_A(X) \xrightarrow{\sim} F_A(X)$. This defines a tensor functor over \mathcal{D} :

$$F_A^G : \begin{cases} \mathcal{C} & \rightarrow \text{MOD}_{\mathcal{C}}(A, \sigma)^G \\ X & \mapsto (F_A(X), (u^g)_{g \in G}). \end{cases}$$

Hence, via the tensor equivalence $\mathcal{D} \simeq \text{mod}_{\mathcal{C}}(A, \sigma)$, we get an action of $\text{Aut}(A, \sigma)$ on the tensor category \mathcal{D} and a tensor functor $F^G : \mathcal{C} \rightarrow \mathcal{D}^G$ such that the following triangle of tensor functors commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F^G} & \mathcal{D}^G \\ & \searrow F & \swarrow \mathcal{U}^G \\ & \mathcal{D} & \end{array}$$

Lemma 6.12. Assume that F is normal. Denote by H its induced Hopf algebra, and by $G(H)$ the group of group-like elements of H . Then $\text{Aut}(A, \sigma)$ is isomorphic to a subgroup of $G(H)$, and in particular its order divides $\dim_{\mathbb{k}} H = \dim_{\mathbb{k}} \text{End}_{\mathcal{C}}(A)$. \square

Proof. Recall that F is monadic and, denoting by T its monad, we have a canonical tensor equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{D}^T$ which sends (A, σ) to the left dual of the induced central coalgebra $(\hat{C}, \hat{\sigma})$ of T . In particular, $\text{Aut}(A, \sigma)$ is isomorphic to the group of automorphisms of $(\hat{C}, \hat{\sigma})$, which is a subgroup of the group $\text{Aut}(\hat{C})$ of automorphisms of the coalgebra \hat{C} . Now, as a coalgebra of \mathcal{D} , $\hat{C} = (T(1), T_2(1, 1), T_0)$, with T -action μ_1 . Since F is normal, we have $T(1) \simeq H \otimes 1$, and, via the canonical tensor equivalence $\langle 1 \rangle \simeq \text{vect}_{\mathbb{k}}$, \hat{C} is just the coalgebra H with H acting by left multiplication, whose group of automorphisms is $G(H)$. Thus, $\text{Aut}(A, \sigma) \subset G(H)$, and since H is free as a $\mathbb{k}G(H)$ module, then the order of $G(H)$ divides $\dim_{\mathbb{k}} H$. We also have by adjunction $H \simeq \text{Hom}_{\mathcal{D}}(1, T(1)) \simeq \text{End}_{\mathcal{D}^T}(\hat{C}) \simeq \text{End}_{\mathcal{C}}(A)$, hence the lemma is proved. ■

Definition 6.13. Let X be a trivial object of a tensor category \mathcal{C} . There is a unique half-braiding $\tau : X \otimes ? \rightarrow ? \otimes X$ such that (X, τ) is a trivial object in the center $\mathcal{Z}(\mathcal{C})$ of \mathcal{C} . This half-braiding is called the *trivial half-braiding for X* . □

Lemma 6.14. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between tensor categories admitting a right adjoint, let (A, σ) be its induced central algebra and T its monad. Denote by τ the trivial half-braiding of the trivial object $F(A)$ of \mathcal{D} . Then the following assertions are equivalent:

- (i) The normal Hopf monad T is cocommutative;
- (ii) For any X object of \mathcal{C} , the following triangle is commutative:

$$\begin{array}{ccc}
 & T_2(1, X) & \\
 T(X) & \xrightarrow{\quad} & T(1) \otimes T(X) \\
 \downarrow T_2(X, 1) & & \searrow \hat{\tau}_{T(X)} \\
 T(X) \otimes T(1) & &
 \end{array}$$

- (iii) For any object X of \mathcal{C} , $F(\sigma_X) = \tau_{F(X)}$. □

Proof. Since $T(1)$ is trivial, (ii) is equivalent to:

$$\text{for all } g : T(1) \rightarrow 1, \quad (\text{id}_{T(X)} \otimes g) T_2(X, 1) = (\text{id}_{T(X)} \otimes g) \hat{\tau}_{T(X)} T_2(1, X),$$

and $(\text{id}_{T(X)} \otimes g)\hat{\tau}_{T(X)}T_2(\mathbf{1}, X) = (g \otimes \text{id}_{T(X)})T_2(\mathbf{1}, X)$ by functoriality, $\hat{\tau}$ being a trivial half-braiding, hence (i) \iff (ii).

Now let us prove (ii) \iff (iii). Since F is monadic, we may assume that $\mathcal{C} = \mathcal{D}^T$, F being the forgetful functor $\mathcal{U} : \mathcal{D}^T \rightarrow \mathcal{D}$. Let $(\hat{C}, \hat{\sigma})$ be the induced central coalgebra of T . Recall that $(\hat{C}, \hat{\sigma})$ is dual to (A, σ) . Thus, denoting by \hat{T} the trivial half-braiding of $\mathcal{U}\hat{C} = T(\mathbf{1})$, (iii) is equivalent to (iii)': $F_{\hat{\sigma}} = \hat{\tau}_F$. Since T is a Hopf monad, for every T -module (M, r) we have isomorphisms

$$\mathbb{H}_{1,(M,r)}^l = (\text{id}_{T(\mathbf{1})} \otimes r)T_2(\mathbf{1}, M) : T(M) \xrightarrow{\sim} T(\mathbf{1}) \otimes M,$$

$$\mathbb{H}_{(M,r),1}^r = (r \otimes \text{id}_{T(\mathbf{1})})T_2(M, \mathbf{1}) : T(M) \xrightarrow{\sim} M \otimes T(\mathbf{1}),$$

and by definition $\hat{\sigma}_{M,r} = \mathbb{H}_{(M,r),1}^r \mathbb{H}_{1,(M,r)}^{l-1}$.

From the definition of $\hat{\sigma}$ and the axioms of a bimonad, we deduce that for X in \mathcal{D} the following diagram is commutative:

$$\begin{array}{ccc} & T_2(\mathbf{1}, T(X)) & \nearrow \\ T(X) & \xrightarrow{T(\eta_X)} & T^2(X) \\ & \searrow & \downarrow \mathbb{H}_{(TX,\mu_X),1}^r \\ & T_2(T(X), \mathbf{1}) & \nearrow \end{array} \quad \begin{array}{c} T(\mathbf{1}) \otimes T(X) \\ \downarrow \hat{\sigma}_{TX,\mu_X} \\ T(X) \otimes T(\mathbf{1}), \end{array}$$

from which we deduce that (iii)' \Rightarrow (ii). Conversely, if (ii) holds, the following diagram commutes:

$$\begin{array}{ccccc} & T(\mathbf{1}) \otimes T(M) & \xrightarrow{\text{id}_{T(\mathbf{1})} \otimes r} & T(\mathbf{1}) \otimes M & \\ T(M) & \begin{array}{c} \nearrow T_2(\mathbf{1}, M) \\ \searrow T_2(M, \mathbf{1}) \end{array} & \downarrow \hat{\tau}_{T(M)} & & \downarrow \hat{\tau}_M \\ & T(M) \otimes T(\mathbf{1}) & \xrightarrow[r \otimes \text{id}_{T(\mathbf{1})}]{} & M \otimes T(\mathbf{1}), & \end{array}$$

so $\hat{\tau}_M = \mathbb{H}_{(M,r),1}^r \mathbb{H}_{1,(M,r)}^{l-1} = \hat{\sigma}_{(M,r)}$, hence (iii)' holds. Hence, the proposition is proved. \blacksquare

Proposition 6.15. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between tensor categories admitting an exact right adjoint, and let (A, σ) be its induced central algebra. Then F is tensor equivalent to an equivariantization if and only if the following conditions are met:

- (1) The tensor functor F is normal, that is, $F(A)$ is trivial;
- (2) For every object X of \mathcal{C} , $F(\sigma_X) = \tau_{F(X)}$, where τ is the trivial half-braiding of $F(A)$;
- (3) The induced Hopf algebra H of F is split semisimple.

If such is the case, $\text{Aut}(A, \sigma)$ is isomorphic to $G(H)$ and $\mathcal{C} \simeq \mathcal{D}^{\text{Aut}(A, \sigma)}$ over \mathcal{D} . □

Proof. Let T be the monad of F . Assume that F is equivalent to an equivariantization, that is, there exists an action ρ of a finite group Γ on \mathcal{D} by tensor autoequivalences, and a tensor equivalence $\mathcal{D}^\Gamma \xrightarrow{\sim} \mathcal{C}$ over \mathcal{D} . We may assume that $T = T^\Gamma$, F being the forgetful functor \mathcal{U}^Γ . Then, by Theorem 5.21, T is normal, cocommutative, and its induced Hopf algebra is $\mathbb{k}\Gamma$. The cocommutativity of T implies Condition (ii) in Lemma 6.14. The induced Hopf algebra of \mathcal{U}^Γ is $\mathbb{k}\Gamma$, and we have $\Gamma = G(H)$. Let (A, σ) be the induced central algebra of T^Γ . According to Lemma 6.12, the group of automorphisms of the algebra A is Γ , and its order is the dimension of $\text{End}_{\mathcal{C}}(A)$. We show that the group of automorphisms of (A, σ) is Γ . Observe first that if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a dominant tensor functor between tensor categories admitting an exact right adjoint R , and $\phi : \mathcal{D} \rightarrow \mathcal{D}'$ is a tensor equivalence, then the induced central algebra of ϕF is canonically isomorphic to that of F , and we may therefore identify them. Secondly, by construction of the equivariantization, we have for each $\gamma \in \Gamma$ a canonical isomorphism $\rho^\gamma \mathcal{U}^\Gamma \simeq \mathcal{U}^\Gamma$. This induces an isomorphism between the induced central algebra of \mathcal{U}^Γ and that of $\rho^\gamma \mathcal{U}^\Gamma$, that is, an automorphism of (A, σ) . Thus, we have $\text{Aut}(A, \sigma) = \text{Aut}(A) \simeq G(H)$ and its order is the dimension of $\text{End}_{\mathcal{C}}(A)$.

Conversely, assume Conditions (i), (ii), and (iii) are satisfied. Condition (ii) implies that T is cocommutative by Lemma 6.14 and, by Theorem 5.24, T is the monad of a group action on \mathcal{C} . ■

6.4 Normal fusion subcategories

Recall that every normal dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{C}''$ between tensor categories gives an exact sequence of tensor categories $\mathfrak{Ker}_F \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}''$. This motivates the following definition.

Definition 6.16. Let \mathcal{C} be a fusion category. A full fusion subcategory \mathcal{C}' of \mathcal{C} is called *normal* if there exists a fusion category \mathcal{C}'' and a normal dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{C}''$ such that the inclusion $\mathcal{C}' \subseteq \mathcal{C}$ induces an equivalence $\mathcal{C}' \simeq \text{Ker}_F$. \square

The next proposition characterizes normal subcategories in term of trivializing algebras.

Proposition 6.17. Let \mathcal{C} be a fusion category over an algebraically closed field \mathbb{k} , and let $\mathcal{C}' \subset \mathcal{C}$ be a full fusion subcategory. Then, \mathcal{C}' is a normal subcategory of \mathcal{C} if and only if there exists commutative central algebra (A, σ) of \mathcal{C} satisfying the following conditions:

- (1) A is a semisimple algebra in \mathcal{C} ;
- (2) $\text{Hom}(1, A) \simeq \mathbb{k}$;
- (3) A belongs to \mathcal{C}' and trivializes all objects of \mathcal{C}' .

In that case, $\text{mod}_{\mathcal{C}} A$ is a fusion category over \mathbb{k} and we have an exact sequence of fusion categories

$$\mathcal{C}' \rightarrow \mathcal{C} \xrightarrow{F} \text{mod}_{\mathcal{C}}(A, \sigma).$$

 \square

Proof. This results immediately from Corollary 6.6 and Proposition 6.7. \blacksquare

6.5 Simple fusion categories

We define a simple fusion category in terms of exact sequences, as follows.

Definition 6.18. A fusion category \mathcal{C} is *simple* if \mathcal{C} is not tensor equivalent to $\text{vect}_{\mathbb{k}}$ and for every exact sequence of fusion categories

$$\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}'',$$

either \mathcal{C}' or \mathcal{C}'' is tensor equivalent to $\text{vect}_{\mathbb{k}}$. \square

Remark 6.19. A fusion category \mathcal{C} is simple if and only if $\mathcal{C} \not\simeq \text{vect}_{\mathbb{k}}$ and for any normal dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we have $\mathcal{D} \simeq \text{vect}_{\mathbb{k}}$ or F is an equivalence. This is because such a functor F fits in an exact sequence $\text{Ker}_F \rightarrow \mathcal{C} \rightarrow \mathcal{D}$. \square

Note that a different notion of a simple fusion category was introduced in [17, Definition 9.10]: a fusion category is simple in the sense of [17] if it has no proper fusion subcategories, that is, $\text{vect}_{\mathbb{k}}$ and \mathcal{C} are the only replete fusion subcategories of \mathcal{C} . The next proposition compares this definition and Definition 6.18.

Proposition 6.20. If a fusion category \mathcal{C} is simple in the sense of [17], it is also simple in the sense of Definition 6.18. \square

Proof. Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be an exact sequence of fusion categories, with $\mathcal{C} \not\simeq \text{vect}_{\mathbb{k}}$. If \mathcal{C} has no proper fusion subcategories, then either $\mathcal{C}' \simeq \text{vect}_{\mathbb{k}}$ or $f: \mathcal{C}' \rightarrow \mathcal{C}$ is an equivalence. One concludes with Lemma 3.8. \blacksquare

The converse of Proposition 6.20 is false. Indeed, we deduce directly from the results of Section 4.1.

Proposition 6.21. Let \mathcal{C} be a pointed fusion category, with Picard group G . Let $\alpha \in H^3(G, \mathbb{k}^\times)$ be the cohomology class defining \mathcal{C} , so that $\mathcal{C} \simeq \mathcal{C}(G, \alpha)$. Then

- (1) The category \mathcal{C} is simple in the sense of [17] if and only if G is a cyclic group of prime order;
- (2) The category \mathcal{C} is simple if and only if there is no proper distinguished subgroup $H \triangleleft G$ such that the restriction of α to H is trivial.

In particular, if G is simple, \mathcal{C} is simple, but it is not simple in the sense of [17] except if $G \simeq \mathbb{Z}_p$, p prime. \square

However, both notions coincide when restricted to categories of representations of finite groups, as follows from the next proposition.

Proposition 6.22. Let G be a finite group such that \mathbb{k} is a splitting field for G and $\text{char}(k)$ does not divide the order of G . Then the following assertions are equivalent:

- (i) The group G is simple.
- (ii) The fusion category $\mathcal{C}(G, 1)$ is simple.
- (iii) The fusion category $\text{rep } G$ is simple.
- (iv) The fusion category $\text{rep } G$ is simple in the sense of [17]. \square

Proof. The equivalence between (i) and (ii) is a special case of Proposition 6.21. We have (iii) \Rightarrow (i) by Corollary 3.14. Now if G is a simple group, then $\text{rep } G$ has no proper fusion subcategories, hence (i) \Rightarrow (iv), and finally (iv) \Rightarrow (iii) by Proposition 6.20, hence the proposition is proved. \blacksquare

7 Braided Fusion Categories of Odd Square-free Dimension

7.1 Braided categories of odd Frobenius–Perron dimension

Let \mathcal{C} be a fusion category over a field \mathbb{k} . Recall that the *Frobenius–Perron dimension* of \mathcal{C} is $\text{FPdim } \mathcal{C} := \sum_{X \in \Lambda_{\mathcal{C}}} (\text{FPdim } X)^2$.

On the other hand, the *global dimension* of \mathcal{C} is defined as $\dim \mathcal{C} := \sum_{X \in \Lambda_{\mathcal{C}}} |X|^2$, where $|X|^2 \in \mathbb{k}^\times$ denotes the squared norm of the simple object X , see [16, Definition 2.2]. When $\text{FPdim } \mathcal{C} = \dim \mathcal{C}$, \mathcal{C} is called *pseudo-unitary*.

If \mathcal{C} is a pivotal fusion category, one defines the the categorical left dimension $\dim^l X$ and right dimension $\dim^r X$ of an object X of \mathcal{C} . For X simple, one has $|X|^2 = \dim^l X \dim^r X$, so that $\dim \mathcal{C} = \sum_{X \in \Lambda_{\mathcal{C}}} \dim^l X \dim^r X$. The category \mathcal{C} is *spherical* if left and right dimensions coincide; in that case, they are denoted by \dim , and we have $\dim_{\mathcal{C}} = \sum_{X \in \Lambda_{\mathcal{C}}} (\dim X)^2$.

A *premodular category* is a braided spherical fusion category. Equivalently, it is a ribbon fusion category, that is, a fusion category equipped with a braiding and a twist (also called a balanced structure) [6].

Assume $\mathbb{k} = \mathbb{C}$. Then \mathcal{C} is *pseudo-unitary* if $\text{FPdim } \mathcal{C} = \dim \mathcal{C}$. If such is the case, then by [16, Proposition 8.23], \mathcal{C} admits a unique spherical structure with respect to which the categorical dimensions of simple objects are all positive. We call it the *canonical spherical structure*. For this structure, the categorical dimension of an object coincides with its Frobenius–Perron dimension.

If \mathcal{C} is a fusion category on \mathbb{C} such that $\text{FPdim } \mathcal{C}$ is an integer, then \mathcal{C} is pseudo-unitary by [16, Proposition 8.24]. Moreover, $\text{FPdim } \mathcal{C}'$ is an integer for any full fusion subcategory $\mathcal{C}' \subseteq \mathcal{C}$, because $\text{FPdim } \mathcal{C}' = \sum_{X \in \Lambda_{\mathcal{C}'}} (\text{FPdim } X)^2$ and for each X , $\text{FPdim } X$ is the square root of a natural integer by [16, Proposition 8.27]. In particular, every full fusion subcategory of \mathcal{C} is pseudo-unitary.

Lemma 7.1. Let \mathcal{C} be a symmetric fusion category over a field \mathbb{k} whose Frobenius–Perron dimension is an odd natural integer. Then any twist on \mathcal{C} is trivial. \square

Proof. Since \mathcal{C} is symmetric, a twist θ on \mathcal{C} is a monoidal automorphism of $\text{id}_{\mathcal{C}}$. Since $\theta^2 = \text{id}_{\mathcal{C}}$, θ defines a $\{\pm 1\}$ -graduation on \mathcal{C} , with $\mathcal{C}_1 \subset \mathcal{C}$ being the full tensor subcategory

of objects X such that $\theta_X = 1$. If $\mathcal{C}_1 \neq \mathcal{C}$, we have $\text{FPdim } \mathcal{C} = 2 \text{FPdim } \mathcal{C}_1$, which contradicts the fact that $\text{FPdim } \mathcal{C}$ is odd, hence $\mathcal{C}_1 = \mathcal{C}$ and $\theta = 1$. \blacksquare

Lemma 7.2. Let \mathcal{C} be a braided fusion category over \mathbb{C} such that $\text{FPdim } \mathcal{C}$ is an odd natural integer. Then \mathcal{C} , endowed with its canonical spherical structure, is modularizable. \square

Proof. The category \mathcal{C} is pseudo-unitary. Equipped with its canonical spherical structure, it is a premodular category and we have $\dim X = \text{FPdim } X \geq 0$ for any object X of \mathcal{C} . Let $\mathcal{T} \subset \mathcal{C}$ be the full tensor subcategory of transparent objects of \mathcal{C} ; it is a symmetric fusion category [6, Section 2]. By [1, Theorem 7.2], the categorical dimensions of the objects of \mathcal{T} are integers. Since $\text{FPdim } \mathcal{C}$ is a natural integer, so is $\text{FPdim } \mathcal{T}$. By [16, Proposition 8.15], $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{T}$ is an algebraic integer, so $\text{FPdim } \mathcal{T}$ divides $\text{FPdim } \mathcal{C}$. Thus $\text{FPdim } \mathcal{T}$ is an odd natural integer as well. By Lemma 7.1, we have $\theta_X = \text{id}_X$ for all X in \mathcal{T} . By [6, Théorème 3.1], \mathcal{C} is modularizable, hence the lemma is proved. \blacksquare

7.2 Proof of Theorem 1.1

In order to prove the theorem, we may assume that the ground field is \mathbb{C} . Indeed, if \mathcal{C} is a fusion category over a field \mathbb{k} of characteristic 0, then \mathcal{C} is defined over the algebraic closure $\bar{\mathbb{Q}} \subset \mathbb{k}$ of \mathbb{Q} , which we may embed into \mathbb{C} . We deal first with the modular case.

Lemma 7.3. Let \mathcal{C} be a modular category whose Frobenius–Perron dimension is a square-free odd integer N . Then there exists an abelian group G of order N such that \mathcal{C} is equivalent to the category of G -graded vector spaces $\mathcal{C}(G, 1)$. \square

Proof. By [17, Theorem 2.11 (ii)], for $X \in \Lambda_{\mathcal{C}}$ we have $\text{FPdim } X = 1$, so X is invertible, hence \mathcal{C} is a pointed category. Thus $\mathcal{C} = \mathcal{C}(G, \alpha)$ for some finite group G and some cohomology class $\alpha \in H^3(G, \mathbb{k}^\times)$.

Now braided and ribbon structures on a pointed fusion category $\mathcal{C} = \mathcal{C}(G, \alpha)$ are classified in [19, 7.5] in terms of group cohomology. (See also [14, 2.4].) The existence of a braiding implies that G is abelian. Moreover, by [19, Proposition 7.5.3 iii)], given a twist θ on \mathcal{C} , the class α is trivial if and only if θ is equal to 1 on the subgroup ${}_2G := \{g \in G : g^2 = 1\}$.

In the present case, we conclude that G is an abelian group of odd order N , so ${}_2G = 1$ and therefore α is trivial, hence the lemma holds. \blacksquare

Lemma 7.4. A braided fusion category \mathcal{C} whose Frobenius–Perron dimension is an odd square-free integer N admits a fiber functor. \square

Proof. By Lemma 7.2 the category \mathcal{C} , endowed with its canonical spherical structure, is modularizable. In particular, the full subcategory $\mathcal{T} \subset \mathcal{C}$ of transparent objects of \mathcal{C} is tannakian; we have $\mathcal{T} \simeq \text{rep } G$ as symmetric tensor categories, G being a finite group, and we have an exact sequence

$$\mathcal{T} \longrightarrow \mathcal{C} \longrightarrow \tilde{\mathcal{C}} ,$$

where $\tilde{\mathcal{C}}$ is a modular category, G acts on $\tilde{\mathcal{C}}$ by braided tensor autoequivalences, and $\mathcal{C} \simeq \tilde{\mathcal{C}}^G$ as braided tensor categories, see Example 5.33.

By Proposition 4.10, $\text{FPdim } \tilde{\mathcal{C}} = N / \text{FPdim } \mathcal{T}$ and $\text{FPdim } \mathcal{T}$ is a natural integer, so $\text{FPdim } \tilde{\mathcal{C}}$ is an odd square-free integer. By Proposition 7.3, $\tilde{\mathcal{C}}$ admits a fiber functor, and so does \mathcal{C} . \blacksquare

So, \mathcal{C} is tensor equivalent to $H\text{-mod}$, where H is a quasitriangular Hopf algebra whose dimension is odd and square-free. By [26, Theorem 1.2], such a Hopf algebra is isomorphic to a group algebra, hence the theorem is proved.

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